REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000-000

ROBUST ESTIMATION OF COMPONENT RELIABILITY BASED ON SYSTEM LIFETIME DATA WITH KNOWN SIGNATURE

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Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- This paper considers the estimation of component reliability based on system lifetime data with known system signature using the minimum density divergence estimation method. Different estimation procedures based on the minimum density divergence estimation method are proposed. Standard error estimation and interval estimation procedures are also studied. Then, a Monte Carlo simulation study is used to evaluate the performance of those proposed procedures and compare those procedures with the maximum likelihood estimation method under different contaminated models. A numerical example is presented to illustrate the effectiveness of the proposed minimum density divergence estimation method. We have shown that the proposed estimation procedures are robust to contamination and model misspecification. Finally, concluding remarks with some possible future research directions are provided.

Key-Words:

- *Censoring; maximum likelihood estimation; minimum density divergence; Monte Carlo simulation; Weibull distribution.*

AMS Subject Classification:

- 62N02, 62N05, 62F35.

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1. INTRODUCTION

System lifetime data are commonly encountered in industrial and engineering settings. In system reliability studies, engineers are always interested in the lifetime distribution of the system as well as the lifetime distribution of the components which make up the system. We consider here the situation that the lifetimes of an n-component system can be observed but not the lifetime of the components. This situation occurs when putting the individual component on a life testing experiment after the n-component system is built is not possible, or when the distribution of the component lifetimes changes while they are used in a specified system. Suppose the lifetimes of the n components in an n-component system are independent and identically distributed (i.i.d.) random variables, denoted as $X_1, X_2, \ldots, X_n$, with probability density function (p.d.f.) $f_X(t; \boldsymbol{\theta})$, cumulative distribution function (c.d.f.) $F_X(t; \boldsymbol{\theta})$ and survival function (s.f.) $\bar{F}_X(t; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the parameter vector. We further denote the ordered component lifetimes within an n-component system as $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ with $X_{i:n}$ be the i-th ordered component lifetime. Although the i.i.d. assumption is restrictive, there are many practical situations in which the i.i.d. assumption is applicable. For instance, Bhattacharya and Samaniego [3] discussed some of the practical examples that the i.i.d. assumption is reasonable such as batteries in a lighting device, wafers in a digital computer, and spark plugs in an automobile, and Jin et al. [13] discussed that the performance of "Redundant Array of Independent Disks (RAID)" computer hardware with n independent disks can be designed to perform like a k-out-of-n system.

When the component lifetime follows an absolutely continuous distribution, the failure time of an n-component system corresponds to the failure time of one of the n components. We consider the coherent system in which every component is relevant and the system has a monotone structure function [7]. In a coherent system consists of n i.i.d. components, the system structure can be described by the system signature defined as an n-element probability vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$, where the i-th element is the probability that the i-th ordered component failure causes the failure of the system [24], i.e.,

$$s_i = \Pr(T = X_{i:n}), i = 1, 2, \ldots, n.$$

Note that the system signature is only depending on the system structure and hence, it is distribution-free. To illustrate the idea of system signature, we consider the 4-component series-parallel III system with system lifetime $T = \min\{X_1, \max\{X_2, X_3, X_4\}\}$ (Figure 1a). For the 4-component series-parallel III system, there are $4! = 24$ possible arrangements of the component lifetimes, The 24 arrangements and their corresponding system lifetimes are presented in

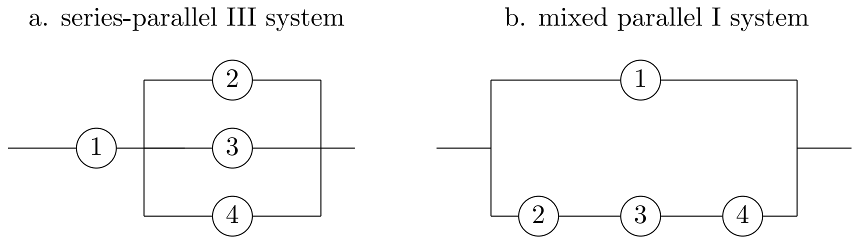

Figure 1: Two 4-component systems for illustration.

Table 1. From Table 1, we can obtain

$$\begin{aligned} s_1 &= \Pr(T = X_{1:4}) = 6/24 = 1/4, \\ s_2 &= \Pr(T = X_{2:4}) = 6/24 = 1/4, \\ s_3 &= \Pr(T = X_{3:4}) = 12/24 = 1/2, \\ \text{and } s_4 &= \Pr(T = X_{4:4}) = 0. \end{aligned}$$

Hence, the system signature of the 4-component series-parallel III system is $\boldsymbol{s} = (1/4, 1/4, 1/2, 0)$. Similarly, for the 4-component mixed parallel I system (Figure 1b), the system signature is $\boldsymbol{s} = (0, 1/2, 1/4, 1/4)$.

Arrangement	System lifetime T	Arrangement	System lifetime T
$X_1 < X_2 < X_3 < X_4$	$X_{1:4}$	$X_3 < X_1 < X_4 < X_2$	$X_{2:4}$
$X_1 < X_2 < X_4 < X_4$	$X_{1:4}$	$X_3 < X_4 < X_1 < X_2$	$X_{3:4}$
$X_1 < X_4 < X_2 < X_3$	$X_{1:4}$	$X_3 < X_1 < X_2 < X_4$	$X_{2:4}$
$X_1 < X_4 < X_3 < X_2$	$X_{1:4}$	$X_3 < X_4 < X_1 < X_2$	$X_{3:4}$
$X_1 < X_3 < X_2 < X_4$	$X_{1:4}$	$X_3 < X_2 < X_1 < X_4$	$X_{3:4}$
$X_1 < X_3 < X_4 < X_2$	$X_{1:4}$	$X_3 < X_2 < X_4 < X_1$	$X_{3:4}$
$X_2 < X_1 < X_3 < X_4$	$X_{2:4}$	$X_4 < X_1 < X_2 < X_3$	$X_{2:4}$
$X_2 < X_1 < X_4 < X_3$	$X_{2:4}$	$X_4 < X_1 < X_3 < X_2$	$X_{2:4}$
$X_2 < X_3 < X_1 < X_4$	$X_{3:4}$	$X_4 < X_2 < X_1 < X_3$	$X_{3:4}$
$X_2 < X_3 < X_4 < X_1$	$X_{3:4}$	$X_4 < X_2 < X_1 < X_1$	$X_{3:4}$
$X_2 < X_4 < X_1 < X_3$	$X_{3:4}$	$X_4 < X_3 < X_1 < X_2$	$X_{3:4}$
$X_2 < X_3 < X_3 < X_1$	$X_{3:4}$	$X_4 < X_3 < X_2 < X_1$	$X_{3:4}$

Table 1: The 24 possible arrangements of the component lifetime in a 4-component series-parallel III system

Given the system signature $\boldsymbol{s}$, the p.d.f. and s.f. of the system lifetime T of an n-component system can be expressed as

$$f_T(t; \boldsymbol{\theta}) = \sum_{i=1}^{n} s_i \binom{n}{i} i f_X(t; \boldsymbol{\theta}) \left[F_X(t; \boldsymbol{\theta})\right]^{i-1} \left[\bar{F}_X(t; \boldsymbol{\theta})\right]^{n-i} \tag{1.1}$$

$$\text{and } \bar{F}_T(t; \boldsymbol{\theta}) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \left[F_X(t; \boldsymbol{\theta})\right]^{j} \left[\bar{F}_X(t; \boldsymbol{\theta})\right]^{n-j}, \tag{1.2}$$

respectively [15]. Based on system lifetimes with known system signatures, the statistical inference of the component lifetime distribution have been discussed in the literature. Balakrishnan et al. [1] developed an exact nonparametric inference for population quantiles and tolerance limits of component lifetime distribution in a system. Balakrishnan et al. [2] derived the best linear unbiased estimator (BLUE) for the parameters in the component lifetime distribution. Navarro et al. [21] discussed the method of moments, the maximum likelihood method and the least squares methods for system lifetime data under a proportional hazard rate model. Chahkandi et al. [8] proposed several nonparametric methods to construct prediction intervals for the lifetime of coherent systems. Zhang et al. [28] proposed a regression-based estimation method for the model parameters of component lifetime distribution based on censored system failure data. Yang et al. [26] proposed a stochastic expectation-maximization (EM) algorithm to obtain an approximation of the maximum likelihood estimates (MLEs) of the parameters in component lifetime distribution. Recently, Yang et al. [27] and Hermanns et al. [12] considered the EM algorithm to obtain the MLEs of the parameters in component lifetime distribution based on system lifetime data when the system structure is unknown. The theory and applications of system signatures are an active research area. For a comprehensive review and bibliometric analysis on system signatures, one can refer to a recent paper by Naqvi et al. [19].

In industrial experiments on systems, there are many situations in which systems are removed from experimentation before the occurrence of the failure of the system. Two common reasons for pre-planned censoring are saving the time on tests and reducing the cost associated with the experiment because failure implies the destruction of a system which may be costly [9, 18]. In this paper, we consider Type-II right censoring scheme in which the number of observed failures is pre-specified as r and the experiment is terminated as soon as the r-th ordered system failure is observed. Several studies on the Type-II censored system lifetime data in a system with system signature have been conducted [2, 12, 21, 28, 26, 27].

In the manufacturing industry, defectives could be induced in the manufacturing process due to different reasons such as human error, insufficient quality control, and failure in addressing reliability aspects during the design stage, etc.. As Raina [22] pointed out, zero-defect is an impossible goal to achieve or cost-prohibitive in the manufacturing process. Manufacturing defects often lead to potential outliers or contamination of the lifetime data. When there is outliers exist in observed lifetime data, the performance of the maximum likelihood or other classical estimation methods may be affected and a poor estimate of the component reliability characteristics may be yielded. Note that the maximum likelihood estimation is sensitive to the outliers as each observation contributes equal information to the estimate. Therefore, it is desired to develop parameter estimation procedures that are less sensitive to contaminated observations. Basu et al. [4] developed a family of density-based divergences measures with a single parameter α that controls the trade-off between robustness and efficiency, and proposed a procedure for estimating model parameters based on minimizing the density divergence. Basu et al. [5] further extended the minimum density divergence procedure to censored survival data with and without contamination, and found that the minimum density divergence estimator (MDE) is superior to the MLE when there is contamination in the censored survival data. Recently, Riani et al. [23] developed an alternative minimum density power divergence estimation procedure using the methods of S-estimation. Basak et al. [6] proposed a procedure to determinate the optimal density power divergence tuning parameter.

In this paper, we study the robust minimum density divergence estimation method for the system lifetime with and without contamination. In Section 2, we introduced the minimum density power divergence and its application to system lifetime data with known system signatures. We also discuss the estimation of the standard error of the estimate and interval estimation, and we show that the bootstrap method for standard error estimation can be adopted for the MDEs. In Section 3, a numerical example is used to illustrate the proposed MDEs. A Monte Carlo simulation study is presented in Section 4 to study the performance of the proposed methodologies. Finally, some concluding remarks and possible extensions are provided in Section 5.

2. MINIMUM DENSITY DIVERGENCE ESTIMATOR FOR SYSTEM LIFETIME DATA

2.1. Minimum density divergence estimator

The density power divergence, proposed by Basu et al. [4], describes a family of density-based divergence measures between two p.d.f.s $g(t)$ and $f(t)$ with a single parameter α. Consider that $f(t;\boldsymbol{\theta})$ is a parametric p.d.f. of the fitted model with parameter vector $\boldsymbol{\theta}$ and $g(t)$ is the target p.d.f., the density power divergence between $f(t;\boldsymbol{\theta})$ and $g(t)$ is defined as

$$d_\alpha(g,f) = \int \left[f^{1+\alpha}(t;\boldsymbol{\theta}) - \left(1+\frac{1}{\alpha}\right) g(t) f^{\alpha}(t;\boldsymbol{\theta}) + \frac{1}{\alpha} g^{1+\alpha}(t) \right] dt, \quad \alpha > 0 \tag{2.1}$$

and

$$d_0(g,f) = \lim_{\alpha\to 0} d_\alpha(g,f) = \int g(t) \ln\left[\frac{g(t)}{f(t;\boldsymbol{\theta})}\right] dt. \tag{2.2}$$

$d_\alpha(g,f) = 0$ when $f(t;\boldsymbol{\theta}) = g(t)$. The MDE of the parameter vector $\boldsymbol{\theta}$ can be obtained by minimizing the density power divergence between $f(t;\boldsymbol{\theta})$ and $g(t)$ with respect to (w.r.t.) $\boldsymbol{\theta}$. Since the term $\int \left[(1/\alpha) g^{1+\alpha}(t)\right] dt$ in Eq. (2.1) does not depend on the parameter vector $\boldsymbol{\theta}$, the minimum divergence estimator of $\boldsymbol{\theta}$ can be obtained by minimizing

$$\int \left[f^{1+\alpha}(t;\boldsymbol{\theta}) - \left(1+\frac{1}{\alpha}\right) g(t) f^{\alpha}(t;\boldsymbol{\theta}) \right] dt \tag{2.3}$$

w.r.t. $\boldsymbol{\theta}$.

The density power divergence reduces to the Kullack-Leibler divergence [16] when $\alpha = 0$, and is the mean squared error when $\alpha = 1$. Hence, the minimum density power divergence procedure is degenerated into the maximum likelihood method when $\alpha = 0$, and becomes the minimization of the mean squared error when $\alpha = 1$. The parameter α in Eq. (2.1) controls the trade-off between robustness and efficiency of the minimum divergence estimator [4, 5]. It has been shown that the typical value of α is in between 0 and 1 and the estimation procedure becomes less efficient as α increases [4]. Hence, in this paper, we consider the value of α in $(0, 1)$.

Basu et al. [5] proposed a method for using the empirical c.d.f. $\hat{G}_n$ to estimate the target distribution G to obtain

$$\begin{aligned} &\int \left[f^{1+\alpha}(t;\boldsymbol{\theta}) - (1+1/\alpha)g(t)f^{\alpha}(t;\boldsymbol{\theta})\right] dt \\ =& \int f^{1+\alpha}(t;\boldsymbol{\theta})dt - \int (1+1/\alpha)f^{\alpha}(t;\boldsymbol{\theta})dG(t) \\ \approx& \int f^{1+\alpha}(t;\boldsymbol{\theta})dt - \int (1+1/\alpha)f^{\alpha}(t;\boldsymbol{\theta})d\hat{G}(t). \end{aligned}$$

Suppose that in a life testing experiment with m independent n-component systems and a Type-II censored system lifetime data $T_{1:m} < T_{2:m} < \ldots < T_{r:m}$ $(r < m)$ is observed, the empirical c.d.f. of the system lifetime, $\hat{G}_T(t)$, can be obtained by using the Kaplan-Meier (K-M) estimator of the survival function $\hat{S}_T(t) = 1 - \hat{G}_T(t)$ [14] based on the Type-II censored system lifetime data. Then, the MDE of $\boldsymbol{\theta}$ can be obtained at the system level by minimizing

$$\hat{d}_\alpha(g,f) = \int f_T^{1+\alpha}(t;\boldsymbol{\theta})dt - \int \left(1+\frac{1}{\alpha}\right) f_T^{\alpha}(t;\boldsymbol{\theta})d\hat{G}_T(t) \tag{2.4}$$

w.r.t. $\boldsymbol{\theta}$. As this minimization is carried out at the system lifetime level, this estimator is named as the MDE at system lifetime level, denoted as MDE_S.

In addition to the MDE at system lifetime level, the MDE can be considered at the component level. Based on the K-M estimator of the survival function of the system lifetime $\hat{S}_T(t)$, a nonparametric empirical distribution of the component lifetime distribution $\hat{G}_X(t)$ can be obtained based on the relationship between F_T and F_X in Eq. (1.2) as

$$\hat{S}_T(t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \left[\hat{G}_X(t)\right]^j \left[1-\hat{G}_X(t)\right]^{n-j}.$$

Then, the model parameter $\boldsymbol{\theta}$ can be estimated by minimizing the density power divergence at component lifetime distribution

$$\hat{d}_\alpha(g,f) = \int f_X^{1+\alpha}(t;\boldsymbol{\theta})dt - \int \left(1+\frac{1}{\alpha}\right) f_X^{\alpha}(t;\boldsymbol{\theta})d\hat{G}_X(t). \tag{2.5}$$

Since the MDE is obtained based on component lifetime distribution, we refer to the estimator obtained by minimizing Eq. (2.5) as the MDE at the component lifetime level, denoted as MDE_C.

Instead of estimating the c.d.f. nonparametrically, we consider the non-parametric kernel density estimator to estimate the p.d.f. of system lifetime $g_T(t)$ [25]. With the observed Type-II censored system lifetime data, the p.d.f. of system lifetime can be estimated using the Gaussian kernel density estimator, denoted as $\hat{g}_T(t)$. Then, the density power divergence function can be expressed as

$$\hat{d}_\alpha(g, f) = \int f_T^{1+\alpha}(t; \boldsymbol{\theta})dt - \int \left(1 + \frac{1}{\alpha}\right) f_T^\alpha(t; \boldsymbol{\theta})\hat{g}_T(t)dt. \tag{2.6}$$

A MDE of $\boldsymbol{\theta}$ can be obtained by minimizing the density power divergence in Eq. (2.6) with the estimated kernel density $\hat{g}_T(t)$ w.r.t. $\boldsymbol{\theta}$. We name the MDE obtained by minimizing Eq. (2.6) as the MDE with estimated p.d.f., denoted as MDE_P.

For comparative purposes, we also consider the MLE of $\boldsymbol{\theta}$ based on Type-II censored system lifetime data. The log-likelihood function based on the observed Type-II censored system lifetime data $t_{1:m} < t_{2:m} < \ldots, t_{r:m}$ is

$$\ln L(\boldsymbol{\theta}|t_{1:m}, t_{2:m}...t_{r:m}) = \sum_{k=1}^{r} \ln f_T(t_{k:m}; \boldsymbol{\theta}) + (m - r) \ln \bar{F}_T(t_{r:m}; \boldsymbol{\theta}), \tag{2.7}$$

where $r \leq m$ is the number of observed system failures and m is the total number of systems on the test. The MLE of $\boldsymbol{\theta}$ can be obtained by maximizing the log-likelihood function in Eq. (2.7) w.r.t. $\boldsymbol{\theta}$.

2.2. Standard error estimation and confidence intervals

2.2.1. Based on the theoretical results from Basu et al. [4]

For the MDE, Theorem 2.2 in [4] proved that under some regularity conditions, the MDE of the parameter $\boldsymbol{\theta}$ (denoted as $\hat{\boldsymbol{\theta}}$) is a consistent estimator for $\boldsymbol{\theta}$, and $n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is asymptotically multivariate normally distributed with zero mean and variance-covariance matrix $J^{-1}KJ^{-1}$, where

$$\begin{aligned} J \;=\; & \int \boldsymbol{u}_\theta(t)\boldsymbol{u}_\theta^T(t) f^{1+\alpha}(t, \boldsymbol{\theta})dt \\ & + \int \left[\boldsymbol{i}_\theta(t) - \alpha \boldsymbol{u}_\theta(t)\boldsymbol{u}_\theta^T(t)\right] \left[g(t) - f(t; \boldsymbol{\theta})\right] f^\alpha(t; \boldsymbol{\theta})dt \end{aligned} \tag{2.8}$$

and

$$
\begin{aligned}
K &= \int \boldsymbol{u}_\theta(t)\boldsymbol{u}_\theta^T(t) f^{2\alpha}(t;\boldsymbol{\theta})g(t)dt \\
&\quad - \left[\int \boldsymbol{u}_\theta f^{\alpha}(t;\boldsymbol{\theta})g(t)dt\right]\left[\int \boldsymbol{u}_\theta f^{\alpha}(t;\boldsymbol{\theta})g(t)dt\right]^T
\end{aligned} \tag{2.9}
$$

with $\boldsymbol{u}_\theta(t) = \partial \ln f(t;\boldsymbol{\theta})/\partial\boldsymbol{\theta}$, and $\boldsymbol{i}_\theta(t) = -\partial \boldsymbol{u}_\theta(t)/\partial\boldsymbol{\theta}$. Basu et al. [5] further proved that the asymptotic property of the MDE holds for censored survival data as well. Based on these results, the variance of the MDE can be approximated by discretizing the integrals in Eqs. (2.8) and (2.9) with the nonparametric estimated c.d.f. $\hat{G}(t)$ or the nonparametric estimated p.d.f. $\hat{g}(t)$. Consider the estimator MDE_S, $\boldsymbol{\theta}_S$, the standard error of the MDE_S can be approximated as

$$
\widehat{SE}_A(\boldsymbol{\theta}_S) = \sqrt{\hat{J}_S^{-1}\hat{K}_S\hat{J}_S^{-1}/n}, \tag{2.10}
$$

where

$$
\begin{aligned}
\hat{J}_S &= \int \left[(1+\alpha)\boldsymbol{u}_{\hat{\theta}_S}(t)\boldsymbol{u}_{\hat{\theta}_S}^T(t) - \boldsymbol{i}_{\hat{\theta}_S}(t)\right] f_T^{1+\alpha}(t,\hat{\boldsymbol{\theta}}_S)dt \\
&\quad + \int \left[i_{\hat{\theta}_S}(t) - \alpha\boldsymbol{u}_{\hat{\theta}_S}(t)\boldsymbol{u}_{\hat{\theta}_S}^T(t)\right] f_T^{\alpha}(t;\hat{\boldsymbol{\theta}}_S)d\hat{G}_T(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{K}_S &= \int \boldsymbol{u}_{\hat{\theta}_S}(t)\boldsymbol{u}_{\hat{\theta}_S}^T(t) f_T^{2\alpha}(t;\hat{\boldsymbol{\theta}}_S)d\hat{G}_T \\
&\quad - \left[\int \boldsymbol{u}_{\hat{\theta}_S}(t) f_T^{\alpha}(t;\hat{\boldsymbol{\theta}}_S)d\hat{G}_T\right]\left[\int \boldsymbol{u}_{\hat{\theta}_S} f_T^{\alpha}(t;\hat{\boldsymbol{\theta}}_S)d\hat{G}_T\right]^T,
\end{aligned}
$$

where

$$
\boldsymbol{u}_{\hat{\theta}_S}(t) = \frac{1}{f_T(t;\boldsymbol{\theta})}\frac{\partial f_T(t;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_S} \quad \text{and } \boldsymbol{i}_{\hat{\theta}_S}(t) = -\frac{\partial \boldsymbol{u}_\theta(t)}{\partial\boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_S}.
$$

The variance-covariance matrices of the estimators MDE_C and MDE_P, $\hat{\boldsymbol{\theta}}_C$ and $\hat{\boldsymbol{\theta}}_P$, can be obtained in a similar manner.

2.2.2. Based on Fisher information matrix of the MLE

In our preliminary study (results are not presented here), we found that the performance of the standard error estimation based on the theoretical results in [5] may not be satisfactory. Therefore, we consider different ways to approximate the standard error of the MDE proposed in this paper. Based on our observations in the preliminary study, the standard error of the MLE and the standard error of the MDE is in the same order of magnitude, especially when the value of α is close to 0. Hence, we consider a standard error estimation method based on the Fisher information matrix similar to using the inverse of observed Fisher information matrix in estimating the standard error of MLE. For the MLE of $\boldsymbol{\theta}$, the asymptotic variance-covariance matrix of the MLE can be approximated by the inverse of the observed Fisher information matrix, i.e.,

$$\widehat{SE}_F(\hat{\boldsymbol{\theta}}) = \sqrt{\widehat{Var}(\hat{\boldsymbol{\theta}})} = \sqrt{diag\left(\left[-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}\right]^{-1}\right)},$$

where $diag(A)$ denotes the diagonal elements of matrix A. According to the asymptotic theory of the MLE, the sampling distribution of $n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is asymptotically multivariate normally distributed with mean zero and variance $Var(\hat{\boldsymbol{\theta}})$. When $\alpha = 0$, the MDE is equivalent to the MLE. Here, we propose to approximate the variance of the MDEs by inverting the observed Fisher information by substituting $\boldsymbol{\theta}$ with its MDE.

2.2.3. Based on bootstrap method

As we expected, when the value of α is far from zero, the performance of the approximation based on the Fisher information matrix may not fulfilling the expectations, therefore, we also consider approximating the standard error of the MDE based on the bootstrap method. Given the estimated parameters, parametric bootstrap samples of system lifetimes are generated with the corresponding censoring proportion. For each bootstrap sample, the MDE is obtained as a bootstrap MDE. Based on B bootstrap MDEs, we compute the standard deviation of those bootstrap MDEs as an approximation of the standard error of the MDE. For instance, consider the MDE based on system-level data, suppose we have B bootstrap samples and the B bootstrap MDEs are $\hat{\boldsymbol{\theta}}_S^{(1)}$, $\hat{\boldsymbol{\theta}}_S^{(2)}$, ..., $\hat{\boldsymbol{\theta}}_S^{(B)}$, the standard error of the estimator $\hat{\boldsymbol{\theta}}_S$ can be approximated as

$$\widehat{SE}_B(\hat{\boldsymbol{\theta}}_S) = \sqrt{\frac{1}{B}\sum_{b=1}^{B}(\hat{\boldsymbol{\theta}}_S^{(b)} - \bar{\hat{\boldsymbol{\theta}}}_S)^2}, \tag{2.11}$$

where $\bar{\hat{\boldsymbol{\theta}}}_S = \sum_{b=1}^{B} \hat{\boldsymbol{\theta}}_S^{(b)}/B$. The size of bootstrap samples needed will be discussed in Section 3 based on a Monte Carlo simulation study.

After obtaining the standard error estimate based on the methods described in Sections 2.2.1, 2.2.2, and 2.2.3, a two-sided $100(1-\alpha)\%$ normal approximated confidence interval of the k-th element of the parameter vector $\boldsymbol{\theta}$ can be obtained as

$$\theta_{kl}, \theta_{ku}] = \left[\hat{\theta}_k - z_{1-\alpha/2}\widehat{SE}(\hat{\theta}_k), \hat{\theta}_k + z_{1-\alpha/2}\widehat{SE}(\hat{\theta}_k)\right],$$

where z_q is the q-th upper percentile of the standard normal distribution. The performance of the standard error estimation methods and the corresponding confidence intervals will be evaluated via a Monte Carlo simulation study in Section 3.

3. MONTE CARLO SIMULATION STUDIES

In this section, Monte Carlo simulation studies are used to evaluate the performance of the proposed estimation methods for different system structures, different sample sizes with different censoring rates, different underlying distributions, and different values of α for the MDE and different contamination proportions. Based on our preliminary study, since similar observations are obtained based on different sample sizes, different system structures, and different underlying distributions, for the sake of simplicity, we only present the simulation results for the 4-component series-parallel III system (namely System I) and the 4-component mixed parallel I system (namely System II) in Figure 1 for sample size $m = 50$ (with different censoring rate) and the component lifetime X follows the two-parameter Weibull distribution with p.d.f.

$$f_X(x; a, b) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} \exp\left[-\left(\frac{x}{a}\right)^{b}\right], \quad x > 0, \tag{3.1}$$

where a is the scale parameter and b is the shape parameter (denoted as $Weibull(a, b)$). The Weibull distribution is considered here as it is one of the commonly used probability models in lifetime data analysis which can be used to model items with increasing, constant, and decreasing failure rates [17, 18]. Moreover, many other commonly used probability distributions such as the exponential distribution and the Rayleigh distribution are special cases of the Weibull distribution. We consider the scale parameters $a = 3$ or $a = 9$ and the shape parameter to be 2 ($b = 2$). For the case that the contaminates have a longer lifetime than the true distribution on average (namely the longer-life contamination model), the $Weibull(3, 2)$ distribution with a mean lifetime of 2.6587 is the true distribution, and the $Weibull(9, 2)$ distribution with mean lifetime 7.9760 is the contaminated distribution. Similarly, for the case that the contaminates have a shorter lifetime than the true distribution on average (namely the shorter-life contamination model), the $Weibull(9, 2)$ distribution is the true distribution and the $Weibull(3, 2)$ distribution is the contaminated distribution. We also consider other parameter settings; however, for the sake of brevity, we only present the results for $Weibull(9, 2)$ and $Weibull(3, 2)$ here. In the simulation study, the contamination proportion is set to be $0\%, 5\%, 10\%$ and 15%, the Type-II censoring rate $(1 - r/m)$ is set to be 0% and 5% (i.e., no censoring and $r = 0.95m$, respectively). The power parameter α in the MDE method is set to be $0.01, 0.1, 0.25, 0.5, 0.75$ and 0.9.

3.1. Results for point estimation

To evaluate the performance of the proposed estimation procedures for point estimation, the three proposed MDEs – MDE_S, MDE_C and MDE_P – are compared with the MLE in terms of their mean squared errors (MSEs) for estimating the mean component lifetime, i.e., $a\Gamma(1+1/b)$, where $\Gamma(\cdot)$ is the gamma function. Specifically, in the ℓ-th simulation, we first estimate the parameter $\boldsymbol{\theta} = (a, b)$ based on different methods, denoted as $\hat{\boldsymbol{\theta}}_{(\ell)} = (\hat{a}_{(\ell)}, \hat{b}_{(\ell)})$ for $Weibull(a, b)$ distribution, and then the estimated mean component lifetime is computed as $\hat{a}_{(\ell)}\Gamma(1 + 1/\hat{b}_{(\ell)})$. The MSE of an estimator is computed as

$$\frac{1}{L}\sum_{\ell=1}^{L}\left[\hat{a}_{(\ell)}\Gamma(1 + 1/\hat{b}_{(\ell)}) - a\Gamma(1 + 1/b)\right]^2.$$

The simulation results in this subsection are computed based on 10000 realizations ($L = 10000$). For comparative purposes, we define the relative efficiency of the MDE to MLE as

$$RE_{MDE} = \frac{MSE(MLE)}{MSE(MDE)}.$$

The value of relative efficiency greater than 1 indicates that the performance of the MDE is better than the MLE. The relative efficiency for different censoring rates, different contamination proportions, and different values of α for the combinations of System I and System II, and the longer-life contamination model and shorter-life contamination model, are plotted in Figures 2–5. From Figures 2–5, we observe that the performance of the MDE_C is the worst among the three proposed MDEs as the relative efficiency is below 1 in many cases. Therefore, we focus the discussion of the results below on the MDE_S and MDE_P.

In Figures 2 and 4, the relative efficiency of MDE_S, MDE_C and MDE_P for System I and System II with longer-life contamination model are presented, respectively. We can observe that MDE_S and MDE_P have similar performance for System I and System II. When there is no contamination (dashed lines with triangles in Figures 2 and 4), the relative efficiency is less than 1 for MDE_S and MDE_P, which indicates that the MLE performs better than MDE_S and MDE_P in terms of MSEs. When the contamination rate increases, the relative efficiency increases and becomes larger than 1 for MDE_S and MDE_P. Moreover, we observe that the performances of MDE_S and MDE_P improve when α gets closer to 1. These observations are consistent in both no censoring case (Figures 2 and 4 (a) – (c)) and the 5% censoring case (Figures 2 and 4 (d) – (f)). However, in the longer-life contamination model, the relative efficiency in the censoring case is smaller than those in the complete sample case. This indicates that Type-II censoring reduces the influence of the contamination in estimating the parameters. It is likely that the contaminated observations with a longer life are censored in the Type-II censoring scheme. For example, the relative efficiency of the MDE_S with $\alpha = 0.9$ is close to 15 when the contamination rate is 15% with no censoring, while the relative efficiency of the MDE_S with $\alpha = 0.9$ reduces to 10 when the contamination rate of 15% with 5% censoring.

In Figures 3 and 5, the relative efficiency of MDE_S, MDE_C and MDE_P for System I and System II with shorter-life contamination model are presented. We can observe that MDE_S and MDE_P have similar performance for System I

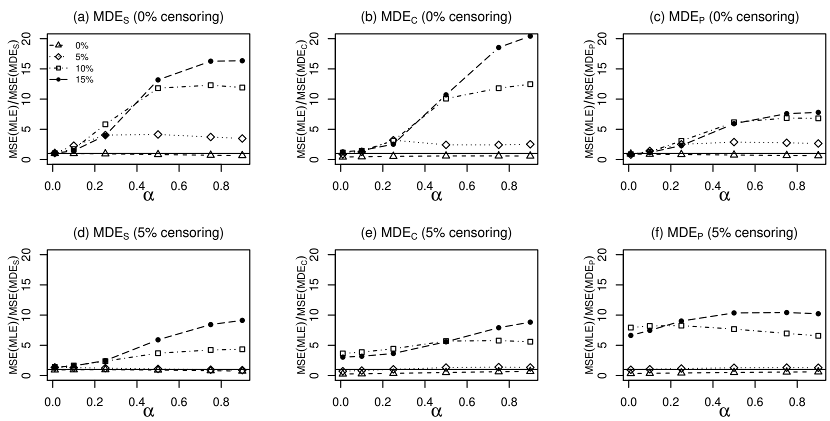

Figure 2: Relative efficiency of estimated mean component lifetime for System I with longer-life contamination model

and System II. In contrast to the longer-life contamination model, MDE_S and MDE_P have different performances in the shorter-life contamination model. In the complete sample case, the MDE_S and MDE_P have relative efficiency greater than 1 when the contamination rate is over 10% in most cases (Figures 3 and 5 (a) and (c)). In the Type-II censoring with 5% censoring case, the MDE_S has relative efficiency greater than 1 when the contamination rate is 15% and the value of α is close to 1 (Figures 3 and 5 (d)), while the MDE_P has relative efficiency less than 1 in most cases (Figures 3 and 5 (f))

In summary, the proposed estimator MDE_S has a better performance compared to MDE_C and MDE_P and it shows an advantage over the MLE when there is contamination present in the data. Moreover, the performance of MDE_S is not much worse than the MLE even when there is no contamination or with a low contamination rate (i.e., relative efficiency less than but close to 1). In the contamination cases, the value of α closer to 1 for the MDE_S has better performance. Therefore, we recommend the use of MDE_S, especially when it is suspected that there is contamination exists in the data. Based on these simulation results and for the simplicity sake, we consider the MDE_S but not the MDE_C and MDE_P in the subsequent study of the performance of standard error estimation and interval estimation.

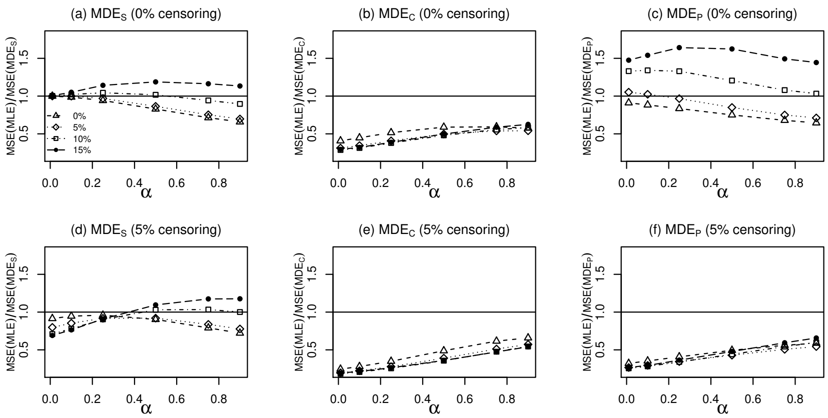

Figure 3: Relative efficiency of estimated mean component lifetime for System I with shorter-life contamination model

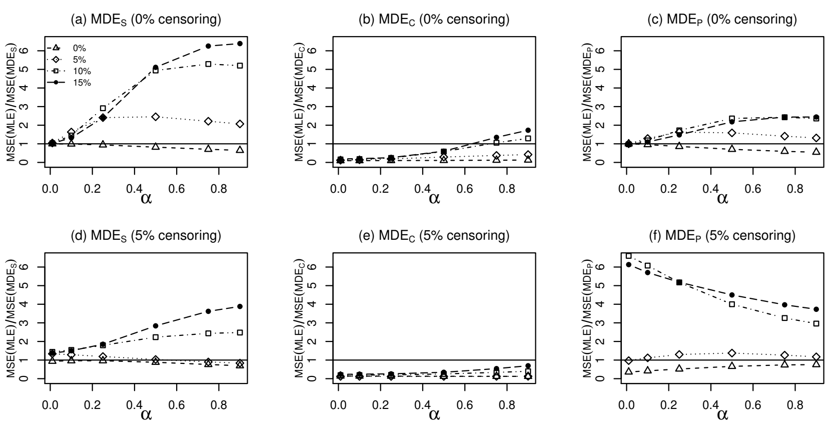

Figure 4: Relative efficiency of estimated mean component lifetime for System II with longer-life contamination model

3.2. Results for standard error estimation and interval estimation

3.2.1. Determining a suitable bootstrap size for standard error estimation

To determine the required bootstrap size B for the standard error estimation for MDE described in Section 2.2.3, following Efron and Tibshirani [10], we consider evaluating the coefficient of variation of the standard error estimates to obtain a reasonable value of the number of bootstrap replicates. We consider the coefficient of variation of the standard error estimates, which is computed as the ratio of the variance of the bootstrap estimate of standard error $\widehat{SE}_B$ to the

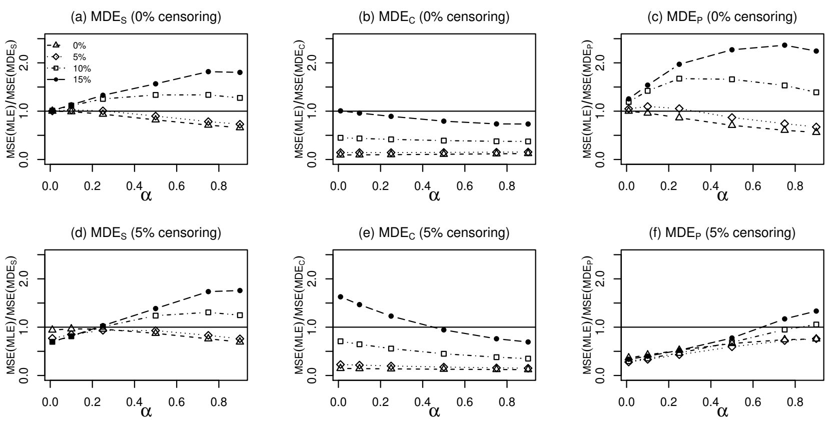

Figure 5: Relative efficiency of estimated mean component lifetime for system II with shorter-life contamination model

expectation of $\widehat{SE}_B$ with different bootstrap size B. The variability of bootstrap estimates can be evaluated by using the coefficient of variation and a suitable value of B is a value such that the variability does not change significantly after increasing the value of B.

A Monte Carlo simulation is carried out to evaluate the coefficient of variation for different bootstrap sizes B in order to determine the proper number of bootstrap replications. We simulate 200 samples of $m = 50$ system lifetimes based on System I (4-component series-parallel III system) with true underlying component lifetime distribution $Weibull(3, 2)$, no contamination, and no censoring. For each simulation, given a bootstrap replication number B, the bootstrap standard error estimate of MDE_S is calculated, denoted as $\hat{SE}_B$. Then, with the 200 bootstrap standard error estimates $\hat{SE}_B^{(1)}$, $\hat{SE}_B^{(2)}$, ..., $\hat{SE}_B^{(200)}$, the simulated coefficient of variation is computed as:

$$\widehat{CV}(\widehat{SE}_B) = \frac{\widehat{Var}(\widehat{SE}_B)}{\hat{E}(\widehat{SE}_B)},$$

where

$$\hat{E}(\widehat{SE}_B) = \frac{1}{B}\sum_{i=1}^{200}\widehat{SE}_B^{(i)},$$

$$\text{and } \widehat{Var}(\widehat{SE}_B) = \frac{1}{B}\sum_{i=1}^{200}(\widehat{SE}_B^{(i)} - \hat{E}(\widehat{SE}_B))^2.$$

Figure 6 presented the simulated coefficient of variation of the standard error of MDE_S. From Figures and 6, we observe that when the bootstrap size B gets above 250, a further increase in the bootstrap size does not bring a substantial reduction in the variation. Hence, we consider the number of bootstrap replications $B = 250$ in the Monte Carlo simulation study for evaluating the performance of confidence intervals.

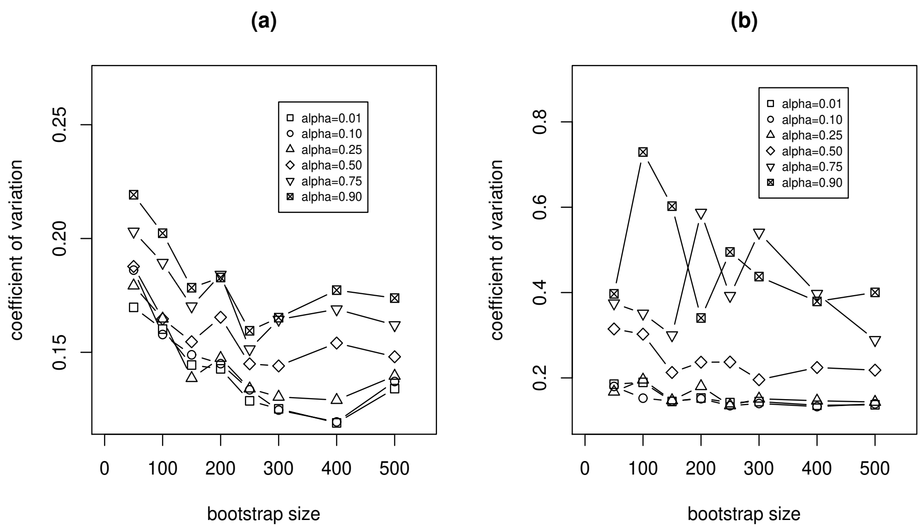

Figure 6: Coefficients of variation of $\widehat{SE}_B$ for (a) shape parameter and (b) scale parameter as functions of the number of bootstrap samples B

3.2.2. Performance of standard error estimates

To evaluate the performance of the three standard error estimation methods for MDE presented in Section 2.2, we compare the simulated standard errors of the MDE based on the system-level data, MDE_S, and the averaged values of the standard error estimates based on the theoretical results from Basu et al. [4] (i.e., $\widehat{SE}_A$), based on observed Fisher information matrix (i.e., $\widehat{SE}_F$), and based on bootstrap method (i.e., $\widehat{SE}_B$) with bootstrap size $B = 250$. We simulate 1000 samples of $m = 50$ system lifetimes based on System 1 (4-component series-parallel III system) with true underlying component lifetime distribution $Weibull(3, 2)$, no contamination, and no censoring. The simulation results are presented in Table 2.

From Table 2, we observe that the standard error estimates based on the theoretical results from Basu et al. [4] can seriously underestimate the standard error of MDE_S, while the standard error estimates based on observed Fisher information matrix provide a reasonable approximation to the standard errors of MDE_S when α is close to 0. Overall, among the three standard error estimation methods for MDE, the bootstrap method with bootstrap size $B = 250$ provides a reasonable approximation to standard error of the MDE_S for all the values of α considered here. Therefore, in the following simulation study for confidence intervals, we use the standard error estimates based on the bootstrap method.

	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
Simulated $\widehat{SE}(\hat{a})$	0.179	0.180	0.184	0.196	0.210	0.219
Average $\widehat{SE}_A(\hat{a})$	0.013	0.011	0.008	0.006	0.005	0.005
Average $\widehat{SE}_F(\hat{a})$	0.214	0.206	0.190	0.172	0.165	0.164
Average $\widehat{SE}_B(\hat{a})$	0.205	0.210	0.197	0.183	0.187	0.189
Simulated $\widehat{SE}(\hat{b})$	0.247	0.248	0.257	0.290	0.326	0.336
Average $\widehat{SE}_A(\hat{b})$	0.009	0.007	0.005	0.003	0.003	0.002
Average $\widehat{SE}_F(\hat{b})$	0.203	0.207	0.215	0.232	0.244	0.248
Average $\widehat{SE}_B(\hat{b})$	0.238	0.230	0.255	0.322	0.375	0.389

Table 2: Simulated standard errors the MDE_S and the averaged standard error estimates based on the theoretical results from [4] ($\widehat{SE}_A$), based on observed Fisher information matrix ($\widehat{SE}_F$), and based on bootstrap method ($\widehat{SE}_B$) with bootstrap size $B = 250$

3.2.3. Performance of confidence intervals

In this subsection, the simulated coverage probabilities and the average widths of 95% confidence intervals of the Weibull parameters a and b for the MLE and the MDE based on system-level data (MDE_S) with different values of α are compared. The two systems (System I and System II) and the longer-life and shorter-life contamination models described in Section 3 are considered here. Specifically, a two-sided $100(1-\alpha)\%$ normal approximated confidence interval of a is constructed as

$$[a_l, a_u] = \left[\hat{a} - z_{1-\alpha/2}\widehat{SE}(\hat{a}), \hat{a} + z_{1-\alpha/2}\widehat{SE}(\hat{a})\right],$$

where the estimated standard error $\widehat{SE}(\hat{a})$ is obtained based on the bootstrap method. Similarly, a two-sided $100(1-\alpha)\%$ normal approximated confidence interval of b is constructed as

$$[b_l, b_u] = \left[\hat{b} - z_{1-\alpha/2}\widehat{SE}(\hat{b}), \hat{b} + z_{1-\alpha/2}\widehat{SE}(\hat{b})\right],$$

where the estimated standard error $\widehat{SE}(\hat{a})$ is obtained based on the bootstrap method. The simulated coverage probability (CP) is computed as the proportion of cases that the true value of the parameter falls within the confidence interval, and the average width (AW) is computed as $2z_{1-\alpha/2}\widehat{SE}(\hat{a})$ and $2z_{1-\alpha/2}\widehat{SE}(\hat{b})$ for parameters a and b, respectively. The simulation results are presented in Tables 3 – 4.

From Tables 3 – 4, we observe that when there is no contamination, the confidence intervals based on MLEs give coverage probabilities close to the nominal 95% for both scale and shape parameters. Compared with MDE_S, the confidence intervals based on MLEs give the highest coverage probabilities and the smallest average widths when there is no contamination (i.e., contamination rate is 0). However, when the contamination rate increases, the coverage probabilities of the confidence intervals based on MLEs decrease for both scale and shape parameters and the average widths increase for the scale parameter but decrease for the shape parameter. These observations are consistent in both the longer-life and shorter-life contamination models with and without censoring for System I and System II. We also observe that the coverage probabilities of the confidence intervals based on MLEs are sensitive to the contamination rate and the type of contamination. In the longer-life contamination model, the coverage probabilities for both scale and shape parameters drop dramatically when the contamination rate increases. For example, in Table 3, when the contamination rate is 15% with no censoring, the simulated coverage probabilities of the confidence intervals based on MLE are only 18% for the scale parameter and 2% for the shape parameter under the longer-life contamination model, while the simulated coverage probabilities of the confidence intervals based on MLE are 80.7% for the scale parameter and 68.7% for the shape parameter under the shorter-life contamination model.

Similar to the MLEs, the coverage probabilities of the confidence intervals based on MDE_S are also sensitive to the contamination rate and the type of the contamination. For the longer-life contamination model, the coverage probabilities of confidence intervals of the scale parameter based on MDE_S with α close to 1 are closer to the nominal levels when the contamination rate is high. However, for the shorter-life contamination model, the coverage probabilities of the confidence intervals of the scale parameter based on MDE_S with α close to 1 are far away from the nominal level when the contamination rate is high (see, Table 3). The confidence intervals based on MDE_S have better coverage probabilities than the confidence intervals based on MLE in the longer-life contamination model when the contamination rate is high. For example, when the contamination rate is 15% under the longer-life contamination model, the coverage probabilities of the confidence intervals based on MDE_S with α close to 1 can still maintain at 95.7% for the scale parameter (Table 3) and 91.6% for shape parameter (Table 4), while the coverage probabilities of the confidence intervals of based on MLE are down to 18% for scale parameter and 2% for shape parameter.

In general, for the longer-life contamination model, compared to the confidence intervals based on MLE, the confidence intervals based on MDE_S have higher coverage probabilities and larger average widths (Tables 3 and 4). Nevertheless, for the shorter-life contamination model, compared to the confidence intervals based on MLE, the confidence intervals based on MDE_S have lower coverage probabilities.

Coverage Probability	Longer-life Contamination Model								Shorter-life Contamination Model							
	No censoring				5% censoring				No censoring				5% censoring			
Contamination Proportion	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%
MLE	93.9	88.1	44.9	18.0	93.3	92.0	58.0	24.5	93.7	92.4	86.3	80.7	93.6	92.3	87.8	82.0
MDE_S ($\alpha = 0.01$)	93.9	92.3	59.1	31.7	87.9	93.1	79.5	50.6	93.6	92.2	86.0	80.0	88.6	84.7	75.3	67.5
MDE_S ($\alpha = 0.10$)	93.9	94.7	73.5	45.9	89.7	93.2	81.3	55.5	93.7	92.5	87.1	81.6	89.8	87.0	79.6	72.1
MDE_S ($\alpha = 0.25$)	93.9	95.2	93.1	82.6	91.3	93.2	86.1	70.2	93.3	91.8	83.3	73.2	90.9	89.2	82.8	73.6
MDE_S ($\alpha = 0.50$)	93.3	94.6	95.9	95.8	92.7	93.3	92.6	91.5	92.6	91.4	82.8	71.3	92.0	91.2	85.2	75.7
MDE_S ($\alpha = 0.75$)	92.9	94.0	95.4	95.9	93.1	93.6	94.8	95.5	92.1	90.5	81.4	68.8	92.2	91.4	85.2	74.8
MDE_S ($\alpha = 0.90$)	92.3	93.8	95.3	95.7	93.2	93.6	95.0	95.9	91.6	90.1	80.2	67.6	92.2	91.3	84.6	73.8

Average Width	Longer-life Contamination Model								Shorter-life Contamination Model							
	No censoring				5% censoring				No censoring				5% censoring			
Contamination Proportion	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%
MLE	0.69	0.99	1.32	1.50	0.70	0.77	1.06	1.33	2.06	2.14	2.24	2.30	2.09	2.18	2.30	2.37
MDE_S ($\alpha = 0.01$)	0.69	1.08	1.45	1.64	0.63	0.73	1.19	1.45	2.06	2.14	2.22	2.26	1.89	1.97	2.06	2.10
MDE_S ($\alpha = 0.10$)	0.69	0.93	1.33	1.54	0.64	0.74	1.14	1.41	2.08	2.17	2.27	2.32	1.92	2.02	2.12	2.18
MDE_S ($\alpha = 0.25$)	0.71	0.82	1.11	1.35	0.66	0.76	1.06	1.31	2.12	2.23	2.34	2.38	1.99	2.10	2.22	2.27
MDE_S ($\alpha = 0.50$)	0.75	0.82	0.98	1.15	0.71	0.80	1.04	1.23	2.24	2.38	2.52	2.57	2.12	2.25	2.40	2.46
MDE_S ($\alpha = 0.75$)	0.80	0.86	0.98	1.11	0.76	0.87	1.08	1.22	2.39	2.53	2.68	2.75	2.27	2.42	2.58	2.65
MDE_S ($\alpha = 0.90$)	0.83	0.89	1.00	1.11	0.80	0.91	1.10	1.21	2.47	2.61	2.78	2.85	2.38	2.52	2.70	2.77

Table 3: Simulated coverage probabilities (in %) and average widths of confidence intervals of the scale parameter computed based on MLE and MDE_S with different values of α under the longer-life and shorter-life contamination models for System I

Coverage Probability	Longer-life Contamination Model								Shorter-life Contamination Model							
	No censoring				5% censoring				No censoring				5% censoring			
Contamination Proportion	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%
MLE	95.2	36.8	6.5	2.0	95.0	93.0	49.8	20.4	95.2	93.7	82.0	68.7	95.3	93.3	81.6	68.7
MDE_S ($\alpha = 0.01$)	92.4	42.3	4.4	1.0	88.7	91.7	73.0	33.8	92.4	92.1	74.8	52.6	88.9	91.9	91.4	82.0
MDE_S ($\alpha = 0.10$)	92.4	70.0	10.3	2.3	89.3	92.2	72.6	33.2	92.5	91.4	72.2	47.7	89.7	92.3	89.4	77.2
MDE_S ($\alpha = 0.25$)	92.5	91.8	70.3	37.5	89.9	92.6	77.7	40.4	92.2	90.0	65.4	37.6	90.3	91.8	83.7	64.2
MDE_S ($\alpha = 0.50$)	92.5	93.0	91.6	87.5	91.1	92.8	88.8	80.0	91.9	89.8	62.2	31.9	91.2	90.8	75.8	49.7
MDE_S ($\alpha = 0.75$)	92.3	93.0	92.8	91.1	92.1	93.4	91.6	89.7	91.4	88.2	60.3	28.9	91.6	89.5	70.0	41.6
MDE_S ($\alpha = 0.90$)	92.3	93.0	92.6	91.6	92.4	93.3	92.3	90.8	91.2	87.3	59.9	28.7	91.4	88.7	67.6	39.0
Average Width	Longer-life Contamination Model								Shorter-life Contamination Model							
	No censoring				5% censoring				No censoring				5% censoring			
Contamination Proportion	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%	0%	5%	10%	15%
MLE	0.95	0.68	0.56	0.53	1.02	0.96	0.77	0.66	0.95	0.90	0.84	0.80	1.02	0.96	0.89	0.85
MDE_S ($\alpha = 0.01$)	1.01	0.78	0.57	0.54	1.25	1.12	0.98	0.79	1.01	0.86	0.71	0.64	1.25	1.04	0.85	0.76
MDE_S ($\alpha = 0.10$)	1.01	0.84	0.60	0.55	1.24	1.11	0.94	0.77	1.01	0.86	0.70	0.62	1.24	1.04	0.84	0.74
MDE_S ($\alpha = 0.25$)	1.05	1.03	0.89	0.74	1.25	1.10	0.91	0.76	1.05	0.89	0.71	0.62	1.25	1.06	0.83	0.73
MDE_S ($\alpha = 0.50$)	1.20	1.18	1.14	1.09	1.33	1.17	1.02	1.00	1.20	1.01	0.75	0.63	1.31	1.12	0.84	0.70
MDE_S ($\alpha = 0.75$)	1.41	1.37	1.31	1.24	1.49	1.34	1.21	1.19	1.39	1.17	0.82	0.65	1.45	1.23	0.90	0.71
MDE_S ($\alpha = 0.90$)	1.51	1.47	1.39	1.32	1.57	1.46	1.31	1.28	1.54	1.28	0.88	0.68	1.55	1.34	0.94	0.73

Table 4: Simulated coverage probabilities and average widths of confidence intervals of the shape parameter computed based on MLE and MDE_S with different values of α under the longer-life and shorter-life contamination models for System I

4. ILLUSTRATIVE EXAMPLE

In this section, a numerical example based on the system lifetime data of the 4-component series-parallel III system with Weibull component lifetimes is used to illustrate the estimation methods proposed in this paper. The system lifetime data was originally presented in [2] and further analyzed by [26]. The data are 10 system lifetimes from the 4-component system with system signature $\boldsymbol{s} = (1/4, 1/4, 1/2, 0)$ with component lifetime follows $Weibull(3, 2)$:

$$0.72717, 1.02050, 1.38633, 1.61244, 1.70590, 1.76789, 2.6786, 3.02676, 3.25943, 3.78497$$

To illustrate the effect of contamination in the statistical inference procedures, we simulated an observation from the $Weibull(9, 2)$ to replace one of the observations in the original data set. Specifically, the observation 1.76789 is replaced by 5.48619. The contaminated data set is as follows.

$$0.72717, 1.02050, 1.38633, 1.61244, 1.70590, 5.48619, 2.6786, 3.02676, 3.25943, 3.78497$$

Based on the original and the contaminated data sets, the MLE and the three proposed MDEs of the Weibull parameters a and b and the corresponding confidence intervals are presented in Tables 5 and 6.

For point estimation, form Tables 5 and 6, the MLE, MDE_S, MDE_C and MDE_P with different values of α provide similar point estimates of the parameters a and b. By comparing the estimates obtained from the data sets with and without contamination, the difference between MDEs (especially α close to 1) obtained from the data sets with and without contamination is smaller than the difference between MLEs obtained from the data sets with and without contamination in general. For example, the MLE of a is 2.695 for the data set without contamination and the MLE of a is 3.249 for the data set with contamination which has a difference 0.554, while the MDE_S with $\alpha = 0.9$ is 2.691 for the data set without contamination and the MDE_S with $\alpha = 0.9$ is 3.105 for the data set with contamination, which has a difference 0.414.

For interval estimation, in both with and without contamination cases (Tables 5 and 6), the confidence intervals for the scale parameter based on the observed Fisher information matrix are very close to the one obtained from the bootstrap method. However, the confidence intervals for the shape parameter based on the bootstrap method is wider than those based on the observed Fisher information matrix. The confidence intervals using MDEs with standard error estimates based on the theoretical results are much narrower than the confidence intervals with standard error estimates based on the observed Fisher information matrix and based on the bootstrap method. This observation agrees with the results in the Monte Carlo simulation that the standard error estimates based on the theoretical results are likely to underestimate the standard errors of the MDEs.

Estimator	$\hat{a}$	95% CI based on $\widehat{SE}_A(\hat{a})$	95% CI based on $\widehat{SE}_F(\hat{a})$	95% CI based on $\widehat{SE}_B(\hat{a})$
MLE	2.695		(1.978, 3.412)	(1.980, 3.410)
MDE_S				
$\alpha = 0.01$	2.696	(2.490, 2.902)	(1.976, 3.416)	(2.014, 3.378)
$\alpha = 0.10$	2.700	(2.504, 2.896)	(1.961, 3.439)	(1.967, 3.433)
$\alpha = 0.25$	2.706	(2.531, 2.881)	(1.937, 3.475)	(1.922, 3.490)
$\alpha = 0.50$	2.710	(2.495, 2.925)	(1.900, 3.520)	(1.780, 3.640)
$\alpha = 0.75$	2.703	(2.507, 2.899)	(1.867, 3.539)	(1.731, 3.675)
$\alpha = 0.90$	2.691	(2.495, 2.887)	(1.850, 3.532)	(1.667, 3.715)
MDE_C				
$\alpha = 0.01$	2.617	(2.421, 2.813)	(2.003, 3.231)	(2.019, 3.215)
$\alpha = 0.10$	2.628	(2.442, 2.814)	(2.002, 3.254)	(2.027, 3.229)
$\alpha = 0.25$	2.647	(2.483, 2.811)	(1.997, 3.297)	(2.024, 3.270)
$\alpha = 0.50$	2.677	(2.538, 2.816)	(1.987, 3.367)	(1.976, 3.378)
$\alpha = 0.75$	2.700	(2.576, 2.824)	(1.972, 3.428)	(1.909, 3.491)
$\alpha = 0.90$	2.709	(2.585, 2.833)	(1.960, 3.458)	(1.896, 3.522)
MDE_P				
$\alpha = 0.01$	2.769	(2.453, 3.085)	(1.892, 3.646)	(1.937, 3.601)
$\alpha = 0.10$	2.782	(2.505, 3.059)	(1.884, 3.680)	(1.877, 3.687)
$\alpha = 0.25$	2.802	(2.579, 3.025)	(1.872, 3.732)	(1.866, 3.738)
$\alpha = 0.50$	2.829	(2.665, 2.993)	(1.851, 3.807)	(1.853, 3.805)
$\alpha = 0.75$	2.848	(2.633, 3.063)	(1.835, 3.861)	(1.815, 3.881)
$\alpha = 0.90$	2.855	(2.649, 3.061)	(1.827, 3.883)	(1.780, 3.930)
Estimator	$\hat{b}$	95% CI based on $\widehat{SE}_A(\hat{b})$	95% CI based on $\widehat{SE}_F(\hat{b})$	95% CI based on $\widehat{SE}_B(\hat{b})$
MLE	2.004		(0.945, 3.063)	(0.566, 3.442)
MDE_S				
$\alpha = 0.01$	1.999	(1.847, 2.151)	(0.942, 3.056)	(0.668, 3.330)
$\alpha = 0.10$	1.946	(1.794, 2.098)	(0.916, 2.976)	(0.430, 3.462)
$\alpha = 0.25$	1.872	(1.733, 2.011)	(0.878, 2.866)	(0.275, 3.469)
$\alpha = 0.50$	1.782	(1.630, 1.934)	(0.832, 2.732)	(0.000, 3.705)
$\alpha = 0.75$	1.718	(1.594, 1.842)	(0.799, 2.637)	(0.000, 5.163)
$\alpha = 0.90$	1.690	(1.566, 1.814)	(0.788, 2.592)	(0.000, 4.666)
MDE_C				
$\alpha = 0.01$	2.340	(2.188, 2.492)	(1.079, 3.601)	(0.310, 4.370)
$\alpha = 0.10$	2.276	(2.124, 2.428)	(1.058, 3.494)	(0.349, 4.203)
$\alpha = 0.25$	2.184	(2.045, 2.323)	(1.024, 3.344)	(0.257, 4.111)
$\alpha = 0.50$	2.065	(1.941, 2.189)	(0.974, 3.156)	(0.000, 4.671)
$\alpha = 0.75$	1.978	(1.854, 2.102)	(0.932, 3.024)	(0.000, 9.267)
$\alpha = 0.90$	1.937	(1.813, 2.061)	(0.911, 2.963)	(0.000, 6.431)
MDE_P				
$\alpha = 0.01$	1.732	(1.462, 2.002)	(0.800, 2.664)	(0.715, 2.749)
$\alpha = 0.10$	1.713	(1.473, 1.953)	(0.787, 2.639)	(0.775, 2.651)
$\alpha = 0.25$	1.688	(1.536, 1.840)	(0.773, 2.603)	(0.373, 3.003)
$\alpha = 0.50$	1.653	(1.501, 1.805)	(0.748, 2.558)	(0.403, 2.903)
$\alpha = 0.75$	1.627	(1.503, 1.751)	(0.731, 2.523)	(0.279, 2.975)
$\alpha = 0.90$	1.615	(1.491, 1.739)	(0.723, 2.507)	(0.028, 3.202)

Table 5: Point and interval estimates for Weibull parameters for the original data set presented in Section 4

5. CONCLUDING REMARKS

In this paper, we study the robust estimation method for the model parameters in the component lifetime distribution based on system lifetime data with known system structure. The minimum density power divergence estimation method is considered and three different MDEs are proposed. Standard error estimation and interval estimation procedures based on the MDEs are also stud-

Estimator	$\hat{a}$	95% CI based on $\widehat{SE}_A(\hat{a})$	95% CI based on $\widehat{SE}_F(\hat{a})$	95% CI based on $\widehat{SE}_B(\hat{a})$
MLE	3.249		(2.172, 4.326)	(2.192, 4.306)
MDE_S				
$\alpha = 0.01$	3.248	(2.851, 3.645)	(2.169, 4.327)	(2.194, 4.302)
$\alpha = 0.10$	3.235	(2.853, 3.617)	(2.154, 4.316)	(2.183, 4.287)
$\alpha = 0.25$	3.210	(2.859, 3.561)	(2.131, 4.289)	(2.129, 4.291)
$\alpha = 0.50$	3.165	(2.861, 3.469)	(2.100, 4.230)	(2.010, 4.320)
$\alpha = 0.75$	3.124	(2.884, 3.364)	(2.072, 4.176)	(1.935, 4.313)
$\alpha = 0.90$	3.105	(2.941, 3.269)	(2.057, 4.153)	(1.855, 4.355)
MDE_C				
$\alpha = 0.01$	3.203	(2.887, 3.519)	(2.261, 4.145)	(2.329, 4.077)
$\alpha = 0.10$	3.207	(2.910, 3.504)	(2.247, 4.167)	(2.292, 4.122)
$\alpha = 0.25$	3.213	(2.965, 3.461)	(2.225, 4.201)	(2.269, 4.157)
$\alpha = 0.50$	3.216	(3.030, 3.402)	(2.194, 4.238)	(2.168, 4.264)
$\alpha = 0.75$	3.206	(2.983, 3.429)	(2.165, 4.247)	(2.090, 4.322)
$\alpha = 0.90$	3.192	(2.969, 3.415)	(2.148, 4.236)	(2.036, 4.348)
MDE_P				
$\alpha = 0.01$	3.368	(2.952, 3.784)	(2.096, 4.640)	(2.168, 4.568)
$\alpha = 0.10$	3.379	(2.973, 3.785)	(2.089, 4.669)	(2.076, 4.682)
$\alpha = 0.25$	3.392	(3.005, 3.779)	(2.080, 4.704)	(2.071, 4.713)
$\alpha = 0.50$	3.406	(3.050, 3.762)	(2.069, 4.743)	(2.008, 4.804)
$\alpha = 0.75$	3.414	(3.080, 3.748)	(2.065, 4.763)	(1.912, 4.916)
$\alpha = 0.90$	3.416	(3.100, 3.732)	(2.062, 4.770)	(1.685, 5.147)
Estimator	$\hat{b}$	95% CI based on $\widehat{SE}_A(\hat{b})$	95% CI based on $\widehat{SE}_F(\hat{b})$	95% CI based on $\widehat{SE}_B(\hat{b})$
MLE	1.607		(0.782, 2.432)	(0.485, 2.729)
MDE_S				
$\alpha = 0.01$	1.604	(1.300, 1.908)	(0.779, 2.429)	(0.404, 2.804)
$\alpha = 0.10$	1.588	(1.284, 1.892)	(0.770, 2.406)	(0.476, 2.700)
$\alpha = 0.25$	1.569	(1.272, 1.866)	(0.761, 2.377)	(0.319, 2.819)
$\alpha = 0.50$	1.550	(1.287, 1.813)	(0.751, 2.349)	(0.000, 3.695)
$\alpha = 0.75$	1.535	(1.411, 1.659)	(0.741, 2.329)	(0.000, 4.102)
$\alpha = 0.90$	1.525	(1.386, 1.664)	(0.736, 2.314)	(0.000, 3.997)
MDE_C				
$\alpha = 0.01$	1.825	(1.593, 2.057)	(0.899, 2.751)	(0.206, 3.444)
$\alpha = 0.10$	1.786	(1.571, 2.001)	(0.879, 2.693)	(0.382, 3.190)
$\alpha = 0.25$	1.732	(1.580, 1.884)	(0.851, 2.613)	(0.162, 3.302)
$\alpha = 0.50$	1.668	(1.504, 1.832)	(0.816, 2.520)	(0.000, 8.867)
$\alpha = 0.75$	1.625	(1.501, 1.749)	(0.791, 2.459)	(0.000, 4.849)
$\alpha = 0.90$	1.606	(1.482, 1.730)	(0.781, 2.431)	(0.000, 5.860)
MDE_P				
$\alpha = 0.01$	1.464	(1.167, 1.761)	(0.697, 2.231)	(0.733, 2.195)
$\alpha = 0.10$	1.455	(1.158, 1.752)	(0.691, 2.219)	(0.626, 2.284)
$\alpha = 0.25$	1.444	(1.147, 1.741)	(0.685, 2.203)	(0.533, 2.355)
$\alpha = 0.50$	1.433	(1.149, 1.717)	(0.676, 2.190)	(0.432, 2.434)
$\alpha = 0.75$	1.428	(1.158, 1.698)	(0.674, 2.182)	(0.292, 2.564)
$\alpha = 0.90$	1.426	(1.170, 1.682)	(0.672, 2.180)	(0.021, 2.831)

Table 6: Point and interval estimates for Weibull parameters for the contaminated data set presented in Section 4.

ied. The three proposed estimation procedures are compared to the maximum likelihood estimation method via a Monte Carlo simulation study. It is shown that the minimum density power divergence estimation method based on system-level data can provide better performance in both point and interval estimation when there is longer-life contamination in the data. We have also shown that the standard error estimates based on the bootstrap method can be adopted for estimating the standard errors of the MDEs.

From our simulation study, for point estimation, the MLE outperforms the

MDEs when there is no contamination in the data. However, we observe that the system-level MDE, MDE_S, is a robust estimation procedure than the MLE when there is contamination in the data. For interval estimation, we observe that the contaminated data considerably affect the coverage probabilities of the confidence intervals based on MLE and MDE_S. The confidence intervals based on MDE_S perform better than those based on MLE for contaminated data, especially when the contamination rate is high (say, 10% or 15%) in the longer-life contamination model. For contamination data with longer lifetimes, MDE_S with large value of α ($\alpha = 0.75$ or 0.9) is recommended. For contamination data with shorter lifetimes, MDE_S with small value of α ($\alpha = 0.01$ or 0.1) is recommended. Since the choice of the value α for the MDE_S affects the results for the interval estimation, it is interesting to study the choice of the value of α in the system-level minimum divergence estimator MDE_S. In practice, the sample size m, the system signature $\boldsymbol{s}$, and the censoring proportion are known, but the underlying component lifetime distribution and the contamination rate are usually unknown. The performance of the estimators with different values of α can be studied under different underlying component lifetime distributions and contamination rates via simulation, and then a reasonable range of the value of α can be obtained.

For future research, a systematic way to choose the value of α for the MDE can be studied. On the other hand, since the simulated coverage probabilities of the confidence intervals based on the estimators studied in this paper can be much lower than the nominal level when there is contamination in the data, it is desired to develop better standard error estimation methods and confidence interval estimation methods which can provide better coverage probabilities when the contamination rates. The current work can be extended to the situation when the lifetime of the unit may be affected by one or more factors/explanatory variables (such as temperature, voltage, load, etc.). For example, consider a Weibull regression model in which K covariates $\boldsymbol{z} = (z_1, z_2, \ldots, z_K)$ affects the scale parameter a in Eq. (3.1), then, we have a parametric proportional hazard model for the lifetime X

$$f_X(x; \boldsymbol{\theta}) = f_X(x; a(\boldsymbol{z}), b),$$

where

$$a(\boldsymbol{z}) = \exp\left(\nu_0 + \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_K z_K\right)$$

and the parameter vector is $\boldsymbol{\theta} = (\nu_0, \nu_1, \ldots, \nu_K, b)$. The proposed minimum density divergence estimation method can be applied to estimate the parameter vector $\boldsymbol{\theta} = (\nu_0, \nu_1, \ldots, \nu_K, b)$. On the other hand, we assume that the system signature is known in this paper, however, for some black box systems, we may not have any knowledge on the system structures. Following the work by Yang et al. [27], one can develop robust procedures for estimating the parameters of the component lifetime distribution and for identifying the system structure based on system-level data simultaneously by assuming the system is a coherent system. We are currently working on these extensions and we hope to report the findings in future work.

ACKNOWLEDGMENTS

The authors sincerely thank the editor and the two anonymous reviewers for their comments and suggestions which greatly improved this article. This work is supported by CUHK Faculty of Science Direct Grant (4053421) and by a grant from the Simons Foundation (#709773 to Tony Ng).

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