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Estimations of Confidence Sets for the Unit Generalized Rayleigh Parameters Using Records Data

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Abstract:

- Estimations of confidence sets are explored for the unit generalized Rayleigh distribution parameters with records data. Using the proposed pivotal quantities, equal-tailed confidence regions of the parameters are constructed. The associated optimal confidence sets with minimum-size are developed by non-linear optimization technique, and various numerical approaches are established for complex computations. For comparison and complementary, the likelihood asymptotic confidence sets of parameters are derived. Extensive simulation studies are conducted to evaluate the performance of all methods, and two practical examples are given for illustrations. Some alternative extensions are provided for pursuing potential higher accuracy confidence sets.

Keywords:

- *unit generalized Rayleigh distribution; records data; confidence set estimation; pivotal quantity; nonlinear optimization.*

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1. INTRODUCTION

Due to practical limitations such as time and(or) budget constraint, it is not easy to obtain complete sample in practice; expecially, when the test units feature character of high reliable and expensive. Therefore, censored data frequently appear during the data collection, where only a portion of exact failure times are observed under such limitated situations, and various censoring schemes are implemented in experimental procedures simulataneously. Common censoring schemes used in experiments include Type-I censoring, Type-II censoring, progressive censoring, as well as hybrid censoring. Interested readers may refer to, for example, the monographes of Balakrishnan and Cramer [6] and Lawless [21] for a comprehensive review. However, besides conventional censored data appeared from aforementioned data collection schemes, there are many other incomplete data types occurred in field and experiment situations such as reliability engineering, survival analysis, hydrology, economics, mining and meteorology among others, and records data is one of popular observation among them. For example, Guo et al.*et al.* [14] gave an example regarding a kind of the rock crushing machine, where the size of the rock being crushed is also obtained when the crush strength is larger than the previously one appearing as record data. Soliman *et al.* [24] investigated a reliability experiment, where the exact measurements of failure under operating stress are observed sequentially and the record-breaking values are only collected in this case due to the practical operating mechanism. The initial conception of records is introduced by Chandler [7] that could be described as follows. Let T_n, $n = 1, 2, ...$ be a series of independent and identically distributed (i.i.d.) random variables. Then an observation T_j is called an upper record, if $T_j > T_i$ for every $j > i$. Due to its wide application in practical fields, records have received wide attention and are discussed by many authors. See, for example, some recent contributions of Asgharzadeh *et al.* [3], Dey *et al.* [9], Singh *et al.* [25], Wang and Ye [30] among others. For more details, one could refer to monographes of Ahsanullah [1] and Nevzorov [23] for a comprehensive review.

In practical lifetime studies, various distributions like exponential, Weibull, gamma, normal, etc., have been proposed for data analysis from various perspectives. One of characteristics of these aforementioned traditional models is that these distributions all feature infinity support $(-\infty, \infty)$ or $(0, \infty)$. However, there are many situations, where observations collected from practical situation are bounded within a specified range, and in turn distributions with finite support may provide better modelling performance than those with infinity support from goodness-of-fitting perspectives. For example, Zhang and Xie [32] used an upper-truncated Weibull distribution to fit the pit depth data of a water pipe where the upper bound of pit depths is the thickness of the water pipe. The same model is further implemented to describe the wind speed data by Kantar and Usta [17]. Vicari *et al.* [27] proposed a generalized Topp–Leone distribution for fitting the V-I indices data of globular clusters with bounded support. Under such aforementioned studies, all of the authors mentioned that the implemented bounded models have better data fitting accuracy than traditional distributions with infinity support in their practical discussions. Therefore, distributions with bounded domain have potential theoretical and practical applications where such models may provide higher weight to the bounded data and give better fitting effect in data analysis, and has been extensively studied by many authors from various perspectives (e.g., [5], [8], [20], [26]).

Among different bounded models, distributions with unit support have attracted considerable attention in practice, where the associated observation within $(0, 1)$ is an important and common occurred data type in reality such as birth rate, mortality data, as well as indices data from fields of energy, reliability and economic among others. There are various distributions with unit bound like beta, Kumaraswamy, Topp–Lenoe models among others. Some discussions and applications for such unit models could be found in the works of Genc [11], Ghitany *et al.* [13], Makouei *et al.* [22] and Wang [31]. Recently, Jha *et al.* [16] proposed another unit generalized Rayleigh distribution (UGRD) as follows. Let T be an UGRD random variable, the associated cumulative distribution function (CDF), probability density function (PDF) and hazard rate function (HRF) of T are respectively given by

$$F(t) = 1 - \left(1 - e^{-(\lambda \ln t)^2}\right)^{\theta}, \quad 0 < t < 1, \tag{1.1}$$

$$f(t) = -\frac{2\theta\lambda^2 \ln t}{t} e^{-(\lambda \ln t)^2} \left(1 - e^{-(\lambda \ln t)^2}\right)^{\theta - 1}, \tag{1.2}$$

and

$$H(t) = \frac{-\frac{2\theta\lambda^2 \ln t}{t} e^{-(\lambda \ln t)^2}}{1 - e^{-(\lambda \ln t)^2}}, \tag{1.3}$$

where $\theta > 0$ and $\lambda > 0$ are shape and scale parameters, respectively. It is noted that the shape parameter θ affects the geometric shape of density curve and the scale parameter λ not only determines the steepness of density curve but also specifically exhibits the value of random variable. Hereafter, the UGRD with parameters θ and λ is denoted by UGRD(θ, λ) for concision. Further, plots of CDF, PDF and HRF of the UGRD are presented in Figure 1 for illustration, and it is noted visually that the UGRD has very flexible fitting ability and may be used as an alternative bound model to traditional Beta, Kumaraswamy and Topp–Leone distributions.

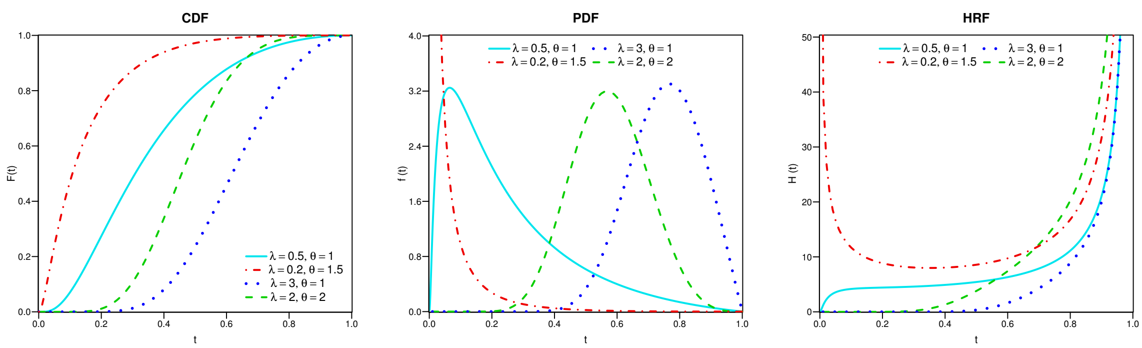

Figure 1: CDF, PDF and HRF of UGRD with different parameters.

In both theoretical and practical studies, point estimation is one of the most used approach in statistical inference. However, point estimation sometimes could not produce robust results, especially when estimates heavily depend on sample size. Since sample size appears frequently as moderate or small due to practical limitations, estimations of confidence sets are proposed alternatively in consequence, and have been discussed by many authors from different perspectives. For instance, Asgharzadeh *et al.* [2] provided the exact confidence intervals and regions when a bathtub-shaped distribution is used, and similar results are also obtained by Kinaci *et al.* [18] for the parameters of the generalized inverted exponential distribution.

Wu [28] constructed the confidence sets for the Weibull parameters under progressively censored data. Based on a modified progressively hybrid censored data, Zhu [33] proposed an adaptive Newton–Raphson algorithm based exact confidence region for a bathtub-shaped distribution. Motivated by such reasons as mentioned above and due to the flexibility and wide applications of the UGRD, the current investigation explores estimations of confidence sets for the UGRD parameters when records data is available, and various approaches are presented for constructing confidence intervals and confidence regions for the UGRD parameters in consequence.

The rest parts of this paper are arranged as follows. In Section 2, various estimates of confidence sets with equal-tailed and minimum-size are established for the UGRD parameters. Extensive numerical simulations are carried out in Section 3 to investigate the performance of different results, and two real life examples are also presented for illustrations. In Section 4, some extended results are further provided for exploring some more potential confidence sets for UGRD parameters with better performance. Finally, some concluding remarks are given in Section 5.

2. ESTIMATIONS OF CONFIDENCE SETS

Based on records data, different confidence sets of the UGRD parameters are established in this section. The equal-tailed confidence intervals and confidence regions are constructed respectively based on the proposed pivotal quantities, and the associated minimum-size confidence sets are also established in consequence. Moreover, conventional asymptotic confidence sets are provided for comparison.

2.1. Equal-tailed confidence sets

The equal-tailed confidence sets (ECSs) are discussed here for UGRD parameters λ and θ including the equal-tailed confidence intervals (ECI) and equal-tailed confidence region (ECR), respectively.

To construct the ECSs, two useful results are provided as follows.

Lemma 2.1. *Let $T = \{T_1, T_2, ..., T_n\}$ be upper records from UGRD(λ, θ). Denote pivotal quantities*

$$\Psi(\lambda) = (n-1)\left[\frac{\ln(1-\exp(-(\lambda \ln T_n)^2))}{\ln(1-\exp(-(\lambda \ln T_1)^2))} - 1\right]^{-1} \tag{2.1}$$

and

$$\Upsilon(\lambda, \theta) = -2\theta \ln(1-\exp(-(\lambda \ln T_n)^2)). \tag{2.2}$$

Then $\Psi(\lambda)$ follows the F distribution with 2 and $2(n-1)$ degrees of freedom, $\Upsilon(\lambda, \theta)$ has a chi-square distribution with $2n$ degree of freedom, and $\Psi(\lambda)$ and $\Upsilon(\lambda, \theta)$ are statistically independent.

Proof: The proof is provided in part A of the Supplementary file. □

Lemma 2.2. *For arbitrary numbers a and b with $0 < b < a < 1$, let*

$$h(\lambda) = \frac{\ln(1 - \exp(-(\lambda \ln a)^2))}{\ln(1 - \exp(-(\lambda \ln b)^2))}, \lambda > 0, \tag{2.3}$$

then function $h(\lambda)$ increases in λ with $\lim_{\lambda \to 0} h(\lambda) = 1$ and $\lim_{\lambda \to \infty} h(\lambda) = \infty$.

Proof: The proof is provided in part B of the Supplementary file. □

Corollary 2.1. *According to Lemma 2.2, function $\Psi(\lambda)$ decreases in λ with range $(0, \infty)$.*

In the following, the ECIs of parameters λ and θ as well as the ECR of parameter vector (λ, θ) are established, respectively.

Theorem 2.1. *Let $T = \{T_1, T_2, ..., T_n\}$ be upper record from UGRD(λ, θ). For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ ECI of λ is given by*

$$\left[\psi\left(F_{\gamma/2}^{2,2(n-1)}\right), \psi\left(F_{1-\gamma/2}^{2,2(n-1)}\right)\right], \tag{2.4}$$

where $\psi(x)$ is the solution of equation $\Psi(\lambda) = x$ w.r.t. λ, and $F_p^{m_1,m_2}$ is the upper $100p\%$ percentile of F distribution with m_1 and m_2 degrees of freedom.

Proof: The proof is provided in part C of the Supplementary file. □

Theorem 2.2. *Let $T = \{T_1, T_2, ..., T_n\}$ be upper record from UGRD(λ, θ). For given λ and arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ ECI of θ can be constructed as*

$$\left[\frac{\chi_{1-\gamma/2}^{2n}}{B(\lambda)}, \frac{\chi_{\gamma/2}^{2n}}{B(\lambda)}\right] \quad \text{with} \quad B(\lambda) = -2\ln(1 - \exp(-(\lambda \ln T_n)^2)), \tag{2.5}$$

where χ_p^m denotes the upper $100p\%$ percentile of chi-square distirbution with m degrees of freedom.

Proof: Using the distribution property of the pivotal quantity $\Upsilon(\lambda, \theta)$ given in Lemma 2.1, the result could be be established directly by following similar line as Theorem 2.1, and the details are omitted here for concision. □

Remark 2.1. It is noted from Theorem 2.2 that the ECI of θ is available with known λ. However, parameter λ is unknown in practical applications. To overcome this drawback, following alternative way is proposed to establish the ECI of parameter θ when parameter λ is unknown.

Let $\psi(Y)$ be the unique solution of λ from equation $\Psi(\lambda) = Y$, where Y is a random sample generating from F distribution with 2 and $2(n-1)$ degrees of freedom. Using the substitution method of Weerahandi [29], a generalized pivotal quantity of θ can be constructed as

$$S = \frac{\Upsilon(\psi(Y), \theta)}{B(\psi(Y))}. \tag{2.6}$$

Correspondingly, an approach termed as Algorithm 1 is provided to obtain the ECI of θ under unknown λ situation.

Algorithm 1: ECI of θ with unknown λ

STEP 1. Generate a random sample Y from the F distribution with 2 and $2(n-1)$ degrees of freedom, then $\psi(Y)$ can be solved through equation $\Psi(\lambda) = Y$.

STEP 2. Generate a random value of $\Upsilon(\psi(Y), \theta)$ from chi-square distirbution with $2n$ degrees of freedom and calculate S in (2.6).

STEP 3. Repeat Steps 1 and 2 M times and obtain a group values of S arranged in the ascending order, $S_1, S_2, ..., S_M$.

STEP 4. For $0 < \gamma < 1$, an ECI of θ with unknown λ can be constructed by

$$\left[S_{\lceil M\frac{\gamma}{2}\rceil}, S_{\lceil M(1-\frac{\gamma}{2})\rceil}\right], \tag{2.7}$$

where '$\lceil\cdot\rceil$' refers to the ceiling function.

Further, an ECR of parameter vector (λ, θ) is established as follows.

Theorem 2.3. *Let $T = \{T_1, T_2, ..., T_n\}$ be upper record from UGRD(λ, θ). For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ ECR of (λ,θ) can be written as*

$$\left\{(\lambda, \theta)\middle|\psi\left(F^{2,2(n-1)}_{\frac{1-\sqrt{1-\gamma}}{2}}\right) < \lambda < \psi\left(F^{2,2(n-1)}_{\frac{1+\sqrt{1-\gamma}}{2}}\right), \frac{\chi^{2n}_{\frac{1+\sqrt{1-\gamma}}{2}}}{B(\lambda)} < \theta < \frac{\chi^{2n}_{\frac{1-\sqrt{1-\gamma}}{2}}}{B(\lambda)}\right\}, \tag{2.8}$$

where associated notations are defined in Theorems 2.1 and 2.2, respectively.

Proof: The proof is provided in part D of the Supplementary file. □

2.2. Minimum-size confidence sets

It is noted from Subsection 2.1 that the proposed ECSs are obtained under equal-tailed approach, and such results someitmes may not have minimum sizes. Alternatively, optimal confidence sets for parameters λ, θ and (λ, θ) are proposed here. Specifically, the minimum-size confidence sets (MCSs) including the minimum-length confidence intervals (MCIs) of λ and θ as well as the minimum-area confidence region (MCR) of (λ, θ) are constructed respectively, and the associated numercial algorithms are also proposed for optimization computation.

Theorem 2.4. *Let $T = \{T_1, T_2, ..., T_n\}$ be upper record from UGRD(λ, θ). For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ MCI of λ is given by*

$$[\psi(x_2^*), \psi(x_1^*)], \tag{2.9}$$

where x_1^* and x_2^* are the solutions of the following non-linear system

$$\begin{cases} \frac{\psi'(x_1)}{\psi'(x_2)} = \frac{P^F_{2,2(n-1)}(x_1)}{P^F_{2,2(n-1)}(x_2)}, \\ F_{2,2(n-1)}(x_2) - F_{2,2(n-1)}(x_1) = 1-\gamma, \end{cases}$$

where $\psi'(x)$ is the derivative of $\psi(x)$ with respect to x, $F_{m_1,m_2}(x)$ is the CDF of F distribution with m_1 and m_2 degrees of freedom and $P^F_{m_1,m_2}(x)$ is the corresponding density function of $F_{m_1,m_2}(x)$.

Proof: The proof is provided in part E of the Supplementary file. □

Clearly, there is no closed form solution (x_1^*, x_2^*) for the MCI of λ in Theorem 2.4, and a numerical approach entitled Algorithm 2 is provided with pre-fixed accuracy level $\sigma > 0$.

Algorithm 2: MCI of λ in Theorem 2.4

STEP 1. Let $\dot{x}_1 = F_\gamma^{2,2(n-1)}$ be the upper bound of x_1 and $N = \lfloor \dot{x}_1/\sigma \rfloor$, where '$\lfloor \cdot \rfloor$' is the floor function.

STEP 2. Obtain a value of $\dot{x}_2$ by computing the equation $F_{2,2(n-1)}(\dot{x}_2) - F_{2,2(n-1)}(\dot{x}_1) = 1 - \gamma$.

STEP 3. Let $\dot{x}_1 = \dot{x}_1 - \sigma$.

STEP 4. Repeat Step 2 and Step 3 until $\dot{x}_1 < 0$ and obtain N groups $(\dot{x}_1^{[i]}, \dot{x}_2^{[i]})$, $i = 1, ..., N$.

STEP 5. The numerical MCI of λ can be constructed as $\left[\psi\left(\dot{x}_2^{[k]}\right), \psi\left(\dot{x}_1^{[k]}\right)\right]$, where k satisfies the equation $\psi(\dot{x}_1^{[k]}) - \psi(\dot{x}_2^{[k]}) = \min_{i=1}^{N}[\psi(\dot{x}_1^{[i]}) - \psi(\dot{x}_2^{[i]})]$.

Theorem 2.5. *Let $T = \{T_1, T_2, ..., T_n\}$ be upper record from UGRD(λ, θ). For given λ and arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ MCI of θ can be constructed as*

$$\left[\frac{y_1^*}{B(\lambda)}, \frac{y_2^*}{B(\lambda)}\right], \tag{2.10}$$

where y_1^* and y_2^* are the solutions of the following non-linear system

$$\begin{cases} P_{2n}^{\chi}(y_2) = P_{2n}^{\chi}(y_1), \\ \chi_{2n}(y_2) - \chi_{2n}(y_1) = 1 - \gamma \end{cases}$$

and both $\chi_m(y)$ and $P_m^{\chi}(y)$ are the CDF and PDF of chi-square distribution with m degree of freedom, respectively.

Proof: For given λ, using pivotal quantity $\Upsilon(\lambda, \theta)$ in Lemma 2.1, the $100(1-\gamma)\%$ MCI of θ can be obtained similarly as in Theorem 2.4 and the details are omitted for concision. □

In addition, a numerical approach called Algorithm 3 are presented for obtaining the MCI of θ.

Remark 2.2. It is noted that the MCI of θ with unknown λ can be still constructed under the alternative approach as

$$\left[S_{\lceil j^*\rceil}, S_{\lceil j^*+M-(M\gamma+1)\rceil}\right], \tag{2.11}$$

where notation $S_{\lceil \cdot \rceil}$ is defined in Algorithm 1 and j^* is an integer satisfying

$$S_{\lceil j^*+M-(M\gamma+1)\rceil} - S_{\lceil j^*\rceil} = \min_{j=1}^{\lceil M\gamma\rceil}\left[S_{\lceil j+M-(M\gamma+1)\rceil} - S_{\lceil j\rceil}\right].$$

Algorithm 3: MCI of θ in Theorem 2.5

STEP 1. Let $p = \sigma$ be the initial value.

STEP 2. Obtain the solutions $\dot{y}_1$ and $\dot{y}_2$ from euqation $P^{\chi}_{2n}(y) = p$, where $0 < \dot{y}_1 < \dot{y}_2$.

STEP 3. Calculate $C = \chi_{2n}(\dot{y}_2) - \chi_{2n}(\dot{y}_1)$, then let $\sigma^* = C - (1-\gamma)$.

STEP 4. If $\sigma^* > \sigma$, then let $p = p + \sigma$, otherwise if $\sigma^* < -\sigma$, let $p = p - \sigma$.

STEP 5. Repeat Steps 2–4 until $|\sigma^*| \leqslant \sigma$, for known λ, the numerical MCI of θ can be given by $[\dot{y}_1/B(\lambda), \dot{y}_2/B(\lambda)]$.

Similarly, the MCR of (λ, θ) is also established as follows.

Theorem 2.6. *Let $T = \{T_1, T_2, ..., T_n\}$ be upper record from UGRD(λ, θ). For an arbitrary $0 < \gamma < 1$, the $100(1-\gamma)\%$ MCR of (λ, θ) is given by*

$$\left\{(\lambda,\theta)\middle|\psi(x_2^*) < \lambda < \psi(x_1^*), \frac{y_1^*}{B(\lambda)} < \theta < \frac{y_2^*}{B(\lambda)}\right\}, \tag{2.12}$$

where $(x_1^*, x_2^*, y_1^*, y_2^*)$ are the solutions of the following non-linear system

$$\begin{cases} P^{\chi}_{2n}(y_1) = P^{\chi}_{2n}(y_2), \\ \left[F_{2,2(n-1)}(x_2) - F_{2,2(n-1)}(x_1)\right][\chi_{2n}(y_2) - \chi_{2n}(y_1)] = 1-\gamma, \\ \dfrac{\psi'(x_2)B(\psi(x_1))}{\psi'(x_1)B(\psi(x_2))} = -\dfrac{P^F_{2,2(n-1)}(x_2)}{P^F_{2,2(n-1)}(x_1)}, \\ \dfrac{\psi'(x_1)\left[F_{2,2(n-1)}(x_2) - F_{2,2(n-1)}(x_1)\right]P^{\chi}_{2n}(y_1)}{[\chi_{2n}(y_2) - \chi_{2n}(y_1)]P^F_{2,2(n-1)}(x_1)\int_{\psi(x_2)}^{\psi(x_1)}\frac{1}{B(\lambda)}d\lambda} = -\dfrac{B(\psi(x_1))}{y_2 - y_1}. \end{cases}$$

Proof: The proof is provided in part F of the Supplementary file. □

For finding solution of MCR, the associated numerical approach termed as Algorithm 4 is provided in consequence.

Algorithm 4: MCR of (λ, θ) in Theorem 2.6

STEP 1. Set $\dot{y}_1 = \sigma$ and obtain $\tilde{y}_1$ and $\tilde{y}_2$ from the following equations

$$\begin{cases} P_{2n}^{\chi}(\tilde{y}_1) = P_{2n}^{\chi}(\tilde{y}_2) \\ \chi_{2n}(\tilde{y}_2) - \chi_{2n}(\tilde{y}_1) = (1-\gamma) \end{cases}, 0 < \tilde{y}_1 < \tilde{y}_2.$$

Then make $M = \lfloor \tilde{y}_1/\sigma \rfloor$.

STEP 2. Obtain a value of $\dot{y}_2 (> \dot{y}_1)$ by equation $P_{2n}^{\chi}(\dot{y}_1) = P_{2n}^{\chi}(\dot{y}_2)$, and calculate $\gamma^* = 1 - (1-\gamma)[\chi_{2n}(\dot{y}_2) - \chi_{2n}(\dot{y}_1)]^{-1}$.

STEP 3. For $i = 1, 2, ..., M$, obtain N_i groups $(\dot{x}_1^{[i_j]}, \dot{x}_2^{[i_j]})$, $j = 1, 2, ..., N_i$ by substituting γ^* for γ in Algorithm 1.

STEP 4. Let $\dot{y}_1 = \dot{y}_1 + \sigma$.

STEP 5. Repeat Step 2 – Step 4 until $\dot{y}_1 > \tilde{y}_1$ and obtain $\sum_{i=1}^{M} N_i$ groups solutions $(\dot{x}_1^{[i_j]}, \dot{x}_2^{[i_j]}, \dot{y}_1^{[i]}, \dot{y}_2^{[i]})$, $i = 1, 2, ..., M$, $j = 1, 2, ..., N_i$.

STEP 6. The numercial MCR of (λ, θ) can be constructed as

$$\left\{ (\lambda, \theta) \middle| \psi\left(\dot{x}_2^{[i^*_{j^*}]}\right) < \lambda < \psi\left(\dot{x}_1^{[i^*_{j^*}]}\right), \frac{\dot{y}_1^{[i^*]}}{B(\lambda)} < \theta < \frac{\dot{y}_2^{[i^*]}}{B(\lambda)} \right\},$$

where $(\dot{x}_1^{[i^*_{j^*}]}, \dot{x}_2^{[i^*_{j^*}]}, \dot{y}_1^{[i^*]}, \dot{y}_2^{[i^*]})$ conform to the following equation

$$\int_{\psi\left(\dot{x}_2^{[i^*_{j^*}]}\right)}^{\psi\left(\dot{x}_1^{[i^*_{j^*}]}\right)} \frac{\dot{y}_2^{[i^*]} - \dot{y}_1^{[i^*]}}{B(\lambda)} d\lambda = \min_{(i,j)=(1,1)}^{(M,N_i)} \left[\int_{\psi\left(\dot{x}_2^{[i_j]}\right)}^{\psi\left(\dot{x}_1^{[i_j]}\right)} \frac{\dot{y}_2^{[i]} - \dot{y}_1^{[i]}}{B(\lambda)} d\lambda \right].$$

2.3. Asymptotic confidence sets

For comparison, traditional asymptotic confidence sets (ACSs) of UGRD parameters are also constructed based on asymptotic theory, where asymptotic confidence intervals (ACIs) of of λ and θ as well as asymptotic confidence region (ACR) of of (λ, θ) are obtained, respectively.

Let $T_1, T_2, ..., T_n$ be upper records from UGRD(λ, θ), and $t_1, t_2, ..., t_n$ be the associated observations. Therefore, log-likelihood function $\ell(\lambda, \theta)$ of λ and θ can be expressed from Ahsanullah [1] as

$$\begin{aligned} \ell(\lambda, \theta) &= n \ln\left(2\theta\lambda^2\right) + \theta \ln\left(1 - e^{-(\lambda \ln t_n)^2}\right) \\ &\quad - \sum_{i=1}^{n} \ln\left[-\left(1 - e^{-(\lambda \ln t_i)^2}\right) t_i^{-1} \ln t_i\right] + (\lambda \ln t_i)^2. \end{aligned} \tag{2.13}$$

By taking derivatives, MLE $\hat{\lambda}$ of λ can be obtained from equation

$$\frac{n}{\lambda^2} - \frac{n(\ln t_n)^2 e^{-(\lambda \ln t_n)^2}}{\left(1 - e^{-(\lambda \ln t_n)^2}\right) \ln\left(1 - e^{-(\lambda \ln t_n)^2}\right)} - \sum_{i=1}^{n} \frac{(\ln t_i)^2}{1 - e^{-(\lambda \ln t_i)^2}} = 0, \tag{2.14}$$

whereas the MLE $\hat{\theta}$ of θ is given by

$$\hat{\theta} = -\frac{n}{\ln\left(1 - e^{-(\hat{\lambda} \ln t_n)^2}\right)}.$$

Remark 2.3. It is worth mentioning that the UGRD MLEs $\hat{\lambda}$ and $\hat{\theta}$ uniquely exist under records situation, and the associated existence and uniqueness are provided in part G of the Supplementary file. Therefore, altough there is no closed form for MLE $\hat{\lambda}$ in equation (2.14), the associated estimate could be obtained in a simple way by using numerical approaches like biasection or fixed-point iteration methods.

Further, let $\boldsymbol{\beta} = (\lambda, \theta)' = (\beta_1, \beta_2)'$ with $\beta_1 = \lambda, \beta_2 = \theta$, the observed information matrix of $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \hat{\beta}_2)'$ is given by

$$\mathbf{I}(\hat{\boldsymbol{\beta}}) = \begin{pmatrix} -\frac{\partial^2 \ell(\lambda,\theta)}{\partial \lambda^2} & -\frac{\partial^2 \ell(\lambda,\theta)}{\partial \lambda \partial \theta} \\ -\frac{\partial^2 \ell(\lambda,\theta)}{\partial \lambda \partial \theta} & -\frac{\partial^2 \ell(\lambda,\theta)}{\partial \theta^2} \end{pmatrix}\Bigg|_{\lambda=\hat{\lambda},\theta=\hat{\theta}}, \tag{2.15}$$

where

$$\frac{\partial^2 \ell(\lambda,\theta)}{\partial^2 \lambda} = -\frac{2n}{\lambda^2} - \omega(t_n) + \theta \sum_{i=1}^{n} \omega(t_i), \quad \frac{\partial^2 \ell(\lambda,\theta)}{\partial^2 \theta} = -\frac{n}{\theta^2},$$

$$\frac{\partial^2 \ell(\lambda,\theta)}{\partial \lambda \partial \theta} = \frac{2\lambda(\ln t_n)^2 e^{-(\lambda \ln t_n)^2}}{1 - e^{-(\lambda \ln t_n)^2}}$$

and

$$\omega(t) = \frac{2(\ln t)^2 e^{-(\lambda \ln t)^2}\left[1 - 2\lambda^2(\ln t)^2 - e^{-(\lambda \ln t)^2}\right]}{\left[1 - e^{-(\lambda \ln t)^2}\right]^2}.$$

Therefore, the variance-covariance matrix of $(\hat{\lambda}, \hat{\theta})$ can be constructed as

$$\mathbf{I}^{-1}(\hat{\boldsymbol{\beta}}) = \begin{pmatrix} \text{Var}(\hat{\lambda}) & \text{Cov}(\hat{\lambda}, \hat{\theta}) \\ \text{Cov}(\hat{\lambda}, \hat{\theta}) & \text{Var}(\hat{\theta}) \end{pmatrix}.$$

Consequently, the asymptotic distribution of $\hat{\boldsymbol{\beta}}$ can be obtained under some mild regularity conditions as $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \to N(0, \mathbf{I}^{-1}(\hat{\boldsymbol{\beta}}))$.

For arbitrary $0 < \gamma < 1$, the $100(1-\gamma)\%$ ACI of β_i can be constructed by

$$\left[\hat{\beta}_i + u_{1-\gamma/2}\sqrt{\text{Var}(\hat{\beta}_i)}, \hat{\beta}_i + u_{\gamma/2}\sqrt{\text{Var}(\hat{\beta}_i)}\right], i = 1, 2, \tag{2.16}$$

where u_γ is the upper $100\gamma\%$ percentile of standard normal distribution. Moreover, the $100(1-\gamma)\%$ ACR of (λ, θ) can be obtained as follows

$$\left\{(\lambda, \theta)\Big|(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{I}(\hat{\boldsymbol{\beta}})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) < \chi^2_\gamma\right\}. \tag{2.17}$$

Remark 2.4. In some cases, the lower confidence bounds of the ACIs (2.16) sometimes may be negative. To overcome this drawback, one could use logarithmic transformation and delta method to obtain the asymptotic normality distribution of $\ln\hat{\beta}_i, i = 1, 2$ as $\ln\hat{\beta}_i - \ln\beta_i \to N\Big(0, \mathrm{Var}\big(\ln\hat{\beta}_i\big)\Big)$, with $\mathrm{Var}(\ln\hat{\beta}_i) = \mathrm{Var}(\hat{\beta}_i)/\hat{\beta}_i^2$. Therefore, the $100(1-\gamma)\%$ modified ACI of β_i can be constructed in this manner as

$$\left[\frac{\hat{\beta}_i}{\exp\left(u_{\gamma/2}\sqrt{\mathrm{Var}\big(\ln\hat{\beta}_i\big)}\right)}, \hat{\beta}_i \exp\left(u_{\gamma/2}\sqrt{\mathrm{Var}\big(\ln\hat{\beta}_i\big)}\right)\right], i = 1, 2. \tag{2.18}$$

3. NUMERICAL ILLUSTRATION

Extensive simulation studies are carried out to investigate the performance of the proposed results. In addition, two real-life examples are also presented to show the applicability of our methods.

3.1. Simulation studies

In simulation studies, performance of ECSs, MCSs and ACSs are compared in terms of criteria quantities including average width (AW) for confidence intervals, average area (AA) for confidence regions and coverage probability (CP) for all confidence sets.

For generating records data, another sampling approach termed as Algorithm 5 is provided as follows.

Algorithm 5: Upper record values from UGRD(λ, θ)

STEP 1. Generate n i.i.d. samples $u_1, u_2, ..., u_n$ from uniform distribution with range $(0, 1)$.

STEP 2. Calculate $v_i = -\ln(1-u_i),\ i = 1, 2, ..., n$.

STEP 3. Take $w_i = 1 - \exp(-\sum_{j=1}^{i} v_j)$, then $w_1, w_2, ..., w_n$ are upper records standard uniform distribution.

STEP 4. Implementing inverse transformation

$$t_i = \exp[-(-\ln(1-(1-w_i)^{1/\theta}))^{1/2}/\lambda],\ i = 1, 2, ..., n$$

then $t_1, t_2, ..., t_n$ are the upper record values from UGRD(λ, θ).

In this simulation, parameter values (λ, θ) are randomly chosen as $(0.5, 1)$, $(0.5, 0.5)$ and $(3, 1)$, sample sizes $n = 3$, 4, 5, 6, 7 and 8 are considered and the significance level is $\gamma = 0.05$.

For all numerical computation in minimum-size confidence sets, the accuracy level is taken to be $\sigma = 0.001$, and the simulations are conducted based 10,000 times of repetitions, where the ECI and MCI of θ are obtained under unknown λ cases by using the strategies provided in Remarks 2.1 and 2.2. The simulated associated criteria quantities AW, AA and CP are tabulated in Tables 1–3. In addition, for complementary and comparison, performance of ECI and MCI for θ given in Theorems 2.2 and 2.5 are also investigated with known λ, the associated criteria quantities are obtained by using the true values of λ in simulation and the associted results are tabulated in Table 4.

Table 1: AWs, AAs and CPs (within brackets) for UGRD confidence sets with $\lambda = 0.5,\ \theta = 1$.

		$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
λ	ACI	1.9864 (0.8226)	1.7601 (0.8835)	1.6818 (0.9043)	1.6299 (0.9244)	1.6118 (0.9370)	1.5961 (0.9434)
	ECI	1.8250 (0.9523)	1.6381 (0.9479)	1.5820 (0.9486)	1.5363 (0.9483)	1.5169 (0.9518)	1.5036 (0.9484)
	MCI	1.6773 (0.9279)	1.5208 (0.9286)	1.4777 (0.9342)	1.4434 (0.9322)	1.4294 (0.9395)	1.4208 (0.9346)
θ	ACI	8.3863 (0.9791)	5.0168 (0.9748)	3.3495 (0.9765)	2.5414 (0.9773)	2.0836 (0.9728)	1.8349 (0.9753)
	ECI	9.0399 (0.9334)	5.2259 (0.9361)	3.1536 (0.9410)	2.3884 (0.9454)	1.9646 (0.9436)	1.7430 (0.9446)
	MCI	5.7770 (0.9496)	4.0595 (0.9443)	2.7762 (0.9508)	2.1864 (0.9557)	1.8336 (0.9509)	1.6346 (0.9489)
(λ,θ)	ACR	12.5066 (0.7982)	8.0762 (0.9010)	5.5025 (0.9369)	4.1949 (0.9559)	3.4725 (0.9629)	3.0587 (0.9626)
	ECR	14.0787 (0.9417)	8.5107 (0.9523)	5.2939 (0.9536)	4.1231 (0.9478)	3.3691 (0.9537)	2.9745 (0.9461)
	MCR	9.0862 (0.9451)	6.3987 (0.9469)	4.5192 (0.9527)	3.5098 (0.9523)	2.9310 (0.9531)	2.6252 (0.9475)

Table 2: AWs, AAs and CPs (within brackets) for UGRD confidence sets with $\lambda = 0.5,\ \theta = 0.5$.

		$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
λ	ACI	5.4513 (0.8705)	5.1275 (0.9194)	5.2425 (0.9465)	4.688 (0.9608)	4.5967 (0.9703)	4.383 (0.9744)
	ECI	4.4116 (0.9489)	4.1500 (0.9490)	4.3133 (0.9492)	3.8481 (0.9476)	3.7679 (0.9491)	3.6012 (0.9500)
	MCI	3.9530 (0.9404)	3.7088 (0.9426)	3.8517 (0.9388)	3.4429 (0.9412)	3.3775 (0.9456)	3.2236 (0.9465)
θ	ACI	4.9927 (0.9815)	2.1162 (0.9783)	1.3507 (0.9752)	1.1324 (0.9807)	0.9712 (0.9835)	0.8753 (0.9908)
	ECI	4.7218 (0.9391)	1.9903 (0.9416)	1.2216 (0.9451)	1.0477 (0.9486)	0.9117 (0.9539)	0.8309 (0.9569)
	MCI	3.4154 (0.9449)	1.6898 (0.9535)	1.1217 (0.9584)	0.9786 (0.9624)	0.8604 (0.9615)	0.7893 (0.9568)
(λ,θ)	ACR	20.0447 (0.8857)	7.6137 (0.9481)	5.7514 (0.9661)	4.4766 (0.9747)	3.8706 (0.9838)	3.3478 (0.9881)
	ECR	22.3566 (0.9517)	6.6294 (0.9519)	4.9081 (0.9524)	3.7994 (0.9514)	3.2875 (0.9622)	2.8493 (0.9587)
	MCR	14.3081 (0.9530)	5.3074 (0.9532)	4.0968 (0.9516)	3.2366 (0.9537)	2.8335 (0.9616)	2.4759 (0.9641)

Table 3: AWs, AAs and CPs (within brackets) for UGRD confidence sets with $\lambda = 3,\ \theta = 1$.

		$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
λ	ACI	11.7857 (0.8416)	10.6667 (0.8856)	10.0701 (0.9086)	9.7815 (0.9252)	9.5784 (0.9352)	9.4684 (0.9422)
	ECI	10.9231 (0.9431)	9.9324 (0.9497)	9.4579 (0.9498)	9.179 (0.9511)	9.0211 (0.9501)	8.9084 (0.953)
	MCI	10.0412 (0.9382)	9.2208 (0.9331)	8.8449 (0.9341)	8.6182 (0.9359)	8.5062 (0.9365)	8.4214 (0.9412)
θ	ACI	11.8480 (0.9799)	5.0491 (0.9799)	3.3021 (0.9748)	2.5118 (0.9734)	2.103 (0.975)	1.8341 (0.9742)
	ECI	12.7022 (0.9425)	4.9763 (0.947)	3.0889 (0.9391)	2.3403 (0.9411)	1.9792 (0.9434)	1.7371 (0.9447)
	MCI	8.2348 (0.9479)	4.0257 (0.9529)	2.7213 (0.9526)	2.1438 (0.9504)	1.8473 (0.9512)	1.639 (0.9499)
(λ,θ)	ACR	96.2407 (0.8087)	49.7994 (0.9003)	32.6615 (0.9376)	24.8758 (0.9530)	20.9596 (0.9606)	18.2307 (0.9645)
	ECR	98.9104 (0.9537)	50.5229 (0.9502)	32.5485 (0.9517)	24.2006 (0.9499)	20.4082 (0.95)	17.6974 (0.9505)
	MCR	73.1492 (0.9467)	38.7621 (0.9504)	26.6496 (0.9522)	20.6069 (0.9508)	17.7536 (0.9523)	15.6203 (0.9513)

Table 4: AWs and CPs (within brackets) of ECIs and MCIs of θ with known λ.

$\lambda = 0.5,\ \theta = 1$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
ECI	3.3429 (0.9499)	2.5739 (0.9487)	2.1547 (0.9516)	1.8892 (0.9492)	1.7095 (0.9518)	1.5629 (0.9525)
MCI	3.0857 (0.9496)	2.4258 (0.9486)	2.0559 (0.9514)	1.8173 (0.9491)	1.6540 (0.9519)	1.5186 (0.9534)
$\lambda = 0.5,\ \theta = 0.5$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
ECI	1.6397 (0.9530)	1.2780 (0.9521)	1.0774 (0.9515)	0.9416 (0.9570)	0.8589 (0.9607)	0.7948 (0.9741)
MCI	1.5136 (0.9526)	1.2045 (0.9518)	1.0281 (0.9524)	0.9058 (0.9534)	0.8309 (0.9615)	0.7722 (0.9728)
$\lambda = 3,\ \theta = 1$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
ECI	3.3284 (0.9440)	2.5828 (0.9463)	2.1577 (0.9533)	1.8856 (0.9507)	1.7096 (0.9498)	1.5634 (0.9515)
MCI	3.0723 (0.9461)	2.4342 (0.9476)	2.0589 (0.9522)	1.8139 (0.9484)	1.6540 (0.9480)	1.5190 (0.9521)

From Tables 1–4, it is noted that

- AWs and AAs of all confidence sets decrease with increase of sample size n. Such phenomenon indicate the consistence property of the proposed results when sample size increases.
- Under fixed sample size n, MCIs of λ have the best performance than ECIs and ACIs in terms of AWs, whereas the ACI estimates of λ feature the largest AWs. In addition, the ACIs for λ have lowest CPs than those of ECIs and MCIs.

- For parameter θ, MCIs of θ are superior to ECIs and ACIs in terms of AWs, whereas although the CPs of ACI are highest, AWs of ACIs are larger than the other two interval estimates in general.
- For confidence region of (λ, θ), MCRs have the smallest AAs, whereas the CPs of all three confidence regions of (λ, θ) are close to the nominal significance level in most cases.
- From Table 4, for given parameter λ, AWs of both ECIs and MCIs of θ are smaller than those of θ with unknown λ shown in Tables 1–3, and the CPs under this case are still close to the nominal significance level.

To sum up, the simulation results indicate that the ECSs and MCSs of different parameters λ, θ and (λ, θ) perform better than traditional likelihood based ACRs in general, and the proposed MCSs are recommended as superior choices in practice.

3.2. Real data illustration

In this subsection, two real life examples are presented to demonstrate the practicality of the proposed methods. For comparison, another three unit bounded distributions namely Beta distribution (BeD), Kumaraswamy distribution (KuD) and Topp–Leone distribution (TLD) are considered as competitors of UGRD in this illustration. The corresponding PDFs of BeD, KuD and TLD are given respectively by:

$$\begin{aligned} BeD:&\ f_1(t) = t^{\alpha-1}(1-x)^{\beta-1}[B(\alpha,\beta)]^{-1},\ \alpha > 0, \beta > 0,\ 0 < t < 1, \\ KuD:&\ f_2(t) = \alpha\beta t^{\alpha-1}(1-t^{\alpha})^{\beta-1},\ \alpha > 0,\ \beta > 0,\ 0 < t < 1, \\ TLD:&\ f_3(t) = 2\alpha(1-t)t^{\alpha-1}(2-t)^{\alpha-1},\ 0 < \alpha < 1,\ 0 < t < 1. \end{aligned}$$

It is noted that above three distributions are also common used unit models that are widely implemented in practical data analysis. (e.g., Arora *et al.* [4], Gupta and Nadarajah [10] and Kohansal [19]).

Example 1. (Reservoir capacity ratio data) In this real life example, a data set from `http://cdec.water.ca.gov/dynamicapp/QueryMonthly?s=SHA` is considered for illustration. The data set is about the water capacities of Shasta reservoir in California, USA for the month of October from 2008 to 2019. Since the maximum capacity of Shasta reservoir is 4552000 acre-foot, the original data were converted to the $(0, 1)$ interval by dividing 4552000 acre-foot. The transformed data are shown as follows:

$$\begin{aligned} &0.28180,\ 0.37520,\ 0.71891,\ 0.70887,\ 0.54177,\ 0.38317 \\ &0.24360,\ 0.31113,\ 0.60748,\ 0.69789,\ 0.48134,\ 0.71859. \end{aligned}$$

To check whether the UGRD could be used to fit the real life data, Kolmogorov–Smirnov (K-S) test is carried out for UGRD, BeD, KuD and TLD respectively under origin complete data, the associated results are tabulated in Table 5. It is noted from Table 5 that comparing with BeD, KuD and TLD, the UGRD seems more proper to fit the reservoir capacity ratio data.

Table 5: MLEs and K-S test of fitted distributions for reservoir capacity ratio data.

	MLE of model parameters	K-S distance	p-value
UGRD	$(\hat{\lambda}, \hat{\theta}) = (1.2369, 1.1284)$	0.1893	0.7166
BeD	$(\hat{\alpha}, \hat{\beta}) = (3.9505, 3.8693)$	0.1949	0.6834
KuD	$(\hat{\alpha}, \hat{\beta}) = (2.9870, 4.7499)$	0.1959	0.6777
TLD	$\hat{\alpha} = 2.8190$	0.2071	0.6111

In addition, the corresponding quantile-quantile (Q-Q) plot, probability-probability (P-P) plot and empirical cumulative distribution (ECD) plot of UGRD are also provided in Figure 2, which also indicates that the UGRD is a reasonable model used here.

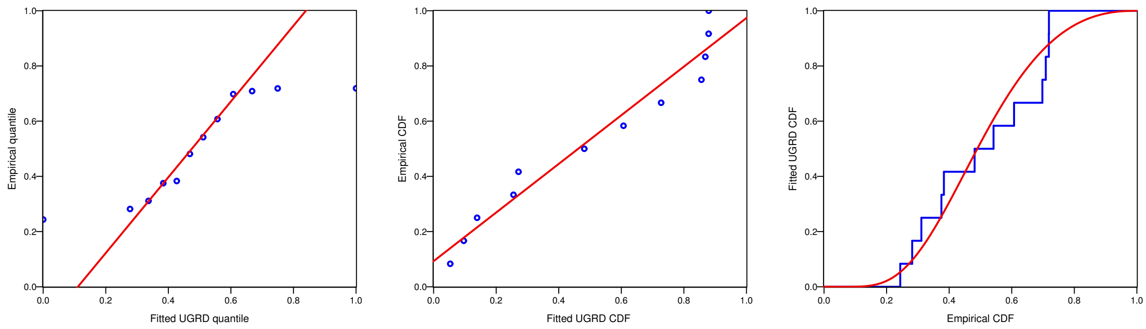

Figure 2: Q-Q, P-P and ECD plots of UGRD under the reservoir capacity ratio data.

Based on the reservoir capacity ratio data, a group of records data of size 3 is obtained as follows:

$$0.28180,\ 0.37520,\ 0.71891.$$

Then different confidence sets are estimated with $\gamma = 0.05$ and $\sigma = 10^{-4}$, and the associated results are listed in Tables 6 and 7 respectively, where the widths of confidence intervals and the areas of confidence regions are provided in parentheses. It is observed that the MCSs outperform the other competitors according to their criteria widths and areas. In Table 6, it is also noted that the lower confidence bounds of ACIs are negative. Using the alternative results in Remark 2.4, the modified ACIs and associated interval widths (within parentheses) of λ and θ are $[0.3264, 2.2162](1.8898)$ and $[0.2652, 5.0985](4.8333)$ respectively.

Table 6: Confidence intervals for λ and θ under reservoir capacity ratio records data.

Parameter	ACI	ECI	MCI
λ	[-0.0326,1.7337] (1.7663)	[0.0007,1.5843] (1.5836)	[0.0001,1.4140] (1.4139)
θ	[-0.4311,2.7566] (3.1877)	[0.1042,3.0771] (2.9729)	[0.0279,2.4935] (2.4656)

Table 7: Confidence regions for (λ, θ) under reservoir capacity ratio records data.

	Confidence regions	Areas
ACR	$\left\{ \begin{pmatrix} 0.8506-\lambda \\ 1.1628-\theta \end{pmatrix}' \begin{pmatrix} 0.2030 & 0.2068 \\ 0.2068 & 0.6613 \end{pmatrix} \begin{pmatrix} 0.8506-\lambda \\ 1.1628-\theta \end{pmatrix} < 5.9915 \right\}$	(5.6939)
ECR	$\left\{ 0.0001 < \lambda < 1.7333, \frac{0.9528}{B(\lambda)} < \theta < \frac{16.2120}{B(\lambda)} \right\}$	(5.4465)
MCR	$\left\{ 0.0002 < \lambda < 1.5389, \frac{0.3790}{B(\lambda)} < \theta < \frac{15.1281}{B(\lambda)} \right\}$	(4.2208)

Moreover, for further illustration, the confidence regions and contour of log-likelihood function (2.13) are plotted in Figure 3 to show the superority of MCR and the uniqueness of MLEs in this real data example.

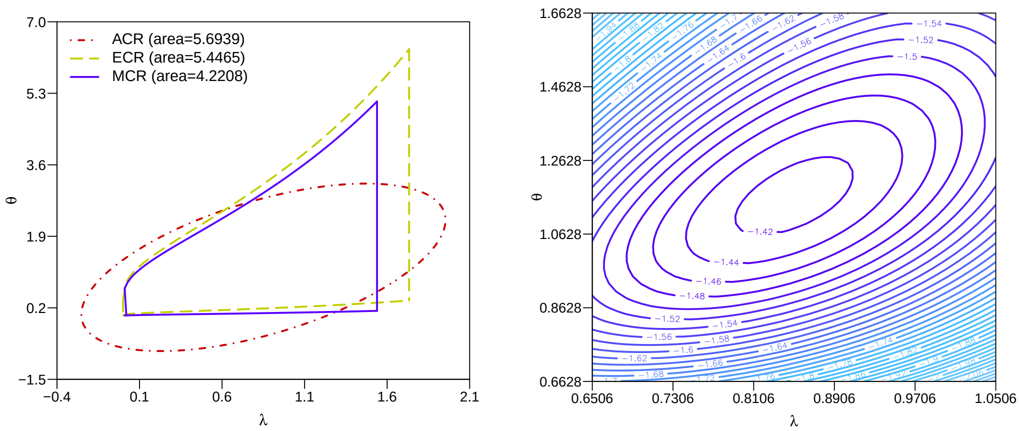

Figure 3: Contour of log-likelihood function (left) and confidence regions (right) under reservoir capacity ratio records data.

Example 2. (Electricity supply rate data) Another real life data set drawn out from `https://data.worldbank.org/indicator/EG.ELC.ACCS.ZS?end=2018&locations=KE&start=2009` is used for illustration. The data set is about the electricity supply rate in Kenya from 2009 to 2018. The original data are shown as follows:

$$0.23000,\ 0.19200,\ 0.38581,\ 0.40793,\ 0.43049,$$
$$0.36000,\ 0.41600,\ 0.65400,\ 0.63589,\ 0.75000.$$

By computation, MLEs and corresponding K-S test results for UGRD, BeD, KuD and TLD are listed in Table 8 under these data. It is also noted that UGRD have best performance among these models to fit the electricity supply rate data. Meanwhile, the associated Q-Q, P-P and ECD plots of UGRD are also shown in Figure 4 for illustration.

Similarly, records from the original observations are given as follows:

$$0.23000,\ 0.38581,\ 0.40793,\ 0.43049,\ 0.65400,\ 0.75000.$$

Table 8: MLEs and K-S test results of fitted distributions for electricity supply rate data.

	MLE of model parameters	K-S distance	p-value
UGRD	$(\hat{\lambda}, \hat{\theta}) = (1.1002, 1.2703)$	0.1971	0.7636
BeD	$(\hat{\alpha}, \hat{\beta}) = (3.4950, 4.3081)$	0.2277	0.6008
KuD	$(\hat{\alpha}, \hat{\beta}) = (2.5636, 5.0465)$	0.2391	0.5404
TLD	$\hat{\alpha} = 2.2064$	0.2790	0.3505

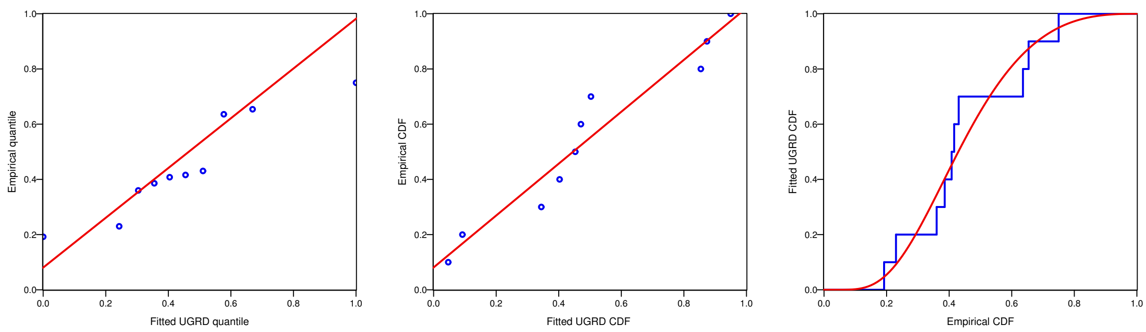

Figure 4: Q-Q plot, P-P plot and ECD plot for the electricity supply rate data of UGRD.

Different ACSs, ECSs and MCSs are shown in Tables 9 and 10 under same setting as Example 1. From the results in Tables 9 and 10, the MCSs still perform best among all estimates.

Table 9: Confidence intervals for λ and θ under electricity supply rate records data.

Parameter	ACI	ECI	MCI
λ	$[0.2086, 1.4980]$ (1.2894)	$[0.1165, 1.4757]$ (1.3592)	$[0.0700, 1.3917]$ (1.3217)
θ	$[0.1008, 4.1260]$ (4.0253)	$[0.5652, 4.5006]$ (3.9351)	$[0.3726, 3.9809]$ (3.6083)

Table 10: Confidence regions for (λ, θ) under electricity supply rate records data.

	Confidence Regions	Areas
ACR	$\left\{ \begin{pmatrix} 0.8533-\lambda \\ 2.1134-\theta \end{pmatrix}' \begin{pmatrix} 13.0909 & -2.2738 \\ -2.2738 & 1.3433 \end{pmatrix} \begin{pmatrix} 0.8533-\lambda \\ 2.1134-\theta \end{pmatrix} < 5.9915 \right\}$	(5.3421)
ECR	$\left\{ 0.0703 < \lambda < 1.5907, \frac{3.7632}{B(\lambda)} < \theta < \frac{25.4910}{B(\lambda)} \right\}$	(5.8436)
MCR	$\left\{ 0.0001 < \lambda < 1.4603, \frac{2.8890}{B(\lambda)} < \theta < \frac{24.1103}{B(\lambda)} \right\}$	(4.9916)

In addition, the plots of confidence regions and contour curve of log-likelihood function are also presented in Figure 5 for illustration and comparison.

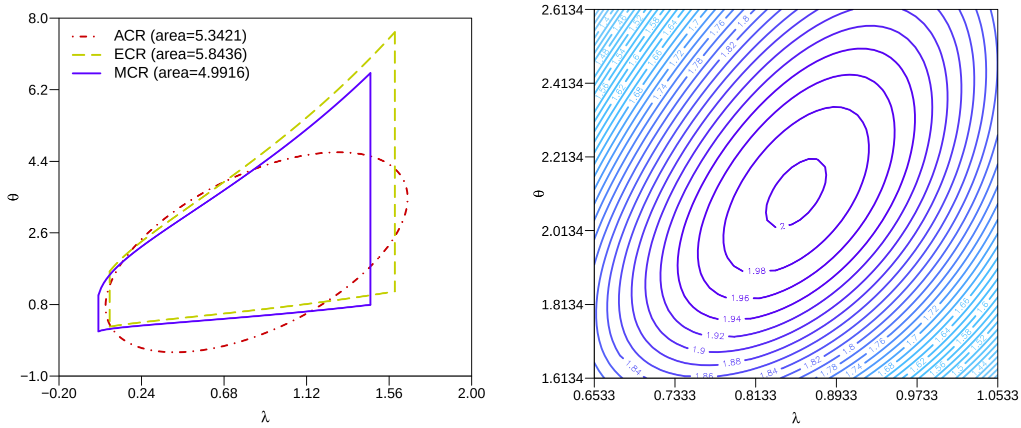

Figure 5: Contour of log-likelihood function (left) and confidence regions (right) under electricity supply rate records data.

4. EXTENSION WORK

In statistical inference, accuracy of confidence sets is one of main concerns in analysis which in turn affects the practical performance of applications. Following similar approach of previous inferential procedure, some extended results are proposed in this section for complementary, where a series of pivotal quantities is constructed, and then alternative generalized confidence sets are provided in consequence.

Using notations $W_1, W_2, ..., W_n$ and associated distribution properties given in Lemma 2.1, let $\xi_k = \sum_{i=1}^{k} W_i$ and $\eta_k = \sum_{i=k+1}^{n} W_i$, $k = 1, 2, ..., n-1$, one has

$$\Psi_k(\lambda) = \frac{2\xi_k/2k}{2\eta_k/2(n-k)} = \frac{(n-k)}{k}\left[\frac{\ln\left(1-\exp\left(-(\lambda \ln T_n)^2\right)\right)}{\ln\left(1-\exp\left(-(\lambda \ln T_k)^2\right)\right)} - 1\right]^{-1} \tag{4.1}$$

and

$$\Upsilon_k(\lambda, \theta) = 2(\xi_k + \eta_k) = -2\theta \ln(1 - \exp(-(\lambda \ln T_n)^2)), k = 1, ..., n-1 \tag{4.2}$$

follow F and chi-square distributions with $(2k, 2(n-k))$ and $2n$ degrees of freedom, respectively. Meanwhile, quantities $\Psi_k(\lambda)$ and $\Upsilon_k(\lambda, \theta)$ are statistically independent. Moreover, it is also noted from Lemma 2.2 that $\Psi_k(\lambda)$ decreases in λ with range $(0, \infty)$.

Using quantities $\Psi_k(\lambda), \Upsilon_k(\lambda, \theta)$ and following similar way as Theorems 2.1 and 2.3, for arbitrary $0 < \gamma < 1$, a series of $100(1-\gamma)\%$ ECIs of λ can be constructed as

$$\left[\psi_k\left(F_{\gamma/2}^{2k,2(n-k)}\right), \psi_k\left(F_{1-\gamma/2k}^{2,2(n-k)}\right)\right], k = 1, 2, ..., n-1, \tag{4.3}$$

where $\psi_k(x)$ refers to the solution of $\Psi_k(\lambda) = x$, and correspondingly a group of $100(1-\gamma)\%$ ECRs of (λ, θ) can be written as

$$\left\{ (\lambda,\theta) \middle| \psi_k\left(F^{2k,2(n-k)}_{\frac{1-\sqrt{1-\gamma}}{2}}\right) < \lambda < \psi_k\left(F^{2k,2(n-k)}_{\frac{1+\sqrt{1-\gamma}}{2}}\right), \frac{\chi^{2n}_{\frac{1+\sqrt{1-\gamma}}{2}}}{B(\lambda)} < \theta < \frac{\chi^{2n}_{\frac{1-\sqrt{1-\gamma}}{2}}}{B(\lambda)} \right\} \tag{4.4}$$

with $k = 1, 2, ..., n-1$.

It is observed that there are $n-1$ confidence intervals and regions obtained under this manner, and their sizes may be different with different k. To find the optimal confidence sets among the proposed results, the following criterions are provided.

Criterion 1. The best ECI of λ is obtained as k^*-th one among proposed ECIs, where k^* satisfies

$$\psi_{k^*}\left(F^{2k^*,2(n-k^*)}_{1-\gamma/2}\right) - \psi_{k^*}\left(F^{2k^*,2(n-k^*)}_{\gamma/2}\right) = \min_{k=1}^{n-1}\left[\psi_k\left(F^{2k,2(n-k)}_{1-\gamma/2}\right) - \psi_k\left(F^{2k,2(n-k)}_{\gamma/2}\right)\right].$$

Criterion 2. The best ECR of (λ,θ) is obtained as k^*-th one among all ECRs, where k^* satisfies

$$\int_{\psi_{k^*}\left(F^{2k^*,2(n-k^*)}_{\frac{1-\sqrt{1-\gamma}}{2}}\right)}^{\psi_{k^*}\left(F^{2k^*,2(n-k^*)}_{\frac{1+\sqrt{1-\gamma}}{2}}\right)} \frac{\chi^{2n}_{\frac{1-\sqrt{1-\gamma}}{2}} - \chi^{2n}_{\frac{1+\sqrt{1-\gamma}}{2}}}{B(\lambda)} d\lambda = \min_{k=1}^{n-1}\left[\int_{\psi_k\left(F^{2k,2(n-k)}_{\frac{1-\sqrt{1-\gamma}}{2}}\right)}^{\psi_k\left(F^{2k,2(n-k)}_{\frac{1+\sqrt{1-\gamma}}{2}}\right)} \frac{\chi^{2n}_{\frac{1-\sqrt{1-\gamma}}{2}} - \chi^{2n}_{\frac{1+\sqrt{1-\gamma}}{2}}}{B(\lambda)} d\lambda\right].$$

Note from (4.2) that, since $\Upsilon_k(\lambda,\theta) = \Upsilon(\lambda,\theta)$ does not change with k and the associated ECI of θ coincides with the results obtained in Theorem 2.2. Further, following the similar approach of Subsection 2.2, a series of MCSs for parameters λ and (λ,θ) could be also obtained based on pivotal quantities $\Psi_k(\lambda)$ and $\Upsilon_k(\lambda,\theta)$, $k = 1, 2, ..., n-1$, the detailed results are omitted for concision and saving space. In addition, the optimal confidence sets for such MCSs could be also selected by using similar method as shown in Criterions 1 and 2.

For illustration, the confidence sets for parameters λ and (λ,θ) are reconstructed by using the records data in Example 2 with $k = 1, 2, 3, 4, 5$, where the significance level is $\gamma = 0.05$ as same as previous. The associated results are tabulated in Tables 11 and 12. From the results, it is seen that for ECSs estimation, the optimal ECI of λ and ECR of (λ,θ) are obtained at $k = 4$, whereas the associated optimal MCI of λ and MCR of (λ,θ) are also obtained at $k = 4$. In addition, one also note that the MCSs perform better than the associated ECSs at given k and that all the confidence sets obtained by using the proposed pivotal quantities $\Psi_k(\lambda), \Upsilon_k(\lambda,\theta), k = 4$ have smaller sizes than the ACSs in Tables 9 and 10. Further, plots of extended ECR, MCR of (λ,θ) with $k = 4$ and traditional ACR are also presented in Figure 6, which indicate that the proposed extended confidence regions have better performance in this manner.

Table 11: ECIs for λ and ECRs for (λ, θ) with different k under electricity supply rate records data.

k	ECIs	ECRs
1	$[0.1165, 1.4757]$ (1.3592)	$\left\{0.0703 < \lambda < 1.5907, \frac{3.7632}{B(\lambda)} < \theta < \frac{25.4910}{B(\lambda)}\right\}$ (5.8436)
2	$[0.0512, 1.6687]$ (1.6175)	$\left\{0.0220 < \lambda < 1.8063, \frac{3.7632}{B(\lambda)} < \theta < \frac{25.4910}{B(\lambda)}\right\}$ (7.4175)
3	$[0.0015, 1.2970]$ (1.2955)	$\left\{0.0002 < \lambda < 1.4326, \frac{3.7632}{B(\lambda)} < \theta < \frac{25.4910}{B(\lambda)}\right\}$ (4.9472)
4	$[0.0001, 0.9519]$ (0.9518)	$\left\{0.0001 < \lambda < 1.0930, \frac{3.7632}{B(\lambda)} < \theta < \frac{25.4910}{B(\lambda)}\right\}$ (3.1794)
5	$[0.0001, 2.2521]$ (2.2520)	$\left\{0.0001 < \lambda < 2.5783, \frac{3.7632}{B(\lambda)} < \theta < \frac{25.4910}{B(\lambda)}\right\}$ (15.0503)

Note: the interval widths and region areas are listed in the parentheses.

Table 12: MCIs for λ and MCRs for (λ, θ) with different k under electricity supply rate records data.

k	MCIs	MCRs
1	$[0.0700, 1.3917]$ (1.3217)	$\left\{0.0001 < \lambda < 1.4603, \frac{2.8890}{B(\lambda)} < \theta < \frac{24.1103}{B(\lambda)}\right\}$ (4.9916)
2	$[0.0101, 1.5144]$ (1.5043)	$\left\{0.0001 < \lambda < 1.6370, \frac{2.8346}{B(\lambda)} < \theta < \frac{24.3403}{B(\lambda)}\right\}$ (6.1677)
3	$[0.0001, 1.1420]$ (1.1419)	$\left\{0.0002 < \lambda < 1.2459, \frac{2.7388}{B(\lambda)} < \theta < \frac{24.7603}{B(\lambda)}\right\}$ (3.9759)
4	$[0.0001, 0.7925]$ (0.7924)	$\left\{0.0001 < \lambda < 0.8702, \frac{2.5663}{B(\lambda)} < \theta < \frac{25.5503}{B(\lambda)}\right\}$ (2.3655)
5	$[0.0001, 1.8748]$ (1.8747)	$\left\{0.0001 < \lambda < 2.0143, \frac{2.4165}{B(\lambda)} < \theta < \frac{26.2903}{B(\lambda)}\right\}$ (10.0198)

Note: the interval widths and region areas are listed in the parentheses.

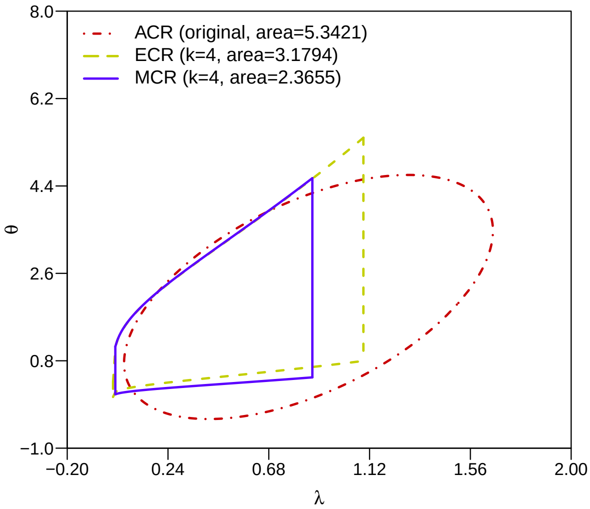

Figure 6: Plots of ECR and MCR with $k = 4$ and traditional ACR.

5. CONCLUSION

In this paper, different confidence sets of parameters from the unit generalized Rayleigh distribution are explored under records data. By constructing pivotal quantities, equal-tailed confidence sets are established for model parameters. Further, the associated minimum-size confidence sets are constructed based on optimization techniques, and the algorithms along with Lagrange multiplier method are also provided for computation. In addition, conventional likelihood based asymptotic confidence sets are also constructed for comparison. Extensive simulation studies and two real life examples are carried out to investigate the performance of different methods, and the results indicate that the proposed pivotal quantities based ECSs and MCSs perform better than common likelihood based confidence sets. Furthermore, a series of confidence sets are also proposed as extension based on constructed alternative pivotal quantities which sometimes may further provide potential better estimates.

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