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Estimation, Prediction and Life Testing Plan for the Exponentiated Gumbel Type-II Progressive Censored Data

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Abstract:

- This article accentuates the estimation and prediction of a three-parameter exponentiated Gumbel type-II (EGT-II) distribution when the data are progressively type-II (PT-II) censored. We obtain maximum likelihood (ML) estimates using expectation maximization (EM) and stochastic expectation maximization (StEM) algorithms. The existence and uniqueness of the ML estimates are discussed. We construct bootstrap confidence intervals. The Bayes estimates are derived with respect to a general entropy loss function. We adopt Lindley's approximation, importance sampling and Metropolis-Hastings (MH) methods. The highest posterior density credible interval is computed based on MH algorithm. Bayesian predictors and associated Bayesian predictive interval estimates are obtained. A real life data set is considered for the purpose of illustration. Finally, we propose different criteria for comparison of different sampling schemes in order to obtain the optimal sampling scheme.

Keywords:

- *EM algorithm; stochastic EM algorithm; Lindley's approximation; importance sampling; MH algorithm; optimal censoring.*

AMS Subject Classification:

- 62F10, 62F12, 62F15, 62F40, 62N02.

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1. INTRODUCTION

In lifetime and reliability studies, an experimenter may not have complete information of the failure times for each and every experimental units. Due to various reasons, it is sometimes required to remove few units from an experiment and as a result, one gets censored data set. There are two most common censoring schemes: (i) type-I and (ii) type-II. The type-I censoring is censored at fixed time, whereas the type-II censoring is censored at a fixed number. These two censoring schemes can not handle the situations, in which we need to remove units at various stages of a test. The removal of experimental units can be done in the progressive censoring scheme. In this paper, we consider PT-II censoring scheme for estimation and prediction for an EGT-II distribution. The PT-II censored scheme is described below. Let n units be placed in a life test. It is pre-decided by the experimenter that m number of failures will be observed. At the time of first failure, we assume that Φ_1 of the remaining $n-1$ surviving units are randomly withdrawn from the experiment. Further, Φ_2 of the remaining $n-\Phi_1-2$ units are removed from the on-going experiment. This procedure continues till the occurrence of m-th failure. We remove all the remaining surviving units $\Phi_m = n-m-\Phi_1-\cdots-\Phi_{m-1}$, when the m-th failure takes place. In PT-II censoring scheme, we denote the m observed failure times as $x_{1:m:n}, ..., x_{m:m:n}$. For simplicity, we use $x_i = x_{i:m:n}$, for $i = 1, ..., m$.

The most popular lifetime models are those with monotone hazard rates (gamma, Weibull), which reflect a wear out or a work hardening behaviour of the population under study. However, there are many other situations, in which the failure pattern is somehow different. When studying the life-cycle of an industrial product or the entire life-span of a biological entity, the three-phase behaviour of the failure rate is likely to be observed. For example, consider a high failure rate in infancy which decreases to a certain level, where it remains fixed for some time, and then increases from a point onwards due to ageing. Thus, in this case, a model having bathtub-shaped hazard rate will be appropriate to study the population's survival capacity. Further, there are also some situations, in which the failure pattern looks like upside-down bathtub. The distributions with upside-down bathtub-shaped hazard rate function is often associated with overload of a component or a subsystem. Intuitively, a lifetime distribution with upside-down shaped hazard rate would suggest a hard stress on the components, leading to fast ageing processes for a part of them but leading to a decreasing failure rate for the surviving items after the stress. There are various real life applications, when the data show upside-down bathtub shape hazard rates. For example, Langlands *et al.* [10] studied cases of breast carcinoma and showed that the associated hazard rate has upside-down bathtub shape. We refer to Efron [5] for more applications of this type of hazard rate functions. The EGT-II distribution has a upside-down bathtub-shaped hazard rate function. Thus, this distribution can be useful in modelling population having bathtub-shaped failure pattern. In addition, this distribution can model various types of data as it can take various shapes (Leptokurtic, platykurtic with thick and thin tails) for various choices of the parameters.

Let X be a random variable following EGT-II distribution, with probability density and cumulative distribution functions respectively given by

$$f_{\boldsymbol{X}}(x:\alpha,\beta,\gamma) \;=\; \alpha\beta\gamma x^{-\beta-1}\exp\{-\gamma x^{-\beta}\}\Big(1-\exp\{-\gamma x^{-\beta}\}\Big)^{\alpha-1} \tag{1.1}$$

and

$$F_{\boldsymbol{X}}(x:\alpha,\beta,\gamma) \;=\; 1-\Big(1-\exp\{-\gamma x^{-\beta}\}\Big)^{\alpha}, \tag{1.2}$$

for $x>0$ and $\alpha,\beta,\gamma>0$. The constants α and β are known as the shape parameters, whereas γ is known as the scale parameter. We denote $X\sim$ EGT-II(α,β,γ) if X has the distribution function given by (1.2). The EGT-II distribution is a generalization of various well known statistical models. When $\alpha=1$, $\beta=1$, $\beta=2$ and $\gamma=1$, the EGT-II distribution reduces to the Gumbel type-II, generalized inverted exponential, inverted exponentiated Rayleigh and exponentiated Frechet distributions, respectively. The EGT-II distribution becomes Frechet distribution for $\alpha=\gamma=1$.

Several authors have considered estimation of parameters and some reliability characteristics of various lifetime distributions based on PT-II censored observations. Maiti and Kayal [13] considered estimation for the generalized Frechet distribution based on the PT-II censored data. Ghanbari *et al.* [7] studied estimation of stress-strength reliability for Marshall-Olkin distributions based on PT-II censored samples. Ren and Gui [17] explored goodness of fit test for Rayleigh distribution based on PT-II censored samples. Tarvirdizade and Nematollahi [20] proposed some inferences for the power-exponential hazard rate distribution under PT-II censored data. To the best of authors' knowledge, nobody has considered the problem of estimation of parameters of EGT-II distribution based on PT-II censored data. It is already seen that this distribution can be considered as an alternative lifetime model since it has upside-down bathtub shaped hazard rate function, which is useful in various places.

The aim of this paper is three-fold. First, we consider statistical inference of EGT-II distribution based on the PT-II censored data. The existence and uniqueness of the maximum likelihood estimates (MLEs) are investigated. Further, we obtain MLEs of the parameters. The closed-form solutions of the likelihood equations can not be obtained. Thus, we apply EM algorithm. We also use stochastic EM algorithm to compute the desired MLEs. Confidence intervals using bootstrap algorithms are obtained. The Bayes estimates are derived. It is noticed that the explicit expressions of the Bayes estimates can not be obtained. So, we use Lindley's approximation and importance sampling methods. The Metropolies-Hastings algorithm is also used for this purpose. Second, we study Bayesian prediction problem, and obtain Bayesian prediction intervals. Third, we consider optimal life testing plan for the present problem.

The paper is organized as follows. In Section 2, we present sufficient condition for the existence and uniqueness of MLEs. For the purpose of computation, two algorithms: EM and stochastic EM are used. In Section 3, we obtain observed Fisher's information matrix. The bootstrap confidence intervals are constructed in Section 4. Section 5 provides the form of Bayes estimates with respect to the entropy loss function. Since explicit expressions of the Bayes estimates do not exist, we use various approximation methods to compute the estimates in Section 6. The prediction problem has been considered in Section 7 from Bayesian point of view. Bayesian predictive intervals are also obtained. Data analysis is carried out in Section 8 based on a real life data set. In Section 9, we propose optimal PT-II censoring plan. Finally, Section 10 concludes the paper.

2. ML ESTIMATES AND THEIR COMPUTATION

In this section, first, we show that MLEs of the parameters exist and unique based on the PT-II censored sample.

2.1. Existence and uniqueness of the MLEs

Consider PT-II censored sample of size m from a sample of size n from EGT-II distribution as $\boldsymbol{X} = (X_1, ..., X_m)$. The likelihood function of α, β and γ is given by

$$L(\alpha,\beta,\gamma|\boldsymbol{x}) \propto \alpha^m \beta^m \gamma^m \prod_{i=1}^{m} x_i^{-(\beta+1)} \exp\{-\gamma x_i^{-\beta}\}\Big(1-\exp\{-\gamma x_i^{-\beta}\}\Big)^{\alpha(\Phi_i+1)-1},$$

where $\boldsymbol{x} = (x_1, ..., x_m)$. Here, $x_1 \leq ... \leq x_m$. The log-likelihood function is denoted by $\ell(\alpha,\beta,\gamma|\boldsymbol{x}) = \ln L(\alpha,\beta,\gamma|\boldsymbol{x})$. The MLEs of α, β and γ can be obtained after solving first order partial derivatives of the log-likelihood function with respect to the parameters equal to zero, simultaneously. The normal equations are

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}(1+\Phi_i)\ln\Big(1-\exp\{-\gamma x_i^{-\beta}\}\Big) = 0, \tag{2.1}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{m}{\beta} - \gamma\sum_{i=1}^{m}\frac{(\alpha(1+\Phi_i)-1)x_i^{-\beta}\exp\{-\gamma x_i^{-\beta}\}\ln x_i}{1-\exp\{-\gamma x_i^{-\beta}\}} + \gamma\sum_{i=1}^{m} x_i^{-\beta}\ln x_i - \sum_{i=1}^{m}\ln x_i = 0, \tag{2.2}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{m}{\gamma} + \sum_{i=1}^{m}\frac{(\alpha(1+\Phi_i)-1)x_i^{-\beta}\exp\{-\gamma x_i^{-\beta}\}}{1-\exp\{-\gamma x_i^{-\beta}\}} - \sum_{i=1}^{m} x_i^{-\beta} = 0. \tag{2.3}$$

Note that the closed forms of the MLEs do not exist. So, to get approximate values of the MLEs, we use EM algorithm, which is presented in the following subsection. An important question always comes out whether the MLEs exist, and unique. To investigate this, note that the domain of $\ell(\alpha,\beta,\gamma|\boldsymbol{x})$ is $(0,\infty)\times(0,\infty)\times(0,\infty)$. So, our goal is to show that for $(\alpha,\beta,\gamma) \in (0,\infty)\times(0,\infty)\times(0,\infty)$, the function $\ell(\alpha,\beta,\gamma|\boldsymbol{x})$ has unique maximum. The second order partial derivatives of ℓ with respect to α, β and γ can be shown to be strictly negative under the following conditions:

$$\frac{\partial^2 \ell}{\partial \alpha^2} < 0, \tag{2.4}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} < 0, \quad \text{if} \quad \alpha(1+\Phi_i) > 1, \tag{2.5}$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} < 0, \quad \text{if} \quad \alpha(1+\Phi_i) > 1. \tag{2.6}$$

Therefore, ℓ is a strictly concave function with respect to one of the parameters keeping other two parameters fixed. For fixed $(\beta,\gamma),(\alpha,\gamma)$ and (α,β), we respectively have

$$\lim_{\alpha\to 0}\ell(\alpha,\beta,\gamma|\boldsymbol{x})=-\infty,\quad \lim_{\alpha\to\infty}\ell(\alpha,\beta,\gamma|\boldsymbol{x})=-\infty,$$
$$\lim_{\beta\to 0}\ell(\alpha,\beta,\gamma|\boldsymbol{x})=-\infty,\quad \lim_{\beta\to\infty}\ell(\alpha,\beta,\gamma|\boldsymbol{x})=-\infty,$$
$$\lim_{\gamma\to 0}\ell(\alpha,\beta,\gamma|\boldsymbol{x})=-\infty,\quad \lim_{\gamma\to\infty}\ell(\alpha,\beta,\gamma|\boldsymbol{x})=-\infty.$$

So, $\ell(\alpha,\beta,\gamma|\boldsymbol{x})$ is a unimodal function with respect to α,β and γ, when other two associated parameters are fixed. Now, proceeding with the similar arguments as in Dey *et al.* [4], we get the following theorem, which provides sufficient conditions for the existence and uniqueness of MLEs.

Theorem 2.1. *The MLEs of α,β and γ when $(\alpha,\beta,\gamma)\in(0,\infty)\times(0,\infty)\times(0,\infty)$ exist and unique based on the PT-II censored sample, provided $\alpha(1+\Phi_i)>1$.*

Remark 2.1. From real data set, which are presented in Section 8, we notice that the sufficient condition in Theorem 2.1 is satisfied. Thus, as stated, the MLEs of the parameters exist and are unique. The profile of the log-likelihood function of α, β and γ for the data set is depicted in Figure 2.

2.2. EM and StEM algorithms

The EM algorithm is very useful iterative process to obtain MLEs of the parameters when the data are censored. For incomplete data problems, the most attractive features of the EM algorithm relative to other optimization techniques are its simplicity and stability. Further, successive iterations of the EM algorithm are guaranteed never to decrease the likelihood function, which is not generally true of gradient methods like Newton-Raphson. Hence, in the case of the unimodal and concave likelihood function, the EM algorithm converges to the global maximizer from any starting value. Due to this, it has been widely used by various authors. One may refer to Singh and Tripathi [19] and Singh *et al.* [18] for computing MLEs of some lifetime distributions using this method. The EM algorithm is described briefly as follows. To start the EM algorithm, the likelihood function of the complete sample which have been put on a test is required. We denote the complete sample by $\boldsymbol{W}=(W_1,...,W_n)$. After conducting the test, we see that the complete sample is a combination of the observe data $\boldsymbol{X}=(X_1,...,X_m)$ and the censored data $\boldsymbol{Z}=(Z_1,...,Z_m)$. Here Z_j is a $1\times\Phi_j$ vector $(Z_{j1},...,Z_{j\Phi_j})$ for $j=1,...,m$. Then, the complete sample is $\boldsymbol{W}=(\boldsymbol{X},\boldsymbol{Z})$. The log-likelihood function of α, β and γ based on the complete sample is given by

$$\begin{aligned}\ell_C(\alpha,\beta,\gamma|\boldsymbol{w}) = {} & n\ln(\alpha\beta\gamma)+\sum_{j=1}^{m}\Bigg[(\alpha-1)\Bigg(\ln(1-\exp\{-\gamma x_j^{-\beta}\}) \\ & +\sum_{k=1}^{\Phi_j}\ln(1-\exp\{-\gamma z_{jk}^{-\beta}\})\Bigg)-\gamma\Bigg(x_j^{-\beta}+\sum_{k=1}^{\Phi_j}z_{jk}^{-\beta}\Bigg) \\ & -(\beta+1)\Bigg(\ln x_j+\sum_{k=1}^{\Phi_j}\ln z_{jk}\Bigg)\Bigg].\end{aligned} \tag{2.7}$$

Further, the pseudo log-likelihood function is obtained in E-step as

$$(2.8)\qquad L_{\mathrm{p}}(\alpha,\beta,\gamma) = n\ln(\alpha\beta\gamma)+(\alpha-1)\sum_{j=1}^{m}\ln(1-\exp\{-\gamma x_j^{-\beta}\})-\gamma\sum_{j=1}^{m}x_j^{-\beta}$$
$$-(\beta+1)\sum_{j=1}^{m}\ln x_j-(\beta+1)\sum_{j=1}^{m}\Phi_j E[\ln Z_{jk}|Z_{jk}>x_j]$$
$$+(\alpha-1)\sum_{j=1}^{m}\Phi_j E[\ln(1-\exp\{-\gamma Z_{jk}^{-\beta}\})|Z_{jk}>x_j].$$

Please see the Appendix A for the expressions of the expectations, which are involved in (2.8). In M-step, we will find the values of the parameters such that the pseudo log-likelihood function is maximum. Let $(\alpha^{(k)},\beta^{(k)},\gamma^{(k)})$ be the value of (α,β,γ) obtained after k-th iteration. Mathematically, at the $(k+1)$-th iteration, $(\alpha^{(k+1)},\beta^{(k+1)},\gamma^{(k+1)})$ has to be computed by maximizing the following function based on $(\alpha^{(k)},\beta^{(k)},\gamma^{(k)})$:

$$(2.9)\qquad L_{\mathrm{p}}^{*}(\alpha,\beta,\gamma) = n\ln(\alpha\beta\gamma)-(\beta+1)\sum_{j=1}^{m}\ln x_j-\gamma\sum_{j=1}^{m}x_j^{-\beta}$$
$$+(\alpha-1)\sum_{j=1}^{m}\ln(1-\exp\{-\gamma x_j^{-\beta}\})$$
$$-(\beta+1)\sum_{j=1}^{m}\Phi_j E[\ln Z_{jk}|Z_{jk}>x_j,\alpha^{(k)},\beta^{(k)},\gamma^{(k)}]$$
$$-\gamma\sum_{j=1}^{m}\Phi_j E[Z_{jk}^{-\beta}|Z_{jk}>x_j,\alpha^{(k)},\beta^{(k)},\gamma^{(k)}]$$
$$+(\alpha-1)\sum_{j=1}^{m}\Phi_j E[\ln(1-\exp\{-\gamma Z_{jk}^{-\beta}\})|Z_{jk}>x_j,\alpha^{(k)},\beta^{(k)},\gamma^{(k)}].$$

The normal equations are given by

$$(2.10)\qquad \frac{n}{\alpha}+\sum_{j=1}^{m}\ln\Big(1-e^{-\gamma x_j^{-\beta}}\Big)+\sum_{j=1}^{m}E_3\Phi_j=0,$$

$$(2.11)\qquad n-(\alpha-1)\beta\gamma\sum_{j=1}^{m}\frac{x_j^{-\beta}e^{-\gamma x_j^{-\beta}}\ln x_j}{1-e^{-\gamma x_j^{-\beta}}}-\beta\bigg(\sum_{j=1}^{m}E_1\Phi_j-\gamma\sum_{j=1}^{m}x_j^{-\beta}\ln x_j+\sum_{j=1}^{m}\ln x_j\bigg)=0,$$

$$(2.12)\qquad \frac{n}{\gamma}+(\alpha-1)\sum_{j=1}^{m}\frac{x_j^{-\beta}e^{-\gamma x_j^{-\beta}}}{1-e^{-\gamma x_j^{-\beta}}}-\sum_{j=1}^{m}x_j^{-\beta}-\sum_{j=1}^{m}E_2\Phi_j=0,$$

where $E_1=E[\ln Z_{jk}|Z_{jk}>x_j,\alpha^{(k)},\beta^{(k)},\gamma^{(k)}]$, $E_2=E[Z_{jk}^{-\beta}|Z_{jk}>x_j,\alpha^{(k)},\beta^{(k)},\gamma^{(k)}]$ and $E_3=E[\ln(1-\exp\{-\gamma Z_{jk}^{-\beta}\})|Z_{jk}>x_j,\alpha^{(k)},\beta^{(k)},\gamma^{(k)}]$. The $(k+1)$-th iteration values of the unknown parameters can be obtained from

$$\alpha^{(k+1)} = -n\left[\sum_{j=1}^{m}\ln\left(1-e^{-\gamma^{(k)}x_j^{-\beta^{(k)}}}\right)+\sum_{j=1}^{m}E_3(x_j;\alpha^{(k)},\beta^{(k)},\gamma^{(k)})\Phi_j\right]^{-1}, \tag{2.13}$$

$$\begin{aligned}\beta^{(k+1)} &= \left(\sum_{j=1}^{m}E_1(x_j,\alpha^{(k+1)},\beta^{(k)},\gamma^{(k)})\Phi_j-\gamma^{(k)}\sum_{j=1}^{m}x_j^{-\beta^{(k)}}\ln x_j+\sum_{j=1}^{m}\ln x_j\right)^{-1}\\ &\quad\times\left(n-(\alpha^{(k+1)}-1)\beta^{(k)}\gamma^{(k)}\sum_{j=1}^{m}\frac{x_j^{-\beta^{(k)}}\exp\{-\gamma^{(k)}x_j^{-\beta^{(k)}}\}\ln x_j}{1-\exp\{-\gamma^{(k)}x_j^{-\beta^{(k)}}\}}\right),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\gamma^{(k+1)} &= n\left(\sum_{j=1}^{m}x_j^{-\beta^{(k+1)}}-(\alpha^{(k+1)}-1)\sum_{j=1}^{m}\frac{x_j^{-\beta^{(k+1)}}\exp\{-\gamma^{(k)}x_j^{-\beta^{(k+1)}}\}}{1-\exp\{-\gamma^{(k)}x_j^{-\beta^{(k+1)}}\}}\right.\\ &\quad\left.+\sum_{j=1}^{m}E_2(x_j,\alpha^{(k+1)},\beta^{(k+1)},\gamma^{(k)})\Phi_j\right)^{-1}.\end{aligned} \tag{2.15}$$

Next, we present the algorithm.

Algorithm-1

Step-1: Set $k=0$. Given the starting value $(\alpha^{(0)},\beta^{(0)},\gamma^{(0)})$, we estimate the parameters α,β and γ.

Step-2: In E-step, let $(\alpha^{(k)},\beta^{(k)},\gamma^{(k)})$ be an estimate of (α,β,γ) at k-th iteration. We compute the required conditional expectations E_1,E_2 and E_3 and then substitute in (2.9).

Step-3: In M-step, we obtain $(\alpha^{(k+1)},\beta^{(k+1)},\gamma^{(k+1)})$ the updated values of the parameters at $(k+1)$-th iteration by solving Equations (2.13 − 2.15).

Step-4: If $|\alpha^{(k+1)}-\alpha^{(k)}|+|\beta^{(k+1)}-\beta^{(k)}|+|\gamma^{(k+1)}-\gamma^{(k)}|\le\epsilon$ for a given $\epsilon>0$ (small tolerance), then we stop the procedure. The latest values will be the MLEs of α,β and γ.

Step-5: If $|\alpha^{(k+1)}-\alpha^{(k)}|+|\beta^{(k+1)}-\beta^{(k)}|+|\gamma^{(k+1)}-\gamma^{(k)}|>\epsilon$, then set $k=k+1$ and go to the Step-1.

Denote the MLEs of α, β and γ by $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$, respectively.

There are various situations, where EM algorithm is difficult to implement due to difficulty in the expectation step. To overcome this, a novel technique has been introduced in the literature called Stochastic EM algorithm. It consists of replacing E-step of the EM algorithm by one iteration of a stochastic approximation procedure. We refer the reader to Nielsen *et al.* [15] for some discussions on this method. The main advantage of StEM algorithm is that it is usually less complicated and gives more appropriate results compared to EM algorithm for many problems (see Tregouet *et al.* [21]). Similar to EM algorithm, the StEM algorithm has two steps: S-step and M-step. In S-step, the missing observations $\boldsymbol{Z}$ are generated from conditional distribution given observed data $\boldsymbol{X}$. We generate Φ_i independent number of censored lifetimes z_{ij} from the condition distribution function $F_{\boldsymbol{Z}|\boldsymbol{X}}(x_j:\alpha,\beta,\gamma)$

for $j = 1, ..., m$, which is given by

$$F_{\boldsymbol{Z}|\boldsymbol{X}}(x_j : \alpha, \beta, \lambda) = \frac{F_{\boldsymbol{Z}}(z_{jk} : \alpha, \beta, \gamma) - F_{\boldsymbol{X}}(x_j : \alpha, \beta, \gamma)}{1 - F_{\boldsymbol{X}}(x_j : \alpha, \beta, \gamma)}. \tag{2.16}$$

The $\boldsymbol{Z}$ is then substituted to (2.7) to form the pseudo log-likelihood function and then this function is optimized in M-step to get $(\alpha^{(k+1)}, \beta^{(k+1)}, \gamma^{(k+1)})$ for the next iteration. These two steps are repeated until a stationary distribution is reached for each parameter. The mean of the stationary distribution is considered as an estimate of the parameters. For brevity, the details are not presented here.

3. OBSERVED FISHER'S INFORMATION MATRIX

In this section, we compute observed Fisher's information matrix, which can be used for construction of the asymptotic confidence intervals. Louis [12] derived Fisher's information matrix using the missing information based on EM algorithm. According to him, the observed information is equal to complete information minus missing information. That is,

$$I_{\boldsymbol{X}}(\alpha, \beta, \gamma) = I_{\boldsymbol{W}}(\alpha, \beta, \gamma) - I_{\boldsymbol{W}|\boldsymbol{X}}(\alpha, \beta, \gamma), \tag{3.1}$$

where $I_{\boldsymbol{X}}(\alpha, \beta, \gamma)$, $I_{\boldsymbol{W}}(\alpha, \beta, \gamma)$ and $I_{\boldsymbol{W}|\boldsymbol{X}}(\alpha, \beta, \gamma)$ are observed, complete and missing informations, respectively. Let $\ell^* = \ell_C(\boldsymbol{w}; \alpha, \beta, \gamma)$ and $a_{kl} = -E[\frac{\partial^2 \ell^*}{\partial \theta_k \partial \theta_l}]$ for $k, l = 1, 2, 3,$ where $\theta_1 = \alpha$, $\theta_2 = \beta$ and $\theta_3 = \gamma$. Then, the complete information matrix $I_{\boldsymbol{W}}(\alpha, \beta, \gamma)$ is given as

$$I_{\boldsymbol{W}}(\alpha, \beta, \gamma) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{3.2}$$

Further, denote $b_{kl} = -\sum_{j=1}^{m} \Phi_j E_{Z_j|X_j}[\frac{\partial^2 \ln f^*}{\partial \theta_k \partial \theta_l}]$ and $f^* = f_{Z_j|X_j}(z_j|x_j, \alpha, \beta, \gamma)$. Thus, the missing information matrix $I_{\boldsymbol{W}|\boldsymbol{X}}(\alpha, \beta, \gamma)$ is

$$I_{\boldsymbol{W}|\boldsymbol{X}}(\alpha, \beta, \gamma) = \sum_{j=1}^{m} \Phi_j I^j_{\boldsymbol{W}|\boldsymbol{X}}(\alpha, \beta, \gamma) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{3.3}$$

where $I^j_{\boldsymbol{W}|\boldsymbol{X}}(\alpha, \beta, \gamma)$ is missing information matrix at the j-th failure time x_j. It is given as

$$I^j_{\boldsymbol{W}|\boldsymbol{X}}(\alpha, \beta, \gamma) = -E_{Z_j|X_j} \begin{pmatrix} \frac{\partial^2 \ln f^*}{\partial \alpha^2} & \frac{\partial^2 \ln f^*}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln f^*}{\partial \alpha \partial \gamma} \\ \frac{\partial^2 \ln f^*}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln f^*}{\partial \beta^2} & \frac{\partial^2 \ln f^*}{\partial \beta \partial \gamma} \\ \frac{\partial^2 \ln f^*}{\partial \gamma \partial \alpha} & \frac{\partial^2 \ln f^*}{\partial \gamma \partial \beta} & \frac{\partial^2 \ln f^*}{\partial \gamma^2} \end{pmatrix}. \tag{3.4}$$

It is worthwhile to mention that the matrices in (3.2) and (3.3) are computed at $(\alpha, \beta, \gamma) = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$. From the 3×3 order matrices given by (3.2) and (3.3), one can easily compute the observed Fisher's information matrix of the model parameters α, β and γ. We obtain the asymptotic variance covariance matrix $(\hat{M})$ for the MLEs of α, β and γ from the inverse of $I_{\boldsymbol{X}}(\alpha, \beta, \gamma)$, which is given by

$$\hat{M} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{cov}(\hat{\alpha}, \hat{\gamma}) \\ \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta}, \hat{\gamma}) \\ \mathrm{cov}(\hat{\alpha}, \hat{\gamma}) & \mathrm{cov}(\hat{\beta}, \hat{\gamma}) & \mathrm{var}(\hat{\gamma}) \end{pmatrix}.$$

The asymptotic confidence intervals of the parameters by using normal approximation (NA) to MLE, and normal approximation of the log-transformed (NL) MLE can be constructed. The derivations are omitted to maintain brevity.

4. BOOTSTRAP CONFIDENCE INTERVALS

In this section, we construct two bootstrap confidence intervals for the parameters. These are the percentile bootstrap (Boot-p) (see Efron and Tibshirani [6]) and the bootstrap-t (Boot-t) (see Hall [8]) methods. The algorithms for these methods are presented below.

Algorithm-2 (Boot-p)

Step-1: From Equations (2.1), (2.2) and (2.3), under the original data sets x_i, $i = 1, ..., m$, we obtain $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$.

Step-2: Based on the values of the estimates of the parameters, generate a bootstrap sample $\boldsymbol{x}^* = (x_1^*, ..., x_m^*)$ for a pre-specified censoring scheme. Then, compute the bootstrap estimates $\hat{\alpha}^*, \hat{\beta}^*$ and $\hat{\gamma}^*$.

Step-3: Repeat Step-2, for $N = 1000$ times to get $(\hat{\alpha}_1^*, ..., \hat{\alpha}_{1000}^*), (\hat{\beta}_1^*, ..., \hat{\beta}_{1000}^*)$ and $(\hat{\gamma}_1^*, ..., \hat{\gamma}_{1000}^*)$.

Step-4: Arrange the values obtained in Step-3 in ascending order and denote $\hat{\alpha}_{(1)}^*, ..., \hat{\alpha}_{(1000)}^*$, $\hat{\beta}_{(1)}^*, ..., \hat{\beta}_{(1000)}^*$ and $\hat{\gamma}_{(1)}^*, ..., \hat{\gamma}_{(1000)}^*$.

Then, for a specified value of σ, the $100(1-\sigma)\%$ Boot-p confidence intervals for α, β and γ are respectively given by

$$\left(\hat{\alpha}_{(N(\frac{\sigma}{2}))}, \hat{\alpha}_{(N(1-\frac{\sigma}{2}))}\right), \ \left(\hat{\beta}_{(N(\frac{\sigma}{2}))}, \hat{\beta}_{(N(1-\frac{\sigma}{2}))}\right) \ \text{and} \ \left(\hat{\gamma}_{(N(\frac{\sigma}{2}))}, \ \hat{\gamma}_{(N(1-\frac{\sigma}{2}))}\right).$$

Algorithm-3 (Boot-t)

Step-1: In analogy to Step-1 and Step-2 as in Boot-p method, obtain bootstrap estimates of the unknown parameters.

Step-2: Compute variance-covariance matrix $I^*(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*)^{-1}$. Write

$$T_{\alpha_i}^* = \frac{\hat{\alpha}_i^* - \hat{\alpha}_i}{\sqrt{\widehat{\mathrm{Var}}(\hat{\alpha}_i^*)}}, \quad T_{\beta_i}^* = \frac{\hat{\beta}_i^* - \hat{\beta}_i}{\sqrt{\widehat{\mathrm{Var}}(\hat{\beta}_i^*)}} \quad \text{and} \quad T_{\gamma_i}^* = \frac{\hat{\gamma}_i^* - \hat{\gamma}_i}{\sqrt{\widehat{\mathrm{Var}}(\hat{\gamma}_i^*)}}$$

for $i = 1, ..., 1000$.

Step-3: Repeat Step-1 and Step-2, $N = 1000$ times and arrange the values in ascending order. Denote

$$T_{\alpha_{(1)}}^*, ..., T_{\alpha_{(1000)}}^*, \quad T_{\beta_{(1)}}^*, ..., T_{\beta_{(1000)}}^* \quad \text{and} \quad T_{\gamma_{(1)}}^*, ..., T_{\gamma_{(1000)}}^*.$$

Thus, for a given σ, the $100(1-\sigma)\%$ Boot-t confidence intervals for α, β and γ are respectively obtained as

$$\left(\hat{T}_{\alpha_{(N(\frac{\sigma}{2}))}}, \hat{T}_{\alpha_{(N(1-\frac{\sigma}{2}))}}\right), \quad \left(\hat{T}_{\beta_{(N(\frac{\sigma}{2}))}}, \hat{T}_{\beta_{(N(1-\frac{\sigma}{2}))}}\right) \quad \text{and} \quad \left(\hat{T}_{\gamma_{(N(\frac{\sigma}{2}))}}, \hat{T}_{\gamma_{(N(1-\frac{\sigma}{2}))}}\right).$$

5. BAYESIAN ESTIMATION

In this section, we focus on obtaining Bayes estimates of α, β and γ with respect to entropy loss function. Let δ be an estimator for the unknown parameter θ. The entropy loss function (ELF) is

$$L_{\mathrm{e}}(\theta,\delta) = \left(\frac{\delta}{\theta}\right)^q - q\ln\left(\frac{\delta}{\theta}\right) - 1, \quad q \neq 0. \tag{5.1}$$

This loss function is asymmetric in nature. The constant q in (5.1) stands for the magnitude and degree of symmetry. The overestimation is dangerous than the underestimation for positive values of q. When q is negative, underestimation is dangerous than the overestimation. The Bayes estimate of θ with respect to this loss function can be obtained using the following tool:

$$\hat{\theta}_{\mathrm{be}} = \left[E_\theta\left(\theta^{-q}|\,\boldsymbol{x}\right)\right]^{-\frac{1}{q}}, \quad q \neq 0. \tag{5.2}$$

Note that the Bayes estimate of the parameter θ under ELF reduces to the Bayes estimates with respect to the squared error loss function (SELF) when $q = -1$. For $q = -2$ and 1, it becomes Bayes estimates under the precautionary loss function (PLF) and weighted squared error loss function (WSELF). Prior distributions play an essential role for derivation of the Bayes estimators. There is no clear method on choosing priors for a particular problem. We refer to Arnold and Press [1] for more details on this. Here, we consider independent gamma prior density functions for the parameters α, β and γ as

$$g_1(\alpha : c_1, c_2) \propto \alpha^{c_1-1}\exp\{-\alpha c_2\}, \tag{5.3}$$

$$g_2(\beta : c_3, c_4) \propto \beta^{c_3-1}\exp\{-\beta c_4\}, \tag{5.4}$$

$$g_3(\gamma : c_5, c_6) \propto \gamma^{c_5-1}\exp\{-\gamma c_6\}, \tag{5.5}$$

where $\alpha, \beta, \gamma > 0$ and $c_i > 0$; $i = 1, 2, 3, 4, 5, 6$. The constants c_i's are known as the hyperparameters. The joint prior distribution of α, β and γ is given by

$$\pi(\alpha,\beta,\gamma) \propto \alpha^{c_1-1}\beta^{c_3-1}\gamma^{c_5-1}\exp\{-(\alpha c_2 + \beta c_4 + \gamma c_6)\}. \tag{5.6}$$

Further, the joint distribution of α, β, γ and $\boldsymbol{X}$ is

$$\begin{aligned}\pi_1(\alpha,\beta,\gamma,\boldsymbol{x}) \propto\; & \alpha^{m+c_1-1}\beta^{m+c_3-1}\gamma^{m+c_5-1}\exp\{-(\alpha c_2 + \beta c_4 + \gamma c_6)\} \\ & \times \prod_{i=1}^{m} x_i^{-(\beta+1)}\exp\{-\gamma x_i^{-\beta}\}(1-\exp\{-\gamma x_i^{-\beta}\})^{\Phi_i+\alpha-1}.\end{aligned} \tag{5.7}$$

The posterior distribution of α, β and γ given $\boldsymbol{X} = \boldsymbol{x}$ is obtained as

$$\begin{aligned}\Pi(\alpha,\beta,\gamma|\boldsymbol{x}) =\; & \frac{1}{k}\alpha^{m+c_1-1}\beta^{m+c_3-1}\gamma^{m+c_5-1}\exp\{-(\alpha c_2 + \beta c_4 + \gamma c_6)\} \\ & \times \prod_{i=1}^{m} x_i^{-(\beta+1)}\exp\{-\gamma x_i^{-\beta}\}(1-\exp\{-\gamma x_i^{-\beta}\})^{\Phi_i+\alpha-1},\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}k =\; & \int_{\alpha=0}^{\infty}\int_{\beta=0}^{\infty}\int_{\gamma=0}^{\infty} \alpha^{m+c_1-1}\beta^{m+c_3-1}\gamma^{m+c_5-1}\exp\{-(\alpha c_2 + \beta c_4 + \gamma c_6)\} \\ & \times \prod_{i=1}^{m} x_i^{-(\beta+1)}\exp\{-\gamma x_i^{-\beta}\}(1-\exp\{-\gamma x_i^{-\beta}\})^{\Phi_i+\alpha-1}\,d\alpha\,d\beta\,d\gamma.\end{aligned} \tag{5.9}$$

Thus, from Equation (5.2), the Bayes estimate of α with respect to the entropy loss function is obtained as

$$(5.10)\quad \hat{\alpha}_{\text{be}} = \left[\frac{1}{k}\int_{\alpha=0}^{\infty}\int_{\beta=0}^{\infty}\int_{\gamma=0}^{\infty}\alpha^{m+c_1-q-1}\beta^{m+c_3-1}\gamma^{m+c_5-1}\exp\{-(\alpha c_2+\beta c_4+\gamma c_6)\}\right.$$
$$\left.\times\prod_{i=1}^{m}\exp\{-\gamma x_i^{-\beta}\}(1-\exp\{-\gamma x_i^{-\beta}\})^{\Phi_i+\alpha-1}x_i^{-(\beta+1)}\,d\alpha\,d\beta\,d\gamma\right]^{-\frac{1}{q}},\quad q\neq 0.$$

Similarly, the Bayes estimates of β and γ with respect to the entropy loss function can be obtained. We omit these for the sake of conciseness. Below, we discuss how to compute Bayes estimates using some well known techniques.

6. COMPUTING METHODS FOR BAYESIAN ESTIMATION

In the previous section, we have seen that the desired Bayes estimates can not be obtained in explicit forms. So, we consider approximation methods in this section. First, we explain Lindley's method (see Lindley [11]).

6.1. Lindley's approximation method

Let θ_1, θ_2 and θ_3 be the unknown parameters of a statistical model and $u(\boldsymbol{\theta})$ be a function of the parameters, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$. It is known that the Bayes estimate of $u(\boldsymbol{\theta})$ is evaluated in terms of expectation, where the expectation is taken with respect to posterior distribution. Let $l(\boldsymbol{\theta}|\boldsymbol{x})$ denote the log-likelihood function and $\rho(\boldsymbol{\theta})$ is the logarithm of the joint prior distribution of θ_1, θ_2 and θ_3. From the Lindley's approximation technique, we obtain (*see* Lindley [11])

$$(6.1)\quad \hat{\delta}_{\text{be}}(\boldsymbol{x}) \approx u(\hat{\boldsymbol{\theta}}) + W(\hat{\boldsymbol{\theta}}) + \rho_1(\hat{\boldsymbol{\theta}})W_{123} + \rho_2(\hat{\boldsymbol{\theta}})W_{213} + \rho_3(\hat{\boldsymbol{\theta}})W_{321}$$
$$+ 0.5\Big[\ell^*_{300}V_{123} + \ell^*_{030}V_{213} + \ell^*_{003}V_{321} + 2\ell^*_{111}(E_{123}+E_{213}+E_{312})$$
$$+ \ell^*_{210}C_{123} + \ell^*_{201}C_{132} + \ell^*_{120}C_{213} + \ell^*_{102}C_{312} + \ell^*_{021}C_{231} + \ell^*_{012}C_{321}\Big],$$

where $W(\hat{\boldsymbol{\theta}}) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}u_{ij}(\hat{\boldsymbol{\theta}})\tau_{ij}(\hat{\boldsymbol{\theta}})$, $\hat{\boldsymbol{\theta}}$ is the MLE of $\boldsymbol{\theta}$, $\ell^*_{ijk} = \frac{\partial^3\ell(\boldsymbol{\theta}|\boldsymbol{x})}{\partial\theta_i\partial\theta_j\partial\theta_k}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$ with $i,j,k = 0,1,2,3$ such that $i+j+k=3$, τ_{ij} is the (i,j)-th element in the inverse matrix of $[-\frac{\partial^2\ell(\boldsymbol{\theta}|\boldsymbol{x})}{\partial\theta_i\partial\theta_j}]|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$. Other unknown terms of (6.1) are given as

$$W_{ijk} = u_i\tau_{ii}(\hat{\boldsymbol{\theta}}) + u_j\tau_{ji}(\hat{\boldsymbol{\theta}}) + u_k\tau_{ki}(\hat{\boldsymbol{\theta}}),$$
$$V_{ijk} = \tau_{ii}(\hat{\boldsymbol{\theta}})(u_i\tau_{ii}(\hat{\boldsymbol{\theta}}) + u_j\tau_{ij}(\hat{\boldsymbol{\theta}}) + u_k\tau_{ik}(\hat{\boldsymbol{\theta}})),$$
$$E_{ijk} = u_i(\tau_{ii}(\hat{\boldsymbol{\theta}})\tau_{jk}(\hat{\boldsymbol{\theta}}) + 2\tau_{ij}(\hat{\boldsymbol{\theta}})\tau_{ik}(\hat{\boldsymbol{\theta}}))$$

and

$$C_{ijk} = 3u_i\tau_{ii}(\hat{\boldsymbol{\theta}})\tau_{ij}(\hat{\boldsymbol{\theta}}) + u_j(\tau_{ii}(\hat{\boldsymbol{\theta}})\tau_{jj}(\hat{\boldsymbol{\theta}}) + 2\tau_{ij}^2(\hat{\boldsymbol{\theta}})) + u_k(\tau_{ii}(\hat{\boldsymbol{\theta}})\tau_{jk}(\hat{\boldsymbol{\theta}}) + 2\tau_{ij}(\hat{\boldsymbol{\theta}})\tau_{ik}(\hat{\boldsymbol{\theta}})).$$

Further, $u_{ij}(\hat{\boldsymbol{\theta}}) = \frac{\partial^2 u(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$, $u_i(\hat{\boldsymbol{\theta}}) = \frac{\partial u(\boldsymbol{\theta})}{\partial\theta_i}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$, $\rho_i(\boldsymbol{\theta}) = \frac{\partial \rho(\boldsymbol{\theta})}{\partial\theta_i}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$, and $\rho(\boldsymbol{\theta})$ is equal to the logarithmic of the joint prior distribution of θ_1, θ_2 and θ_3, where $i, j, k = 1, 2, 3$. Now, we provide approximate Bayes estimate for the unknown parameter α with respect to the entropy loss function. In order to write the Bayes estimate of α with respect to the entropy loss function, we have $u(\alpha, \beta, \gamma) = \alpha^{-q}$, $u_1 = -q\alpha^{-(q+1)}$, $u_{11} = q(q+1)\alpha^{-(q+2)}$ and $u_2 = u_3 = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$. Thus, from (6.1), the approximate Bayes estimate of α with respect to the entropy loss function is obtained as

$$(6.2)\qquad \hat{\alpha}_{\text{be}}^{LI} = \Big[\alpha^{-q} + 0.5\Big[q(q+1)\alpha^{-(q+2)}\tau_{11} - q\alpha^{-(q+1)}\Big\{\ell^*_{300}\tau_{11}^2 + \ell^*_{030}\tau_{21}\tau_{22} + \ell^*_{003}\tau_{31}\tau_{33} + 2\ell^*_{111}(\tau_{11}\tau_{23} + 2\tau_{13}\tau_{12}) + \ell^*_{120}(\tau_{11}\tau_{22} + 2\tau_{21}^2) + \ell^*_{102}(\tau_{33}\tau_{11} + 2\tau_{31}^2) + \ell^*_{021}(\tau_{22}\tau_{31} + 2\tau_{23}\tau_{21}) + \ell^*_{012}(\tau_{33}\tau_{21} + 2\tau_{32}\tau_{31}) + 2\rho_1\tau_{11} + 2\rho_2\tau_{12} + 2\rho_3\tau_{13}\Big\}\Big]\Big]^{-\frac{1}{q}}\Bigg|_{(\alpha,\beta,\gamma)=(\hat{\alpha},\hat{\beta},\hat{\gamma})}.$$

Similarly, we can obtain the Bayes estimates of β and γ with respect to the entropy loss function. The expressions are omitted here.

6.2. Importance sampling method

In the subsection, we consider another approximation technique known as importance sampling method to obtain the Bayes estimates for the parameters. We rewrite the posterior distribution of α, β and γ given in (5.8) as

$$(6.3)\qquad \Pi(\alpha, \beta, \gamma|\boldsymbol{x}) \propto G_\beta\Big(m + c_3, c_4 + \sum_{i=1}^m \ln x_i\Big).G_{\gamma|\beta}\Big(m + c_5, c_6 + \sum_{i=1}^m x_i^{-\beta}\Big) \times G_{\alpha|\beta,\gamma}\Big(m + c_1, c_2 - \sum_{i=1}^m \ln(1 - \exp\{-\gamma x_i^{-\beta}\})\Big).\psi(\alpha, \beta, \gamma),$$

where

$$(6.4)\qquad \psi(\alpha, \beta, \gamma) = \frac{(c_4 + \sum_{i=1}^m \ln x_i)^{-(m+c_3)}}{(c_2 - \sum_{i=1}^m \ln(1 - \exp\{-\gamma x_i^{-\beta}\}))^{(m+c_1)}} \times \frac{\exp\{\sum_{i=1}^m (\Phi_i + 1)\ln(1 - \exp\{-\gamma x_i^{-\beta}\})\}}{(c_6 + \sum_{i=1}^m x_i^{-\beta})^{m+c_5}\exp\{\sum_{i=1}^m \ln x_i\}}.$$

Below, we present the steps which will be used for the implementation of importance sampling technique.

Algorithm-4

Step-1: Generate β from $G_\beta(m + c_3, c_4 + \sum_{i=1}^m \ln x_i)$, that is, from a gamma distribution with shape parameter $(m + c_3)$ and scale parameter $(c_4 + \sum_{i=1}^m \ln x_i)$).

Step-2: For a given β as obtained in Step-1, we generate γ from $G_{\gamma|\beta}(m + c_5,\ c_6 + \sum_{i=1}^m x_i^{-\beta})$.

Step-3: For β and γ as generated in Step-1 and Step-2, we will generate parameter α from $G_{\alpha|\beta,\gamma}(m + c_1, c_2 - \sum_{i=1}^{m} \ln(1 - \exp\{-\gamma x_i^{-\beta}\}))$.

Step-4: Repeat Steps-1,2 and 3, $N = 1000$ times to obtain $(\alpha_1, \beta_1, \gamma_1), ..., (\alpha_N, \beta_N, \gamma_N)$.

Finally, the Bayes estimate of a parametric function $g(\alpha, \beta, \gamma)$ with respect to entropy loss function is obtained as

$$\hat{g}_{\text{be}}^{IS}(\alpha, \beta, \gamma) = \left[\frac{\sum_{i=1}^{N} g(\alpha_i, \beta_i, \gamma_i)^{-q} \psi(\alpha_i, \beta_i, \gamma_i)}{\sum_{i=1}^{N} \psi(\alpha_i, \beta_i, \gamma_i)} \right]^{-\frac{1}{q}}. \tag{6.5}$$

Substituting $q = -1, 1$ and $q = -2$ in the above expression, we obtain Bayes estimates with respect to the SELF, WSELF and PLF, respectively. Further, to get the Bayes estimates of α, β and γ, one needs to respectively replace α, β and γ in place of $g(\alpha, \beta, \gamma)$ in (6.5).

6.3. Metropolis-Hastings algorithm

In this subsection, we use an alternative method to get Bayes estimates of α, β and γ using Gibbs sampling method and Metropolis-Hastings algorithm. The MH algorithm is also used for the construction of credible intervals. After analysing the posterior distribution given by (5.8), the marginal posterior distribution of α given β, γ and $\boldsymbol{x}$ is obtained as

$$\Pi_1(\alpha|\beta, \gamma, \boldsymbol{x}) \propto G\left(m + c_1, \left(c_1 - \sum_{i=1}^{m} \ln(1 - \exp\{-\gamma x_i^{-\beta}\}) \right) \right). \tag{6.6}$$

Similarly, the marginal posterior distributions of β given α, γ and $\boldsymbol{x}$; and γ given α, β and $\boldsymbol{x}$ can be obtained as

$$\Pi_2(\beta|\alpha, \gamma, \boldsymbol{x}) \propto \beta^{m+c_3-1} \exp\{-\beta c_4\} \prod_{i=1}^{m} \frac{x_i^{-\beta} \exp\{-\gamma x_i^{-\beta}\}}{(1 - \exp\{-\gamma x_i^{-\beta}\})} \tag{6.7}$$

and

$$\Pi_3(\gamma|\alpha, \beta, \boldsymbol{x}) \propto \gamma^{m+c_5-1} \exp\{-\gamma c_6\} \prod_{i=1}^{m} \frac{\exp\{-\gamma x_i^{-\beta}\}}{(1 - \exp\{-\gamma x_i^{-\beta}\})}, \tag{6.8}$$

respectively. Note that the marginal posterior distribution in (6.6) is gamma distribution. But, other two marginal posterior distributions in (6.7) and (6.8) do not follow any know models. Thus, one has to generate random samples for β and γ from the normal proposal distribution. The following algorithm is useful for the generation of the posterior samples.

Algorithm-5

Step-1: Set an initial value $(\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)})$ and set $j = 1$.

Step-2: Generate β^* and γ^* from the proposal distributions $N(\beta^{(j-1)}, \text{var}(\beta))$ and $N(\gamma^{(j-1)}, \text{var}(\gamma))$, respectively. Then, generate α^* from $G(m + c_1, (c_1 - \sum_{i=1}^{m} \ln(1 - \exp\{-\gamma^{(j-1)} x_i^{-\beta^{(j-1)}}\})))$.

Step-3: Compute

(6.9)
$$\omega_\beta = \min\left\{1, \frac{\Pi_2(\beta^*|\alpha^{(j)}, \gamma^{(j)}, \boldsymbol{x})}{\Pi_2(\beta^{(j-1)}|\alpha^{(j-1)}, \gamma^{(j-1)}, \boldsymbol{x})}\right\}$$
$$\text{and} \quad \omega_\gamma = \min\left\{1, \frac{\Pi_3(\gamma^*|\alpha^{(j)}, \beta^{(j)}, \boldsymbol{x})}{\Pi_3(\gamma^{(j-1)}|\alpha^{(j-1)}, \beta^{(j-1)}, \boldsymbol{x})}\right\}.$$

Step-4: Generate samples u_2 and u_3 from uniform distribution $U(0, 1)$.

Step-5: If $u_2 \le \omega_\beta$ and $u_3 \le \omega_\gamma$ then $\beta^{(j)} \longleftarrow \beta^*$, else $\beta^{(j)} \longleftarrow \beta^{(j-1)}$ and $\gamma^{(j)} \longleftarrow \gamma^*$, else $\gamma^{(j)} \longleftarrow \gamma^{(j-1)}$, respectively. Further, set $j = j + 1$.

Step-6: Repeat Steps $(3 - 5)$, $N = 1000$ times to obtain MCMC samples. These are denoted as $(\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}), ..., (\alpha^{(N)}, \beta^{(N)}, \gamma^{(N)})$.

Now, the Bayes estimate of α with respect to entropy loss function based on MCMC samples is given by

(6.10)
$$\hat{\alpha}_{be}^{MH} = \left[\frac{1}{N}\sum_{j=1}^{N}\left(\alpha^{(j)}\right)^{-q}\right]^{-\frac{1}{q}}.$$

Similarly, the Bayes estimates of β and γ under entropy loss function can be obtained. Next, we compute HPD credible intervals of α, β and γ by using the method due to Chen and Shao [3]. Here, we use MH algorithm to generate samples from the posterior density. After that we arrange $\hat{\alpha}^{(j)}, \hat{\beta}^{(j)}$ and $\hat{\gamma}^{(j)}$ in ascending order, and denote $\hat{\alpha}^{(1)}, ..., \hat{\alpha}^{(N)}$, $\hat{\beta}^{(1)}, ..., \hat{\beta}^{(N)}$ and $\hat{\gamma}^{(1)}, ..., \hat{\gamma}^{(N)}$, respectively. Thus, the $100(1 - \sigma)\%$ credible intervals for α, β and γ are respectively given by

$$\left(\hat{\alpha}^{\left(N\left(\frac{\sigma}{2}\right)\right)}, \ \hat{\alpha}^{\left(N\left(1-\frac{\sigma}{2}\right)\right)}\right), \quad \left(\hat{\beta}^{\left(N\left(\frac{\sigma}{2}\right)\right)}, \ \hat{\beta}^{\left(N\left(1-\frac{\sigma}{2}\right)\right)}\right) \quad \text{and} \quad \left(\hat{\gamma}^{\left(N\left(\frac{\sigma}{2}\right)\right)}, \ \hat{\gamma}^{\left(N\left(1-\frac{\sigma}{2}\right)\right)}\right).$$

6.4. Computation of hyper-parameters

Here, we briefly discuss the procedure how to calculate the hyper-parameters when informative priors are known to us. The hyper-parameters are c_1, c_2, c_3, c_4, c_5 and c_6. These are obtained from gamma prior distributions as given in Section 5. Suppose r samples are available from the EGT-II distribution. The MLEs of the parameters α, β and γ are $\hat{\alpha}^j, \hat{\beta}^j$ and $\hat{\gamma}^j$ for $j = 1, ..., r$, respectively for each of these r number of samples. Note that these hyper-parameter values are evaluated from the past data set. First, we calculate hyper-parameters c_1 and c_2. The mean and variance of the gamma prior of α are c_1/c_2 and c_1/c_2^2, respectively. Further, the mean and variance of the MLEs of α for r samples are $\frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j$ and $\frac{1}{r-1}\sum_{j=1}^{r}(\hat{\alpha}^j - \frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j)^2$, respectively. Therefore, $\frac{c_1}{c_2} = \frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j$ and $\frac{c_1}{c_2^2} = \frac{1}{r-1}\sum_{j=1}^{r}(\hat{\alpha}^j - \frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j)^2$. Solving these equations, we get

(6.11)
$$c_1 = \frac{(\frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j)^2}{\frac{1}{r-1}\sum_{j=1}^{r}(\hat{\alpha}^j - \frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j)^2}$$
$$\text{and} \quad c_2 = \frac{\frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j}{\frac{1}{r-1}\sum_{j=1}^{r}(\hat{\alpha}^j - \frac{1}{r}\sum_{j=1}^{r}\hat{\alpha}^j)^2}.$$

Similarly, other hyper-parameters c_3, c_4 and c_5, c_6 can be obtained from (6.11) replacing $\hat{\beta}^j$ and $\hat{\gamma}^j$ in place of $\hat{\alpha}^j$, respectively.

7. BAYESIAN PREDICTION AND INTERVAL ESTIMATION

In this section, we discuss Bayesian prediction for the future observations depending upon the PT-II censored sample. We assume that the sample is taken from the EGT-II distribution. We also compute the corresponding prediction intervals. Many authors have studied prediction problems related to Bayesian prediction and interval estimation. We refer to Bdair *et al.* [2] and Maiti and Kayal [13] for some references. We illustrate the one-sample prediction problem. Suppose n independent life testing units are put in an experiment. Let $\boldsymbol{x} = (x_1, ..., x_m)$ be the observed PT-II censored sample. Further, assume that the censoring scheme is taken as $\boldsymbol{\Phi} = (\Phi_1, ..., \Phi_m)$. Let $y_i = (y_{i1}, ..., y_{i\Phi_i})$ represent the ordered lifetimes of the units which are censored at the i-th failure x_i. Our goal is to predict the future observations based on $\boldsymbol{x}$. We assume that these are $y = (y_{ip}\ ; i = 1, ..., m; p = 1, ..., \Phi_i)$. The conditional density of y under the given information can be obtained as

$$(7.1)\qquad \begin{aligned} f_1(y|\boldsymbol{x}, \alpha, \beta, \gamma) \;=\; & \alpha\beta\gamma p\binom{\Phi_i}{p}\sum_{k=0}^{p-1}(-1)^{p-k-1}\binom{p-1}{k}y^{-(\beta+1)} \\ & \times \exp\{-\gamma y^{-\beta}\}(1-\exp\{-\gamma y^{-\beta}\})^{\alpha(\Phi_i-k)-1} \\ & \times(1-\exp\{-\gamma x_i^{-\beta}\})^{\alpha(k-\Phi_i)}, \qquad y > x_i. \end{aligned}$$

The conditional distribution function is

$$(7.2)\qquad \begin{aligned} F_1(y|\boldsymbol{x}, \alpha, \beta, \gamma) \;=\; & p\binom{\Phi_i}{p}\sum_{k=0}^{p-1}\frac{(-1)^{p-k-1}}{\Phi_i - k}\binom{p-1}{k} \\ & \times\left[1-(1-\exp\{-\gamma x_i^{-\beta}\})^{\alpha(k-\Phi_i)}(1-\exp\{-\gamma y^{-\beta}\})^{\alpha(\Phi_i-k)}\right]. \end{aligned}$$

Notice that the posterior predictive density and the distribution functions are respectively given by

$$(7.3)\qquad f_1^*(y|\boldsymbol{x}) = \int_0^\infty\int_0^\infty\int_0^\infty f_1(y|\boldsymbol{x}, \alpha, \beta, \gamma)\Pi(\alpha, \beta, \gamma|\boldsymbol{x})\ d\alpha\, d\beta\, d\gamma$$

and

$$(7.4)\qquad F_1^*(y|\boldsymbol{x}) = \int_0^\infty\int_0^\infty\int_0^\infty F_1(y|\boldsymbol{x}, \alpha, \beta, \gamma)\Pi(\alpha, \beta, \gamma|\boldsymbol{x})\ d\alpha\, d\beta\, d\gamma.$$

Thus, the Bayesian predictive estimate of y under the entropy loss function is obtained as

$$\hat{y}_{\text{be}} \;=\; [E(P_1(\alpha, \beta, \gamma)|\boldsymbol{x})]^{-\frac{1}{q}},$$

where

$$P_1(\alpha, \beta, \gamma) = \int_{x_i}^\infty y^{-q}f_1(y|\boldsymbol{x}, \alpha, \beta, \gamma)\,dy.$$

Note that the above integrals can not be evaluated analytically. Therefore, we have to adopt numerical technique for the computation of the predictive estimates. In this purpose, we use importance sampling method which is mentioned in Subsection 6.2. Equation (7.5) can be computed using the importance sampling method as

$$(7.5)\qquad \hat{y}_{\text{be}}^{BP} \;=\; \left[\frac{\sum_{i=1}^{1000}P_1(\alpha_i, \beta_i, \gamma_i)\psi(\alpha_i, \beta_i, \gamma_i)}{\sum_{i=1}^{1000}\psi(\alpha_i, \beta_i, \gamma_i)}\right]^{-1/q}.$$

Now, we obtain the Bayesian predictive interval (BPI). The prior predictive survival function is obtained as

$$S_1(t|\boldsymbol{x},\alpha,\beta,\gamma) = \frac{P(y>t|\boldsymbol{x},\alpha,\beta,\gamma)}{P(y>x_i|\boldsymbol{x},\alpha,\beta,\gamma)} = \frac{\int_t^\infty f_1(u|\boldsymbol{x},\alpha,\beta,\gamma)\,du}{\int_{x_i}^\infty f_1(u|\boldsymbol{x},\alpha,\beta,\gamma)\,du}.$$

The posterior survival function is

$$S_1^*(t|\boldsymbol{x}) = \int_0^\infty\int_0^\infty\int_0^\infty S_1(t|\boldsymbol{x},\alpha,\beta,\gamma)\Pi(\alpha,\beta,\gamma|\boldsymbol{x})\; d\alpha\, d\beta\, d\gamma. \tag{7.6}$$

Using (7.6), we obtain two-sided $100(1-\sigma)\%$ equal-tail symmetric predictive interval (L,U) by solving the following non-linear equations

$$S_1^*(L|\boldsymbol{x}) = 1-\frac{\sigma}{2} \quad \text{and} \quad S_1^*(U|\boldsymbol{x}) = \frac{\sigma}{2}. \tag{7.7}$$

For the algorithm to obtain L and U from the above equations, we refer to Singh and Tripathi [19].

8. REAL DATA ANALYSIS

In this section, we analyze a real life data set to illustrate our established results. We consider real life data set representing the window strength in a life test. The data set was provided by Ed Fuller of the NICT Ceramics Division in December 1993. It contains polished window strength data. The data set was introduced by Pepi [16]. The data set is presented below:

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52
25.8	26.69	26.77	26.78	27.05	27.67	29.9	31.11	33.2
33.73	33.76	33.89	34.76	35.75	35.91	36.98	37.08	37.09
39.58	44.045	45.29	45.381					

For the purpose of goodness of fit test, we consider various methods such as Bayesian information criterion (BIC), Akaikes-information criterion (AIC), the associated second-order information criterion (AICc), negative log-likelihood criterion and Kolmogorov-Smirnov (KS) statistic. Five distributions such as exponential (Exp), half-logistic (HL), inverse Weibull (InWE), Weibull (WE) and EGT-II distributions. The values of the MLEs and the five goodness of fit test statistics are presented in Table 1. It is observed that the values of test statistics corresponding to the EGT-II distribution are smaller comparing to the other distributions. Thus, it can be assumed that the given data set follows EGT-II distribution.

Next, we consider the PT-II censoring sample and two different censoring schemes (CS) as CS-I and CS-II with the failure sample size $m = 20$ in Table 2. The CS-I is progressive type-II censoring and CS-II is conventional type-II censoring schemes.

In Table 3, we present the values of the proposed estimates of α, β and γ for different censoring schemes. Note that CS-III represents for the case of the complete sample. We assume $c_1 = 2, c_2 = c_3 = 4, c_4 = 3, c_5 = 2$ and $c_6 = 4$ while computing the Bayes estimates.

Table 1: The MLE, BIC, AICc, AIC, negative log-likelihood and KS values for the real data set.

Method	Parameter		Exp	HL	InWE	WE	EGT-II
MLE	Shape	α					55.68475
		β			17.18068	4.63630	1.11743
	Scale	γ	0.03247	0.04961	0.58803	33.67241	198.992
BIC			277.9629	266.8936	260.3572	218.8458	218.6415
AICc			276.6668	265.5976	257.9178	215.4064	215.2285
AIC			276.5289	265.4596	257.4892	214.9779	214.3396
−InL			137.2645	131.7298	126.7446	105.4889	104.1698
KS			0.45878	0.44230	0.47472	0.15257	0.13645

Table 2: PT-II censored data and censoring schemes for the real data set.

(n, m)									
(31, 20)	x_i		18.83	20.80	21.657	24.05	24.321	25.8	26.78
			27.05	27.67	29.9	33.73	33.89	34.76	35.75
			35.91	36.98	37.08	37.09	39.58	45.381	
	Φ_i	(CS-I)	2	0	0	0	0	0	0
			0	0	2	2	0	0	0
			0	5	0	0	0	0	
		(CS-II)	0	0	0	0	0	0	0
			0	0	0	0	0	0	0
			0	0	0	0	0	11	

The Bayes estimates with respect to the ELF are computed for two distinct values of q, say -0.5 and 0.5, which are denoted by $\hat{(\cdot)}^{\text{EN}}_{\text{be}}|-0.5$ and $\hat{(\cdot)}^{\text{EN}}_{\text{be}}|0.5$, respectively. Further, under the squared error, weighted squared error and precautionary loss functions, the Bayes estimates are presented, which are respectively denoted by $\hat{(\cdot)}^{\text{SE}}_{\text{be}}, \hat{(\cdot)}^{\text{WS}}_{\text{be}}$ and $\hat{(\cdot)}^{\text{PL}}_{\text{be}}$. We use $\hat{(\cdot)}_{\text{EM}}$ and $\hat{(\cdot)}_{\text{StEM}}$ for the MLEs by the EM and StEM algorithms, respectively. The sixth column presents three different methods such as Lindley's approximation (LI), importance sampling (IS) and Metropolies-Hastings algorithm. In Table 4, the 95% various confidence and credible intervals for α, β and γ are presented. These are the asymptotic (asy) confidence intervals based on the NA to MLE and NL, the bootstrap (t and p) confidence intervals and the HPD credible intervals. Table 5 reports one-sample predictive observations and 95% predictive interval estimates of the lifetime of first two units at i-th failure. The following points can be pointed out from Tables 3, 4 and 5:

- From Table 3, we notice that the estimated values of the parameters obtained based on MH algorithm are smaller compared to that obtained using LI and IS methods. The Lindley's method provides largest Bayes estimates with respect to WSELF. For PLF, we get largest Bayes estimates when IS method is used. Under the ELF with $q = -0.5$, MH method yields largest estimates. It is also observed that the estimated values for $q = 0.5$ are always smaller than that for $q = -0.5$.

Table 3: Estimates of the parameters α, β and γ for the real data set.

(n,m) Schemes		$\hat{(\cdot)}_{\text{EM}}$ $\hat{(\cdot)}_{\text{StEM}}$		$\hat{(\cdot)}_{\text{be}}^{\text{SE}}$	$\hat{(\cdot)}_{\text{be}}^{\text{WS}}$	$\hat{(\cdot)}_{\text{be}}^{\text{PL}}$	$\hat{(\cdot)}_{\text{be}}^{\text{EN}}\|-0.5$	$\hat{(\cdot)}_{\text{be}}^{\text{EN}}\|0.5$
(31, 20) (CS-I)	α	92.76551	LI	91.16425	92.34685	91.94697	91.46521	91.13645
		92.64304	MH	90.26894	90.13469	91.23567	92.2641	91.56770
			IS	93.10641	94.34077	96.11584	92.36499	91.82408
	β	0.96694	LI	0.94365	0.99643	0.94895	0.93482	0.91142
		0.99004	MH	0.90876	0.90397	0.93499	0.96315	0.94315
			IS	0.92465	0.95157	0.95708	0.93145	0.91349
	γ	147.2729	LI	144.26052	146.48510	143.05582	143.89825	143.63215
		147.5308	MH	141.68496	140.99953	142.53064	143.10289	142.94891
			IS	144.70046	147.83406	148.31648	144.32213	141.70526
(31, 20) (CS-II)	α	29.74175	LI	28.16496	29.03008	27.69961	27.16431	26.92482
		29.31584	MH	26.64515	26.31584	27.00948	27.67598	27.20806
			IS	28.16845	31.06412	31.47601	28.10648	26.16004
	β	0.78636	LI	0.73168	0.765461	0.74524	0.72886	0.71145
		0.79857	MH	0.68065	0.641328	0.69546	0.70094	0.69088
			IS	0.71094	0.71643	0.71948	0.68315	0.66081
	γ	66.33405	LI	65.10594	65.84694	63.40869	64.28256	63.81425
		66.94850	MH	61.16764	60.46131	63.16512	63.84247	61.23548
			IS	63.16185	67.99107	68.57093	66.91354	64.09728
(31, 31) (CS-III)	α	55.68475	LI	54.06889	54.76121	53.16428	53.81254	53.46942
		55.40823	MH	51.68434	51.20809	53.46849	55.64813	51.65741
			IS	53.64794	54.58215	54.61348	52.16310	50.82622
	β	1.11743	LI	1.10144	1.12465	1.10412	1.09526	1.09034
		1.13526	MH	1.07164	1.04316	1.08797	1.08964	1.05152
			IS	1.11364	1.15049	1.15942	1.11310	1.10034
	γ	198.992	LI	194.56894	197.16421	196.46852	192.84542	192.08806
		199.35641	MH	191.47964	191.08871	194.94568	195.81774	192.67880
			IS	195.39486	196.03513	196.37460	193.81345	191.10130

- Table 4 shows that among the asymptotic intervals, estimates obtained via NA method performs better than that obtained using NL method. Here, performance has been measured in terms of the length. In boot type intervals, Boot-t provides better confidence interval estimates than Boot-p method. Considering all the five methods together, it is observed that the HPD method outperforms others. Further, the lengths of the confidence and credible intervals decrease when effective sample size increases. The length of the interval estimates under CS-I is smaller than that under CS-II. Also, the lengths in the scheme CS-III is smaller compared to the other schemes. When progressive type-II censoring and type-II censoring plans are compared, the progressive type-II plan provides better result.

- From Table 5, we see that the values of the predictive estimates and prediction lengths increase as i and p increase. Further, when the effective sample size (m) increases, the predictive estimate values and predictive interval lengths decrease. The PT-II plan provides smaller length of the interval estimates compared to the type-II scheme.

Table 4: 95% confidence and credible intervals of α, β and γ for the real data set.

(n,m) Schemes	Methods		α	β	γ
(31, 20) (CS-I)	Asy	NA	(87.96352, 97.39403)	(0.50153, 2.85852)	(138.13641, 159.88704)
		NL	(88.31042, 97.82276)	(0.61582, 3.12609)	(135.99003, 158.96112)
	Boot	t	(85.10587, 96.93401)	(0.68148, 3.44237)	(137.46815, 161.96911)
		p	(88.73142, 99.93547)	(0.70824, 3.64435)	(134.69740, 161.75805)
	HPD		(89.50718, 98.16640)	(0.83540, 2.71149)	(138.74876, 158.76592)
(31, 20) (CS-II)	Asy	NA	(24.16824, 33.83525)	(0.46815, 3.12882)	(23.16784, 45.18696)
		NL	(25.34025, 35.07839)	(0.51672, 3.33375)	(25.43152, 48.55777)
	Boot	t	(25.76253, 36.60352)	(0.54157, 3.57718)	(22.08264, 45.74319)
		p	(23.15687, 34.76890)	(0.55481, 3.71807)	(23.56840, 47.79902)
	HPD		(25.84508, 34.94702)	(0.76109, 2.86216)	(25.94206, 47.15132)
(31, 31) (CS-III)	Asy	NA	(47.16494, 55.78248)	(0.81995, 2.68102)	(189.18240, 205.44600)
		NL	(49.15205, 57.99300)	(0.86454, 2.76610)	(183.49482, 201.24983)
	Boot	t	(46.85641, 56.57569)	(0.84099, 2.79355)	(185.10064, 204.99083)
		p	(47.52215, 57.84276)	(0.84672, 2.89734)	(185.76185, 206.85296)
	HPD		(48.77806, 55.97009)	(0.92187, 2.21199)	(188.64287, 202.75298)

Table 5: One-sample predicted values and 95% prediction intervals for future observations for the real data set.

(n,m) Scheme	i	p	$\hat{(\cdot)}^{SE}_{be}$	$\hat{(\cdot)}^{WS}_{be}$	$\hat{(\cdot)}^{PL}_{be}$	$\hat{(\cdot)}^{EN}_{be}\|-0.5$	$\hat{(\cdot)}^{EN}_{be}\|0.5$	Interval
(31, 20) (CS-I)	1	1	0.09415	0.11180	0.13157	0.11097	0.10894	(0.00769, 0.13465)
		2	0.13642	0.13758	0.14064	0.12771	0.11756	(0.02482, 0.17559)
	10	1	0.74288	0.81256	0.88157	0.80526	0.79157	(0.60408, 1.02431)
		2	0.77051	0.78109	0.78698	0.75281	0.75101	(0.66826, 1.14794)
(31, 20) (CS-II)	1	1	0.13065	0.16848	0.17582	0.14033	0.11005	(0.07534, 0.26626)
		2	0.14359	0.18157	0.18278	0.18072	0.16121	(0.09077, 0.31233)
	10	1	0.70204	0.76241	0.76587	0.76112	0.71089	(0.48262, 1.08883)
		2	0.74885	0.79485	0.79948	0.74158	0.71826	(0.53170, 1.16397)
(31, 31) (CS-III)	1	1	0.02465	0.03110	0.03345	0.03197	0.03128	(0.00894, 0.04656)
		2	0.06004	0.06422	0.06784	0.05682	0.04997	(0.01348, 0.09229)
	10	1	0.70909	0.71184	0.71648	0.70482	0.67158	(0.51582, 0.77149)
		2	0.72187	0.75001	0.75389	0.72807	0.72554	(0.62877, 0.91890)

Figure 1(a) presents the histogram and fitted probability density plots of five models based on real data set. From the graphs, we visualize that the EGT-II distribution covers the maximum area of the data set comparing to other distributions. The scaled total time on test (TTT) plot reveal that the hazard rate function of the fitted distribution is upside-down bathtub in Figure 1(b). The profile of the log-likelihood function of α, β and γ for real data set is shown in Figure 2(a, b, c).

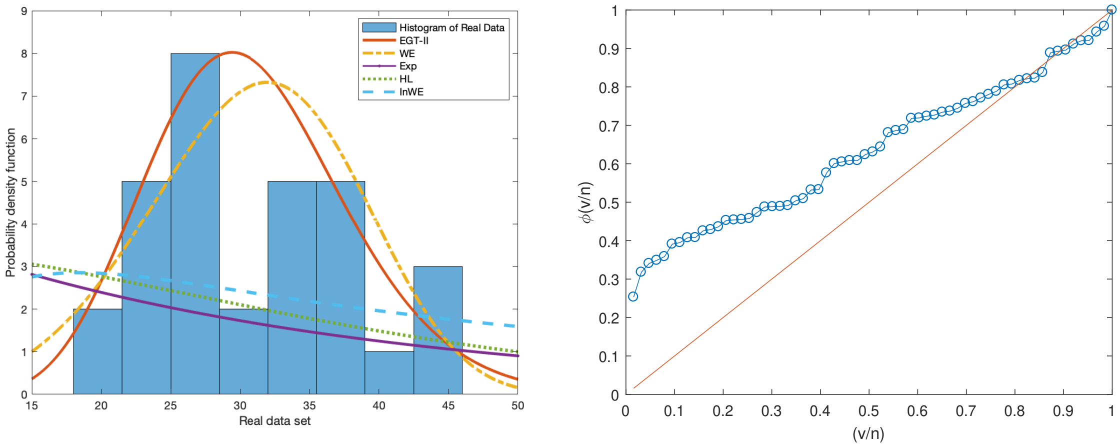

Figure 1: The first figure (in left) is the plots of the histogram and probability density functions of the fitted EGT-II, WE, Exp, HL, InWE models for the real data set. The second figure (in right) is for the scaled TTT plot.

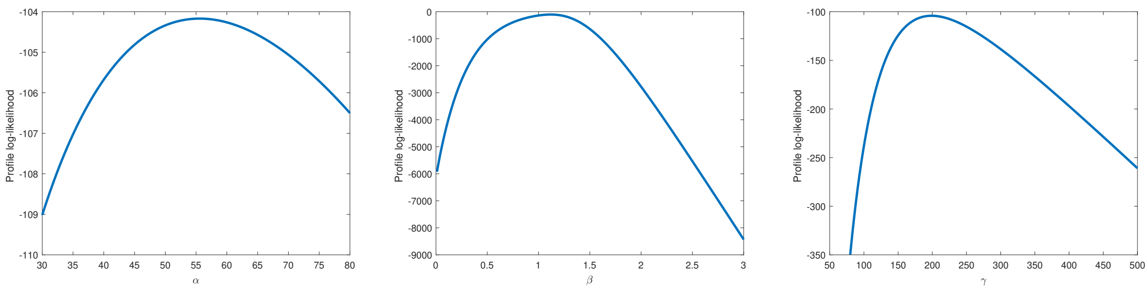

Figure 2: The profile log-likelihood plots for α (left), β (middle) and γ (right) for real data set.

9. OPTIMAL PT-II CENSORING SCHEME

In this section, we obtain optimum progressive censoring scheme from different censoring schemes for which the value of the chosen criterion is minimum. At first, we need to define a criterion. Define

$$Cr_1(\Phi) = E_{\mathrm{D}}\{V_{\mathrm{Pos}(\Phi)}(\ln T_p)\},$$

where $V_{\mathrm{Pos}(\Phi)}(\ln T_p)$ is the posterior variance of $\ln T_p$, $\Phi = (\Phi_1, ..., \Phi_m)$ is the censoring scheme and E_{D} is the expectation with respect to the data set. Further, T_p is the p-th quantile of the EGT-II distribution, which is given by

$$T_p = \left[-\left(\frac{1}{\gamma}\right)\ln\left(1-(1-p)^{\frac{1}{\alpha}}\right)\right]^{-\left(\frac{1}{\beta}\right)}. \tag{9.1}$$

Note that the total number of possible censoring schemes, given by $\binom{n-1}{m-1}$ is finite and large for fixed n and m. For example, when for $n = 30$ and $m = 20$, it is equal to 20030010, which is quite large. We say that a scheme $\Phi^{(1)} = (\Phi_1^{(1)}, ..., \Phi_m^{(1)})$ is better than another scheme $\Phi^{(2)} = (\Phi_1^{(2)}, ..., \Phi_m^{(2)})$, if $\Phi^{(1)}$ gives more information about the parameters than $\Phi^{(2)}$.

Mathematically, this is equivalent to $Cr_1(\Phi^{(1)}) < Cr_1(\Phi^{(2)})$. We refer to Kundu and Pradhan [9] and Singh and Tripathi [19] for more discussions in this direction. It is easy to see that the explicit expressions of the criterion are hard to obtain. Therefore, we use Lindley's approximation method. In the criterion, we compute approximated value of $V_{\text{Pos}(\Phi)}(\ln(T_p))$. We know that

$$V_{\text{Pos}(\Phi)}(\ln T_p) = E_{\text{Pos}(\Phi)}[\ln T_p]^2 - (E_{\text{Pos}(\Phi)}[\ln T_p])^2. \tag{9.2}$$

For simulation purpose, we generate all the parameters α, β and γ from $Gamma(5,5)$ distribution. To evaluate both terms in the RHS of (9.2), we apply Lindley's approximation method, which is explained in Section 6.1. To approximate $E_{\text{Pos}(\Phi)}[\ln T_p]^2$, we have $u(\alpha,\beta,\gamma) = (\ln T_p)^2$. Further,

$$\begin{aligned}
u_1 &= -\frac{2(1-p)^{\frac{1}{\alpha}}\ln(1-p)\ln T_p}{\alpha^2\beta\gamma\Big((1-p)^{\frac{1}{\alpha}}-1\Big)\exp\{-\beta\ln T_p\}}, \quad u_2 = -\frac{2u(\alpha,\beta,\gamma)}{\beta},\\
u_3 &= \frac{2\ln T_p}{\beta\gamma}, \quad u_{12} = u_{21} = \frac{4(1-p)^{\frac{1}{\alpha}}\ln(1-p)\ln T_p}{\alpha^2\beta^2\gamma\Big((1-p)^{\frac{1}{\alpha}}-1\Big)\exp\{-\beta\ln T_p\}},\\
u_{31} &= u_{13} = -\frac{2(1-p)^{\frac{1}{\alpha}}\ln(1-p)}{\alpha^2\beta^2\gamma^2\Big((1-p)^{\frac{1}{\alpha}}-1\Big)\exp\{-\beta\ln T_p\}},\\
u_{32} &= u_{32} = \frac{4\ln T_p}{\beta^2\gamma}, \quad u_{22} = \frac{6u(\alpha,\beta,\gamma)}{\beta^2}, \quad u_{33} = \frac{2(1-\beta\ln T_p)}{\beta^2\gamma^2},\\
u_{11} &= -\frac{2(1-p)^{\frac{1}{\alpha}}\ln(1-p)}{\alpha^4\beta^2\gamma^2\Big((1-p)^{\frac{1}{\alpha}}-1\Big)^2(\exp\{-\beta\ln T_p\})^2}\\
&\quad\times\bigg((1-p)^{\frac{1}{\alpha}}\Big((2\alpha\gamma\exp\{-\beta\ln T_p\})+\ln(1-p)\Big)(-\beta\ln T_p)+\ln(1-p)\bigg)\\
&\quad-\beta\gamma\exp\{-\beta\ln T_p\}\Big(2\alpha+\ln(1-p)\Big)\ln T_p.
\end{aligned}$$

Other terms in (6.1) are same. In this way, $E_{\text{Pos}(\Phi)}[\ln T_p]^2$ can be approximated. Similarly, to compute $E_{\text{Pos}(\Phi)}[\ln T_p]$, we have $u(\alpha,\beta,\gamma) = \ln T_p$. Furthermore,

$$\begin{aligned}
u_1 &= -\frac{(1-p)^{\frac{1}{\alpha}}\ln(1-p)}{\alpha^2\beta\gamma\Big((1-p)^{\frac{1}{\alpha}}-1\Big)\exp\{-\beta\ln T_p\}}, \quad u_2 = -\frac{\ln T_p}{\beta}, \quad u_3 = \frac{1}{\beta\gamma},\\
u_{11} &= \frac{\Big((1-p)^{\frac{1}{\alpha}}\ln(1-p)-\gamma\exp\{-\beta\ln T_p\}\Big(\ln(1-p)-2\alpha\Big((1-p)^{\frac{1}{\alpha}}-1\Big)\Big)\Big)}{\alpha^4\beta\gamma^2\Big((1-p)^{\frac{1}{\alpha}}-1\Big)^2(\exp\{-\beta\ln T_p\})^2}\\
&\quad\times(1-p)^{\frac{1}{\alpha}}\ln(1-p), \quad u_{31} = u_{13} = 0,\\
u_{22} &= \frac{2\ln T_p}{\beta^2}, \quad u_{33} = -\frac{1}{\beta\gamma^2}, \quad u_{12} = u_{21} = -\frac{(1-p)^{\frac{1}{\alpha}}\ln(1-p)}{\alpha^2\beta^2\gamma\Big((1-p)^{\frac{1}{\alpha}}-1\Big)\exp\{-\beta\ln T_p\}}.
\end{aligned}$$

From Table 6, we observe that $\Phi^{(3)}$ plan gives maximum information compared to other plans, when $p = 0.25$ and $(n,m) = (25,15)$. So, plan $\Phi^{(3)}$ is optimal. Similarly, when $(n,m) = (25,15)$, $\Phi^{(1)}$ and $\Phi^{(2)}$ plans are optimal for $p = 0.5, 0.9$ and $p = 0.75$, respectively. In each censoring scheme, p increases, then the value of criterion increases. Next, for $(n,m) = (25,20)$, the plans $\Phi^{(2)}, \Phi^{(3)}, \Phi^{(1)}$ and $\Phi^{(3)}$ are optimal for $p = 0.25, 0.5, 0.75$ and $p = 0.9$, respectively.

Table 6: The values of $Cr_1(\Phi)$ for different censoring schemes Φ.

(n,m)	Φ	$(\Phi_1,...,\Phi_m)$	$p=0.25$	$p=0.5$	$p=0.75$	$p=0.9$
$(25,15)$	$\Phi^{(1)}$	(10,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	0.74826	0.76445	0.90050	0.92418
	$\Phi^{(2)}$	(5,0,0,0,0,0,0,0,0,0,0,0,0,0,5)	0.85085	0.86170	0.87642	0.98054
	$\Phi^{(3)}$	(1,1,1,1,0,1,1,0,0,0,0,1,1,1,1)	0.66523	0.81064	0.91135	0.97068
	Type-II	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,10)	1.09417	1.14158	1.15333	1.20081
$(25,20)$	$\Phi^{(1)}$	(5,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	0.55135	0.61053	0.67182	0.72992
	$\Phi^{(2)}$	(2,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0,1)	0.39408	0.51581	0.71540	0.76204
	$\Phi^{(3)}$	(1,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,1)	0.45826	0.47643	0.69471	0.70648
	Type-II	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,5)	0.63105	0.68846	0.72283	0.78524

10. CONCLUDING REMARKS

In this paper, we studied the problem of estimation and prediction when the lifetime data follow EGT-II distribution under the constraint that the sample is progressively censored. First, we proved that the MLEs exist and are unique. Further, it was seen that the closed form expressions of the MLEs do not exist. Thus, we used EM algorithm. The process of EM algorithm is little complicated since it requires integrations which need to be computed numerically. So, we next used stochastic version of the EM algorithm for the purpose of computation of the MLEs. In numerical study, it has been noticed that the performance of the stochastic EM algorithm is better than that of the EM algorithm. The observed Fisher's information matrix was also calculated. This is useful for obtaining the asymptotic confidence intervals. In addition, we used Boot-t and p algorithms for the computation of the confidence intervals. Bayes estimates were derived. Like the MLEs, the explicit forms of the Bayes estimates are difficult to obtain. Thus, we adopted three approximation techniques: (i) Lindley's approximation method, (ii) Importance sampling method and (iii) Metropolis-Hastings algorithm. The HPD credible intervals were also proposed. In data analysis, it was seen that the HPD credible intervals outperform others. The discussions on the elicitation of the hyper-parameters have been presented. Next, we presented the prediction problem. Here, we obtained Bayes prediction estimates and the associated Bayesian predictive interval estimates. Finally, we proposed the use of a criteria for the comparison of different sampling schemes, and then, pointed out the optimal sampling scheme for the given criterion.

A. APPENDIX

Theorem A.1. *The conditional distribution of* z_{jk} *for* $k = 1, ..., R_j$ *given* $X_1 = x_1, ...,$ $X_j = x_j$ *has the form*

$$f_{Z|X}(z_j|X_1 = x_1, ..., X_j = x_j) = f_{Z|X}(z_j|X_j = x_j) = \begin{cases} \dfrac{f(z_j : \alpha, \beta, \gamma)}{1 - F(x_j : \alpha, \beta, \gamma)}, & z_j > x_j \\ 0, & \textit{elsewhere.} \end{cases}$$

Proof: The proof is straightforward. For details, see Ng *et al.* [14]. □

Using Theorem A.1, we can write

$$E[\ln Z_{jk}|Z_{jk} > x_j, \alpha, \beta, \gamma] =$$

$$= \frac{\alpha\beta\gamma}{1 - F_X(x_j : \alpha, \beta, \gamma)} \int_{x_j}^{\infty} t^{-\beta-1} \exp\{-\gamma t^{-\beta}\} (1 - \exp\{-\gamma t^{-\beta}\})^{\alpha-1} \ln t \; dt$$

$$= \frac{\alpha}{\beta\Big(1 - \exp\{-\gamma x_j^{-\beta}\}\Big)^{\alpha}} \int_{1-\exp\{-\gamma x_j^{-\beta}\}}^{0} u^{\alpha-1} \ln\left(\frac{\ln(u-1)}{\gamma}\right) du,$$

$$E[Z_{jk}^{-\beta}|Z_{jk} > x_j, \alpha, \beta, \gamma] =$$

$$= \frac{\alpha\beta\gamma}{1 - F_X(x_j : \alpha, \beta, \gamma)} \int_{x_j}^{\infty} t^{-2\beta-1} \exp\{-\gamma t^{-\beta}\} (1 - \exp\{-\gamma t^{-\beta}\})^{\alpha-1} dt$$

$$= \frac{\alpha}{\gamma(1 - \exp\{-\gamma t^{-\beta}\})^{\alpha}} \int_{1-\exp\{-\gamma x_j^{-\beta}\}}^{0} u^{\alpha-1} \ln(1-u) \; du,$$

$$E[\ln(1 - \exp\{-\gamma Z_{jk}^{-\beta}\})|Z_{jk} > x_j, \alpha, \beta, \gamma] =$$

$$= \frac{\alpha\beta\gamma}{1 - F_X(x_j : \alpha, \beta, \gamma)} \int_{x_j}^{\infty} t^{-\beta-1} \exp\{-\gamma t^{-\beta}\} (1 - \exp\{-\gamma t^{-\beta}\})^{\alpha-1} \ln(1 - \exp\{-\gamma t^{-\beta}\}) \; dt$$

$$= \frac{\alpha}{\Big(1 - \exp\{-\gamma x_j^{-\beta}\}\Big)^{\alpha}} \int_{0}^{1-\exp\{-\gamma x_j^{-\beta}\}} u^{\alpha-1} \ln u \; du.$$

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