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On q-Generalized Extreme Values under Power Normalization with Properties, Estimation Methods and Applications to COVID-19 Data

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Abstract:

- This paper introduces the q-analogues of the generalized extreme value distribution and its discrete counterpart under power normalization. The inclusion of the parameter q enhances modeling flexibility. The continuous extended model can produce various types of hazard rate functions, with supports that can be finite, infinite, or bounded above or below. Additionally, these new models can effectively handle skewed data, particularly those with highly extreme observations. Statistical properties of the proposed continuous distribution are presented, and the model parameters are estimated using various approaches. A simulation study evaluates the performance of the estimators across different sample sizes. Finally, three distinct real datasets are analyzed to demonstrate the versatility of the proposed model.

Keywords:

- *extreme value theory; survival discretization approach; entropy; estimation methods; simulation; COVID-19.*

AMS Subject Classification:

- 60E05, 62E10, 62E15, 62F10, 62N05.

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1. INTRODUCTION

In the last two decades, there has been an increasing interest in building statistical models for estimating the probability of rare and extreme events. These models involving extreme value theory (EVT) are of a great interest in environmental sciences, engineering, finance, insurance, and many other disciplines. Especially in finance, extreme price movement of a financial asset or a market index can be defined as the lowest and highest costs in an observed period (see Gilli, 2006 [19]). EVT shows that the asymptotic minimum and maximum returns have a definite shape that is independent of the return process itself. The EVT deals with the probabilistic description of the extremes of a stochastic sequence. The fundamental results of Fisher and Tippett (1928) [17] constitute the backbone of the classical EVT. The fundamental theorem states that maxima of independent and identically distributed random variables have one of the three extreme value distributions: Fréchet distribution, with infinite upper and heavy tail, Gumbel distribution, whose upper tail is also infinite, but lighter than the Fréchet distribution, and inverse Weibull distribution with finite upper tail. The three previous models can be gathered in the following family

$$G_\xi(x;\mu,\sigma,\xi)=\begin{cases}\exp\{-(1+\xi(\frac{x-\mu}{\sigma}))^{\frac{-1}{\xi}}\}; & \xi\neq 0,\\ \exp\{-\exp(-\frac{x-\mu}{\sigma})\}; & \xi\to 0,\end{cases} \tag{1.1}$$

and

$$g_\xi(x;\mu,\sigma,\xi)=\begin{cases}\frac{1}{\sigma}\exp\{-(1+\xi(\frac{x-\mu}{\sigma}))^{\frac{-1}{\xi}}\}(1+\xi(\frac{x-\mu}{\sigma}))^{\frac{-1}{\xi}-1}; & \xi\neq 0,\\ \frac{1}{\sigma}\exp\{-\exp(-(\frac{x-\mu}{\sigma}))\}\exp(-(\frac{x-\mu}{\sigma})); & \xi\to 0,\end{cases} \tag{1.2}$$

where μ is a location parameter, σ is a positive scale parameter, and ξ is the shape parameter, for more detail (see De Haan and Ferreira, 2007 [14]). The cumulative distribution function (CDF) and probability density function (PDF) in Equations (1.1) and (1.2), respectively, are known as the generalized extreme value distribution under linear normalization (GEVL). Another reason for using the power normalization in EVT is concerning the possibility of getting a better rate of convergence in EVT (see Barakat *et al.*, 2010 [2]). The CDF F is said to belong to the max stable model under power normalization or simply p-max domain of attraction of a non-degenerate CDF H, denote by $F\in D_p(H)$, if for some norming constants $\alpha_n>0$ and $\beta_n>0$, we have

$$P(|\frac{X_{n:n}}{\alpha_n}|^{1/\beta_n}\operatorname{sign}(X_{n:n})\le x)=F^n(\alpha_n|x|^{\beta_n}\operatorname{sign}(x))\xrightarrow[n]{w}H(x), \tag{1.3}$$

where $\operatorname{sign}(x)=-1$, or 0, or 1, according as $x<0$, or $x=0$, or $x>0$. Pantcheva (1985) [22] proved that $H(x)$ belongs to one p-type of the following six classes of extreme value distributions

$$\text{Type-I}:\quad H_{1,\beta}(x)=\begin{cases}0; & x\le 1,\\ \exp\{-(\log x)^{-\beta}\}; & x>1,\beta>0,\end{cases}$$

$$\text{Type-II}:\quad H_{2,\beta}(x)=\begin{cases}0; & x\le 0,\\ \exp\{-(-\log x)^{\beta}\}; & -1\le x\le 1,\\ 1; & x>1,\end{cases}$$

$$\text{Type-III}: \quad H_{3,\beta}(x) = \begin{cases} 0; & x \leq -1, \\ \exp\{-(-\log(-x)^{-\beta}\}; & -1 \leq x \leq 0, \\ 1; & x > 0, \end{cases}$$

$$\text{Type-IV}: \quad H_{4,\beta}(x) = \begin{cases} \exp\{-(\log(-x))^{\beta}\}; & x \leq -1, \\ 1; & x > -1, \end{cases}$$

$$\text{Type-V}: \quad H_5(x) = \begin{cases} 0; & x \leq 0, \\ \exp\{-x^{-1}\}; & x > 0, \end{cases}$$

and

$$\text{Type-VI}: \quad H_6(x) = \begin{cases} \exp\{x\}; & x \leq 0, \\ 1; & x > 0. \end{cases} \tag{1.4}$$

Nasri-Roudsari (1999) [28] demonstrated that the six p-max stable laws in can be represented as two families. We call them log-GEVL distribution in positive support, and negative log-GEVL distribution in negative support, i.e.:

(**1**) For $x^0 > 0$, $x > 0$ and $1 + \frac{\xi}{\sigma}\log(e^{-\mu}x) > 0$

$$H_{\xi,1}(x;\mu,\sigma) = \begin{cases} \exp\{-(1 + \frac{\xi}{\sigma}\log(e^{-\mu}x))^{\frac{-1}{\xi}}\}; & \xi \neq 0, \\ \exp\{-(xe^{-\mu})^{\frac{1}{\sigma}}\}; & \xi \to 0. \end{cases} \tag{1.5}$$

(**2**) For $x^0 \leq 0$, $x \leq 0$ and $1 - \frac{\xi}{\sigma}\log(-e^{-\mu}x) > 0$

$$H_{\xi,2}(x;\mu,\sigma) = \begin{cases} \exp\{-(1 - \frac{\xi}{\sigma}\log(-e^{-\mu}x))^{\frac{-1}{\xi}}\}; & \xi \neq 0, \\ \exp\{-(-xe^{-\mu})^{\frac{-1}{\sigma}}\}; & \xi \to 0. \end{cases} \tag{1.6}$$

The corresponding density function to Equation (1.5) can be formulated as

$$h_\xi(x;\mu,\sigma,\xi) = \begin{cases} \frac{1}{\sigma x}\exp\{-(1 + \frac{\xi}{\sigma}\log(xe^{-\mu})\operatorname{sign}(x))^{\frac{-1}{\xi}}\}((1 + \frac{\xi}{\sigma}\log(xe^{-\mu})\operatorname{sign}(x)))^{\frac{-1}{\xi}-1}; & \xi \neq 0, \\ \exp\{-(xe^{-\mu}\operatorname{sign}(x))^{\frac{-1}{\sigma}}\}\,\frac{e^{-\mu}}{\sigma}(xe^{-\mu}\operatorname{sign}(x))^{\frac{-1}{\sigma}-1}; & \xi \to 0. \end{cases} \tag{1.7}$$

The results of Gnedenko *et al.* (1943) [21] and De Haan (1971) [13] concerning linear normalization were extended to p-max stable laws. They showed that every CDF attracted to linear max stable law is necessarily attracted to some p-max stable, and that p-max stable laws, in fact attract more. For more information about the extreme under power normalization and its applications, see Galambos (1987) [18], Nasri-Roudsari (1999) [28], Barakat *et al.* (2010 [2], 2013 [3], 2014a [4], 2014b [5], 2015 [6], 2019 [7]), among others.

In mathematical physics and probability, the q-distribution is more general than classical distribution. It was introduced by Diaz and Pariguan (2009) [12] and Diaz *et al.* (2010) [11] in the continuous case, and by Charalambides (2010) [9] in the discrete version. The construction of a q-distribution is the construction of a q-analogue of ordinary distribution. Mathai and Provost (2006) [27] introduced the q-analogue of the gamma distribution with respect to Lebesgue measure. Recently, several q-type super statistical distributions such as the

q-exponential, q-Weibull, and q-logistic were developed in the context of statistical mechanics, information theory and reliability modelling, as discussed for instance in Chung *et al.* (1994) [10], Picoli *et al.* (2003) [24], Gauchman (2004) [20], De Sole and Kac (2003) [31], Mathai (2005) [26], Srivastava and Choi (2012) [30], among others. Provost *et al.* (2018) [23] introduced the CDF and PDF of q-generalized extreme value under linear normalization (q-GEVL) and q-Gumbel distributions as

$$F(x;\mu,\sigma,\xi,q)=\begin{cases}[1+q(\xi(sx-m)+1)^{-\frac{1}{\xi}}]^{-\frac{1}{q}}; & \xi\neq 0,\quad q\neq 0,\\ (1+qe^{-(sx-m)})^{-\frac{1}{q}}; & \xi\to 0,\quad q\neq 0,\end{cases} \tag{1.8}$$

and

$$f(x;\mu,\sigma,\xi,q)=\begin{cases}s(1+\xi(sx-m))^{\frac{-1}{\xi}-1}[1+q(\xi(sx-m)+1)^{-\frac{1}{\xi}}]^{-\frac{1}{q}-1}; & \xi\neq 0,\quad q\neq 0,\\ (1+qe^{-(sx-m)})^{-\frac{1}{q}-1}se^{-(sx-m)}; & \xi\to 0,\quad q\neq 0,\end{cases} \tag{1.9}$$

where $s=\frac{1}{\sigma}$ and $m=\frac{\mu}{\sigma}$. In this paper, we propose the q-analogues of the generalized extreme value under power normalization (q-GEVP) to construct heavy-tailed distributions for modeling real data; to propose various types of the hazard rate function; and to generate flexible distributions with left-skewed and right-skewed shape, which can be utilized effectively in modeling extreme observations.

The paper is organized as follows. In Section 2, the q-GEVP model is reported. Some mathematical properties such as quantile function, moments, moment generating function and Shanon entropy are derived in Section 3. Section 4, explains how to determine the maximum likelihood, Cramer-von Mises minimum distance, ordinary and weighted least-square estimators of the model parameters. A Monte Carlo simulation study is carried out in Section 5, to compare the behavior of the different estimation techniques which used in the estimation of the unknown parameters of the model. In Section 6, we fit some models to COVID-19 in three countries, Japan, Saudi Arabia and Romania. Also, some statistics are employed in order to assess goodness of fit. Finally, some concluding remarks are introduced in the last section.

2. ON q-GENERALIZED EXTREME DISTRIBUTION UNDER POWER NORMALIZATION

The CDF and PDF of the q-GEVP model and q-distribution “$\xi\to 0$” are, respectively, given by:

(1) For $x^0>0$, $x>0$ and $1+\frac{\xi}{\sigma}\log(e^{-\mu}x)>0$

$$H_{q,\xi,1}(x;\mu,\sigma)=\begin{cases}(1+q(1+\frac{\xi}{\sigma}\log(e^{-\mu}x))^{\frac{-1}{\xi}})^{\frac{-1}{q}}; & \xi\neq 0,\quad q\neq 0,\\ (1+q(xe^{-\mu})^{\frac{-1}{\sigma}})^{\frac{-1}{q}}; & \xi\to 0,\quad q\neq 0,\end{cases} \tag{2.1}$$

and
(2.2)

$$h_{q,\xi,1}(x;\mu,\sigma,\xi)=\begin{cases}\left(1+q\left(1+\frac{\xi}{\sigma}\log(e^{-\mu}x)\right)^{\frac{-1}{\xi}}\right)^{\frac{-1}{q}-1}\frac{1}{\sigma x}\left(1+\frac{\xi}{\sigma}\log(e^{-\mu}x)\right)^{\frac{-1}{\xi}-1}; & \xi\neq 0,\ q\neq 0,\\ \left(1+q(xe^{-\mu})^{\frac{-1}{\sigma}}\right)^{\frac{-1}{q}-1}\frac{e^{-\mu}}{\sigma}(xe^{-\mu})^{\frac{-1}{\sigma}-1}; & \xi\to 0,\ q\neq 0.\end{cases}$$

(**2**) For $x^0\leq 0$, $x\leq 0$ and $1-\frac{\xi}{\sigma}\log(-e^{-\mu}x)>0$

$$H_{q,\xi,2}(x;\mu,\sigma)=\begin{cases}\left(1+q\left(1-\frac{\xi}{\sigma}\log(-xe^{-\mu})\right)^{\frac{-1}{\xi}}\right)^{\frac{-1}{q}}; & \xi\neq 0,\ q\neq 0,\\ \left(1+q(-xe^{-\mu})^{\frac{1}{\sigma}}\right)^{\frac{-1}{q}}; & \xi\to 0,\ q\neq 0,\end{cases}\tag{2.3}$$

and
(2.4)

$$h_{q,\xi,2}(x;\mu,\sigma,\xi)=\begin{cases}\left(1+q\left(1-\frac{\xi}{\sigma}\log(-xe^{-\mu})\right)^{\frac{-1}{\xi}}\right)^{\frac{-1}{q}-1}\frac{1}{\sigma x}\left(1-\frac{\xi}{\sigma}\log(-xe^{-\mu})\right)^{\frac{-1}{\xi}-1}; & \xi\neq 0,\ q\neq 0,\\ \left(1+q(-xe^{-\mu})^{\frac{1}{\sigma}}\right)^{\frac{-1}{q}-1}\frac{e^{-\mu}}{\sigma}(-xe^{-\mu})^{\frac{1}{\sigma}-1}; & \xi\to 0,\ q\neq 0,\end{cases}$$

where $x^0=\sup\{x:F(x)<1\}$. Figures 1 and 2 show the PDF of the q-GEVP model in case of $\xi\neq 0$ and $\xi\to 0$, respectively, for various values of the parameters.

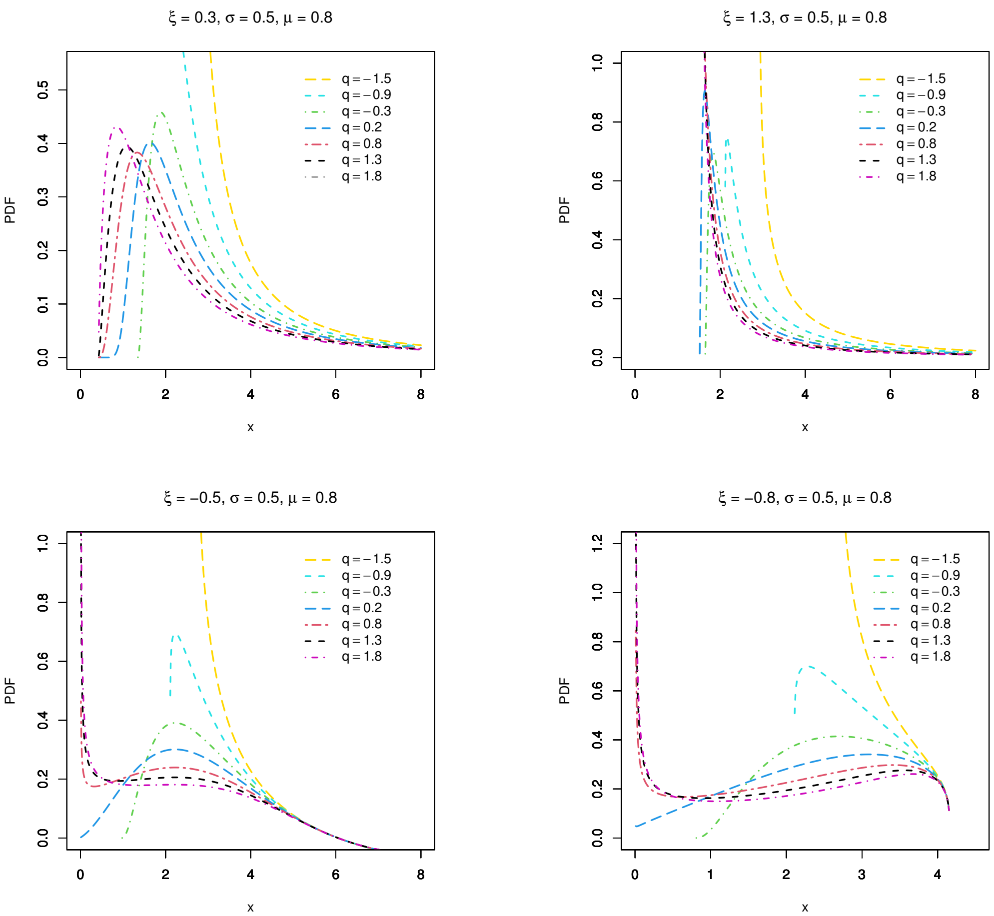

Figure 1: The PDF plots of the q-GEVP distribution in case of $\xi\neq 0$.

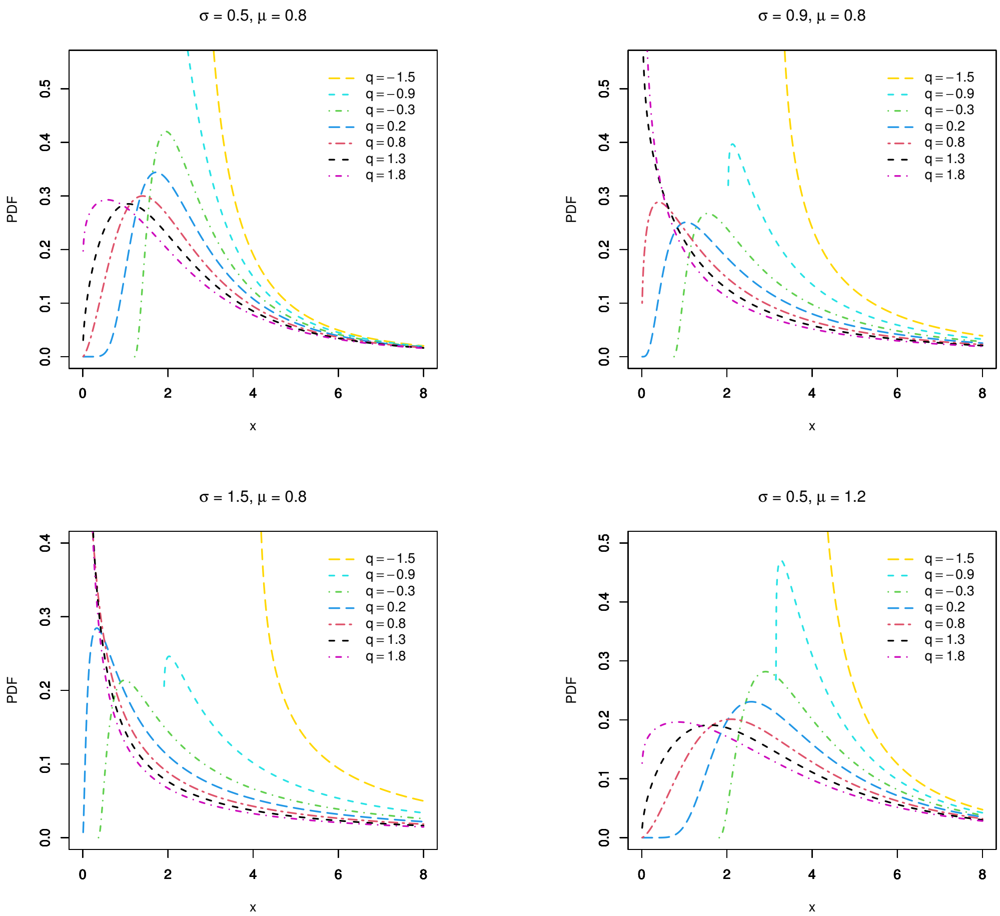

Figure 2: The PDF plots of the q-GEVP distribution in case of $\xi \to 0$.

According to Figures 1 and 2, it is noted that the proposed distribution can be used to model left and right skewed data. Moreover, the shape of the PDF can be unimodal and bimodal, which makes the proposed model can be utilized for modeling various data in different fields.

The hazard function (also called the force of mortality, instantaneous failure rate, instantaneous death rate, or age-specific failure rate) is a way to model data distribution in survival analysis. The most common use of the function is to model a participant's chance of death as a function of their age. However, it can be used to model any other time-dependent event of interest. The hazard function (HF) is defined as $\frac{h(x)}{1-H(x)}$. Figures 3 and 4 display the HF for the proposed model for $\xi \neq 0$ and $\xi \to 0$, respectively, and it is noted that the HF has various shapes including increasing, decreasing, unimodal, or bathtub.

In several cases, lifetimes need to be recorded on a discrete scale rather than on a continuous analogue. Due to the previous reason, discretizing continuous distributions has received much attention in the statistical literature. See for example, Bebbington *et al.* (2012) [8], Nekoukhou and Bidram (2015) [29], El-Morshedy *et al.* (2020) [15], Eliwa *et al.* (2020) [16], Altun *et al.* (2020) [1], and references cited therein. Based on discretization survival approach, the CDF and probability mass function (PMF) of the discrete q-GEVP

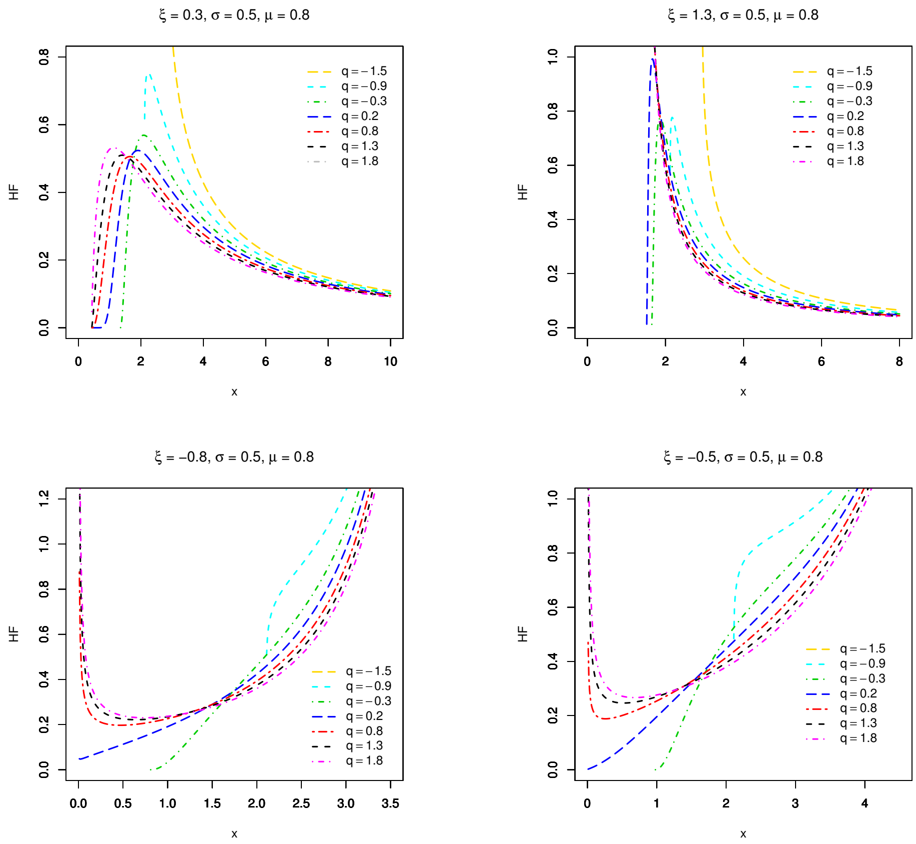

Figure 3: The HRF plots of the q-GEVP distribution in case of $\xi \neq 0$.

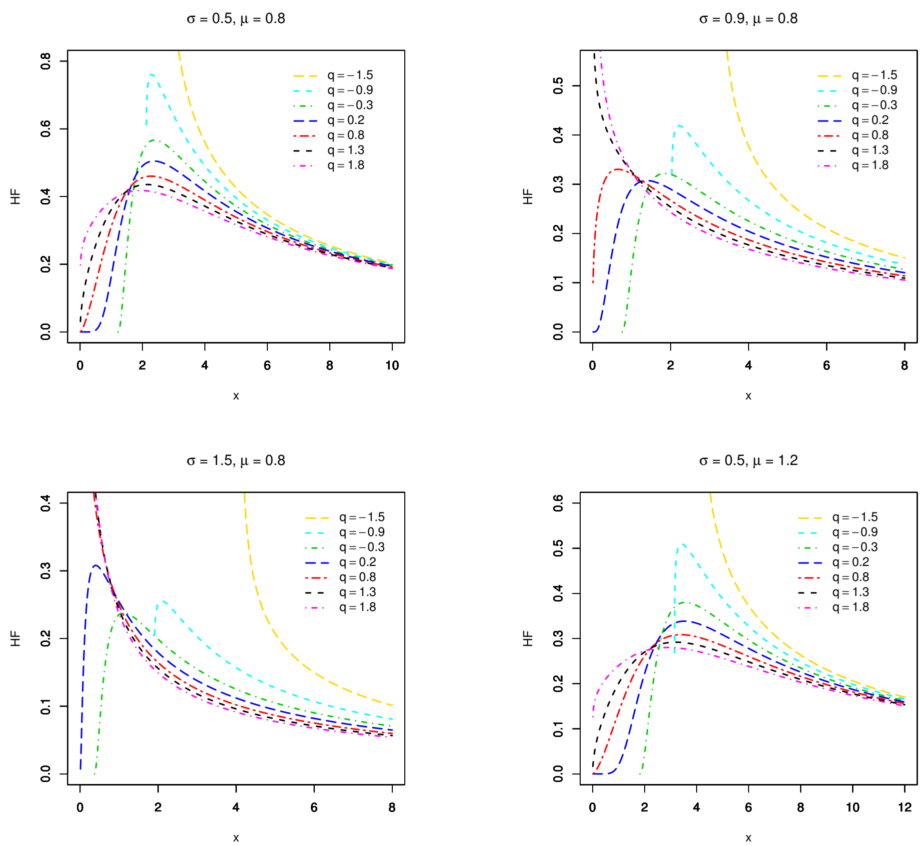

Figure 4: The HRF plots of the q-GEVP distribution in case of $\xi \to 0$.

(Dq-GEVP) model can be formulated as

$$H_{q,\xi,1}(x;\mu,\sigma)=\begin{cases}(1+q(1+\frac{\xi}{\sigma}\log(e^{-\mu}(x+1)))^{\frac{-1}{\xi}})^{\frac{-1}{q}}; & \xi\neq 0,\ \ q\neq 0,\\ (1+q((x+1)e^{-\mu})^{\frac{-1}{\sigma}})^{\frac{-1}{q}}; & \xi\to 0,\ \ q\neq 0,\end{cases} \tag{2.5}$$

and

$$f(x;\mu,\sigma,\xi,q)=\begin{cases}(1+q(1+\frac{\xi}{\sigma}\log(e^{-\mu}(x+1)))^{\frac{-1}{\xi}})^{\frac{-1}{q}}-(1+q(1+\frac{\xi}{\sigma}\log(e^{-\mu}x))^{\frac{-1}{\xi}})^{\frac{-1}{q}}; \\ \qquad\qquad \xi\neq 0,\ \ q\neq 0,\\ (1+qe^{-(sx-m)})^{-\frac{1}{q}-1}se^{-(sx-m)}-(1+q(xe^{-\mu})^{\frac{-1}{\sigma}})^{\frac{-1}{q}}; \\ \qquad\qquad \xi\to 0,\ \ q\neq 0,\end{cases} \tag{2.6}$$

respectively. Figure 18 shows the PMF and HRF of the Dq-GEVP models for various values of the model parameters, and it is found that the PMF can be used to model asymmetric data which have extreme observations. Further, the HRF can be utilized to model data with have decreasing failure shape.

3. STATISTICAL PROPERTIES

3.1. Quantile function and moments

The quantile function (QF) is frequently utilized for determining confidence intervals or eliciting certain properties of a distribution. In order to obtain the QF of a random variable (RV) X, that is, one has to solve the equation $F(x)=p$ with respect to x for some fixed $p\in(0,1)$, where $F(x)$ denotes the CDF of X. The QFs of the q-GEVP ($\xi\neq 0$) and q-distribution ($\xi\to 0$) can be listed as

$$x_p=H^{-1}(q,\xi,1)=\begin{cases}e^{\frac{\sigma}{\xi}(q^{\xi}(p^{-q}-1)^{-\xi}-1)+\mu}; & \xi\neq 0,\ \ q\neq 0,\\ q^{\sigma}(p^{-q}-1)^{-\sigma}e^{\mu}; & \xi\to 0,\ \ q\neq 0,\end{cases} \tag{3.1}$$

and

$$x_p=H^{-1}(q,\xi,2)=\begin{cases}-e^{\frac{\sigma}{\xi}(1-q^{\xi}(p^{-q}-1)^{-\xi})+\mu}; & \xi\neq 0,\ \ q\neq 0,\\ -q^{-\sigma}(p^{-q}-1)^{\sigma}e^{\mu}; & \xi\to 0,\ \ q\neq 0,\end{cases} \tag{3.2}$$

respectively. Assume non-negative RV have a q-GEVP model, then the n-th moment, and moment generating function of X, are given, respectively, as follows:

$$\begin{aligned}\mathbf{E}(X^n)&=\int_0^{\infty}x^n h(x;\mu,\sigma,\xi)\ dx\\ &=\Upsilon_{(n,\mu)}^{(\sigma,\xi,q)}\sum_{j=0}^{\infty}\left(\frac{n\sigma q^{\xi}}{\xi}\right)^j\frac{\Gamma(1-\xi j)\Gamma\left(\frac{1}{q}+\xi j\right)}{j!\Gamma\left(\frac{1}{q}+1\right)}\end{aligned}$$

and

$$\begin{aligned}\mathbf{M}_X(t) &= \int_0^\infty \exp(tx)h(x;\mu,\sigma,\xi)\; dx \\ &= \Theta_{(\mu)}^{(\sigma,\xi)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} t^j \left(\frac{\sigma q^\xi j}{\xi}\right)^k \frac{\Gamma(1-\xi k)\Gamma\left(\frac{1}{q}+\xi k\right)}{j!k!\Gamma\left(\frac{1}{q}+1\right)},\end{aligned}$$

where $\Theta_{(\mu)}^{(\sigma,\xi)} = \frac{1}{2}\exp\Big(t\exp\Big\{\mu - \frac{\sigma}{\xi}\Big\}\Big)$, $\Upsilon_{(n,\mu)}^{(\sigma,\xi,q)} = \frac{1}{q}\exp\Big\{n\Big(\mu - \frac{\sigma}{\xi}\Big)\Big\}$, and the terms $(1-\xi j)$, $\left(\frac{1}{q}+\xi j\right)$, $\left(\frac{1}{q}+1\right)$, $(1-\xi k)$, $\left(\frac{1}{q}+\xi k\right)$ and $\left(\frac{1}{q}+1\right)$ should be greater than 0. Figure 5 shows the skewness and kurtosis under different values of the model parameters "$\xi = -0.5$ and $\sigma = 0.2$" in the left panel, and "$\xi = -1.5$ and $\sigma = 1.2$" in the right panel, respectively.

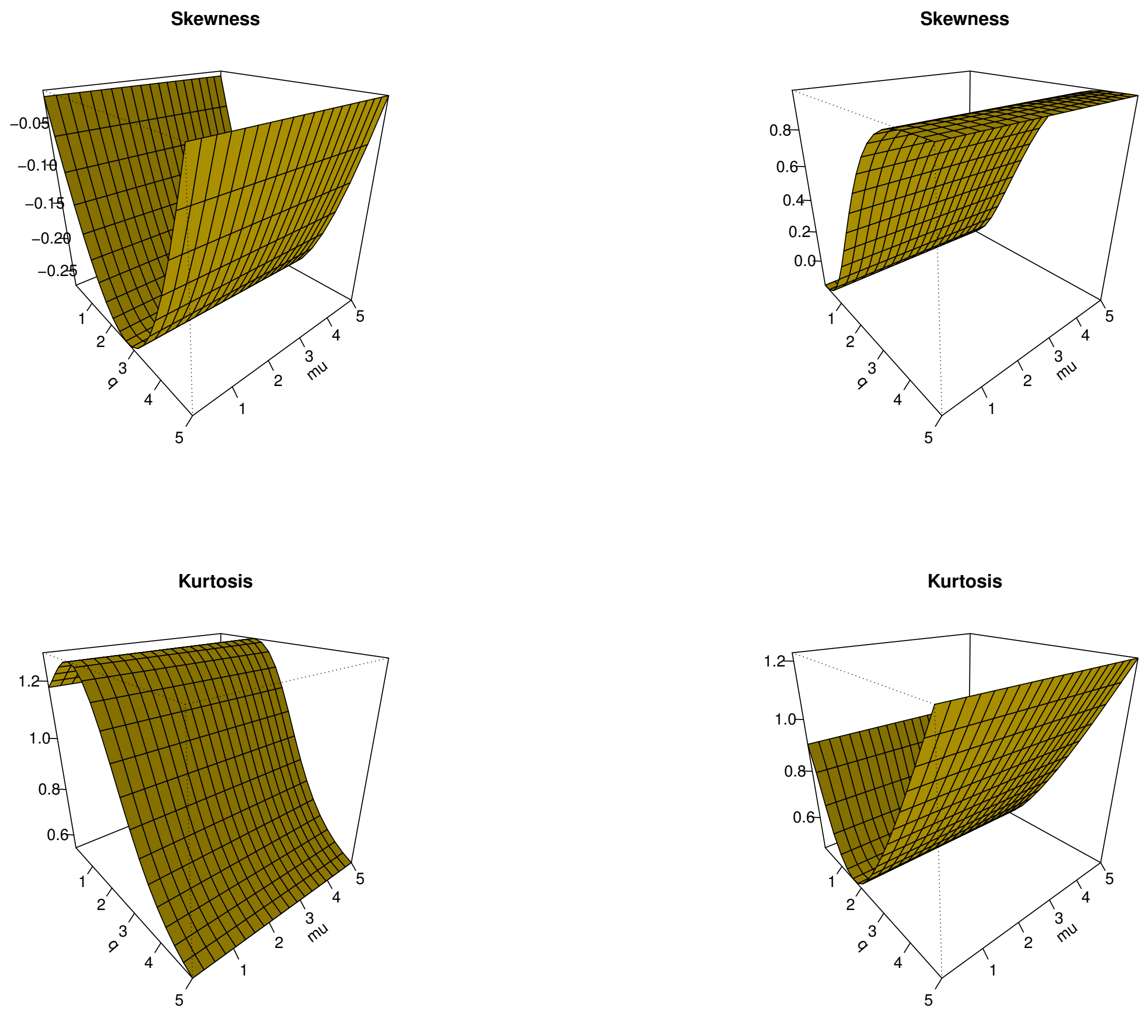

Figure 5: The skewness and kurtosis of the q-GEVP distribution in case of $\xi \neq 0$.

Figure 6 shows the skewness and kurtosis in case of $\xi \to 0$ with $\sigma = 0.5$ "left panel" and $\sigma = 2.5$ "right panel", respectively, which support our results.

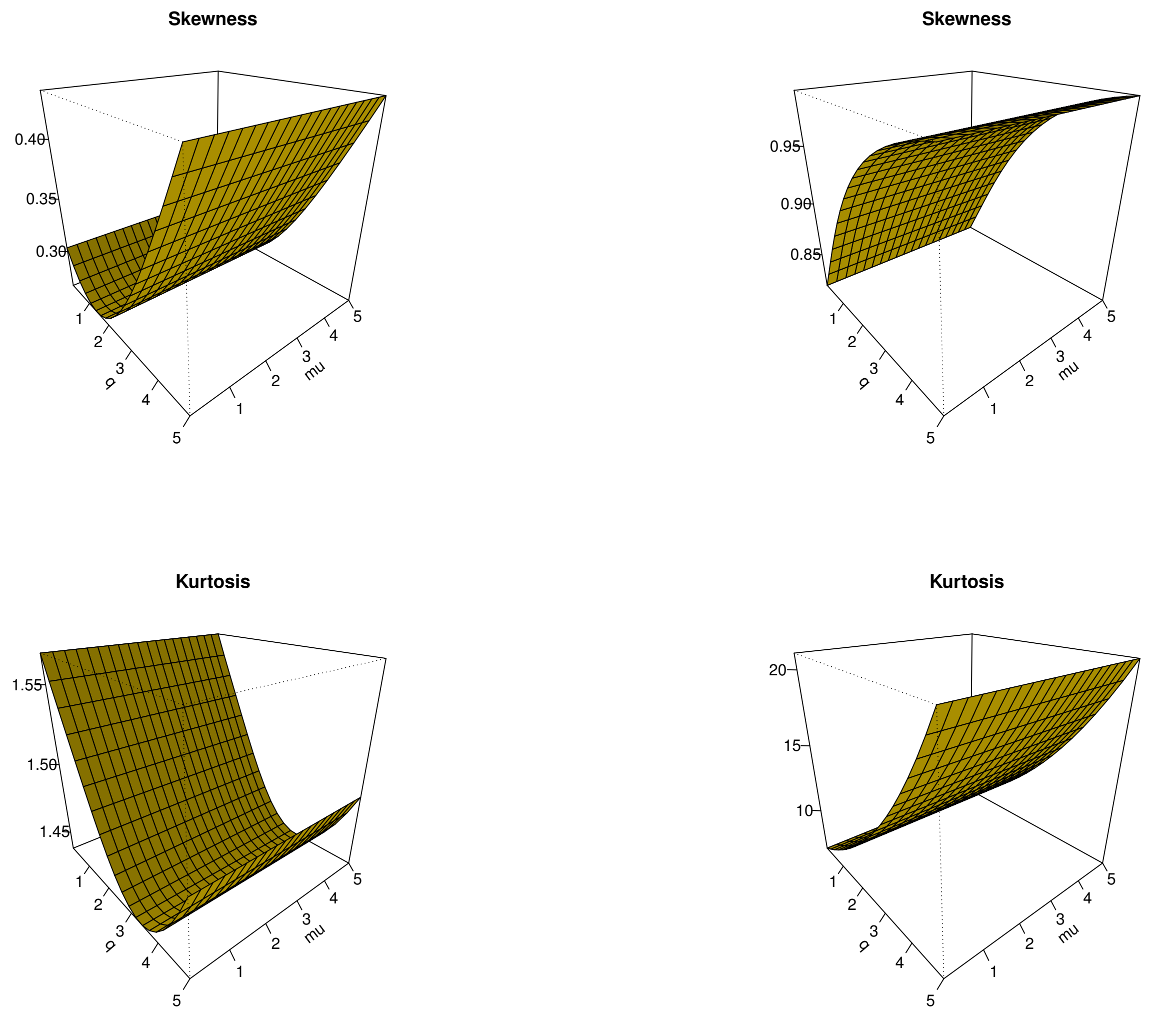

Figure 6: The skewness and kurtosis of the q-GEVP distribution in case of $\xi \to 0$.

3.2. Entropy

An entropy of RV X is a measure of variation of the uncertainty. Shannon entropy (SnEy) is defined by

$$H(X) = -\int_A f(x) \log f(x) dx, \tag{3.3}$$

where $A = x : f(x) > 0$. The SnEy of the GEVL family can be expressed as

$$H(X) = \log \hat{\sigma} + (\hat{\xi} + 1)\gamma + 1. \tag{3.4}$$

The SnEy of six classes of extreme value distributions which was mentioned in Section 1, is evaluated by Ravi and Saeb (2012) [25]. Herein, the SnEy of GEVP, q-GEVL and q-GEVP families are listed in the following theorems.

Theorem 3.1. *If X is a RV with CDF GEVP for $\xi<0$, then the SnEy of X is given by*

$$H(X)=\mu+\log\hat{\sigma}+(\hat{\xi}+1)\gamma+\frac{\widehat{\sigma}}{\widehat{\xi}}[\Gamma(1-\widehat{\xi})-1]+1. \tag{3.5}$$

Proof: From Equations (1.7) and (3.3), we have

$$H(X)=-\mathbf{E}(\log h_{\mu,\xi,\sigma,q})=\log\sigma+\mathbf{E}(\log|X|)+\Delta_{(\mu,\xi,\sigma)}+\Delta^{(\mu,\xi,\sigma)},$$

where $\Delta_{(\mu,\xi,\sigma)}=\mathbf{E}(1+\frac{\xi}{\sigma}\log|X|e^{-\mu})^{\frac{-1}{\xi}}$ and $\Delta^{(\mu,\xi,\sigma)}=(1+\frac{1}{\xi})\mathbf{E}(\log(1+\frac{\xi}{\sigma}\log|X|e^{-\mu}))$. Let $Y=(1+\frac{\xi}{\sigma}\log|x|e^{-\mu}))^{\frac{-1}{\xi}}$, which have the standard exponential (StEx) distribution, then

$$\mathbf{E}(\log|X|)=\frac{\sigma}{\xi}\mathbf{E}(Y^{-\xi}-1)+\mu=\frac{\sigma}{\xi}[\Gamma(1-\xi)-1]+\mu, \tag{3.6}$$

$$\Delta_{(\mu,\xi,\sigma)}=\mathbf{E}(Y)=1 \tag{3.7}$$

and

$$\Delta^{(\mu,\xi,\sigma)}=(1+\frac{1}{\xi})\mathbf{E}(-\xi\log Y)=-(1+\xi)\mathbf{E}(\log Y)=(1+\xi)\gamma, \tag{3.8}$$

where $\gamma=-\int_0^\infty\log ye^{-y}dy$. From Equations (3.6)–(3.8), Equation (3.5) can be derived. □

Theorem 3.2. *If X is a RV with CDF q-GEVL for $\xi<0$, then the SnEy of X is given by*

$$H(X)=\log\hat{\sigma}+(\hat{\xi}+1)\gamma+(1+q)\left[1-\sum_{n=2}^{\infty}(-1)^{n+1}q^{n-1}\Gamma(n-1)\right]. \tag{3.9}$$

Proof: Since the PDF of the q-GEVL model can be listed as

$$f_X(x)=\frac{1}{\sigma}(1+\frac{\xi}{\sigma}(x-\mu))^{-\frac{1}{\xi}-1}[1+q(1+\frac{\xi}{\sigma}(x-\mu))^{-\frac{1}{\xi}}]^{-\frac{1}{q}-1}.$$

Then,

$$H(X)=-\mathbf{E}(\log f_X(X))=\log\sigma+\Delta^*_{(\mu,\xi,\sigma)}+\Delta_*^{(\mu,\xi,\sigma)},$$

where $\Delta^*_{(\mu,\xi,\sigma)}=(1+\frac{1}{\xi})\mathbf{E}(\log(1+\frac{\xi}{\sigma}(X-\mu)))$ and $\Delta_*^{(\mu,\xi,\sigma)}=(1+\frac{1}{q})\mathbf{E}(\log(1+q(1+\frac{\xi}{\sigma}(X-\mu))^{\frac{-1}{\xi}})$. Assume $Y=(1+\frac{\xi}{\sigma}(x-\mu))^{\frac{-1}{\xi}}$, which have the StEx distribution, then Equation (3.9) can be derived. □

Theorem 3.3. *If X is a RV with CDF q-GEVP for $\xi<0$, then the SnEy of X is given by*

$$H(X)=\mu+\log\hat{\sigma}+(\hat{\xi}+1)\gamma+\frac{\xi}{\sigma}\mathbf{E}\{\mathbf{sign}(X)[\Gamma(1-\xi)-1]\}+(1+q)\left[1-\sum_{n=2}^{\infty}(-1)^{n+1}q^{n-1}\Gamma(n-1)\right]. \tag{3.10}$$

Proof: Since the RV have the q-GEVP distribution, then

$$H(X) = \log\sigma + \mathbf{E}(\log|X|) + \mathbf{E}\left(1+\frac{\xi}{\sigma}\log|X|e^{-\mu}\right)^{\frac{-1}{\xi}} + \left(1+\frac{1}{\xi}\right)\Omega_{(\mu,\xi,\sigma)} + \left(1+\frac{1}{q}\right)\Omega^{(\mu,\xi,\sigma)},$$

where $\Omega_{(\mu,\xi,\sigma)} = \mathbf{E}\Big(\log(1+\frac{\xi}{\sigma}\log|X|e^{-\mu})\Big)$ and $\Omega^{(\mu,\xi,\sigma)} = \mathbf{E}(\log(1+q(1+\frac{\xi}{\sigma}\log|X|e^{-\mu})^{\frac{-1}{\xi}})$.

Let $Y = \left(1+\frac{\xi}{\sigma}\log|x|e^{-\mu}\right)^{\frac{-1}{\xi}}$ which have the StEx distribution, then

$$\mathbf{E}(\log|X|) = \frac{\sigma}{\xi}\mathbf{E}(Y^{-\xi}-1)+\mu = \frac{\sigma}{\xi}[\Gamma(1-\xi)-1]+\mu, \tag{3.11}$$

$$(1+\frac{1}{\xi})\Omega_{(\mu,\xi,\sigma)} = (1+\frac{1}{\xi})\mathbf{E}(-\xi\log Y) = (1+\xi)\gamma \tag{3.12}$$

and

$$\begin{aligned}\left(1+\frac{1}{q}\right)\Omega^{(\mu,\xi,\sigma)} &= (1+\frac{1}{q})\int_0^\infty \log(1+qy))e^{-y}dy\\ &= (1+\frac{1}{q})\left[1-\sum_{n=2}^{\infty}(-1)^{n+1}q^{n-1}\Gamma(n-1)\right].\end{aligned} \tag{3.13}$$

From Equations (3.11)–(3.13), Equation (3.10) can be derived. □

Hint: If $q \to 0$ in Equations (3.9) and (3.10), we get Equations (3.4) and (3.5).

4. VARIOUS ESTIMATION APPROACHES

4.1. Maximum likelihood estimation

In order to estimate the parameters of the q-GEVP model and q-distribution whose density functions are in (2.2), one has to maximize their respective log-likelihood functions with respect to the model parameters. Given the observations $x_i, i = 1, ..., n$, the log-likelihood functions of the q-GEVP model and q-distribution are, respectively, given by

$$\ell(\mu,\ \sigma,\ \xi,\ q) = -n\log\sigma - \sum_{i=1}^{n}\log x_i - (1+\frac{1}{q})\sum_{i=1}^{n}\log[1+q\ A_i^{\frac{-1}{\xi}}] - (1+\frac{1}{\xi})\sum_{i=1}^{n}\log A_i \tag{4.1}$$

and

$$\ell^*(\mu,\ \sigma,\ q) = -n\log\sigma + \frac{n\mu}{\sigma} - (1+\frac{1}{\sigma})\sum_{i=1}^{n}\log x_i - (1+\frac{1}{q})\sum_{i=1}^{n}\log[1+q\ B_i^{\frac{-1}{\sigma}}], \tag{4.2}$$

where $A_i = 1+\frac{\xi}{\sigma}\log B_i$ and $B_i = x_i e^{-\mu}$. The associated log-likelihood system of equations are, respectively,

$$\frac{\partial \ell}{\partial \mu} = (\frac{\xi+1}{\sigma})\sum_{i=1}^{n} A_i^{\frac{1}{\xi}} - (\frac{q+1}{\sigma})\sum_{i=1}^{n}\frac{A_i^{\frac{-1}{\xi}-1}}{1+qA_i^{\frac{-1}{\xi}}},$$

$$\frac{\partial \ell}{\partial \sigma} = -\frac{n}{\sigma} + (\frac{\xi+1}{\sigma^2})\sum_{i=1}^{n}\frac{\log B_i}{A_i} - (\frac{q+1}{\sigma^2})\sum_{i=1}^{n}\frac{A_i^{\frac{-1}{\xi}-1}\log B_i}{1+qA_i^{\frac{-1}{\xi}}},$$

$$\frac{\partial \ell}{\partial \xi} = \frac{1}{\xi^2}\sum_{i=1}^{n}\log A_i - (\frac{\xi+1}{\xi\sigma})\sum_{i=1}^{n}\frac{\log B_i}{A_i} + (\frac{q+1}{\xi\sigma})\sum_{i=1}^{n}\frac{A_i^{\frac{-1}{\xi}-1}\log B_i}{1+qA_i^{\frac{-1}{\xi}}} - (\frac{q+1}{\xi^2})\sum_{i=1}^{n}\frac{A_i^{\frac{-1}{\xi}}\log A_i}{1+qA_i^{\frac{-1}{\xi}}},$$

$$\frac{\partial \ell}{\partial q} = \frac{1}{q^2}\sum_{i=1}^{n}\log[1+qA_i^{\frac{-1}{\xi}}] - (\frac{1}{q}+1)\sum_{i=1}^{n}\frac{A_i^{\frac{-1}{\xi}}}{1+qA_i^{\frac{-1}{\xi}}} \tag{4.3}$$

and

$$\frac{\partial \ell^*}{\partial \mu} = \frac{n}{\sigma} - (\frac{1+q}{\sigma})\sum_{i=1}^{n}\frac{B_i^{\frac{-1}{\sigma}}}{1+qB_i^{\frac{-1}{\sigma}}},$$

$$\frac{\partial \ell^*}{\partial \sigma} = -\frac{n}{\sigma} - \frac{n\mu}{\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n}\log x_i - (\frac{q+1}{\sigma^2})\sum_{i=1}^{n}\frac{B_i^{\frac{-1}{\sigma}}\log B_i}{1+qB_i^{\frac{-1}{\sigma}}},$$

$$\frac{\partial \ell^*}{\partial q} = \frac{1}{q^2}\sum_{i=1}^{n}\log[1+qB_i^{\frac{-1}{\sigma}}] - (\frac{1}{q}+1)\sum_{i=1}^{n}\frac{B_i^{\frac{-1}{\sigma}}}{1+qB_i^{\frac{-1}{\sigma}}}. \tag{4.4}$$

Solving the nonlinear systems specified by the sets of equations yields the maximum likelihood estimates (MLE's) of the parameters of the q-GEVP model and q-distribution. Since these equations cannot be solved analytically; iterative method such as the Newton–Raphson technique is required.

4.2. Ordinary and weighted least-square estimators

Let $x_{(1)}, x_{(2)}, ..., x_{(r)}$ be the order statistics (OS) of the random sample of size r from $F(x; q, \xi, \sigma, \mu)$. The least square estimators (LSEs) of the q-GEVP parameters, say, $\widehat{q}_{LS}$, $\widehat{\xi}_{LS}$, $\widehat{\sigma}_{LS}$ and $\widehat{\mu}_{LS}$ can be obtained by solving the non-linear equations

$$\sum_{d=1}^{r}\left[F\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big) - \frac{d}{r+1}\right]\Delta_{\varrho}\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big) = 0, \quad \varrho = 1,2,3,4,$$

where

$$\begin{cases} \Delta_1\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big) = \frac{\partial}{\partial q}F\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big), & \Delta_2\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big) = \frac{\partial}{\partial \xi}F\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big), \\ \Delta_3\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big) = \frac{\partial}{\partial \sigma}F\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big), & \Delta_4\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big) = \frac{\partial}{\partial \mu}F\big(x_{(d)}\,|\,q,\xi,\sigma,\mu\big). \end{cases} \tag{4.5}$$

Whereas the weighted least squares estimators (WLSEs), say, $\widehat{q}_{WLS}$, $\widehat{\xi}_{WLS}$, $\widehat{\sigma}_{WLS}$ and $\widehat{\mu}_{WLS}$ can be reported by solving the non-linear equations

$$\sum_{d=1}^{r} \frac{(r+1)^2(r+2)}{d(r-d+1)} \left[F\left(x_{(d)} \,|\, q, \xi, \sigma, \mu\right) - \frac{d}{r+1} \right] \Delta_{\varrho}\left(x_{(d)} \,|\, q, \xi, \sigma, \mu\right) = 0, \quad \varrho = 1, 2, 3, 4,$$

where $\Delta_1(\cdot \,|\, q, \xi, \sigma, \mu)$, $\Delta_2(\cdot \,|\, q, \xi, \sigma, \mu)$, $\Delta_3(\cdot \,|\, q, \xi, \sigma, \mu)$ and $\Delta_4(\cdot \,|\, q, \xi, \sigma, \mu)$ are provided in Equation (4.5).

4.3. Cramer-von Mises minimum distance estimators

The CVMEs of the q-GEVP parameters are derived by solving the non-linear equations

$$\sum_{d=1}^{r} \left[F\left(x_{(d)} \,|\, q, \xi, \sigma, \mu\right) - \frac{2d-1}{2r} \right] \Delta_{\varrho}\left(x_{(d)} \,|\, q, \xi, \sigma, \mu\right) = 0, \quad \varrho = 1, 2, 3,$$

where $\Delta_1(\cdot \,|\, q, \xi, \sigma, \mu)$, $\Delta_2(\cdot \,|\, q, \xi, \sigma, \mu)$, $\Delta_3(\cdot \,|\, q, \xi, \sigma, \mu)$ and $\Delta_4(\cdot \,|\, q, \xi, \sigma, \mu)$ are defined in Equation (4.5).

5. THE MONTE CARLO SIMULATION STUDY

Here, we have conducted a Monte Carlo simulation study to compare the behavior of the different estimation techniques (MLEs, LSEs, WLSEs, and CVMEs) used in the estimation of the unknown parameters of the q-GEVP model in case of $\xi \neq 0$, and $\xi \to 0$. We have drawn 1000 samples of size $n = 20, 50, 100, 150, 200, 250, 300, 500$ from q-GEVP$(0.5, 0.5, 0.8, 0.5)$ and q-GEVP$(0.8, \xi \to 0, 0.5, 0.3)$, respectively, through the **R** software. We have calculated the MLEs, LSEs, WLSEs, and CVMEs for each of the 1000 samples, say, $\widehat{q}_k, \widehat{\xi}_k, \widehat{\sigma}_k$ and $\widehat{\mu}_k$ for $k = 1, 2, ..., 1000$. We have calculated the biases and mean-squared errors (MSEs) for $\Upsilon = q, \xi, \sigma$, and μ through the following formulas

$$\text{Bias} = \frac{1}{1000} \sum_{k=1}^{1000} \left(\widehat{\Upsilon}_k - \Upsilon \right) \text{ and MSE} = \frac{1}{1000} \sum_{k=1}^{1000} \left(\widehat{\Upsilon}_k - \Upsilon \right)^2.$$

The empirical results are given in Figures 7 and 8.

From Figures 7 and 8 the following observations can be made:

1. As the value of n increases, the magnitude of the bias decreases towards zero.
2. The MSEs of all the estimators decrease when we increase the value of the sample size n. This finding supports the first-order asymptotic theory.
3. In view of MSEs, clearly, MLE, LSE, WLSE, and CVME techniques perform satisfactorily in the estimation of q-GEVP parameters in case of $\xi \neq 0$, and $\xi \to 0$.

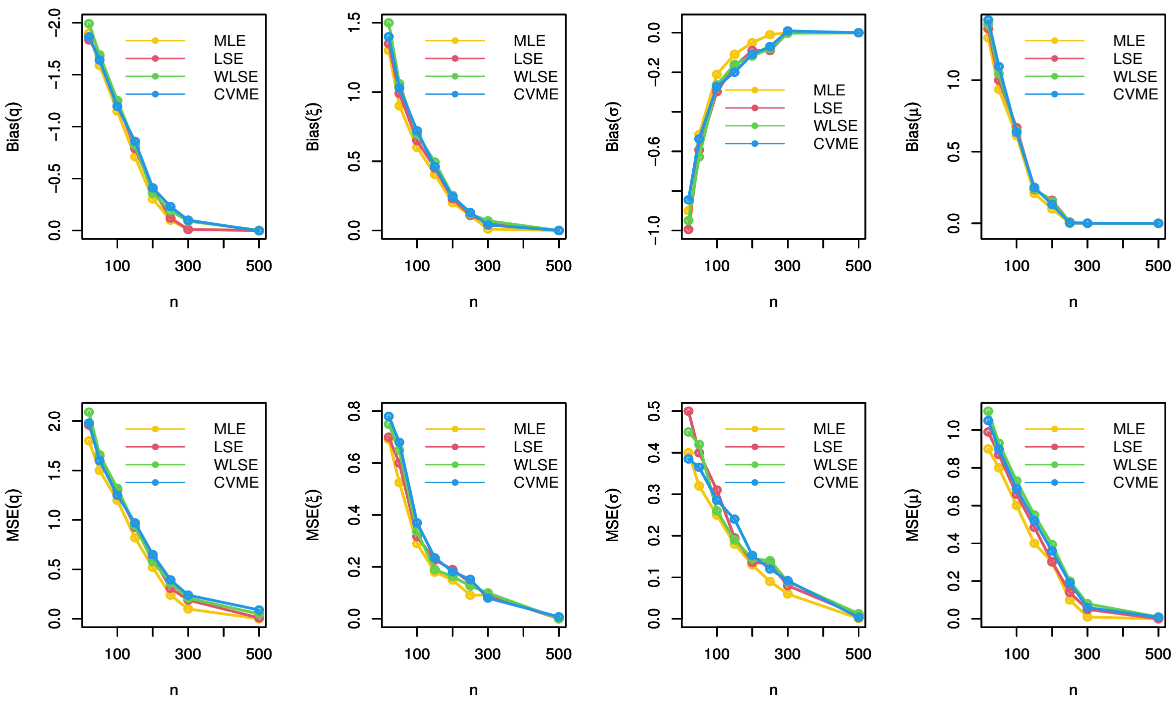

Figure 7: The bias of $\widehat{q}, \widehat{\xi}, \widehat{\sigma}$ and $\widehat{\mu}$ versus n for the q-GEVP$(0.5, 0.5, 0.8, 0.5)$ model.

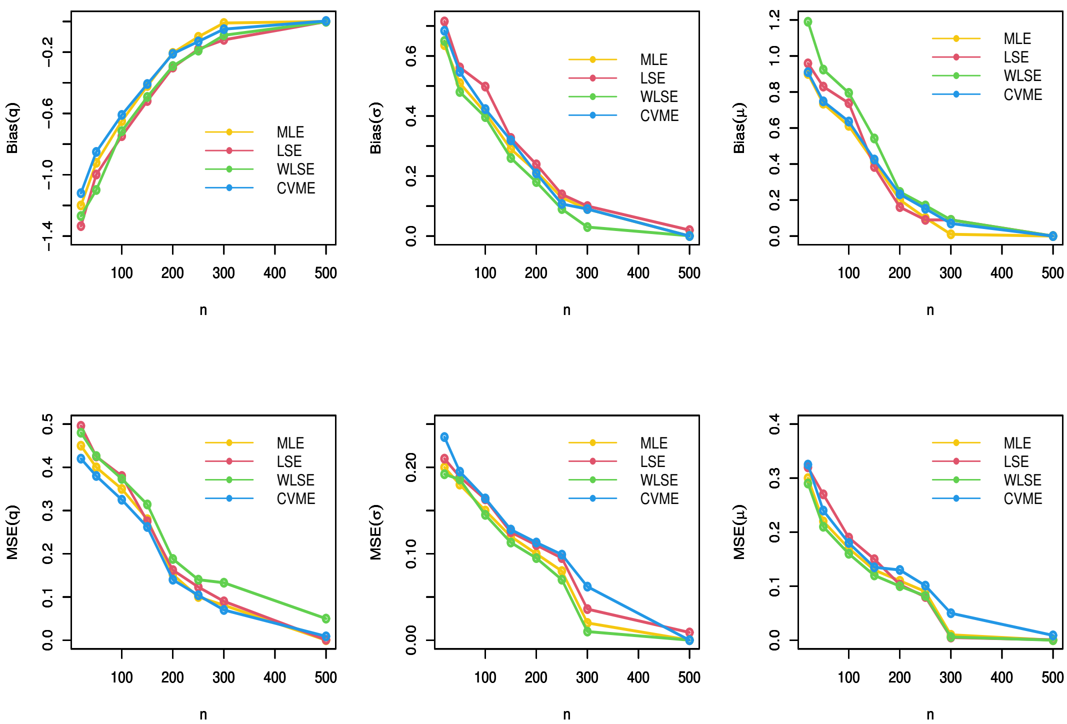

Figure 8: The bias of $\widehat{q}, \widehat{\xi}, \widehat{\sigma}$ and $\widehat{\mu}$ versus n for the q-GEVP$(0.8, \xi \to 0, 0.5, 0.3)$ model.

6. DATA ANALYSIS

In this section, we discuss the empirical importance of the q-GEVP model in case of $\xi \neq 0$, and $\xi \to 0$ for a positive random variable by using three applications to COVID-19 data. The fitted distributions are compared utilizing some criteria namely, Cramér-von Mises (CM), Anderson-Darling (AD) statistics, and Kolmogorov-Smirnov (KS) statistic with their p-values. Moreover, Akaike information criterion (AIC) with its corrected value (CAIC) beside Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) have been used as a part from these criteria. We shall compare the fits of the q-GEVP distribution with some competitive models like GEVP-type I (GEVP-I), inverse Weibull (IW), Gumbel (Gu), Weibull (W), generalized inverse Weibull (GIW), Gumbel inverse Weibull (GuIW), and type I generalized exponential inverse Weibull (T1GEIW) in case of $\xi \neq 0$ "see data sets I and II", and $\xi \to 0$ "see data set III".

6.1. Data set I: COVID-19 in Japan

This data is listed in (`https://www.worldometers.info/coronavirus/country/japan/`) which represents the maximum value of the new deaths per a week due to COVID-19 in Japan from 7 Mar 2020 up to 20 Feb 2021. Initial density shape is explored using the nonparametric "Kernel density estimation (KDE)" approach in Figure 9, and it is noted that the density is asymmetric and multimodal functions. The "normality" condition is checked via the "quantile-quantile (Q-Q) plot" in Figure 9. The extremes are spotted from the "box plot" in Figure 9, and it is showed that some extreme observations were founded.

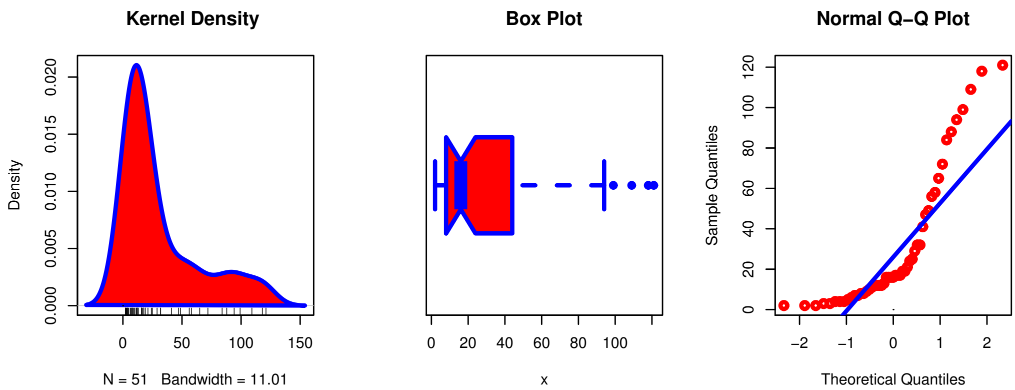

Figure 9: The KDE, Q-Q, and box plots for data set I.

Table 1 lists the MLEs with its standard errors (SE) in parentheses, whereas the goodness-of-fit (GOF) measures have been reported in Table 2 for data sets I.

From Table 2, it is noted that the q-GEVP model provides the best fit among all competitive distributions because it has the smallest value of CM, AD, KS, AIC, CAIC, BIC, and HQIC as well as it has the highest p-value. The empirical PDF, CDF, SF and P-P plots for data set I are displayed in Figure 10, which indicates that the data set plausibly came from q-GEVP model.

Table 1: The MLEs with its SE in parentheses for data set I.

Model	MLEs(SE)			
q-**GEVP**(q, ξ, σ, μ)	$-0.8659(0.5492)$	$-0.9892(0.0236)$	$3.5633(0.1379)$	$1.1944(0.2913)$
GEVP-I(α, β)	$-0.6454(0.0632)$	$1.0229(0.1503)$	–	–
IW(α, β)	$9.6646(1.5479)$	$0.9269(0.0971)$	–	–
Gu(μ, σ)	$17.0872(2.8921)$	$19.9133(2.4818)$	–	–
W(α, β)	$30.1519(4.6867)$	$0.9545(0.1020)$	–	–
GIW(α, β, γ)	$1.9247(407.6033)$	$0.9269(0.0971)$	$4.4629(876.0921)$	–
GuIW$(\gamma, \delta, \alpha, \beta)$	$3.3919(0.5459)$	$8.9611(0.9268)$	$2.3058(0.0273)$	$7.8184(0.0301)$
T1GEIW$(\gamma, \delta, \alpha, \beta)$	$388.4329(2.4 \times 10^3)$	$0.1385(1.5242)$	$0.1299(1.5230)$	$0.9254(0.0974)$

Table 2: The GOF measures for data set I.

GOF	Model							
	q-**GEVP**	**GEVP-I**	**IW**	**Gu**	**W**	**GIW**	**GuIW**	**T1GEIW**
KS	0.0981	0.3458	0.1079	0.2047	0.1333	0.1079	0.1141	0.1166
p-value	0.7109	≤ 0.001	0.5929	0.0279	0.3252	0.5929	0.5206	0.4916
$\mathbf{A}^*$	0.6679	3.1846	0.7298	3.0261	0.8439	0.7298	0.8678	1.5855
p-value	0.5855	0.1510	0.5337	0.0267	0.4498	0.5337	0.4340	0.1574
$\mathbf{W}^*$	0.0842	0.6189	0.0982	0.4801	0.1421	0.0982	0.1160	0.1846
p-value	0.6701	0.1172	0.5958	0.0444	0.4157	0.5958	0.5134	0.2999
-L	217.4310	252.9968	226.6616	238.7913	225.7776	226.6616	225.3522	229.6592
AIC	442.8620	509.9936	457.3232	481.5826	455.5552	459.3232	458.7045	467.3185
CAIC	443.7316	510.2436	457.5732	481.8326	455.8052	459.8338	459.5741	468.1881
BIC	450.5893	513.8573	461.1868	485.4462	459.4188	465.1186	466.4318	475.0458
HQIC	445.8148	511.4700	458.7996	483.059	457.0316	461.5378	461.6573	470.2713

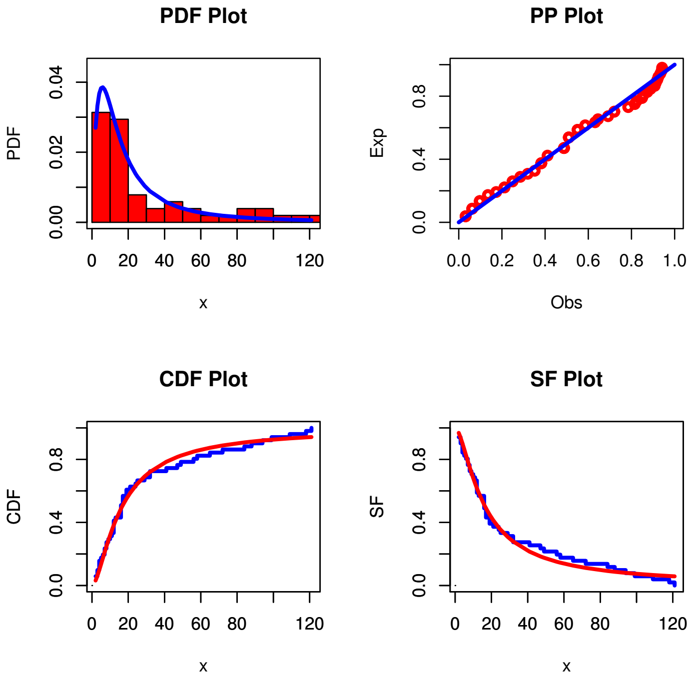

Figure 10: The fitted PDF, P-P, estimated CDF, and empirical SF plots for data set I.

Table 3 lists the estimates of the unknown parameters using three estimation methods for data set I.

Table 3: Various estimators of the q-GEVP model for data set I.

Parameters and GOF	Methods			
	MLE	**LSE**	**WLSE**	**CVME**
q	−0.8659	0.5785	0.2283	0.5634
ξ	−0.9892	−0.0333	−0.3006	−0.0338
σ	3.5633	0.8705	1.1325	0.8480
μ	1.1944	2.6773	2.5464	2.6744
KS	0.0981	0.0744	0.0808	0.0693
p-value	0.7109	0.9405	0.8929	0.9672
$\mathbf{A}^*$	0.6679	0.3633	0.4353	0.3454
p-value	0.5855	0.8837	0.8124	0.9002
$\mathbf{W}^*$	0.0842	0.0395	0.0554	0.0381
p-value	0.6701	0.9375	0.8454	0.9444

Table 3 illustrates that all estimation methods work quite well beside the MLE method, but the CVME approach is the best for data set I. Figure 11 shows the fitted PDFs, estimated CDFs, and empirical SF plots for data set I utilizing the estimators in Table 3, which support our results.

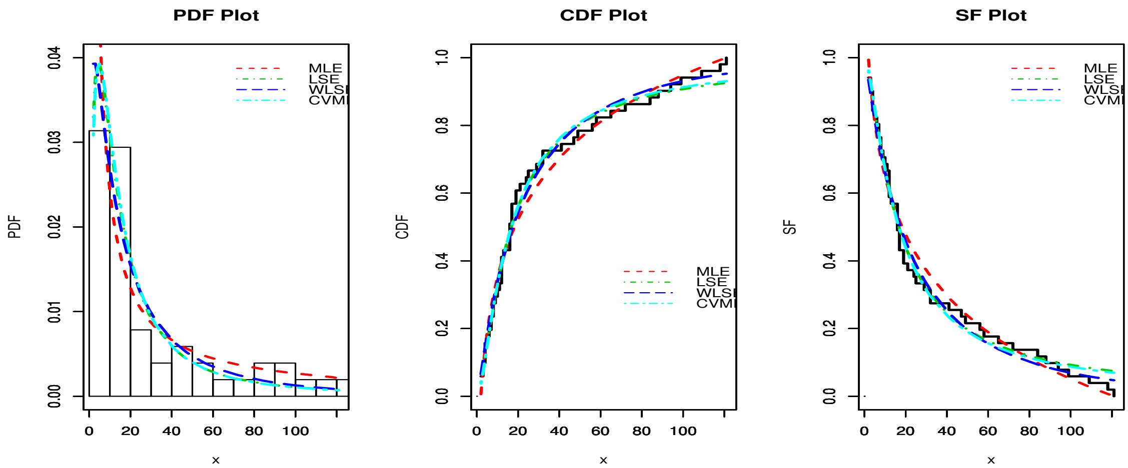

Figure 11: The fitted PDF, estimated CDF, and empirical SF plots based on various estimators for data set I.

6.2. Data set II: COVID-19 in Saudi Arabia

This data is reported in (`https://www.worldometers.info/coronavirus/country/saudi-arabia/`) which represents the maximum value of the new deaths per a week due to COVID-19 in Saudi Arabia from 28 Mar 2020 up to 20 Feb 2021. Initial density shape is explored utilizing the KDE approach in Figure 12, and it is noted that the density is asymmetric and bimodal functions. Further, the Q-Q and box plots are displayed in the same Figure.

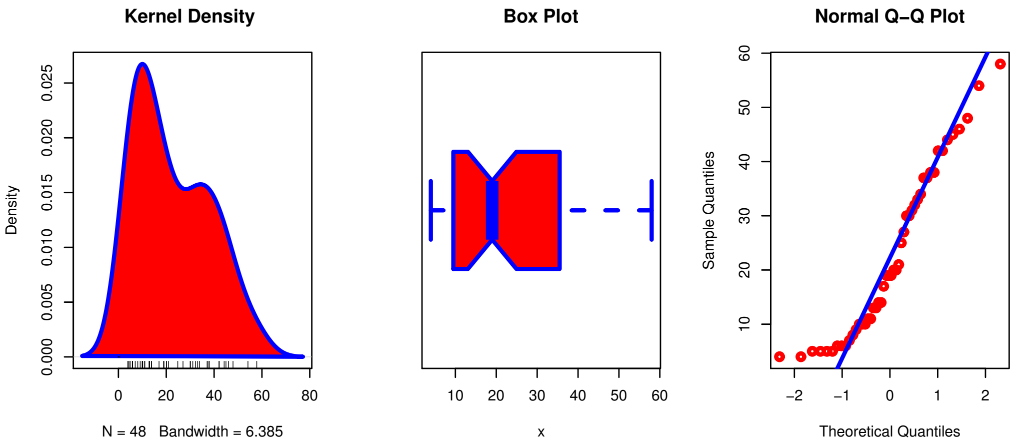

Figure 12: The KDE, Q-Q, and box plots for data set II.

Tables 4 and 5 list the MLEs, SE, and GOF measures for data sets II.

Table 4: The MLEs with its SE in parentheses for data set II.

Model	**MLEs(SE)**			
q-**GEVP**(q, ξ, σ, μ)	−0.7945(0.0231)	−0.9354(0.1254)	2.0494(0.2415)	1.8730(0.1478)
GEVP-I(α, β)	−1.4687(0.1341)	1.1769(0.2136)	–	–
IW(α, β)	24.0124(7.9202)	1.3077(0.1431)	–	–
Gu(μ, σ)	15.3915(1.8026)	11.8673(1.4129)	–	–
W(α, β)	1.5059(0.1742)	25.0505(2.5344)	–	–
GIW(α, β, γ)	3.5549(910.6226)	1.3077(0.1431)	4.5719(1531.5631)	–
GuIW$(\gamma, \delta, \alpha, \beta)$	3.0714(0.4487)	9.4339(0.9552)	4.4974(NaN)	11.7043(NaN)
T1GEIW$(\gamma, \delta, \alpha, \beta)$	213.3478(1411.1180)	0.2929(6.3869)	0.4758(8.0940)	1.3044(0.1445)

Table 5: The GOF measures for data set II.

GOF	**Model**							
	q-**GEVP**	**GEVP-I**	**IW**	**Gu**	**W**	**GIW**	**GuIW**	**T1GEIW**
KS	0.1049	0.3303	0.1417	0.1218	0.1057	0.1417	0.1398	0.1420
p-value	0.6659	≤ 0.001	0.2898	0.4751	0.6568	0.2898	0.3053	0.2875
$\mathbf{A}^*$	0.5464	8.6403	1.383	1.1593	0.8067	1.3830	1.3421	1.3869
p-value	0.6992	≤ 0.001	0.2070	0.2833	0.4756	0.2070	0.2191	0.2059
$\mathbf{W}^*$	0.0724	1.7523	0.21031	0.1802	0.1243	0.2103	0.2049	0.2111
p-value	0.7394	≤ 0.001	0.2487	0.3101	0.4798	0.2487	0.2585	0.2473
-L	184.2950	215.4770	196.8051	195.6623	192.2741	196.8051	195.3561	196.8159
AIC	376.5900	434.9540	397.6103	395.3245	388.5482	399.6103	398.7121	401.6319
CAIC	377.5202	435.2207	397.8769	395.5912	388.8149	400.1557	399.6423	402.5621
BIC	384.0748	438.6964	401.3527	399.0669	392.2906	405.2239	406.1969	409.1167
HQIC	379.4185	436.3683	399.0245	396.7388	389.9625	401.7316	401.5406	404.4604

From Table 5, it is noted that the q-GEVP distribution provides the best fit among all competitive models. The empirical PDF, CDF, SF and P-P plots for data set II are displayed in Figure 13.

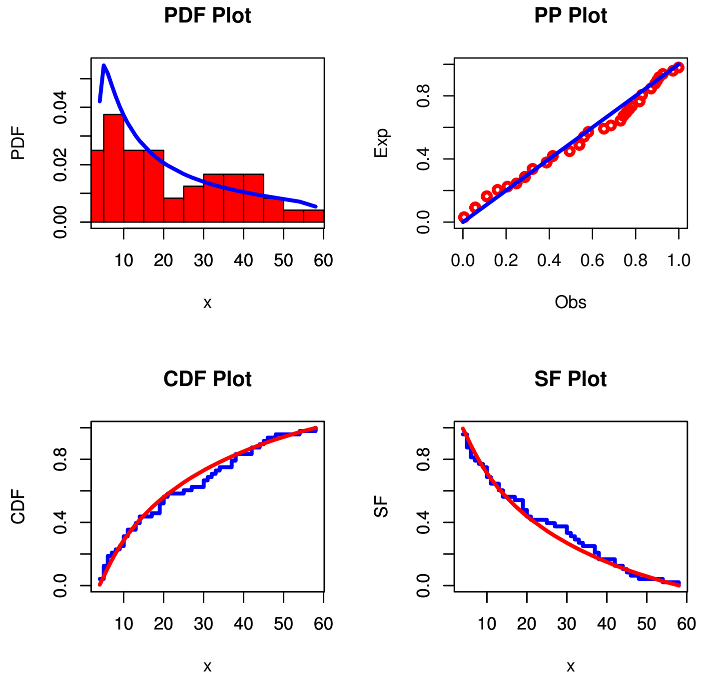

Figure 13: The fitted PDF, P-P, estimated CDF, and empirical SF plots for data set II.

Table 6 lists the estimates of the unknown parameters using various estimation methods for data set II.

Table 6: Various estimators of the q-GEVP model for data set II.

Parameters and GOF	Methods			
	MLE	LSE	WLSE	CVME
q	-0.7945	0.1733	0.1982	0.1614
ξ	-0.9354	-0.5311	-0.6364	-0.5547
σ	2.0494	0.9121	0.9321	0.9113
μ	1.8730	2.6956	2.7251	2.6978
KS	0.1049	0.0949	0.1097	0.0939
p-value	0.6659	0.7793	0.6107	0.7911
$\mathbf{A}^*$	0.5464	0.5914	0.5793	0.5514
p-value	0.6992	0.6552	0.6669	0.6942
$\mathbf{W}^*$	0.0724	0.0582	0.0616	0.0569
p-value	0.7394	0.8279	0.8060	0.8353

From Table 6, it is clear that all estimation techniques work quite well beside the MLE method, but the CVME approach is the best for data set II. Figure 14 shows the fitted PDFs, estimated CDFs, and empirical SF plots for data set II by using the estimators in Table 6, which support our results.

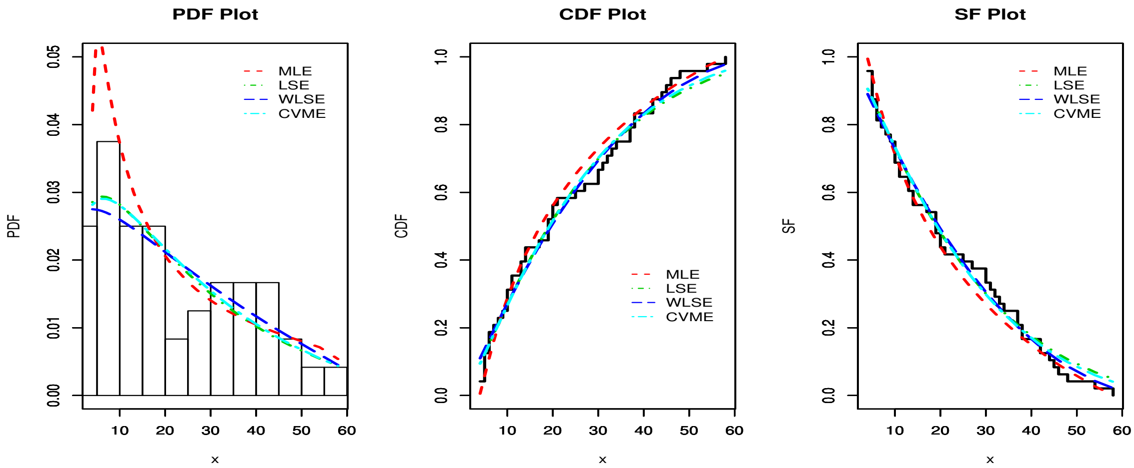

Figure 14: The fitted PDF, estimated CDF, and empirical SF plots based on various estimators for data set II.

6.3. Data set III: COVID-19 in Romania

This data is reported in (`https://www.worldometers.info/coronavirus/country/romania/`) which represents the maximum value of the new deaths per a week due to COVID-19 in Romania from 7 Mar 2020 up to 20 Feb 2021. Initial density shape is explored using the KDE method in Figure 15, and it is clear that the density is asymmetric and bimodal functions. Moreover, the Q-Q and box plots are displayed in the same Figure.

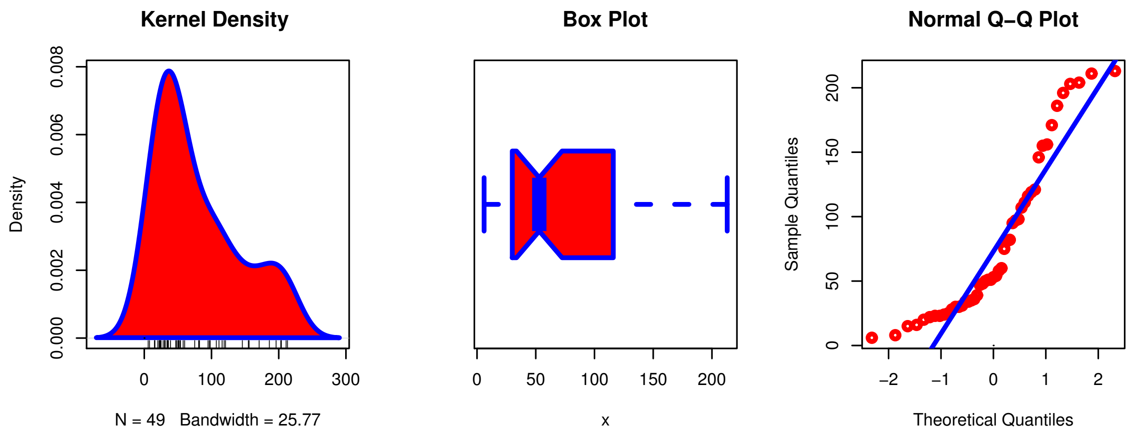

Figure 15: The KDE, Q-Q, and box plots for data set III.

Tables 7 and 8 report the MLEs, SE, and the GOF measures for data sets III.

From Table 8, it is noted that the q-GEVP model provides the best fit among all competitive distributions. The empirical PDF, CDF, SF and P-P plots for data set III are displayed in Figure 16.

Table 7: The MLEs with its SE in parentheses for data set III.

Model	MLEs(SE)			
q-GEVP(q, σ, μ)	1.1182(0.7912)	0.4969(0.1408)	4.0881(0.2569)	–
GEVP-I(α, β)	−1.5821(0.0265)	0.7867(0.1254)	–	–
IW(α, β)	52.0870(19.6569)	1.1022(0.1117)	–	–
Gu(μ, σ)	51.7563(6.5298)	43.6582(5.2948)	–	–
W(α, β)	1.3167(0.1469)	86.5845(9.9331)	–	–
GIW(α, β, γ)	5.6304(2654.1739)	1.1022(0.1117)	7.7531(4028.5284)	–
GuIW$(\gamma, \delta, \alpha, \beta)$	9.7866(2.2779)	6.4026(0.6714)	4.5341(0.1529)	7.0421(0.1946)
T1GEIW$(\gamma, \delta, \alpha, \beta)$	1115.2175(5733.0614)	0.1679(2.2473)	0.3115(3.9096)	1.1012(0.1116)

Table 8: The GOF measures for data set III.

GOF	Model							
	q-GEVP	GEVP-I	IW	Gu	W	GIW	GuIW	T1GEIW
KS	0.0900	0.4171	0.1318	0.1438	0.1151	0.1318	0.1322	0.1319
p-value	0.8221	≤ 0.001	0.3621	0.2627	0.5351	0.3621	0.3584	0.3611
A^*	0.5499	12.1030	0.9625	1.4830	0.7140	0.9625	0.9693	0.9652
p-value	0.6957	≤ 0.001	0.3771	0.1806	0.5464	0.3771	0.3733	0.3756
W^*	0.0821	2.5283	0.1097	0.2313	0.1172	0.1097	0.1106	0.1101
p-value	0.6817	≤ 0.001	0.5411	0.2145	0.5084	0.5411	0.5372	0.5392
-L	262.1081	311.5210	266.2566	265.1471	260.7146	266.2566	266.2533	266.2694
AIC	530.2162	627.0420	536.5132	534.2943	525.4292	538.5132	540.5066	540.5388
CAIC	530.7495	627.3029	536.7741	534.5551	525.6901	539.0465	541.4157	541.4479
BIC	535.8916	630.8256	540.2968	538.0779	529.2128	544.1886	548.0739	548.1061
HQIC	532.3694	628.4775	537.9487	535.7298	526.8647	540.6664	543.3777	543.4098

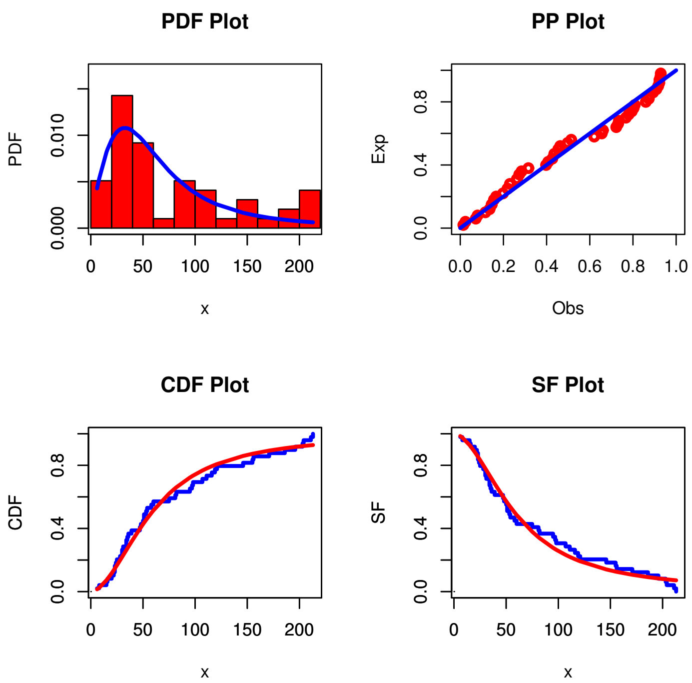

Figure 16: The fitted PDF, P-P, estimated CDF, and empirical SF plots for data set III.

Table 9 reports the estimates of the unknown parameters using various estimation approaches for data set III.

Table 9: Various estimators of the q-GEVP model for data set III.

Parameters and GOF	Methods			
	MLE	**LSE**	**WLSE**	**CVME**
q	1.1182	0.6959	2.2430	0.6642
σ	0.4969	0.6599	0.4102	0.6448
μ	4.0881	3.9588	4.3003	3.9499
KS	0.0900	0.1084	0.0859	0.1024
p-value	0.8221	0.6128	0.8620	0.6827
$\mathbf{A}^*$	0.5499	0.5272	0.6501	0.4884
p-value	0.6957	0.7184	0.6011	0.7578
$\mathbf{W}^*$	0.0821	0.0550	0.0793	0.0535
p-value	0.6817	0.8476	0.6979	0.8571

Table 9 illustrates that all estimation methods work quite well besides the MLE method. Figure 17 shows the fitted PDFs, estimated CDFs empirical SF plots for data set III using the estimators in Table 9, which support our results.

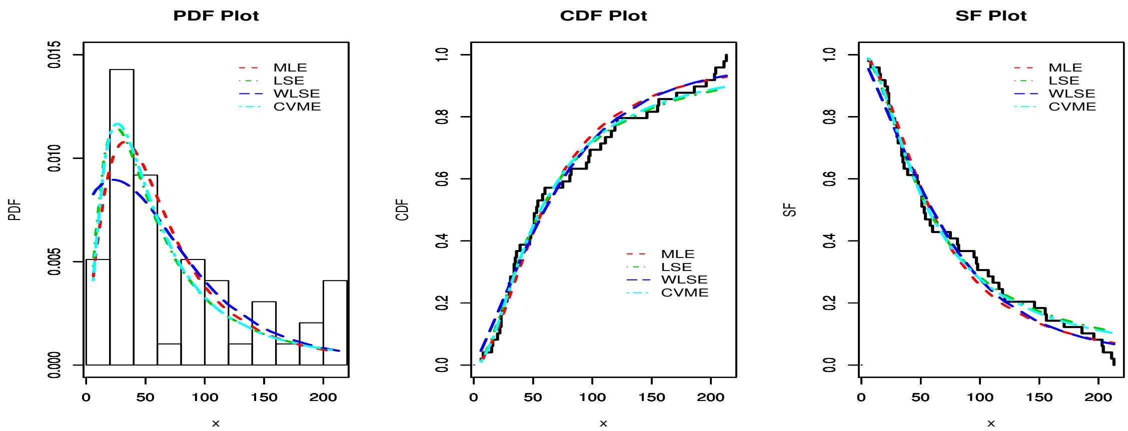

Figure 17: The fitted PDF, estimated CDF, and empirical SF plots based on various estimators for data set III.

7. CONCLUSIONS

In this paper, we proposed q-generalized extreme values model and its discrete version under power normalization technique. Its various statistical features have been derived in detail. It was found that the proposed models are a proper for modelling skewed data sets, especially which have very extreme observations. Moreover, the new model provides a wide variation in the shape of the HRF, including decreasing, increasing, unimodal, and bathtub shapes, and consequently the proposed distribution can be utilized in modelling several different kinds of data. The model parameters have been estimated using four different estimation approaches, namely, MLE, LSE, WLSE, and CVME. A simulation has been performed based on different sample sizes, and it was found that the four methods work quit effectively in estimating the model parameters. Three distinctive data sets "COVID-19" have been analyzed to illustrate and prove the flexibility of the proposed model. Finally, the q-generalized extreme values model under power normalization technique would be a better alternative to other lifetime models available in existing literature, especially, in extreme values field.

APPENDIX

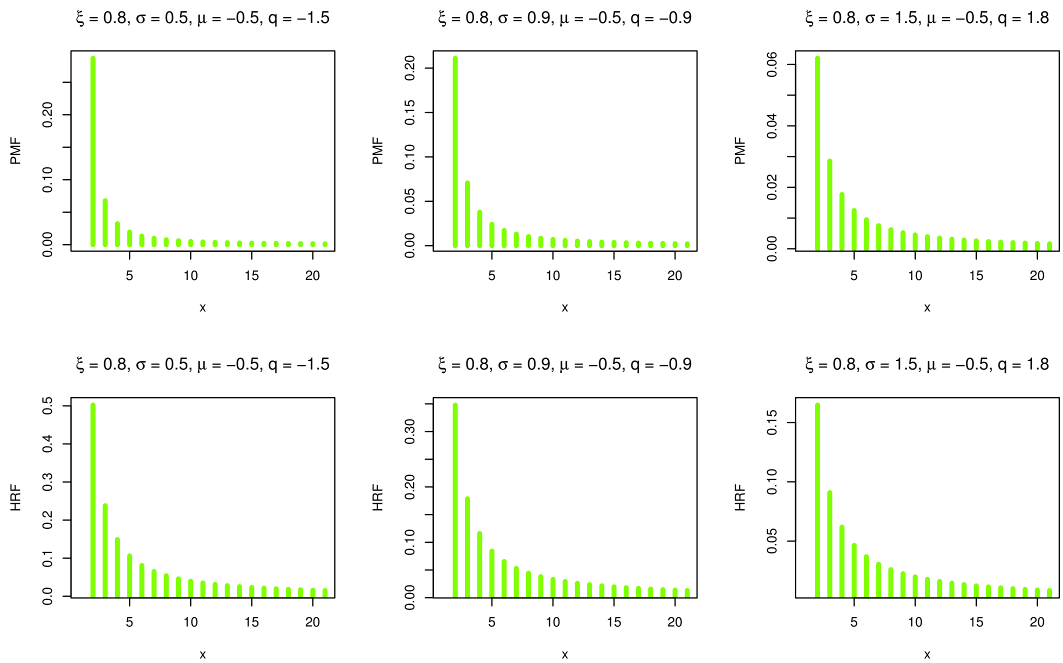

Figure 18: The PMF and HRF plots of the Dq-GEVP model for some parameter values.

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