REVSTAT – Statistical Journal
Volume 23, Number 1, January 2025, 19–45
https://doi.org/10.57805/revstat.v23i1.453

Statistical Inferences to the Parameter and Reliability Characteristics of Gamma-mixed Rayleigh Distribution under Progressively Censored Data with Application

Authors: Kousik Maiti ✉
– Department of Computer Science and Engineering,
Indian Institute of Technology Roorkee,
Roorkee-247667, India
kousikulu@gmail.com

Suchandan Kayal
– Department of Mathematics, National Institute of Technology Rourkela,
Rourkela-769008, India
suchandan.kayal@gmail.com , kayals@nitrkl.ac.in

Received: April 2022 Revised: December 2022 Accepted: December 2022

Abstract:

- We consider estimation of the model parameters and the reliability characteristics of a gamma-mixed Rayleigh distribution based on progressively type-II censored sample (PT-IICS). The sufficient condition for existence and uniqueness of the maximum likelihood estimates (MLE) is obtained. We compute MLEs using the expectation maximization (EM) algorithm. Asymptotic confidence intervals are constructed. Confidence intervals using the bootstrap-p and bootstrap-t methods are constructed. Bayes estimates are derived. Highest posterior density (HPD) credible intervals are derived using the importance sampling method. Prediction estimates and associated prediction equal-tail intervals under one-sample and two-sample frameworks are obtained. A simulation study is conducted. Finally, a real dataset is considered and analyzed.

Keywords:

- *EM algorithm; observed Fisher information matrix; Bayes estimates; Bayesian prediction estimates; HPD credible interval.*

AMS Subject Classification:

- 62F10, 62F15, 62F40, 62N01.

✉ Corresponding author

1. INTRODUCTION

In recent years, life testing experiments are less preferred because of being time consuming and expensive. In many situations, use of complete sample is neither possible nor desirable. In such cases, the sample needs to be censored. Censoring is a condition in which the value of observation is partially known and incomplete. Among different types of censoring schemes, the two basic censoring schemes are type-I and type-II. In the type-I censoring scheme, the life testing experiment terminates at a pre-specified time T, whereas, the type-II censoring scheme terminates when one has m number of failures. For applications and importance of these schemes, we refer to Lawless (2011) and Cohen (2016). The main drawback of these censoring schemes is that they do not allow removal of the items in between other than the termination point. To overcome such drawback, a more general censoring scheme, known as the progressive censoring was introduced in the literature. It can be classified into progressive type-I and progressive type-II censoring schemes. In the progressive type-I censoring scheme, let the number of items used in a life testing experiment be n. In this scheme, $R_1, R_2, ..., R_m$ items are randomly withdrawn at pre-specified time points $T_1, T_2, ..., T_m$, respectively. The test will be terminated at prefixed time point T_m in this scheme. Now, we describe the PT-IICS. Consider n number of total units at initial time on an experiment. We remove randomly R_1 number of survival units when first failure time $X_{1:m:n}$ is observed. This process continues till the m-th failure occurs. We assume that the m-th failure takes place at time $X_{m:m:n}$ and the remaining number of surviving units is $R_m = n - (m + \sum_{i=1}^{m-1} R_i)$. Henceforth, we denote $R = (R_1, R_2, ..., R_m)$ and $X = (X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ for the censoring scheme and the PT-IICS, respectively. Due to several applications, various inferential procedures based on PT-IICS have been established for many lifetime distributions. For instance, see Muhammed and Almetwally (2020), Nik *et al.* (2021), Albalawi *et al.* (2022) and the references contained therein.

A random variable X is said to follow a gamma-mixed Rayleigh distribution if its probability density and cumulative distribution functions are respectively given by $(\alpha, \beta > 0)$

$$f_X(x; \alpha, \beta) = \frac{\alpha \beta^\alpha x}{(x^2+\beta^2)^{(\alpha/2)+1}} \quad \text{and} \quad F_X(x; \alpha, \beta) = 1 - \frac{\beta^\alpha}{(x^2+\beta^2)^{\alpha/2}}, \tag{1.1}$$

where $x > 0$. Here, α is known as the shape parameter and β is known as the scale parameter. The reliability function and the hazard function of this distribution are respectively obtained as

$$r(x; \alpha, \beta) = \frac{\beta^\alpha}{(x^2+\beta^2)^{\alpha/2}} \quad \text{and} \quad h(x; \alpha, \beta) = \frac{x\alpha}{x^2+\beta^2}, \tag{1.2}$$

where $x > 0$ and $\alpha, \beta > 0$. Various shapes of the probability density, reliability and hazard functions of the gamma-mixed Rayleigh distribution are depicted in Figures 1(a), 1(b) and 1(c), respectively. Differentiating $h(x; \alpha, \beta)$ with respect to x, we obtain

$$\frac{dh(x; \alpha, \beta)}{dx} = \frac{\alpha(\beta+x)(\beta-x)}{(x^2+\beta^2)^2} = \begin{cases} > 0, & \text{for } x < \beta \\ < 0, & \text{for } x > \beta \\ = 0, & \text{for } x = \beta. \end{cases} \tag{1.3}$$

Thus, the hazard function of the gamma-mixed Rayleigh distribution is increasing for $x < \beta$ and decreasing for $x > \beta$, for any value of $\alpha > 0$. Figure 1(c) shows that the hazard of the gamma-mixed Rayleigh distribution is hump-shaped, that is, the hazard is increasing early and eventually begins declining. One may refer to Sarhan *et al.* (2013) for similar study on the exponentiated generalized linear exponential distribution. This type of hazard is often used in modeling data related to survival after successful surgery, where there is an initial increase in risk due to infection or other complications just after the procedure, followed by a steady decline in risk as the patient recovers (see Klein and Moeschberger, 1997).

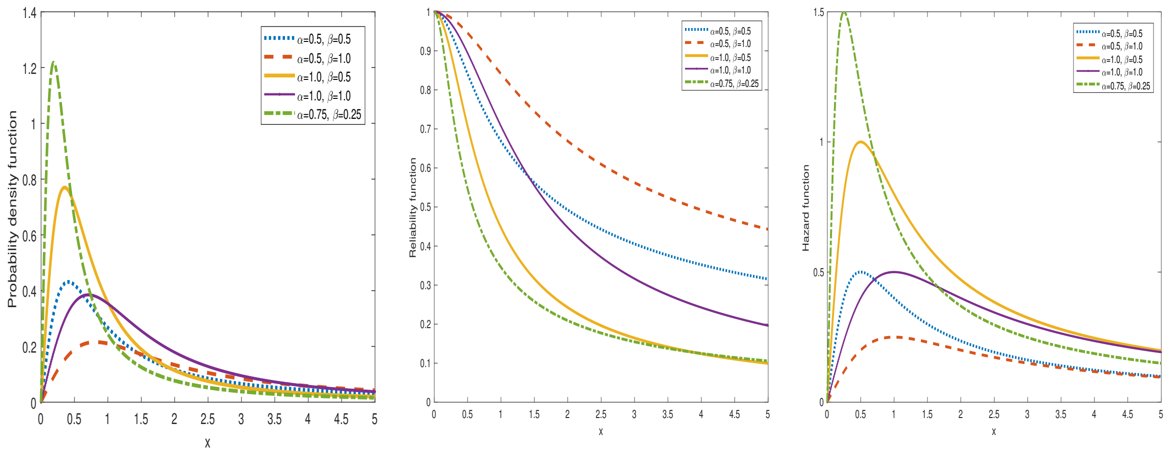


Figure 1: The plots of the (a) density (b) reliability and (c) hazard functions based on different values of the parameters.

The Bayesian prediction of the unknown observation is an important problem. Various authors have studied prediction problems based on the PT-IICS. Kayal *et al.* (2017) obtained the prediction intervals and estimates for future observations in one-sample and two-sample problems for the Chen distribution. Similar problem was studied by Arabi *et al.* (2019) for the Poisson-exponential distribution when PT-IICS is available. For flexible Weibull distribution, Bdair *et al.* (2019) considered Bayesian prediction problem based on the progressive type-II censored data. Very recently, Maiti and Kayal (2019) obtained prediction estimates and intervals for future observations in one-sample and two-sample problems for the generalized Fréchet distribution from Bayesian point of view. To the best of our knowledge, nobody has considered the gamma-mixed Rayleigh distribution with distribution function given by (1.1) for the purpose of statistical inference and Bayesian prediction based on the PT-IICS. In this paper, we address the problem of inference and prediction when the PT-IICS is available from gamma-mixed Rayleigh distribution.

The rest of the paper is organized as follows. In the next section, we obtain MLEs for the unknown parameters, reliability and hazard functions. The existence and uniqueness of the MLEs have been studied. The EM algorithm is described to compute the proposed MLEs. Section 3 deals with the construction of various interval estimates. In Section 4, we derive Bayes estimates with respect to three loss functions. Two approaches are adopted to compute approximate Bayes estimates. Importance sampling method is used to compute HPD credible intervals. Further, in Section 5, we derive Bayesian prediction and interval estimates. In Section 6, we carry out a simulation study to compare the performance of the proposed estimates. A real life dataset is considered for the illustration purpose. Finally, Section 7 concludes the paper.

2. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we derive MLEs of α and β of the gamma-mixed Rayleigh distribution based on the PT-IICS. Using invariance property of the MLE, the MLEs of $r(x)$ and $h(x)$ can be obtained. The likelihood function of α and β is given by

$$L(\alpha,\beta \mid x) = K\prod_{i=1}^{m}\big(1-F_X(x_{i:m:n};\alpha,\beta)\big)^{R_i} f_X(x_{i:m:n};\alpha,\beta), \tag{2.1}$$

where the constant $K = n(n-(R_1+1))(n-(\sum_{j=1}^{2}R_j+2))\cdots(n-\sum_{j=1}^{m-1}(R_j+1))$ and $x=(x_{1:m:n},x_{2:m:n},...,x_{m:m:n})$. The log-likelihood function of α and β is obtained as

$$\begin{aligned}\ell \;=\; \ell(\alpha,\beta\mid x) &\propto m\ln\alpha + m\alpha\ln\beta + \sum_{i=1}^{m}\ln x_{i:m:n} + \alpha\ln\beta\sum_{i=1}^{m}R_i \\ &\quad - \sum_{i=1}^{m}\Big(\frac{\alpha}{2}(1+R_i)+1\Big)\ln(x_{i:m:n}^2+\beta^2).\end{aligned} \tag{2.2}$$

The likelihood equations of α and β are

$$m\Big(\frac{1}{\alpha}+\ln\beta\Big) + \ln\beta\sum_{i=1}^{m}R_i - \frac{1}{2}\sum_{i=1}^{m}(R_i+1)\ln\big(\beta^2+x_{i:m:n}^2\big) = 0 \tag{2.3}$$

and

$$\alpha\Bigg(\sum_{i=1}^{m}R_i+m\Bigg) - 2\beta^2\sum_{i=1}^{m}\frac{\big(\frac{\alpha}{2}(R_i+1)+1\big)}{\beta^2+x_{i:m:n}^2} = 0, \tag{2.4}$$

respectively. The MLEs of α and β can be obtained after solving (2.3) and (2.4) simultaneously. These are difficult to obtain in explicit form. The above system of nonlinear equations can be solved by solving a two-dimensional optimization problem. In this case, one may use the Newton-Raphson algorithm. However, the standard Newton-Raphson method does not converge in some cases. We use EM algorithm to compute the MLEs of α and β, which is described below. Note that the EM algorithm was introduced by Dempster *et al.* (1977). Prior to the computation, we discuss the condition under which the MLEs exist and are unique.

Theorem 2.1. *The MLEs of α and β for $(\alpha,\beta)\in(0,\infty)\times(0,\infty)$ exist and are unique under the PT-IICS, provided $x_{i:m:n}>\beta$ holds, for $i=1,...,m$.*

Proof: We show that the maximum value of the log-likelihood function $\ell(\alpha,\beta\mid x)$ exists and also unique for $(\alpha,\beta)\in(0,\infty)\times(0,\infty)$. One may refer to the papers by Cancho *et al.* (2011) and Khan and Mitra (2019) for similar study in other estimation problems. The second order partial derivatives of the log-likelihood function ℓ with respect to α and β are given by

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{m}{\alpha^2} < 0, \tag{2.5}$$

$$\frac{\partial^2\ell}{\partial\beta^2} = -\frac{\alpha(\sum_{i=1}^{m}R_i+m)}{\beta^2} - \sum_{i=1}^{m}(\alpha(R_i+1)+2)\frac{(x_{i:m:n}^2-\beta^2)}{(x_{i:m:n}^2+\beta^2)^2} < 0, \tag{2.6}$$

if $x_{i:m:n}>\beta$. Therefore, for fixed $\alpha(\beta)$, ℓ is a strictly concave function with respect to $\beta(\alpha)$.

For fixed β, we get

$$\lim_{\alpha\to 0} \ell(\alpha,\beta \mid x) = -\infty \quad \text{and} \quad \lim_{\alpha\to\infty} \ell(\alpha,\beta \mid x) = -\infty.$$

Similarly, for fixed α, we have $\lim_{\beta\to 0} \ell(\alpha,\beta \mid x) = -\infty$ and $\lim_{\beta\to\infty} \ell(\alpha,\beta \mid x) = -\infty$. So, for fixed $\alpha(\beta)$, ℓ is a unimodal function with respect to $\beta(\alpha)$. Again,

$$\lim_{\alpha\to 0,\ \beta\to 0} \ell(\alpha,\beta \mid x) = -\infty, \quad \lim_{\alpha\to \infty,\ \beta\to 0} \ell(\alpha,\beta \mid x) = -\infty,$$
$$\lim_{\alpha\to 0,\ \beta\to \infty} \ell(\alpha,\beta \mid x) = -\infty, \quad \lim_{\alpha\to \infty,\ \beta\to \infty} \ell(\alpha,\beta \mid x) = -\infty.$$

Let $(\alpha_0,\beta_0) \in (0,\infty)\times(0,\infty)$ and $\ell(\alpha_0,\beta_0 \mid x) = \rho$. Further, set

$$D = \Big\{(\alpha,\beta) : (\alpha,\beta) \in (0,\infty)\times(0,\infty),\ \ell(\alpha,\beta \mid x) \geq \rho\Big\}.$$

So, D is a closed and bounded set, hence D is compact set. Note that the function ℓ is continuous with respect to (α,β). Thus, ℓ has a maximum value for some $(\alpha,\beta)\in D$. Suppose that at $(\alpha_1,\beta_1) \in (0,\infty)\times(0,\infty)$, the function ℓ has maximum. Now, we have to show that (α_1,β_1) is unique. We observe that

$$\ell(\alpha_1,\beta_1 \mid x) > \ell(\alpha_1,\beta \mid x) > \ell(\alpha,\beta \mid x),$$

for $(\alpha,\beta) \in (0,\infty)\times(0,\infty)$, which ensures the desired uniqueness. □

2.1. EM algorithm

The EM algorithm is mainly used to compute the MLEs of the unknown parameters in cases where the likelihood equations cannot be solved explicitly. EM algorithm has two steps: the expectation (E) step and the maximization (M) step. The E-step involves computation of the pseudo log-likelihood function. The M-step involves maximization of the pseudo log-likelihood function. Let the observed sample and censored data be denoted by $X = (X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ and $Z = (Z_1, Z_2, ..., Z_m)$, respectively, where Z_j is a $1\times R_j$ vector $(Z_{j1}, Z_{j2}, ..., Z_{jR_j})$, for $j = 1,2,...,m$. Note that the complete sample is a combination of the observed sample and the censored data. Denote the complete sample by $W = (X,Z)$. The likelihood function of the complete sample (see Ng *et al.*, 2002) is given by

$$L_C(W;\alpha,\beta) = \prod_{j=1}^{m}\left[f_X(x_{j:m:n};\alpha,\beta)\prod_{k=1}^{R_j} f_Z(z_{jk};\alpha,\beta)\right]. \tag{2.7}$$

Then, the log-likelihood function for the complete sample is

$$\ell_C(W;\alpha,\beta) = n\ln(\alpha\beta^{\alpha}) + \sum_{j=1}^{m}\left[\ln x_{j:m:n} + \sum_{k=1}^{R_j}\ln z_{jk} - \left(\frac{\alpha}{2}+1\right)\left(\sum_{k=1}^{R_j}\ln(z_{jk}^2+\beta^2) + \ln(x_{j:m:n}^2+\beta^2)\right)\right]. \tag{2.8}$$

In the E-step, the conditional expectation of the log-likelihood function $\ell_C(W;\alpha,\beta)$ is obtained. This is known as the pseudo log-likelihood function. This can be obtained from $\ell_C(W;\alpha,\beta)$ by replacing any function of z_{jk} say $\psi(z_{jk})$ with $E[\psi(Z_{jk})|Z_{jk} > x_{j:m:n}]$. Thus, the pseudo log-likelihood function is obtained as

$$\begin{aligned}(2.9)\qquad \ell_s(\alpha,\beta) &= n(\ln\alpha\beta^{\alpha}) + \sum_{j=1}^{m}\ln x_{j:m:n} + \sum_{j=1}^{m} R_j A(x_{j:m:n};\alpha,\beta)\\ &\quad - \left(\frac{\alpha}{2}+1\right)\sum_{j=1}^{m}\Big(R_j B(x_{j:m:n};\alpha,\beta) + \ln(x_{j:m:n}^2+\beta^2)\Big),\end{aligned}$$

where

$$\begin{aligned}(2.10)\qquad A(x_{j:m:n};\alpha,\beta) &= E[\ln Z_{jk}|Z_{jk} > x_{j:m:n}]\\ &= \alpha(x_j^2+\beta^2)^{\alpha/2}\int_{x_{j:m:n}}^{\infty}\frac{t\ln t}{(t^2+\beta^2)^{(\alpha/2)+1}}\,dt\end{aligned}$$

and

$$\begin{aligned}(2.11)\qquad B(x_{j:m:n};\alpha,\beta) &= E\big[\ln(Z_{jk}^2+\beta^2)|Z_{jk} > x_{j:m:n}\big]\\ &= \ln(x_{j:m:n}^2+\beta^2) + \frac{2}{\alpha}.\end{aligned}$$

In the M-step, we maximize the pseudo log-likelihood function given by (2.9) obtained in E-step after substituting the values of (2.10) and (2.11) in (2.9). Let $(\alpha^{(p)},\beta^{(p)})$ be an estimate of (α,β) at p-th stage. The corresponding updated estimate $(\alpha^{(p+1)},\beta^{(p+1)})$ can be obtained by maximizing

$$\begin{aligned}(2.12)\qquad \ell_s^{*}(\alpha,\beta) &= n(\ln\alpha\beta^{\alpha}) + \sum_{j=1}^{m}\ln x_{j:m:n} + \sum_{j=1}^{m} R_j A(x_{j:m:n};\alpha^{(p)},\beta^{(p)})\\ &\quad - \left(\frac{\alpha}{2}+1\right)\sum_{j=1}^{m}\Big(R_j B(x_{j:m:n};\alpha^{(p)},\beta^{(p)}) + \ln(x_{j:m:n}^2+\beta^2)\Big)\end{aligned}$$

with respect to α and β. Now, we compute $\beta^{(p+1)}$ using fixed point iteration method (see Kundu and Pradhan, 2009). The corresponding estimate is obtained by solving the equation

$$(2.13)\qquad \exp\left\{\frac{1}{2n}\sum_{j=1}^{m}(BR_j+\ln(\beta^2+x_{j:m:n}^2)) - \frac{1}{\hat{\alpha}(\beta)}\right\} = \beta,$$

where

$$(2.14)\qquad \hat{\alpha}(\beta) = \left(n - \sum_{j=1}^{m}\frac{\beta^2}{\beta^2+x_{j:m:n}^2}\right)^{-1}\sum_{j=1}^{m}\frac{2\beta^2}{\beta^2+x_{j:m:n}^2}$$

with $B = B(x_{j:m:n};\alpha^{(p)},\beta^{(p)})$. We estimate $\beta^{(p+1)}$. The updated estimate $\alpha^{(p+1)}$ can be obtained from $\alpha^{(p+1)} = \hat{\alpha}(\beta^{(p+1)})$ using (2.14). The algorithm is provided below.

Step-1: Set $p = 0$. Based on the starting value $(\alpha^{(0)}, \beta^{(0)})$, we estimate the parameters α and β.

Step-2: Calculate $B = B(x_{j:m:n}; \alpha^{(p)}, \beta^{(p)})$ from the observed sample $X = x$ and the parameters $\alpha^{(p)}, \beta^{(p)}$.

Step-3: Update (α, β) as $(\alpha^{(p+1)}, \beta^{(p+1)})$.

Step-4: If $|(\alpha^{(p+1)}, \beta^{(p+1)}) - (\alpha^{(p)}, \beta^{(p)})| \leq \epsilon$ ($\epsilon > 0$ very small tolerance), then we get the MLEs of the parameters α and β.

Step-5: If $|(\alpha^{(p+1)}, \beta^{(p+1)}) - (\alpha^{(p)}, \beta^{(p)})| > \epsilon$, then set $p = p + 1$ and go to the step 1.

Denote the MLEs of α and β by $\hat{\alpha}$ and $\hat{\beta}$. Replacing α and β with $\hat{\alpha}$ and $\hat{\beta}$, the MLEs of the reliability and hazard functions are respectively obtained as $(x > 0)$

$$\hat{r}(x) = \frac{\beta^{\alpha}}{(x^2 + \beta^2)^{\alpha/2}}\Big|_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} \quad \text{and} \quad \hat{h}(x) = \frac{x\alpha}{x^2 + \beta^2}\Big|_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})}. \tag{2.15}$$

Remark 2.1. The main advantage of the EM algorithm is that computations are straightforward and does not require second and higher order derivatives.

3. INTERVAL ESTIMATES

In this section, we obtain $100(1 - \varphi)\%$ confidence intervals for the parameters, reliability and hazard functions based on PT-IICS. Two techniques are used. First, we discuss the construction of asymptotic confidence intervals. It is noted that to apply this procedure, we need the concept of observed Fisher information matrix. Louis (1982) first derived the observed Fisher information matrix using missing information based on the EM algorithm. The observed Fisher information matrix is used to construct the asymptotic confidence intervals. According to Louis, the observed information equals to the complete information minus the missing information. That is, $I_X(\theta) = I_W(\theta) - I_{W|X}(\theta)$, where $I_X(\theta)$, $I_W(\theta)$ and $I_{W|X}(\theta)$ are the observed information, complete information and missing information, respectively. Denote $\theta = (\alpha, \beta)$. The complete information matrix $I_W(\theta)$ is given as

$$I_W(\theta) = -E\left[\frac{\partial^2 \ell_C(W; \theta)}{\partial \theta^2}\right] = \begin{pmatrix} \frac{n}{\alpha^2} & -\frac{2n}{\beta(\alpha+2)} \\ -\frac{2n}{\beta(\alpha+2)} & \frac{4n\alpha}{\beta^2(\alpha+4)} \end{pmatrix}. \tag{3.1}$$

Again, the missing information $I_{W|X}(\theta)$ at j-th failure time $x_{j:m:n}$ is obtained as

$$I_{W|X}^{j:m:n}(\theta) = \begin{pmatrix} -b_{20}(x_{j:m:n}; \alpha, \beta) & -b_{11}(x_{j:m:n}; \alpha, \beta) \\ -b_{11}(x_{j:m:n}; \alpha, \beta) & -b_{02}(x_{j:m:n}; \alpha, \beta) \end{pmatrix},$$

where

$$b_{20}(x_{j:m:n}; \alpha, \beta) = -\frac{1}{\alpha^2}, \qquad b_{11}(x_{j:m:n}; \alpha, \beta) = \frac{2\beta}{(\alpha + 2)(x_{j:m:n}^2 + \beta^2)},$$

$$b_{02}(x_{j:m:n}; \alpha, \beta) = \frac{\alpha}{(x_{j:m:n}^2 + \beta^2)}\left[\frac{(x_{j:m:n}^2 - \beta^2)}{(x_{j:m:n}^2 + \beta^2)} + \frac{2(\alpha + 2)\beta^2}{(\alpha + 4)(x_{j:m:n}^2 + \beta^2)} - 1\right].$$

Thus, the total missing information $I_{W|X}(\theta)$ is given as

$$I_{W|X}(\theta) = \sum_{j=1}^{m} R_j I_{W|X}^{j:m:n}(\theta). \tag{3.2}$$

From the 2×2 order matrices given by (3.1) and (3.2), we compute the observed Fisher information matrix of α and β as

$$I_X(\theta) = \begin{pmatrix} d_{20} & d_{11} \\ d_{11} & d_{02} \end{pmatrix}, \tag{3.3}$$

where

$$d_{20} = \frac{1}{\alpha^2}\left(n - \sum_{j=1}^{m} R_j\right), \qquad d_{11} = -\frac{2}{\beta(\alpha+2)}\left[n - \frac{\beta^2 \sum_{j=1}^{m} R_j}{(x_{j:m:n}^2 + \beta^2)}\right] \quad \text{and}$$

$$d_{02} = \frac{4n\alpha}{\beta^2(\alpha+4)} + \frac{\alpha \sum_{j=1}^{m} R_j}{(x_{j:m:n}^2 + \beta^2)}\left[\frac{(x_{j:m:n}^2 - \beta^2)}{(x_{j:m:n}^2 + \beta^2)} + \frac{2(\alpha+2)\beta^2}{(\alpha+4)(x_{j:m:n}^2 + \beta^2)} - 1\right].$$

In this part, we obtain asymptotic confidence intervals using (i) normal approximation (NA) of the MLE and (ii) the log-transformed (NL) MLE methods. We omit the details of this method to maintain brevity. For the formulas for the NA and NL approaches, see Lee and Cho (2017) and Maiti and Kayal (2020, 2021).

3.1. Bootstrap confidence intervals

It is seen in the previous subsection that to obtain the approximate confidence intervals of the unknown model parameters, it is required to derive second order derivatives which is cumbersome. So, we consider bootstrap technique, which is simpler than NA and NL methods. In particular, we adopt percentile bootstrap (Boot-p) and bootstrap-t (Boot-t) techniques. Here, we describe the procedure how to obtain confidence intervals using Boot-p method. First, we obtain the MLEs of $\eta = (\alpha, \beta, r(x), h(x))$. Denote the MLEs of η by $\hat{\eta} = (\hat{\alpha}, \hat{\beta}, \hat{r}(x), \hat{h}(x))$. Now, based on $\hat{\alpha}$ and $\hat{\beta}$, the bootstrap sample $x^* = (x_1^*, x_2^*, ..., x_n^*)$ has to be generated. We compute $\hat{\eta}^* = (\hat{\alpha}^*, \hat{\beta}^*, \hat{r}^*(x), \hat{h}^*(x))$ based on x^*. Repeat this procedure for 1000 times to get $\hat{\eta}_1^*, \hat{\eta}_2^*, ..., \hat{\eta}_{1000}^*$, where $\hat{\eta}_i^* = (\hat{\alpha}_i^*, \hat{\beta}_i^*, \hat{r}_i^*(x), \hat{h}_i^*(x)),\ i = 1, 2, ..., 1000$. Next, we arrange $\hat{\eta}_i^*$'s in ascending order and denote $\hat{\eta}_{(1)}^* \le \hat{\eta}_{(2)}^* \le \cdots \le \hat{\eta}_{(1000)}^*$. Thus, the $100(1-\varphi)\%$ approximate bootstrap-p confidence interval for η is obtained as (L, U), where $L = \hat{\eta}_{(\frac{i\varphi}{2})}^*$ and $U = \hat{\eta}_{(i(1-\frac{\varphi}{2}))}^*$. The percentile bootstrap confidence interval of η at 95% level of confidence is $(\hat{\eta}_{(25)}^*, \hat{\eta}_{(975)}^*)$. For small sample size, the Boot-p method does not perform well. In this subsection, we discuss Boot-t method, which is simple to apply compared to Boot-p method. We obtain $\hat{\eta}^* = (\hat{\alpha}^*, \hat{\beta}^*, \hat{r}^*(x), \hat{h}^*(x))$ similar to the procedure as mentioned in Boot-p method. Then, based on the bootstrap sample $x^* = (x_1^*, x_2^*, ..., x_n^*)$, we compute the variance-covariance matrix $I_X^{*-1}(\hat{\alpha}^*, \hat{\beta}^*)$. For $i = 1, 2, ..., 1000$, calculate the value of the statistic $T_{\eta_i}^* = (\hat{\eta}_i^* - \hat{\eta}_i)/\sqrt{\widehat{var}(\hat{\eta}_i^*)}$. Then, we arrange in the ascending order and get $T_{\eta_{(1)}}^* \le T_{\eta_{(2)}}^* \le \cdots \le T_{\eta_{(1000)}}^*$. Now, the $100(1-\varphi)\%$ approximate bootstrap-t confidence interval for η is given by (L, U), where $L = T_{\eta_{(\frac{i\varphi}{2})}}^*$ and $U = T_{\eta_{(i(1-\frac{\varphi}{2}))}}^*$. The approximate Boot-$t$ confidence interval of η at 95% level of confidence is $(T_{\eta_{(25)}}^*, T_{\eta_{(975)}}^*)$.

4. BAYESIAN ESTIMATION

In this section, we obtain Bayes estimates of the unknown parameters α, β and the reliability characteristics $r(t), h(t)$ of the gamma-mixed Rayleigh distribution based on PT-IICS. Three loss functions have been considered: (i) squared error loss (SEL) function, (ii) LINEX loss function and (iii) entropy loss function. The SEL function is a balance type loss function. That is, when this loss function is used, the overestimation as well as underestimation do not have any effect on the estimation problem. However, there are situations, where the squared error loss function is not suitable. For example, when we estimate reliability of a rocket, the underestimation is dangerous than the overestimation. Further, the overestimation is severe than the underestimation when estimating the water level of bank of river in a flood-prone area. We also consider two asymmetric loss functions (LINEX and entropy) which are useful to deal with this type of situations. Let δ be an estimator of the unknown parameter ϕ. Then, Table 1 represents Bayes estimates of ϕ under the squared error, LINEX and entropy loss functions. In Table 1, ω and κ are both non-zero real numbers. For $\kappa = -1$, the Bayes estimate with respect to the entropy loss function reduces to that under the squared error function. To obtain the Bayes estimates, one needs to consider prior distributions for the unknown model parameters. It is well known that the joint conjugate prior is not available when both the parameters are not known. Further, there is no clear methodology to choose an appropriate prior (see Arnold and Press, 1983) for a Bayesian estimation problem.

Table 1: Loss functions and the corresponding form of the Bayes estimates.

Name of the loss functions	Form of the loss functions	Form of the Bayes estimates
SEL	$l_s(\phi,\delta) = (\delta-\phi)^2$	$E_\phi(\phi\mid x)$
LINEX	$l_\ell(\phi,\delta) = \exp\{\omega(\delta-\phi)\} - \omega(\delta-\phi) - 1$	$-\frac{1}{\omega}\ln(E_\phi(\exp\{-\omega\phi\}\mid x))$
Entropy	$l_e(\phi,\delta) = (\delta/\phi)^\kappa - \kappa\ln(\delta/\phi) - 1$	$\left[E_\phi(\phi^{-\kappa}\mid x)\right]^{-\frac{1}{\kappa}}$

Note that the gamma distribution is versatile for adjusting different shapes of the density function. It has a log-concave density function in the interval $(0,\infty)$. Jeffery's prior can be obtained as a special case of the gamma prior. Due to these facts, various authors have considered independent gamma distributions as the priors for different Bayesian estimation problems. See, for instance, Kundu (2008), Huang and Wu (2012) and Maiti and Kayal (2020). Here, we assume independent gamma priors for α and β. Let $\alpha \sim \text{Gamma}(a_1,a_2)$ and $\beta \sim \text{Gamma}(a_3,a_4)$, when $\text{Gamma}(a_1,a_2)$ and $\text{Gamma}(a_3,a_4)$ represent gamma distributions with scale and shape parameters $1/a_2, a_1$ and $1/a_4, a_3$, respectively. The probability density functions of $\text{Gamma}(a_1,a_2)$ and $\text{Gamma}(a_3,a_4)$ are given by

$$g_1(\alpha;a_1,a_2) \propto \alpha^{a_1-1}\exp\{-\alpha a_2\} \quad \text{and} \quad g_2(\beta;a_3,a_4) \propto \beta^{a_3-1}\exp\{-\beta a_4\},$$

respectively, where $\alpha,\beta > 0$ and $a_1,a_2,a_3,a_4 > 0$. The hyper-parameters in the prior distributions are assumed to be known. After some simplification, the posterior distribution of

α, β given $X = x$ is obtained as

$$\Pi(\alpha, \beta|x) \propto \frac{\Pi_1(\alpha, \beta, x)}{\int_0^\infty \int_0^\infty \Pi_1(\alpha, \beta, x) d\alpha d\beta}, \tag{4.1}$$

where the joint distribution of α, β and X is given by

$$\Pi_1(\alpha, \beta, x) \propto \alpha^{m+a_1-1} \beta^{m\alpha+a_3-1} \exp\{-(\alpha a_2 + \beta a_4)\} \prod_{i=1}^{m} \frac{x_{i:m:n} \beta^{\alpha R_i}}{(x_{i:m:n}^2 + \beta^2)^{\frac{\alpha}{2}(1+R_i)+1}}. \tag{4.2}$$

Thus, for any arbitrary estimand $g(\alpha, \beta)$, the Bayes estimates with respect to the LINEX and entropy loss functions are respectively obtained as

$$\hat{g}_{bl} = -\frac{1}{\omega} \ln \left[\frac{\int_0^\infty \int_0^\infty \exp\{-\omega g(\alpha, \beta)\} \Pi_1(\alpha, \beta, x) d\alpha d\beta}{\int_0^\infty \int_0^\infty \Pi_1(\alpha, \beta, x) d\alpha d\beta} \right] \quad \text{and} \tag{4.3}$$

$$\hat{g}_{be} = \left[\frac{\int_0^\infty \int_0^\infty g^{-\kappa}(\alpha, \beta) \Pi_1(\alpha, \beta, x) d\alpha d\beta}{\int_0^\infty \int_0^\infty \Pi_1(\alpha, \beta, x) d\alpha d\beta} \right]^{-\frac{1}{\kappa}}. \tag{4.4}$$

As mentioned before, the Bayes estimate with respect to the SEL function can be obtained from (4.4) when $\kappa = -1$. Note that the required Bayes estimates of $\alpha, \beta, r(x)$ and $h(x)$ with respect to the LINEX and entropy loss functions can be computed after substituting $\alpha, \beta, r(x)$ and $h(x)$ in the place of $g(\alpha, \beta)$ in (4.3) and (4.4), respectively. Choosing values of the hyper-parameters is always an important task from Bayesian point of view. Below, we propose a method in this purpose.

Remark 4.1. We generate m samples from a gamma-mixed Rayleigh distribution with distribution function given by (1.1). For each of this m samples, we obtain the MLEs of the model parameters, which are denoted by $\hat{\alpha}^j$ and $\hat{\beta}^j$, $j = 1, 2, ..., m$. The mean and variance of the gamma prior distribution with density function $g_1(\alpha; a_1, a_2)$ are $\frac{a_1}{a_2}$ and $\frac{a_1}{a_2^2}$, respectively. Further, the mean and variance of the MLEs of α for m samples are $\frac{1}{m}\sum_{j=1}^m \hat{\alpha}^j$ and $\frac{1}{m-1}\sum_{j=1}^m (\hat{\alpha}^j - \frac{1}{m}\sum_{j=1}^m \hat{\alpha}^j)^2$, respectively. Therefore, the mean and variance of the MLEs are equal to $\frac{a_1}{a_2}$ and $\frac{a_1}{a_2^2}$, respectively. That is,

$$\frac{a_1}{a_2} = \frac{1}{m} \sum_{j=1}^m \hat{\alpha}^j \quad \text{and} \quad \frac{a_1}{a_2^2} = \frac{1}{m-1} \sum_{j=1}^m \left(\hat{\alpha}^j - \frac{1}{m} \sum_{j=1}^m \hat{\alpha}^j \right)^2.$$

Solving these equations, we get

$$a_1 = \frac{\left(\frac{1}{m}\sum_{j=1}^m \hat{\alpha}^j\right)^2}{\frac{1}{m-1}\sum_{j=1}^m \left(\hat{\alpha}^j - \frac{1}{m}\sum_{j=1}^m \hat{\alpha}^j\right)^2} \quad \text{and} \quad a_2 = \frac{\frac{1}{m}\sum_{j=1}^m \hat{\alpha}^j}{\frac{1}{m-1}\sum_{j=1}^m \left(\hat{\alpha}^j - \frac{1}{m}\sum_{j=1}^m \hat{\alpha}^j\right)^2}.$$

In a similar manner, the hyper-parameters a_3 and a_4 can be obtained from the above equations by replacing $\hat{\alpha}^j$ with $\hat{\beta}^j$.

4.1. Computational methods

In the above section, we see that the proposed Bayes estimates are in the form of the ratio of two integrals. These integrals can not be evaluated in terms of some closed-form expressions. So, we use two approaches in order to get approximate values of the Bayes estimates. One of these is proposed by Lindley (1980). Other is due to Chen and Shao (1999).

4.1.1. Lindley's approximation method

In this subsection, we discuss the Bayes estimates of $\alpha, \beta, r(x)$ and $h(x)$ using Lindley's approximation technique. The detailed derivations are omitted to maintain brevity. We refer to Lee and Cho (2017) and Maiti and Kayal (2021) for detailed derivation of the Bayes estimates using this method. First, we consider LINEX loss function. With respect to this loss function, the Bayes estimate of α is given by

$$\hat{\alpha}_{bl} = -\frac{1}{\omega}\ln\Big[\exp\{-\omega\alpha\} + (1/2)\omega\exp\{-\omega\alpha\}\left[\omega\tau_{11} - A(\alpha,\beta)\right]\Big]\Big|_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})}, \tag{4.5}$$

where $A(\alpha,\beta) = \{l_{30}\tau_{11}^2 + l_{03}\tau_{21}\tau_{22} + 3l_{21}\tau_{11}\tau_{12} + l_{12}(\tau_{11}\tau_{22} + 2\tau_{21}^2) + 2p_1\tau_{11} + 2p_2\tau_{12}\},\ l_{ij} = \frac{\partial^{i+j}l}{\partial\alpha^i\partial\beta^j};\ i,j = 0,1,2,3;\ i+j=3,\ p_1 = \frac{\partial p}{\partial\alpha},\ p_2 = \frac{\partial p}{\partial\beta}$ and p is equal to the logarithm of joint prior distribution of α and β. The Bayes estimate of α with respect to the entropy loss function is

$$\hat{\alpha}_{be} = \Big[\alpha^{-\kappa} + (1/2)\kappa\alpha^{-(\kappa+1)}\big[(\kappa+1)\alpha^{-1}\tau_{11} - A(\alpha,\beta)\big]\Big]^{-\frac{1}{\kappa}}\Big|_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})}. \tag{4.6}$$

The Bayes estimate of α with respect to the squared error loss function can be obtained from (4.6) substituting $\kappa = -1$. Further, the Bayes estimates of $\beta, r(x)$ and $h(x)$ with respect to the squared error, LINEX and entropy loss functions can be derived similarly.

4.1.2. Importance sampling method

In the previous subsection, we obtain the Bayes estimates using Lindley's approximation method. One disadvantage of this method is that it requires higher order partial derivatives of the log-likelihood function. Further, the Lindley's approximation can not be used to construct highest posterior density (HPD) credible intervals. In this subsection, we describe importance sampling method which is free from the higher order partial derivatives. It is also used to compute HPD credible intervals. To apply importance sampling method, we need to rewrite the joint posterior distribution of α, β given $X = x$ in (4.1) as

$$\Pi(\alpha,\beta|X=x) \propto Gamma_{\alpha}(m+a_1,a_2)Gamma_{\beta|\alpha}(m\alpha+a_3,a_4)h(\alpha,\beta) \tag{4.7}$$

where

$$h(\alpha,\beta) = a_4^{-(m\alpha+a_3)}\prod_{i=1}^{m}\beta^{\alpha R_i}x_{i:m:n}(x_{i:m:n}^2+\beta^2)^{-(\frac{\alpha}{2}(1+R_i)+1)}.$$

At first, we generate α from gamma distribution $Gamma_{\alpha}(m+a_1,a_2)$. Next, β is generated from the $Gamma_{\beta|\alpha}(m\alpha+a_3,a_4)$ distribution. We repeat this procedure 1000 times to obtain $(\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_{1000},\beta_{1000})$. Thus, the Bayes estimates of a parametric function $g(\alpha,\beta)$ under LINEX and entropy loss functions are respectively given by

$$\hat{g}_{bl} = -\frac{1}{\omega}\ln\left[\frac{\sum_{i=1}^{1000}\exp\{-\omega g(\alpha_i,\beta_i)\}h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000}h(\alpha_i,\beta_i)}\right] \tag{4.8}$$

and

$$\hat{g}_{be} = \left[\frac{\sum_{i=1}^{1000} g(\alpha_i,\beta_i)^{-\kappa} h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000} h(\alpha_i,\beta_i)}\right]^{-\frac{1}{\kappa}}. \tag{4.9}$$

We compute the Bayes estimates of $\alpha, \beta, r(x)$ and $h(x)$ substituting $\alpha, \beta, r(x)$ and $h(x)$ in place of $g(\alpha,\beta)$, respectively in (4.8) and (4.9) under LINEX and entropy loss functions. Using the concept of importance sampling method, one can derive HPD credible intervals for the unknown parameters α, β and reliability characteristics $r(x), h(x)$. The derivation of the credible intervals have been skipped from this paper due to sake of conciseness. One may refer to Kundu and Raqab (2015) and Rastogi and Tripathi (2014) for elaborate discussion on the derivation of the HPD credible interval for some lifetime distributions.

5. BAYESIAN PREDICTION AND INTERVAL ESTIMATION

In the previous section, we study the Bayesian estimation for the unknown parameters, reliability and the hazard functions. Here, we discuss Bayesian prediction for the future observations based on the PT-IICS taken from the gamma-mixed Rayleigh distribution. We compute the corresponding prediction intervals. There have been a lot of efforts from various authors in prediction problems. For some recent references, please refer to Dey *et al.* (2018) and Bdair *et al.* (2019). This section is divided into two subsections. The following subsection deals with one-sample prediction problem.

5.1. One-sample prediction and Bayesian prediction interval (BPI)

Suppose n number of total independent life testing units are subjected to an experiment. Let $x = (x_{1:m:n}, x_{2:m:n}, ..., x_{m:m:n})$ be the observed progressively type-II censored sample. The censoring scheme is taken as $R = (R_1, R_2, ..., R_m)$. Let $y_i = (y_{i1}, y_{i2}, ..., y_{iR_i})$ represent the ordered lifetimes of the units which are censored at the i-th failure $x_{i:m:n}$. The future observations to be predicted based on x are $y = (y_{ip}; i = 1, 2, ..., m; p = 1, 2, ..., R_i)$. The conditional density y under the given information can be obtained as

$$\begin{aligned} f_1(y|x,\alpha,\beta) \;&=\; p\binom{R_i}{p}\sum_{k=0}^{p-1}(-1)^{p-k-1}\binom{p-1}{k} f(y)(1-F(y))^{R_i-k-1} \\ &\quad\times(1-F(x_i))^{k-R_i}, \qquad y > x_{i:m:n}. \end{aligned} \tag{5.1}$$

The distribution function is

$$F_1(y|x,\alpha,\beta) \;=\; p\binom{R_i}{p}\sum_{k=0}^{p-1}\frac{(-1)^{p-k-1}}{R_i-k}\binom{p-1}{k}\left[1-(1-F(x_i))^{k-R_i}(1-F(y))^{R_i-k}\right]. \tag{5.2}$$

Notice that the posterior predictive density and distribution functions are respectively given by

$$f_1^*(y|x) = \int_0^\infty \int_0^\infty f_1(y|x,\alpha,\beta)\Pi(\alpha,\beta|x)d\alpha d\beta \tag{5.3}$$

and

$$F_1^*(y|x) = \int_0^\infty \int_0^\infty F_1(y|x,\alpha,\beta)\Pi(\alpha,\beta|x)d\alpha d\beta. \tag{5.4}$$

The Bayesian predictive estimate of y under LINEX and entropy loss functions are respectively given by

$$\widehat{y}_l = -\frac{1}{\omega}\ln\left[\int_{x_i}^\infty \exp\{-\omega y\}f_1^*(y|x)dy\right] = -\frac{1}{\omega}\ln[E(P_1(\alpha,\beta)|x)] \tag{5.5}$$

and

$$\widehat{y}_e = \left[\int_{x_i}^\infty y^{-\kappa}f_1^*(y|x)dz\right]^{-\frac{1}{\kappa}} = [E(P_2(\alpha,\beta)|x)]^{-\frac{1}{\kappa}}, \tag{5.6}$$

where

$$P_1(\alpha,\beta) = \int_{x_i}^\infty \exp\{-\omega y\}f_1(y|x,\alpha,\beta)dy \quad \text{and} \quad P_2(\alpha,\beta) = \int_{x_i}^\infty y^{-\kappa}f_1(y|x,\alpha,\beta)dy.$$

Note that above integrals can not be computed analytically. Thus, one needs to use numerical technique in order to compute the predictive estimates. In this purpose, we use importance sampling methods as mentioned in Subsection 4.1.2. Equations (5.5) and (5.6) can be evaluated using importance sampling method as

$$\begin{aligned} \widehat{y}_l &= -\left(\frac{1}{\omega}\right)\ln\left[\frac{\sum_{i=1}^{1000} P_1(\alpha_i,\beta_i)h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000} h(\alpha_i,\beta_i)}\right] \quad \text{and} \\ \widehat{y}_e &= \left[\frac{\sum_{i=1}^{1000} P_2(\alpha_i,\beta_i)h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000} h(\alpha_i,\beta_i)}\right]^{-1/\kappa}, \end{aligned} \tag{5.7}$$

respectively. Next, Bayesian prediction interval is obtained. The prior predictive survival function $S_1(t|x,\alpha,\beta)$ is obtained as

$$S_1(t|x,\alpha,\beta) = \frac{P(y > t|x,\alpha,\beta)}{P(y > x_{i:m:n}|x,\alpha,\beta)} = \frac{\int_t^\infty f_1(u|x,\alpha,\beta)du}{\int_{x_{i:m:n}}^\infty f_1(u|x,\alpha,\beta)du}.$$

The posterior survival function is given by

$$S_1^*(t|x) = \int_0^\infty \int_0^\infty S_1(t|x,\alpha,\beta)\Pi(\alpha,\beta|x)d\alpha d\beta. \tag{5.8}$$

Equation (5.8) can be evaluated using importance sampling method under SEL function as

$$S_1^*(t|x) = \frac{\sum_{i=1}^{1000} S_1(t|x,\alpha_i,\beta_i)h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000} h(\alpha_i,\beta_i)}. \tag{5.9}$$

We obtain two sided $100(1-\varphi)\%$ equal-tail symmetric predictive interval (L,U) by solving the following non-linear equations

$$S_1^*(L|x) = 1 - \frac{\varphi}{2} \quad \text{and} \quad S_1^*(U|x) = \frac{\varphi}{2}. \tag{5.10}$$

The algorithm to obtain the lower bound L and the upper bound U from $S_1^*(t|x) = \eta$, where t is L or U and $\eta = (1 - \frac{\varphi}{2})$ or $\frac{\varphi}{2}$ is described below.

Step-1: Set initial value $t = t_0$.

Step-2: Calculate $S_1^*(t|x) = \frac{\sum_{i=1}^{1000} S_1(t|x,\alpha_i,\beta_i)h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000} h(\alpha_i,\beta_i)}$.

Step-3: If $S_1^*(t|x) < \eta$, then increase t value otherwise decrease the value of t.

Step-4: Repeat steps 2 and 3 until $S_1^*(t|x) \simeq \eta$.

5.2. Two-sample prediction and BPI

In this section, we derive Bayesian two-sample prediction estimate for future observation based on the PT-IICS. It is noted that the two-sample plan is applied in which the observed sample is the PT-IICS and $Z_1 < Z_2 < \cdots < Z_T$ be the unobserved future observations from the same sample, yet to be observed. The predictive density function of Z_j can be written as

$$f(z_j|\alpha,\beta) = j\binom{T}{j}\sum_{p=0}^{j-1}(-1)^{j-1-p}\binom{j-1}{p}[1-F(z_j)]^{T-1-p}f(z_j). \tag{5.11}$$

Again, the posterior prediction density function is obtained as

$$f^*(z_j|x) = \int_0^\infty \int_0^\infty f(z_j|\alpha,\beta)\Pi(\alpha,\beta|x)d\alpha d\beta.$$

Further, the Bayesian predictive estimate of Z_j under LINEX and entropy loss functions are respectively obtained as

$$\widehat{z}_{j_l} = -\left(\frac{1}{\omega}\right)\ln\left[\frac{\sum_{i=1}^{1000} T_1(\alpha_i,\beta_i)h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000} h(\alpha_i,\beta_i)}\right]$$

and

$$\widehat{z}_{j_e} = \left[\frac{\sum_{i=1}^{1000} T_2(\alpha_i,\beta_i)h(\alpha_i,\beta_i)}{\sum_{i=1}^{1000} h(\alpha_i,\beta_i)}\right]^{-1/\kappa},$$

where

$$T_1(\alpha,\beta) = \int_0^\infty \exp\{-\omega z_j\}f(z_j|\alpha,\beta)dz_j \quad \text{and} \quad T_2(\alpha,\beta) = \int_0^\infty z_j^{-\kappa}f(z_j|\alpha,\beta)dz_j.$$

Next, Bayesian prediction interval is obtained. The predictive posterior survival function is given by

$$S_1^*(z_j|x) = \int_0^\infty \int_0^\infty S_1(z_j|x,\alpha,\beta)\Pi(\alpha,\beta|x)d\alpha d\beta,$$

where

$$S_1(z_j|x,\alpha,\beta) = \frac{\int_{z_j}^\infty f_1(u|x,\alpha,\beta)du}{\int_{x_{i:m:n}}^\infty f_1(u|x,\alpha,\beta)du}.$$

The above integration can be approximated using importance sampling method. Further, to obtain the two-sided $100(1-\varphi)\%$ equal-tail symmetric prediction interval (L, U) for Z_j, we have to solve the non-linear equations given by

$$S_1^*(L|x) = 1 - \frac{\varphi}{2} \quad \text{and} \quad S_1^*(U|x) = \frac{\varphi}{2}. \tag{5.12}$$

6. SIMULATION RESULTS AND REAL DATA ANALYSIS

In this section, we first carry out simulation study to observe the performance of the proposed estimates. Next, we consider a real dataset for illustrative purpose.

6.1. Simulation results

This subsection is devoted to the comparative study of the proposed estimates. For this purpose, we generate 1000 progressive type-II censored samples from gamma-mixed Rayleigh distribution. We consider various combinations of (n, m) as $(35, 20), (35, 35), (50, 45)$ and $(50, 50)$. The actual values of α and β are taken as 0.5 and 0.25, respectively. The actual values of $r(x)$ and $h(x)$ are 0.405461 and 0.324324, respectively for $x = 1.5$. There is no reason of taking the value of x as 1.5. One may consider other values of x too. The simulation study has been carried out for other values of x, but not presented here for brevity. For other values of x, similar behaviour of the proposed methods have been observed. The simulation is carried out using the statistical software R (Vienna, Austria; `https://www.r-project.org/`), version 4.1.0. In Table 2, the estimated values of the hyper-parameters are presented for different values of m. For the purpose of the Bayesian estimates, we take $\omega = -0.25, 0.001$ and $\kappa = -0.5, 0.5$ for LINEX and entropy loss functions, respectively. Further, for each n, three different censoring schemes such as progressive type-II, type-II and complete sample have been used for simulation study. These schemes are presented in Table 3. It is known that the type-II censoring scheme is a special case of the progressive type-II censoring scheme.

Table 2: Values of the hyper-parameters for different m.

(α, β)	m	a_1	a_2	a_3	a_4
(0.5, 0.25)	20	0.29516	0.10370	0.29308	0.19850
	35	0.79587	0.50618	0.78481	0.98676
	45	1.41808	1.18262	1.38071	2.30699
	50	1.88165	1.75665	1.84799	3.42774

Table 3: Different censoring schemes (CS).

Scheme	Category	m	$(R_1, R_2, ..., R_m)$
Progressive type-II censoring	Pr-IIc	Odd	$\left(R_{\frac{m+1}{2}} = n - m,\ R_i = 0;\ \ i \neq \frac{m+1}{2}\right)$
		Even	$\left(R_{m/2} = n - m,\ R_i = 0;\ i \neq \frac{m}{2}\right)$
Type-II censoring	Ty-IIc		$(R_m = n - m,\ R_i = 0;\ i \neq m)$
Complete case	Cc		$(R_i = 0;\ i = 1 \sim m)$

Tables 4 and 5 present the average and mean squared error (MSE) values of the MLEs and the Bayes estimates for (α, β) and $(r(x), h(x))$, respectively. The 1st column is for (n, m), the 2nd column is for various censoring schemes (CS), 3rd column is for the estimands. Here, estimands are the unknown parameters α, β and the reliability characteristics $r(x), h(x)$.

Table 4: Average and MSE values of estimates for the parameters α and β.

(n,m)	CS	Parameter	EM	Method	SEL	LINEX		EL	
						$\omega=-0.25$	$\omega=0.001$	$\kappa=-0.5$	$\kappa=0.5$
			Avg (MSE)		Avg (MSE)	Avg (MSE)	Avg (MSE)	Avg (MSE)	Avg (MSE)
(35,20)	Pr-IIc	α	0.577322 (0.045870)	Lin	0.604350 (0.010889)	0.609516 (0.011994)	0.604329 (0.010884)	0.586034 (0.007402)	0.550292 (0.002529)
				Imp	0.615036 (0.012560)	0.617727 (0.012989)	0.615021 (0.012558)	0.608165 (0.010962)	0.601033 (0.008704)
		β	0.294180 (0.018021)	Lin	0.305454 (0.003075)	0.306914 (0.003239)	0.305448 (0.003074)	0.295391 (0.002060)	0.276136 (0.000683)
				Imp	0.320065 (0.003605)	0.323106 (0.004121)	0.320040 (0.003603)	0.305264 (0.003116)	0.303128 (0.001605)
	Ty-IIc	α	0.650948 (0.166828)	Lin	0.739859 (0.057532)	0.748100 (0.061553)	0.739825 (0.057516)	0.712266 (0.045057)	0.652946 (0.023393)
				Imp	0.742216 (0.064523)	0.748506 (0.067051)	0.742177 (0.064507)	0.0.73152 (0.060087)	0.730816 (0.053506)
		β	0.322548 (0.041284)	Lin	0.356579 (0.011359)	0.358578 (0.011789)	0.356571 (0.011357)	0.343506 (0.008743)	0.316460 (0.004417)
				Imp	0.358136 (0.018642)	0.360861 (0.021394)	0.358132 (0.018637)	0.351134 (0.016752)	0.342618 (0.011306)
(,35)	Cc	α	0.517253 (0.015562)	Lin	0.537146 (0.001380)	0.538660 (0.001495)	0.537140 (0.001379)	0.531160 (0.000971)	0.518915 (0.000358)
				Imp	0.541207 (0.002163)	0.546251 (0.002237)	0.541206 (0.002162)	0.537645 (0.002088)	0.524005 (0.001463)
		β	0.259399 (0.006465)	Lin	0.277225 (0.000741)	0.278008 (0.000784)	0.277222 (0.000741)	0.270984 (0.000440)	0.258125 (0.000066)
				Imp	0.284130 (0.001053)	0.287056 (0.001134)	0.284087 (0.001051)	0.282670 (0.001041)	0.278009 (0.000915)
(50,45)	Pr-IIc	α	0.527374 (0.014947)	Lin	0.561924 (0.003834)	0.563149 (0.003988)	0.561918 (0.003834)	0.557074 (0.003257)	0.546668 (0.002178)
				Imp	0.564010 (0.003952)	0.567732 (0.004139)	0.564008 (0.003948)	0.558507 (0.003760)	0.556072 (0.003427)
		β	0.276102 (0.008370)	Lin	0.309227 (0.003508)	0.309901 (0.003588)	0.309224 (0.003507)	0.303973 (0.002913)	0.292121 (0.001774)
				Imp	0.312564 (0.003567)	0.318007 (0.003644)	0.312561 (0.003565)	0.307715 (0.003340)	0.296405 (0.003197)
	Ty-IIc	α	0.528481 (0.015516)	Lin	0.564288 (0.004133)	0.565528 (0.004294)	0.564284 (0.004132)	0.559387 (0.003527)	0.548837 (0.002385)
				Imp	0.570806 (0.004215)	0.577130 (0.004362)	0.570806 (0.004210)	0.561010 (0.004100)	0.560377 (0.003761)
		β	0.276786 (0.008606)	Lin	0.310822 (0.003699)	0.311502 (0.003782)	0.310819 (0.003699)	0.305526 (0.003083)	0.293530 (0.001895)
				Imp	0.315542 (0.003720)	0.320566 (0.003935)	0.315537 (0.003720)	0.312147 (0.003565)	0.307081 (0.003416)
(,50)	Cc	α	0.527521 (0.011298)	Lin	0.573868 (0.005456)	0.574784 (0.005592)	0.573864 (0.005456)	0.570163 (0.004923)	0.561857 (0.003826)
				Imp	0.579013 (0.005521)	0.581451 (0.005640)	0.579011 (0.005520)	0.560891 (0.005314)	0.560071 (0.005281)
		β	0.279152 (0.008391)	Lin	0.339271 (0.007969)	0.339828 (0.008069)	0.339268 (0.007969)	0.334464 (0.007134)	0.3216673 (0.005136)
				Imp	0.341553 (0.007974)	0.346086 (0.008213)	0.341550 (0.007971)	0.338880 (0.007718)	0.320799 (0.007428)

The average values and the MSEs of the MLEs are presented in 4th column. Note that the MLEs are computed based on EM algorithm. We present two methods Lindley's approximation (Lin) and importance sampling (Imp) in fifth column. In 6–10th columns, the average and MSE values of the Bayes estimates with respect to the squared error, LINEX and entropy loss functions are presented. The MSE values of each estimate are placed inside the parenthesis.

Table 5: Average and MSE values of the estimates for $r(x)$ and $h(x)$.

(n,m)	CS	Parameter	EM Avg (MSE)	Method	SEL Avg (MSE)	LINEX $\omega=-0.25$ Avg (MSE)	LINEX $\omega=0.001$ Avg (MSE)	EL $\kappa=-0.5$ Avg (MSE)	EL $\kappa=0.5$ Avg (MSE)
(35,20)	Pr-IIc	$r(x)$	0.398127 (0.006980)	Lin	0.392919 (0.001572)	0.393820 (0.001364)	0.392916 (0.001571)	0.388221 (0.002977)	0.378922 (0.007040)
				Imp	0.421631 (0.002423)	0.425102 (0.002608)	0.421630 (0.002416)	0.405181 (0.002335)	0.402219 (0.002286)
		$h(x)$	0.365147 (0.014572)	Lin	0.382504 (0.003385)	0.384415 (0.003611)	0.382496 (0.003384)	0.372067 (0.002279)	0.351927 (0.000762)
				Imp	0.386712 (0.003461)	0.405312 (0.003516)	0.386710 (0.003460)	0.381207 (0.003070)	0.364315 (0.001004)
	Ty-IIc	$r(x)$	0.390138 (0.008294)	Lin	0.359881 (0.002078)	0.360923 (0.001984)	0.359876 (0.002078)	0.354171 (0.002631)	0.343389 (0.001853)
				Imp	0.361864 (0.003105)	0.367701 (0.002281)	0.361861 (0.003104)	0.346105 (0.002845)	0.331611 (0.002506)
		$h(x)$	0.396838 (0.034886)	Lin	0.459716 (0.018331)	0.462730 (0.019156)	0.459704 (0.018328)	0.444301 (0.014394)	0.411965 (0.007681)
				Imp	0.466071 (0.02160)	0.463102 (0.02377)	0.466070 (0.02159)	0.446265 (0.016492)	0.413509 (0.015423)
(,35)	Cc	$r(x)$	0.405129 (0.003941)	Lin	0.403232 (0.000497)	0.403732 (0.000299)	0.40323 (0.000495)	0.400726 (0.000224)	0.395755 (0.000142)
				Imp	0.414506 (0.000534)	0.418187 (0.000436)	0.414501 (0.000530)	0.409461 (0.000303)	0.407196 (0.000287)
		$h(x)$	0.332928 (0.005467)	Lin	0.344692 (0.000415)	0.345272 (0.000439)	0.344689 (0.000415)	0.341173 (0.000284)	0.334047 (0.000145)
				Imp	0.347236 (0.000521)	0.356043 (0.000540)	0.347230 (0.000518)	0.347991 (0.000477)	0.340564 (0.000420)
(50,45)	Pr-IIc	$r(x)$	0.409594 (0.003151)	Lin	0.409331 (0.000150)	0.409721 (0.000181)	0.409331 (0.000150)	0.407411 (0.000138)	0.403577 (0.000113)
				Imp	0.416460 (0.000213)	0.418805 (0.000227)	0.416456 (0.000212)	0.415643 (0.000186)	0.408037 (0.000172)
		$h(x)$	0.337868 (0.005060)	Lin	0.357965 (0.001132)	0.358433 (0.001163)	0.357965 (0.001132)	0.355143 (0.000950)	0.349185 (0.000618)
				Imp	0.371653 (0.001276)	0.366203 (0.001315)	0.371652 (0.001276)	0.364966 (0.001219)	0.361500 (0.001207)
	Ty-IIc	$r(x)$	0.409480 (0.003157)	Lin	0.408961 (0.000224)	0.409349 (0.000251)	0.408960 (0.000222)	0.407044 (0.000202)	0.403218 (0.000150)
				Imp	0.415008 (0.000415)	0.417648 (0.000430)	0.415000 (0.000415)	0.410651 (0.000407)	0.409472 (0.000389)
		$h(x)$	0.338443 (0.005220)	Lin	0.359309 (0.001224)	0.359779 (0.001257)	0.359307 (0.001224)	0.356457 (0.001033)	0.350424 (0.000681)
				Imp	0.376511 (0.001672)	0.385205 (0.001842)	0.376510 (0.001670)	0.364006 (0.001258)	0.362380 (0.001029)
(,50)	Cc	$r(x)$	0.409277 (0.003093)	Lin	0.421297 (0.000251)	0.421678 (0.000263)	0.421296 (0.000251)	0.419397 (0.000194)	0.415464 (0.000100)
				Imp	0.425031 (0.000271)	0.428014 (0.000284)	0.425030 (0.000269)	0.423330 (0.000253)	0.421643 (0.000234)
		$h(x)$	0.337988 (0.003660)	Lin	0.362957 (0.001492)	0.363316 (0.001520)	0.362956 (0.001492)	0.360747 (0.001327)	0.355944 (0.000910)
				Imp	0.366172 (0.001781)	0.368008 (0.001845)	0.366171 (0.001780)	0.364031 (0.001542)	0.358813 (0.001325)

Table 6 represents average lengths of 95% confidence intervals and the HPD credible intervals for $\alpha, \beta, r(x)$ and $h(x)$. We have tabulated one-sample and two-sample Bayesian prediction estimates and 95% prediction intervals for future observation in Tables 7 and 8, respectively.

Table 6: Average lengths of 95% interval estimates of α, β, $r(x)$ and $h(x)$.

(n,m)	CS		Asymptotic		Bootstrap		HPD
			NA	NL	Boot-p	Boot-t	
(35,20)	Pr-IIc	α	0.648062	0.682627	0.576212	0.543509	0.267133
		β	0.329015	0.346433	0.311642	0.300701	0.186809
		$r(x)$	0.322288	0.331721	0.277547	0.246013	0.167990
		$h(x)$	0.418470	0.441055	0.378052	0.320005	0.210808
	Ty-IIc	α	0.858970	0.922660	0.761805	0.721656	0.531964
		β	0.400653	0.426912	0.346010	0.300891	0.222437
		$r(x)$	0.338097	0.350497	0.294644	0.260079	0.190059
		$h(x)$	0.580588	0.629159	0.510660	0.493112	0.301243
(,35)	Cc	α	0.420716	0.432409	0.375604	0.310064	0.176117
		β	0.301196	0.318404	0.265203	0.228561	0.133991
		$r(x)$	0.252868	0.257088	0.210705	0.189620	0.117867
		$h(x)$	0.276372	0.284284	0.236081	0.194051	0.140660
(50,45)	Pr-IIc	α	0.427858	0.439689	0.346033	0.310446	0.188656
		β	0.360034	0.386089	0.306770	0.264126	0.160533
		$r(x)$	0.220637	0.223362	0.176136	0.143088	0.100362
		$h(x)$	0.246997	0.252462	0.184461	0.136504	0.086420
	Ty-IIc	α	0.433796	0.446078	0.379100	0.341444	0.213064
		β	0.364740	0.391709	0.310788	0.306171	0.175757
		$r(x)$	0.220169	0.222880	0.150991	0.123404	0.068944
		$h(x)$	0.250175	0.255833	0.197005	0.158817	0.100888
(,50)	Cc	α	0.374461	0.382373	0.334671	0.280062	0.133785
		β	0.292170	0.305690	0.245508	0.224999	0.145871
		$r(x)$	0.211673	0.214052	0.131106	0.108841	0.043649
		$h(x)$	0.230374	0.234809	0.169507	0.120889	0.056643

In the both sample prediction problems, the values of the parameters, hyper-parameters, ω and κ are taken same. Here, we consider $p = 1, 2, 3$ for 1st and 7th failure stages in one-sample prediction, and $j = 1, 2, 3$ for $T = m$ in two-sample prediction. From the numerical values, the following discussions can be drawn.

1. From Table 2, it is observed that with increasing values of m, values of the hyper-parameters a_1, a_2, a_3 and a_4 increase.

2. From the tabulated values in Table 4, we notice that the Bayes estimates perform better than the MLE in terms of the MSE. Further, the Bayes estimates for positive values of ω and κ are better than that for negative values of ω and κ in terms of the average values and MSEs. The simulated average values of the estimates approach towards the true value when (n, m) increases. Further, MSE decreases when (n, m) increases. Similar observation is noticed for the case of complete sample. As expected, the behavior of the Bayes estimates under SEL function and the LINEX loss function is approximately same for small values of ω (here $\omega = 0.001$). It is seen that in general, the progressive type-II censoring schemes produces better result than type-II censoring scheme in terms of the average (Avg) and MSE values. Similar behavior of the estimates of $r(x)$ and $h(x)$ (presented in Table 5) can be pointed out. The abbreviation EL is used for entropy loss function.

3. In Table 6, it is observed that for asymptotic confidence intervals, the NA method provides better estimates than NL method. For the case of bootstrap confidence intervals, Boot-t method performs better than Boot-p method. However, among the computed five intervals, HPD credible intervals give the best performance. Also, it is noticed that the average length decreases when effective sample size increases. When comparing progressive type-II censoring and type-II censoring plans, the progressive type-II plan provides better result.

4. From Table 7, we observe that the values of the predictive estimates based on progressive type-II censoring scheme are larger than that for type-II censoring scheme. The values of the predictive estimates and prediction lengths increase as i, p increase. When the effective sample size (m) increases, the predictive estimate values and predictive interval lengths decrease. Similar observation can be noticed from Table 8 for two-sample prediction problem.

Table 7: One-sample prediction values and 95% prediction intervals for future observations.

(n,m)	CS	i	p	SEL	LINEX		EL		L	U	Width
					$\omega=-0.25$	$\omega=0.001$	$\kappa=-0.5$	$\kappa=0.5$			
(35,20)	Pr-IIc	1	1	0.108452	0.117826	0.108417	0.104236	0.102013	0.033642	0.30036	0.266718
			2	0.124011	0.152041	0.124010	0.110628	0.109972	0.080750	0.421644	0.340894
			3	0.177314	0.182622	0.177306	0.154082	0.147880	0.019462	0.430405	0.410943
		7	1	0.240586	0.285261	0.240585	0.231178	0.200864	0.108235	0.671046	0.562811
			2	0.293324	0.374406	0.293319	0.274152	0.261997	0.152083	0.782603	0.630520
			3	0.315852	0.418077	0.315847	0.300919	0.281485	0.072634	0.766852	0.694218
	Ty-IIc	1	1	0.075164	0.106172	0.075158	0.052800	0.046281	0.008285	0.394668	0.386383
			2	0.108273	0.134867	0.108266	0.095076	0.052997	0.052347	0.460809	0.408462
			3	0.152972	0.160723	0.152897	0.123699	0.120758	0.089046	0.501897	0.412851
		7	1	0.192640	0.248671	0.192578	0.164284	0.128670	0.097825	0.840869	0.743044
			2	0.228068	0.278526	0.227972	0.196099	0.180907	0.130884	0.956252	0.825368
			3	0.276291	0.286070	0.276188	0.241046	0.211220	0.172691	1.068736	0.896045
(,35)	Cc	1	1	0.059483	0.080963	0.059476	0.024317	0.018605	0.016180	0.156838	0.140658
			2	0.089255	0.118047	0.089250	0.070965	0.059672	0.019941	0.192223	0.172282
			3	0.136974	0.145052	0.136970	0.114441	0.118064	0.056427	0.279560	0.223133
		7	1	0.149753	0.215093	0.149748	0.137570	0.119329	0.082594	0.275598	0.193004
			2	0.191426	0.255305	0.191422	0.158973	0.152834	0.113426	0.339898	0.226472
			3	0.248593	0.276009	0.248497	0.216852	0.208440	0.064285	0.324402	0.260117
(50,45)	Pr-IIc	1	1	0.087745	0.102351	0.087742	0.061882	0.026954	0.007920	0.183156	0.175236
			2	0.116562	0.139095	0.116558	0.098726	0.072609	0.025834	0.235675	0.209841
			3	0.150768	0.164964	0.150760	0.130999	0.109556	0.072440	0.366447	0.294007
		7	1	0.174109	0.245108	0.173981	0.167168	0.150699	0.100826	0.583458	0.482632
			2	0.228347	0.264052	0.228337	0.184623	0.159082	0.119950	0.650769	0.530819
			3	0.298067	0.345223	0.298060	0.265214	0.221704	0.075301	0.685883	0.610582
	Ty-IIc	1	1	0.067653	0.899425	0.067647	0.050715	0.041532	0.006715	0.249486	0.242771
			2	0.075989	0.125008	0.075988	0.071324	0.068227	0.028600	0.423756	0.395156
			3	0.126706	0.140764	0.123803	0.093587	0.074553	0.053428	0.464197	0.410769
		7	1	0.176572	0.207206	0.176568	0.120975	0.096408	0.031407	0.584353	0.552946
			2	0.205034	0.235607	0.205027	0.185209	0.172136	0.125855	0.758676	0.632821
			3	0.246174	0.264084	0.246172	0.208461	0.174507	0.131252	0.797154	0.665902
(,50)	Cc	1	1	0.038965	0.064027	0.038957	0.034208	0.028497	0.006783	0.116741	0.109958
			2	0.061794	0.097659	0.061788	0.058993	0.037806	0.009397	0.192057	0.182660
			3	0.106455	0.128709	0.106449	0.097808	0.070845	0.053129	0.296908	0.243779
		7	1	0.130846	0.198432	0.130840	0.100975	0.074588	0.034628	0.16496	0.130332
			2	0.174050	0.226741	0.174043	0.164317	0.130894	0.093459	0.234368	0.140909
			3	0.215686	0.231606	0.215679	0.196309	0.164320	0.116237	0.32676	0.210523

Table 8: Two-sample prediction values and 95% prediction intervals for future observations.

(n,m)	CS	j	SEL	LINEX		EL		L	U	Width
				$\omega=-0.25$	$\omega=0.001$	$\kappa=-0.5$	$\kappa=0.5$			
(35,20)	Pr-IIc	1	0.035652	0.084621	0.035649	0.029411	0.016425	0.004351	0.562487	0.558136
		2	0.118036	0.140993	0.118030	0.084403	0.042692	0.009422	0.700663	0.691241
		3	0.156008	0.162308	0.155792	0.110699	0.097183	0.013140	0.733492	0.720352
	Ty-IIc	1	0.010546	0.043119	0.010537	0.007234	0.004977	0.000782	0.712836	0.712054
		2	0.061582	0.075408	0.061561	0.038947	0.029425	0.005279	0.84835	0.843071
		3	0.109776	0.158223	0.109770	0.102432	0.092564	0.006233	0.889221	0.882988
(,35)	Cc	1	0.007642	0.009751	0.007636	0.004318	0.003707	0.000824	0.388925	0.388101
		2	0.021274	0.048030	0.021259	0.016799	0.012973	0.005348	0.441291	0.435943
		3	0.057293	0.132947	0.057288	0.033741	0.030912	0.009425	0.497781	0.488356
(50,45)	Pr-IIc	1	0.026423	0.053190	0.026418	0.024083	0.014654	0.000693	0.36197	0.361277
		2	0.080145	0.117522	0.080140	0.055920	0.046728	0.008432	0.448753	0.440321
		3	0.129506	0.127001	0.129489	0.107418	0.086947	0.026488	0.540278	0.513790
	Ty-IIc	1	0.008824	0.015283	0.008819	0.005271	0.004725	0.000707	0.484861	0.484154
		2	0.041672	0.049382	0.041672	0.028526	0.021310	0.008262	0.559655	0.551393
		3	0.010243	0.129743	0.010240	0.091253	0.058291	0.003714	0.616722	0.613008
(,50)	Cc	1	0.005281	0.006994	0.005274	0.002867	0.002173	0.000848	0.302409	0.301561
		2	0.017126	0.024907	0.017121	0.004892	0.003282	0.000437	0.361289	0.360852
		3	0.022809	0.037615	0.022800	0.019264	0.014066	0.002640	0.424486	0.421846

6.2. Real data analysis

In this subsection, we consider real life dataset representing the times to breakdown of an insulating fluid between electrodes recorded at the voltage of 34 kV (minutes) in a life test. The dataset is introduced by Nelson (2016) and used by Soliman (2005). The dataset is presented below.

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67
4.85	6.50	7.3	8.01	8.27	12.06	31.75	32.52
33.91	36.71	72.89					

For the purpose of goodness of fit test, we consider various methods such as log-likelihood criterion, Kolmogorov-Smirnov (KS) statistic, Akaike's-information criterion (AIC), the associated second-order information criterion (AICc) and Bayesian information criterion (BIC). The values of the MLEs and the five goodness of fit test statistics are presented in Table 9.

Table 9: The MLEs, KS, log-likelihood, AIC, AICc and BIC values for the real dataset.

Distribution	MLEs		KS	$\ln L$	BIC	AICc	AIC
	Shape	Scale					
G-MR(α,β)	$\hat{\alpha}=0.795210$	$\hat{\beta}=2.392015$	0.135509	-70.34277	146.5744	145.4355	144.6855
HL(λ)		$\hat{\lambda}=0.088745$	0.332880	-71.97299	146.8904	146.1813	145.946
IExpHL(α,θ)	$\hat{\alpha}=0.426676$	$\hat{\theta}=0.801178$	0.264552	-74.03980	153.9685	152.8296	152.0796
IW(α,λ)	$\hat{\alpha}=2.038295$	$\lambda=1.119888$	0.329144	-75.25765	156.4042	155.2653	154.5153
GF (α,λ,σ)	$\hat{\alpha}=7.465586$, $\hat{\lambda}=0.354321$	$\hat{\sigma}=7.260802$	0.667178	-95.1172	199.0677	197.8344	196.2344

The numerical values in Table 9 suggest that the gamma-mixed Rayleigh (G-MR) distribution fits the data well compared to the half-logistic (HL), inverted exponentiated half-logistic (IExpHL), inverse Weibull (IW) and generalized Fréchet (GF) distributions. Now, we compute the proposed estimates for the unknown parameters, reliability and hazard functions. In Table 10, we consider progressive type-II censored sample with total sample size $n = 19$, the failure sample size $m = 14$. We adopt various schemes for the purpose of computation. Here, we consider three schemes say Pr-IIc, Ty-IIc and Cc, that is $(R_1, R_2, ..., R_m)$ =(0*6,5,0*7), (0*13,5) and (0*19), respectively. Note that (0*3,2) denotes the censoring scheme $(0, 0, 0, 2)$.

Table 10: Progressive type-II censored data for the real dataset.

i	1	2	3	4	5	6	7
$x_{i:m:n}$	0.19	0.78	1.31	2.78	4.15	4.67	4.85
i	8	9	10	11	12	13	14
$x_{i:m:n}$	8.01	8.27	12.06	31.75	33.91	36.71	72.89

We take all the hyperparameter values as zero. We present the average values of the proposed estimates of α, β in Table 11 and $r(x)$ and $h(x)$ in Table 12. Table 13 represents 95% interval estimates of α, β, $r(x)$ and $h(x)$. Further, we have tabulated one-sample and two-sample predicted values and 95% prediction intervals in Tables 14 and 15, respectively. Here, we obtain one-sample prediction estimates of the lifetime of first three units at i-th failure and two-sample prediction estimates of the lifetime of first three units and size of sample $T = 10$. The plots of the probability density functions of five different models and histogram for real dataset are presented in Figure 2. In Figures 3 and 4, the plots of the density, distribution, reliability and hazard functions of gamma-mixed Rayleigh distribution under Pr-IIc, Ty-IIc and Cc schemes are depicted.

Table 11: Estimates of α and β for the real dataset.

(n, m)	CS		EM	Method	SEL	LINEX		EL	
						$\omega = -0.25$	$\omega = 0.001$	$\kappa = -0.5$	$\kappa = 0.5$
			Avg		Avg	Avg	Avg	Avg	Avg
(19,14)	Pr-IIc	α	0.776441	Lin	0.399239	0.399847	0.399239	0.411029	0.437988
				Imp	0.367051	0.376001	0.367050	0.381976	0.387007
		β	3.356965	Lin	1.362572	1.455717	1.362637	1.463404	1.652837
				Imp	1.335429	1.389600	1.335427	1.412632	1.486753
	Ty-IIc	α	0.294786	Lin	0.206869	0.207328	0.206867	0.205371	0.205485
				Imp	0.176532	0.193725	0.176530	0.162305	0.167035
		β	1.12691	Lin	0.502474	0.537084	0.502364	0.495457	0.531081
				Imp	0.464582	0.499007	0.464578	0.446396	0.468757
(,19)	Cc	α	0.795210	Lin	0.570614	0.575673	0.570595	0.563006	0.558460
				Imp	0.523781	0.568766	0.523780	0.512525	0.504817
		β	2.392015	Lin	1.465001	1.477988	1.464669	1.429069	1.429344
				Imp	1.400864	1.459764	1.400860	1.387562	1.385258

Table 12: Estimates of $r(x)$ and $h(x)$ for the real dataset.

(n, m)	CS		EM	Method	SEL	LINEX		EL	
						$\omega = -0.25$	$\omega = 0.001$	$\kappa = -0.5$	$\kappa = 0.5$
			Avg		Avg	Avg	Avg	Avg	Avg
(19,14)	Pr-IIc	$r(x)$	0.931769	Lin	0.867124	0.868877	0.867125	0.867672	0.868731
				Imp	0.846209	0.847669	0.846207	0.847008	0.847460
		$h(x)$	0.086149	Lin	0.152817	0.152628	0.152818	0.151136	0.148734
				Imp	0.186422	0.187994	0.186421	0.177537	0.153480
	Ty-IIc	$r(x)$	0.860480	Lin	0.773932	0.773973	0.773932	0.773969	0.774245
				Imp	0.748209	0.748867	0.748208	0.748452	0.748666
		$h(x)$	0.125622	Lin	0.132865	0.133130	0.132864	0.128547	0.120052
				Imp	0.164263	0.168117	0.164261	0.123786	0.119007
(,19)	Cc	$r(x)$	0.876466	Lin	0.801888	0.801855	0.801888	0.802035	0.802425
				Imp	0.774826	0.780074	0.774824	0.775314	0.776174
		$h(x)$	0.149630	Lin	0.211578	0.211800	0.211577	0.206734	0.189875
				Imp	0.230761	0.236482	0.230760	0.214776	0.206782

Table 13: 95% interval estimates of α, β, $r(x)$ and $h(x)$ based on the real dataset.

(n, m)	CS		Asymptotic		Bootstrap		HPD
			NA	NL	Boot-p	Boot-t	
(19,14)	Pr-IIc	α	(0.03697, 1.5159)	(0.29957, 2.01244)	(0.11275, 1.32586)	(0.08563, 1.16478)	(0.00794, 0.86115)
		β	(0.00000, 7.20239)	(1.06772, 10.5544)	(2.86452, 9.98328)	(3.05617, 9.06852)	(0.09908, 4.43365)
		$r(x)$	(0.83953, 1.02401)	(0.84394, 1.02873)	(0.80581, 0.95486)	(0.82068, 0.96246)	(0.84105, 0.94857)
		$h(x)$	(0.00000, 0.19222)	(0.02515, 0.29512)	(0.05946, 0.24653)	(0.07563, 0.23213)	(0.07884, 0.18947)
	Ty-IIc	α	(0.08674, 0.50284)	(0.14554, 0.59706)	(0.10466, 0.47148)	(0.09462, 0.41656)	(0.12035, 0.2984)
		β	(0.00000, 2.65557)	(0.29025, 4.37535)	(0.12630, 2.24429)	(0.21364, 2.06421)	(0.41286, 1.51018)
		$r(x)$	(0.68785, 1.03311)	(0.70406, 1.05165)	(0.62482, 0.90287)	(0.66946, 0.88035)	(0.72451, 0.87172)
		$h(x)$	(0.03414, 0.21710)	(0.06065, 0.26020)	(0.07567, 0.22233)	(0.08745, 0.19816)	(0.10526, 0.1824)
(,19)	Cc	α	(0.21443, 1.37599)	(0.38309, 1.65070)	(0.33456, 1.43563)	(0.27664, 1.35645)	(0.42784, 1.31559)
		β	(0.00000, 4.81147)	(0.86994, 6.57719)	(1.01503, 4.92624)	(1.21536, 4.02611)	(1.36485, 2.53242)
		$r(x)$	(0.73417, 1.01876)	(0.74512, 1.03096)	(0.68356, 0.90118)	(0.70599, 0.88412)	(0.75086, 0.88837)
		$h(x)$	(0.00223, 0.29703)	(0.05587, 0.40071)	(0.04312, 0.25221)	(0.10537, 0.26529)	(0.13458, 0.25346)

Table 14: One-sample prediction values and 95% prediction intervals for future observations.

(n,m)	CS	i	p	SEL	LINEX		EL		BPI
					$\omega=-0.25$	$\omega=0.001$	$\kappa=-0.5$	$\kappa=0.5$	
(19,14)	Pr-IIc	1	1	4.864253	5.283172	4.864247	4.064281	3.642513	(3.410321, 5.572681)
			2	5.123467	6.017240	5.123460	4.618267	4.042580	(3.843775, 6.333201)
			3	5.507412	6.724315	5.507403	4.970824	4.386421	(4.210341, 7.262675)
		5	1	7.129901	9.058243	7.129882	6.527802	6.201358	(5.462305, 10.305467)
			2	8.105347	11.25680	8.105339	7.201634	7.053412	(6.76250, 11.738928)
			3	8.753105	12.88402	8.753087	7.836405	7.643187	(7.335213, 12.599454)
	Ty-IIc	1	1	2.560428	3.526411	2.557964	2.134526	2.074091	(1.761855, 3.837296)
			2	4.068825	4.672553	4.068822	2.760048	2.496253	(2.142392, 5.058906)
			3	4.748263	4.958240	4.748257	3.209992	2.897385	(2.437218, 5.321908)
		5	1	3.580742	5.272538	3.580739	3.336142	2.582773	(2.161007, 5.509298)
			2	5.23189	6.856842	5.231817	4.219138	4.000876	(3.496125, 7.196189)
			3	7.100923	7.594625	7.100916	6.735364	5.436582	(4.430564, 8.493101)
(,19)	Cc	1	1	2.182731	2.382467	2.182726	1.766528	1.623407	(0.942521, 2.676136)
			2	3.067269	3.631854	3.067261	2.578532	2.247716	(1.432610, 4.259329)
			3	3.854727	5.582428	3.854718	2.891759	2.374685	(1.796821, 6.011103)
		5	1	2.978550	4.297582	2.978544	2.432080	2.178441	(1.800672, 4.810808)
			2	3.352725	5.317162	3.352716	2.70553	2.484336	(2.134255, 5.565777)
			3	5.034900	5.924856	5.033875	3.885664	3.100858	(2.704073, 7.589788)

Table 15: Two-sample prediction values and 95% prediction intervals for future observations.

(n,m)	CS	j	SEL	LINEX		EL		BPI
				$\omega=-0.25$	$\omega=0.001$	$\kappa=-0.5$	$\kappa=0.5$	
(19,14)	Pr-IIc	1	1.724826	2.074824	1.724813	1.376145	1.265064	(1.077346, 2.385841)
		2	2.064354	2.467026	2.064348	1.750517	1.305119	(1.213075, 2.826527)
		3	4.846177	3.794121	4.846169	3.526443	3.344056	(2.19428, 4.92259)
	Ty-IIc	1	1.255674	1.699764	1.255670	1.096628	0.946491	(0.631672, 1.886354)
		2	1.462812	2.152316	1.462805	1.224056	1.064583	(0.816055, 2.28461)
		3	2.803286	3.45821	2.803179	2.437182	2.175564	(1.430592, 3.802253)
(,19)	Cc	1	0.860765	1.180676	0.860761	0.681231	0.620711	(0.462854, 1.257712)
		2	1.113446	1.445251	1.113437	0.846616	0.726489	(0.371066, 1.623857)
		3	2.045084	2.152647	2.045076	1.615233	1.086961	(0.794823, 2.343845)

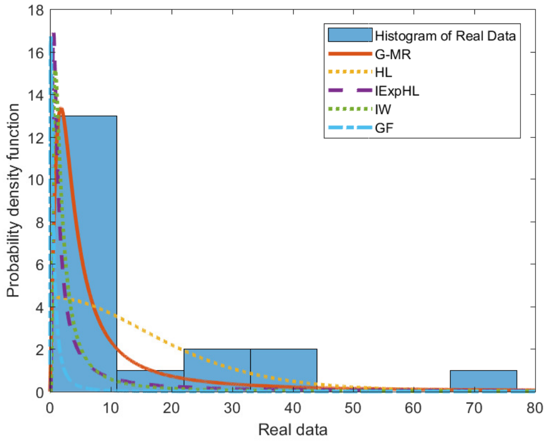
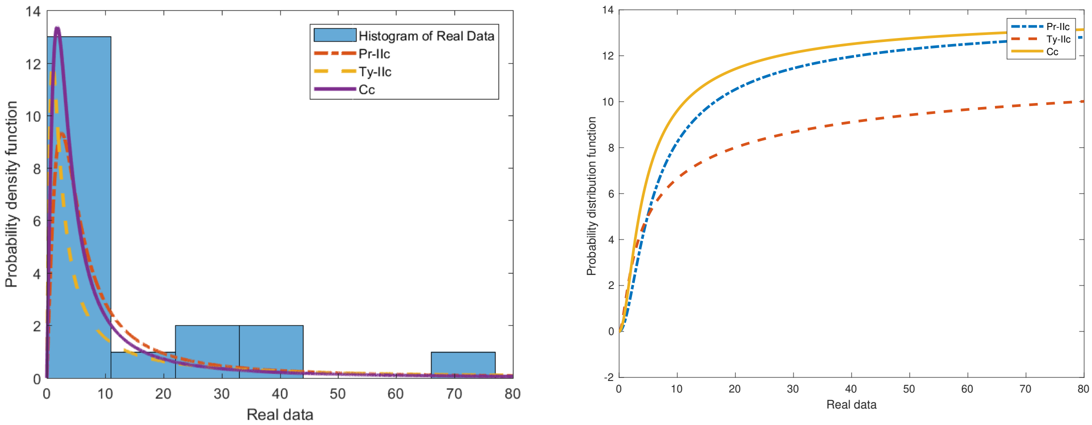
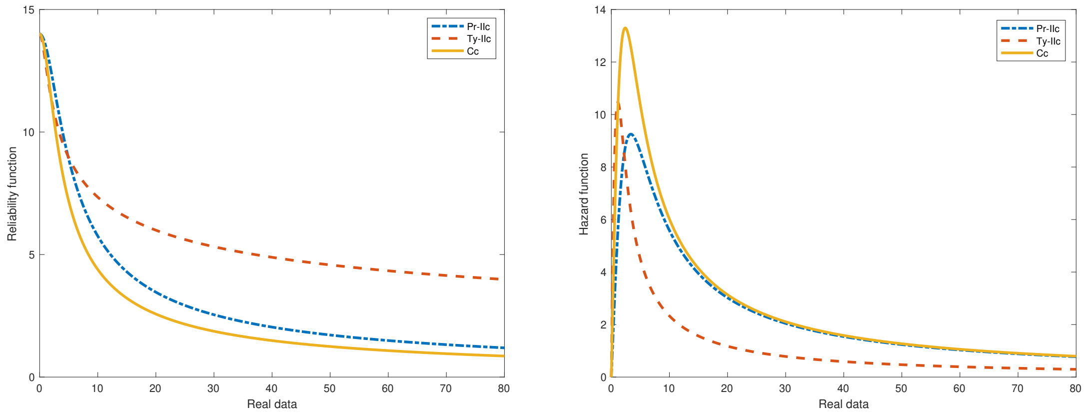


Figure 2: The histogram of the real dataset and the plots of the probability density functions of the fitted G-MR, HL, IExpHL, IW, GF models.



(a) (b)

Figure 3: The plots of the (a) density and (b) distribution functions of the gamma-mixed Rayleigh distribution based on different censoring schemes.



(a) (b)

Figure 4: The plots of the (a) reliability and (b) hazard functions based on different censoring schemes.

7. CONCLUDING REMARKS

In industrial life tests, reliability analysis and clinical trials, the type-II progressive censoring methodology has become quite popular for analyzing lifetime data. It allows for random removals of the remaining survival units at each failure time. In this article, we considered inference and prediction problems for the gamma-mixed Rayleigh distribution when progressive type-II censored sample is available. We obtained conditions under which the MLEs exist and are unique, then derived the MLEs using EM algorithm. The Bayes estimates have been computed with respect to three loss functions, such as squared error, LINEX and entropy loss functions. Two approximations say Lindley approximation and importance sampling method have been used for the computation of the Bayes estimates. We also derived confidence and credible intervals using various methods. Specifically, we have obtained asymptotic, bootstrap-p and bootstrap-t confidence intervals and highest posterior density credible interval. Further, we discussed Bayesian prediction problems. One-sample and two-sample prediction problems have been considered. An elaborate simulation study was conducted for the comparison of the proposed estimates. From the simulation study, it has been observed that the Bayes estimates perform better than the MLEs in terms of the MSE values. Further, the Bayes estimates for positive values of ω and κ are better than that for negative values of ω and κ in terms of the average values and MSEs. For the present problem, we recommend the Bayes estimates to use for the case of point estimation. It has been observed that for asymptotic confidence intervals, the NA method provides better estimates than NL method in the sense of the average lengths. For the case of bootstrap confidence intervals, Boot-t method performs better than Boot-p method. However, among the computed five intervals, HPD credible intervals give the best performance. Among all the interval estimates, we recommend HPD credible interval estimate. In addition to these, we have also computed predictive estimates. It has been noticed that when the effective sample size increases, the predictive estimates and predictive interval lengths decrease. Finally, we considered a real life dataset representing the times to breakdown of an insulating fluid between electrodes recorded at the voltage of 34 kV (minutes) in a life test for illustrative purposes.

ACKNOWLEDGMENTS

The authors would like to thank the Editor-in-chief, an associate editor and an anonymous referee for careful reading and valuable comments, which have led to this improved version. The authors also wish to thank Dr. Sugata Gangopadhyay for his valuable suggestions while preparing the draft of this manuscript. One of the authors Kousik Maiti thanks the Ministry of Electronics & Information Technology (MeitY), Government of India for the financial assistantship received to carry out this research work.

REFERENCES

Albalawi, O., Kabdwal, N. C., Azhad, Q. J., Hora, R., and Alsaedi, B. S. (2022). Estimation of the generalized logarithmic transformation exponential distribution under progressively Type-II censored data with application to the COVID-19 mortality rates. *Mathematics*, 10(7):1015.

Arabi, B. R., Noori, A. M., Gurunlu, A. O., Singh, S., and Vasfi, M. (2019). Estimation and prediction for the Poisson-exponential distribution based on Type-II censored data. *American Journal of Mathematical and Management Sciences*, 38(1):96–115.

Arnold, B. C. and Press, S. J. (1983). Bayesian inference for Pareto populations. *Journal of Econometrics*, 21(3):287–306.

Bdair, O., Awwad, R. A., Abufoudeh, G., and Naser, M. (2019). Estimation and prediction for flexible Weibull distribution based on progressive Type II censored data. *Communications in Mathematics and Statistics*, `https://doi.org/10.1007/s40304-018-00173-0`, 1–23.

Cancho, V. G., Louzada, N. F., and Barriga, G. D. (2011). The Poisson-exponential lifetime distribution. *Journal of Econometrics*, 55(1):677–686.

Chen, M. H. and Shao, Q. M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computational and Graphical Statistics*, 8(1):69–92.

Cohen, A. C. (2016). *Truncated and Censored Samples: Theory and Applications*, CRC press.

Dempster, A. P., Laird, N. M., and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society: Series B (Methodological)*, 39(1):1–38.

Dey, S., Nassar, M., Maurya, R. K., and Tripathi, Y. M. (2018). Estimation and prediction of Marshall–Olkin extended exponential distribution under progressively type-II censored data. *Journal of Statistical Computation and Simulation*, 88(12):2287–2308.

Huang, S. R. and Wu, S. J. (2012). Bayesian estimation and prediction for Weibull model with progressive censoring. *Journal of Statistical Computation and Simulation*, 82(11):1607–1620.

Kayal, T., Tripathi, Y. M., Singh, D. P., and Rastogi, M. K. (2017). Estimation and prediction for Chen distribution with bathtub shape under progressive censoring. *Journal of Statistical Computation and Simulation*, 87(2):348–366.

Khan, R. A. and Mitra, M. (2019). Estimation issues in the Exponential–Logarithmic model under hybrid censoring. *Statistical Papers*, 1–32.

Klein, J. P. and Moeschberger, M. L. (1997). Statistics for biology and health. *Stat. Biol. Health*, New York, 27238.

Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics*, 50(2):144–154.

Kundu, D. and Pradhan, B. (2009). Estimating the parameters of the generalized exponential distribution in presence of hybrid censoring. *Communications in Statistics – Theory and Methods*, 38(12):2030–2041.

Kundu, D. and Raqab, M. Z. (2015). Estimation of $R = P[Y < X]$ for three-parameter generalized Rayleigh distribution. *Journal of Statistical Computation and Simulation*, 85(4):725–739.

Lawless, J. F. (2011). *Statistical Models and Methods for Lifetime Data*. Vol. 362, John Wiley & Sons.

Lee, K. and Cho, Y. (2017). Bayesian and maximum likelihood estimations of the inverted exponentiated half logistic distribution under progressive Type II censoring. *Journal of Applied Statistics*, 44(5):811–832.

Lindley, D. V. (1980). Approximate Bayesian methods. *Trabajos de Estadistica Y de Investigacion Operativa*, 31(1):223–245.

Louis, T. A. (1982). Finding the observed information matrix when using the EM algorithm. *Journal of the Royal Statistical Society: Series B (Methodological)*, 44(2):226–233.

Maiti, K. and Kayal, S. (2019). Estimation for the generalized Fréchet distribution under progressive censoring scheme. *International Journal of System Assurance Engineering and Management*, 10(5):1276–1301.

Maiti, K. and Kayal, S. (2020). Estimating reliability characteristics of the log-logistic distribution under progressive censoring with two applications. *Annals of Data Science*, `https://doi.org/10.1007/s40745-020-00292-y`, 1–40.

Maiti, K. and Kayal, S. (2021). Estimation of parameters and reliability characteristics for a generalized Rayleigh distribution under progressive type-II censored sample. *Communications in Statistics – Simulation and Computation*, 50(11):3669–3698.

Muhammed, H. Z. and Almetwally, E. M. (2020). Bayesian and non-Bayesian estimation for the bivariate inverse Weibull distribution under progressive type-II censoring. *Annals of Data Science*, `https://doi.org/10.1007/s40745-020-00316-7`, 1–32.

Nelson, W. B. (2016). *Applied Life Data Analysis*, Vol. 577, John Wiley & Sons.

Ng, H., Chan, P., and Balakrishnan, N. (2002). Estimation of parameters from progressively censored data using EM algorithm. *Computational Statistics & Data Analysis*, 39(4):371–386.

Nik, A. S., Asgharzadeh, A., and Raqab, M. Z. (2021). Estimation and prediction for a new Pareto-type distribution under progressive type-II censoring. *Mathematics and Computers in Simulation*, 190:508–530.

Rastogi, M. K. and Tripathi, Y. M. (2014). Parameter and reliability estimation for an exponentiated half-logistic distribution under progressive type II censoring. *Journal of Statistical Computation and Simulation*, 84(8):1711–1727.

Sarhan, A. M., Abd, E. A. A., and Alasbahi, I. A. (2013). Exponentiated generalized linear exponential distribution. *Applied Mathematical Modelling*, 37(5):2838–2849.

Soliman, A. A. (2005). Estimation of parameters of life from progressively censored data using Burr-XII model. *IEEE Transactions on Reliability*, 54(1):34–42.