REVSTAT – Statistical Journal
Supplementary file

Supplement — Estimations of confidence sets for the unit generalized Rayleigh parameters using records data

Authors: Xuanjia Zuo, Liang Wang ✉
– School of Mathematics, Yunnan Normal University, Kunming, PR China
zo_syunka@163.com, liang610112@163.com

Yuhlong Lio
– Department of Mathematical Sciences, University of South Dakota, Vermillion, USA
Yuhlong.Lio@usd.edu

Yogesh Mani Tripathi
– Department of Mathematics, Indian Institute of Technology Patna, Patna, India
yogesh@iitp.ac.in

This is the supplementary file for the paper "Estimations of Confidence Sets for the Unit Generalized Rayleigh Parameters Using Records Data" published in journal REVSTAT, and all detailed proofs of the main contents are provided in this seperate file.

✉ Corresponding author

A. Proof of Lemma 2.1

Denote $V_i = -\log(1 - F(T_i)) = -\theta \log(1 - \exp(-(\lambda \log T_i)^2)), \quad i = 1, 2, ..., n,$ then $V_1, V_2, ..., V_n$ are upper records from the standard exponential distribution with mean one. Let $W_1 = V_1$, $W_i = V_i - V_{i-1}$, $i = 2, ..., n$, follow the Lemma 1 in Wang and Ye [30], $W_1, W_2, ..., W_n$ are the independent random variables of standard exponential distribution. Taking $\xi = W_1$ and $\eta = \sum_{i=2}^{n} W_i$, then 2ξ follows chi-square distribution with 2 degree of freedom,and 2η follows chi-square distribution with $2(n-1)$ degrees of freedom, respectively. Finally, let

$$\Psi(\lambda) = \frac{2\xi/2}{2\eta/2\,(n-1)} = (n-1)\left[\frac{\log(1 - \exp(-(\lambda \log T_n)^2))}{\log(1 - \exp(-(\lambda \log T_1)^2))} - 1\right]^{-1}$$

and

$$\Upsilon(\lambda, \theta) = 2(\xi + \eta) = -2\theta \log(1 - \exp(-(\lambda \log T_n)^2)),$$

then it is noted that $\Psi(\lambda)$ follows F distribution with 2 and $2(n-1)$ degrees of freedom, $\Upsilon(\lambda, \theta)$ follows chi-square distribution with $2n$ degrees of freedom. Moreover, from Johnson et al. [15], one also has $\Psi(\lambda)$ and $\Upsilon(\lambda, \theta)$ are statistically independent.

B. Proof of Lemma 2.2

Denote $c(x) = 1 - \exp\left(-(\lambda \ln x)^2\right)$ and taking drivetive of $h(\lambda)$, one has

$$h'(\lambda) = -\frac{2}{\lambda}\frac{\ln(c(a))}{\ln(c(b))}\left[\frac{(1-c(a))\ln(1-c(a))}{c(a)\ln(c(a))} - \frac{(1-c(b))\ln(1-c(b))}{c(b)\ln(c(b))}\right]$$

with $0 < c(a) < c(b) < 1$. To show $h(\lambda)$ increases in λ is equavelent to prove that function $g(x) = (1-x)\ln(1-x)/(x\ln x)$, $0 < x < 1$ increases in x.

Taking derivative of $g(x)$ with respect to x, one has

$$g'(x) = \frac{-x\ln x - (1-x)\ln(1-x) - \ln x \ln(1-x)}{(x\ln x)^2},$$

let $g_1(x) = \ln(1-x)$, $g_2(x) = \ln x$ and using Lagrange's mean value theorem, there exist two numbers $0 < \varepsilon_1 < x < \varepsilon_2 < 1$ satisfying

$$g_1(x) - g_1(0) = -\frac{x}{1-\varepsilon_1} \quad \text{and} \quad g_2(1) - g_2(x) = \frac{1-x}{\varepsilon_2}.$$

Then function $g'(x)$ can be rewritten as $g'(x) = \frac{x(1-x)\left[\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2(1-\varepsilon_1)}\right]}{(x\ln x)^2} > 0$. In addition, the limitation results of $h(\lambda)$ could be obtained directly. Therefore, the assertion is comleted.

C. Proof of Theorem 2.1

Based on **Lemma 2.1** and **Corollary 2.1**, for arbitrary $0 < \gamma < 1$, it has

$$P\left\{F_{1-\gamma/2}^{2,2(n-1)} < \Psi(\lambda) < F_{\gamma/2}^{2,2(n-1)}\right\} = P\left\{\psi\left(F_{\gamma/2}^{2,2(n-1)}\right) < \lambda < \psi\left(F_{1-\gamma/2}^{2,2(n-1)}\right)\right\} = 1 - \gamma,$$

then the result is proved.

D. Proof of Theorem 2.3

Since that $\Psi(\lambda)$ and $\Upsilon(\lambda,\theta)$ are statistically independent, using the similar manner with **Theorem 2.1**, for arbitrary $0<\gamma<1$, one has

$$\begin{aligned}
&P\left\{F^{2,2(n-1)}_{\frac{1+\sqrt{1-\gamma}}{2}}<\Psi(\lambda)<F^{2,2(n-1)}_{\frac{1-\sqrt{1-\gamma}}{2}},\chi^{2n}_{\frac{1+\sqrt{1-\gamma}}{2}}<\Upsilon\left(\lambda,\theta\right)<\chi^{2n}_{\frac{1-\sqrt{1-\gamma}}{2}}\right\}\\
=&P\left\{\psi\left(F^{2,2(n-1)}_{\frac{1-\sqrt{1-\gamma}}{2}}\right)<\lambda<\psi\left(F^{2,2(n-1)}_{\frac{1+\sqrt{1-\gamma}}{2}}\right)\right\}\\
\times&P\left\{\left(\chi^{2n}_{\frac{1+\sqrt{1-\gamma}}{2}}/B(\lambda)\right)<\theta<\left(\chi^{2n}_{\frac{1-\sqrt{1-\gamma}}{2}}/B(\lambda)\right)\right\}\\
=&\sqrt{1-\gamma}\times\sqrt{1-\gamma}=1-\gamma.
\end{aligned}$$

Therefore, the MCR for (λ,θ) is obtained.

E. Proof of Theorem 2.4

Following similar line of **Theorem 2.1**, for arbitrary $0<\gamma<1$, assume that x_1 and x_2 are two upper percentiles in F distribution with 2 and $2(n-1)$ degrees of freedom satisfying $P\{x_1<\Psi(\lambda)<x_2\}=1-\gamma$, then a $100(1-\gamma)\%$ confidence interval of λ can be constructed as $[\psi\left(x_2\right),\psi\left(x_1\right)]$, and the associated interval length can be expressed as $L\left(x_1,x_2\right)=\psi\left(x_1\right)-\psi\left(x_2\right)$.

In order to find the MCI of λ, consider following optimization problem

$$\begin{aligned}
\min\quad & L\left(x_1,x_2\right),\\
s.t.\quad & F_{2,2(n-1)}(x_2)-F_{2,2(n-1)}(x_1)=1-\gamma,\\
& x_1<x_2.
\end{aligned}$$

implying that the Lagrange function can be constructed as

$$\mathcal{L}\left(x_1,x_2,z\right)=\psi\left(x_1\right)-\psi\left(x_2\right)+z\left[F_{2,2(n-1)}\left(x_2\right)-F_{2,2(n-1)}\left(x_1\right)-(1-\gamma)\right],$$

with z being the lagrange multiplier.

By taking derivatives of $\mathcal{L}\left(x_1,x_2,z\right)$ and equating them to zero, the optimal solutions (x_1^*,x_2^*) of (x_1,x_2) can be obtained from following equations

$$\begin{cases}\frac{\psi'(x_1)}{\psi'(x_2)}=\frac{P^F_{2,2(n-1)}(x_1)}{P^F_{2,2(n-1)}(x_2)},\\ F_{2,2(n-1)}\left(x_2\right)-F_{2,2(n-1)}\left(x_1\right)=1-\gamma.\end{cases}$$

Therefore, the $100(1-\gamma)\%$ MCI of λ can be constructed as $[\psi\left(x_2^*\right),\psi\left(x_1^*\right)]$.

F. Proof of Theorem 2.6

Using same notations as in **Theorems 2.4** and **2.5**, and assume that x_1 and x_2 are the two upper percentiles in F distribution with 2 and $2(n-1)$ degrees of freedom, and y_1 and y_2 are the two upper percentiles in chi-square distribution with $2n$ degrees of freedom, then one has that

$$P\left\{x_1 < \Psi(\lambda) < x_2, y_1 < \Upsilon(\lambda,\theta) < y_2\right\} = 1-\gamma,$$

implying that a $100(1-\gamma)\%$ confidence region of (λ,θ) can be obtained as

$$\left\{(\lambda,\theta)\,\middle|\,\psi(x_2) < \lambda < \psi(x_1), \frac{y_1}{B(\lambda)} < \theta < \frac{y_2}{B(\lambda)}\right\}$$

with area $S(x_1,x_2,y_1,y_2) = \int_{\psi(x_2)}^{\psi(x_1)} \frac{y_2-y_1}{B(\lambda)} d\lambda$.

To find MCR of (λ,θ), following optimization problem is considered

$$\begin{aligned} &\min \;\; S(x_1,x_2,y_1,y_2), \\ &s.t. \;\; \left[F_{2,2(n-1)}(x_2) - F_{2,2(n-1)}(x_1)\right]\cdot\left[\chi_{2n}(y_2) - \chi_{2n}(y_1)\right] = 1-\gamma, \\ &\qquad x_1 < x_2, y_1 < y_2. \end{aligned}$$

and the Lagrange function can be obtained as follows

$$\mathcal{A}(x_1,x_2,y_1,y_2,z) = \int_{\psi(x_2)}^{\psi(x_1)} \frac{y_2-y_1}{B(\lambda)} d\lambda + z\left\{\begin{array}{l}\left[F^{2,2(n-1)}(x_2) - F^{2,2(n-1)}(x_1)\right] \\ \times\left[\chi^{2n}(y_2) - \chi^{2n}(y_1)\right] - (1-\gamma)\end{array}\right\}.$$

Following similar approach of the proof in **Theorem 2.4**, the optimal solution $(x_1^*, x_2^*, y_1^*, y_2^*)$ of (x_1,x_2,y_1,y_2) minimizing the area of the confidence region could be obtained via the Lagrangian multiplier method which is the solution of following equations

$$\begin{cases} P_{2n}^{\chi}(y_1) = P_{2n}^{\chi}(y_2), \\ \left[F_{2,2(n-1)}(x_2) - F_{2,2(n-1)}(x_1)\right]\left[\chi_{2n}(y_2) - \chi_{2n}(y_1)\right] = 1-\gamma, \\ \frac{\psi'(x_2)B(\psi(x_1))}{\psi'(x_1)B(\psi(x_2))} = -\frac{P_{2,2(n-1)}^{F}(x_2)}{P_{2,2(n-1)}^{F}(x_1)}, \\ \frac{\psi'(x_1)\left[F_{2,2(n-1)}(x_2) - F_{2,2(n-1)}(x_1)\right]P_{2n}^{\chi}(y_1)}{\left[\chi_{2n}(y_2)-\chi_{2n}(y_1)\right]P_{2,2(n-1)}^{F}(x_1)\int_{\psi(x_2)}^{\psi(x_1)}\frac{1}{B(\lambda)}d\lambda} = -\frac{B(\psi(x_1))}{y_2-y_1}. \end{cases}$$

Therefore, the $100(1-\gamma)\%$ MCR of (λ,θ) can be established as

$$\left\{(\lambda,\theta)\,\middle|\,\psi(x_2^*) < \lambda < \psi(x_1^*), \frac{y_1^*}{B(\lambda)} < \theta < \frac{y_2^*}{B(\lambda)}\right\},$$

and the assertion is completed.

G. Uniqueness and existence of MLEs $\hat{\lambda}$ and $\hat{\theta}$

It is noted that the uniqueness and existence of MLEs $\hat{\lambda}$ and $\hat{\theta}$ is equivalent to show that equation (2.14) has an unique solution for $\lambda > 0$.

Fon convience, let $\mu = \lambda^2$ and $a_i = (\ln t_i)^2$, $i = 1, 2, ..., n$ satisfying $a_1 > a_2 > \cdots > a_n$. Then equation (2.14) could be rewritten as a function of μ as follows

$$H(\mu) = \frac{n}{\mu} - \frac{n a_n e^{-\mu a_n}}{(1 - e^{-\mu a_n}) \ln(1 - e^{-\mu a_n})} - \sum_{i=1}^{n} \frac{a_i}{1 - e^{-\mu a_i}} = 0.$$

For showing equaiton $H(\mu) = 0$, some useful results from Ghitany et al. [12] are provided as follows.

$$(a).\ \lim_{y \to 0} \frac{y e^{-y}}{1 - e^{-y}} = 1, \quad (b).\ \lim_{y \to 0} y \left|\ln\left(1 - e^{-y}\right)\right| = 0, \quad (c).\ \lim_{y \to \infty} \frac{|\ln(1 - e^{-y})|}{e^{-y}} = 1,$$
$$(d).\ y^k e^{-y} < \left(1 - e^{-y}\right)^k, k = 1, 2, y > 0, \quad (e).\ \ln y \leqslant y - 1, y > 0,$$
$$(f).\ \ln\left(1 - e^{-y}\right) < -e^{-y}, y > 0.$$

Using results (a), (b) and (c), one directly has $\lim_{\mu \to 0^+} H(\mu) = \infty > 0$ and $\lim_{\mu \to \infty} H(\mu) = \sum_{i=1}^{n} (a_n - a_i) < 0$. Further, based on results (d), (e) and (f), one could also observe that

$$\frac{\partial H(\mu)}{\partial \mu} < -\frac{2 n a_n^2}{[(e^{\mu a_n} - 1) \ln(1 - e^{-\mu a_n})]^2} < 0.$$

Therefore, MLE $\hat{\lambda}$ of $\lambda = \sqrt{\mu}$ uniquely exists and the assertion is completed.