REVSTAT – Statistical Journal
Volume 21, Number 4, October 2023, Supplementary Material, 1–6

Supplementary Material for “New Members of the Johnson Family of Probability Distributions: Properties and Application”

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Received: June 2021 Revised: February 2022 Accepted: February 2022

R codes

```
#eta - square side in SKS measure, ymax - maximum value of kurtosis
#ile - number of skewness and kurtosis values
#gamma12[ile,1] - values of skewness, gamma12[ile,2] - values of kurtosis
SKS=function(eta, ymax, gamma12){
xmin = -sqrt(ymax^2-1); xmax = sqrt(ymax^2-1); ymin = 1
w = (ymax-ymin)/eta; k=(xmax-xmin)/eta
X=numeric(k+1); Y=numeric(w+1); Yw=numeric(k+1)
n=matrix(NA, nrow=w+1, ncol=k+1)
for (j in 1:(k+1)){X[j]=xmin+(j-1)*eta; Yw[j]=X[j]^2+1}
for (i in 1:(w+1)) Y[i]=ymin+(i-1)*eta
a = 0
for (i in 2:(w+1)) {
for(j in 2:(k/2)) if (Y[i]<Yw[j]) a=a+1}
for (i in 2:(w+1)){
for(j in 2:(k+1)) n[i,j]=0}
for(u in 1:ile){
g1=gamma12[u,1]; g2=gamma12[u,2]
for (j in 2:(k+1)){
for(i in 2:(w+1)){
if (g1<= X[j-1]&g1<X[j]&g2>= Y[i-1]&g2<Y[i]&n[i,j]==0) n[i,j]=1}}}
SI=0
ST=w*k-2*a
for (j in 2:(k+1)) {for(i in 2:(w+1)) if (n[i,j]==1) SI=SI+1}
return(SI/ST)
}
```

```
dEN=function (x,a1,b1,a2,b2,c){
return((exp((a1-x)/b1)/b1+exp((x-a2)/b2)/b2)*
dnorm(c-exp((a1-x)/b1)+exp((x-a2)/b2),0,1))}

pEN=function (x,a1,b1,a2,b2,c){
return(pnorm(c-exp((a1-x)/b1)+exp((x-a2)/b2),0,1))}

qEN=function(p,a1,b1,a2,b2,c){
return (uniroot(function(x)
c-exp((a1-x)/b1)+exp((x-a2)/b2)-qnorm(p,0,1),lower=-10,
upper=10, extendInt = "yes", tol=1e-9)$root)}

rEN=function(n,a1,b1,a2,b2,c){
x=numeric(n)
for (i in 1:n){
x[i]=qEN(runif(1,0,1),a1,b1,a2,b2,c)} }
return(sort(x))}
```

1. Introduction

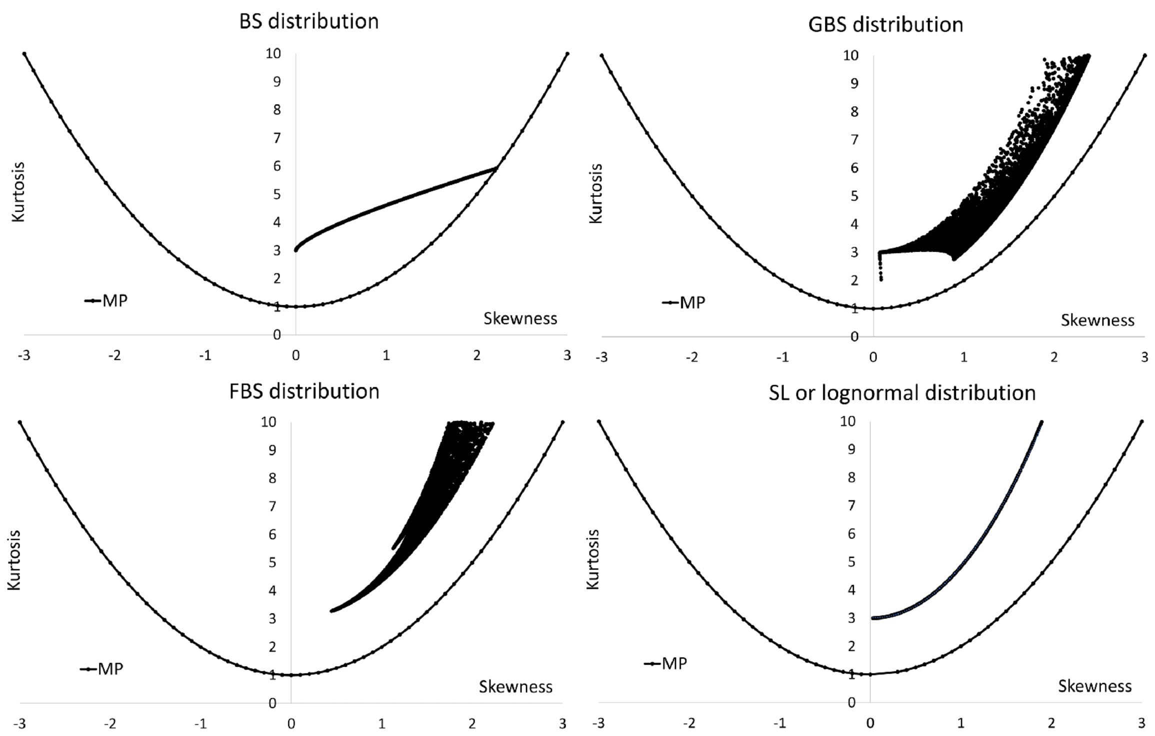

Figure 1: Skewness and kurtosis for the the BS, GBS, FBS, SL distributions.

2.4. Moments and moment generating function

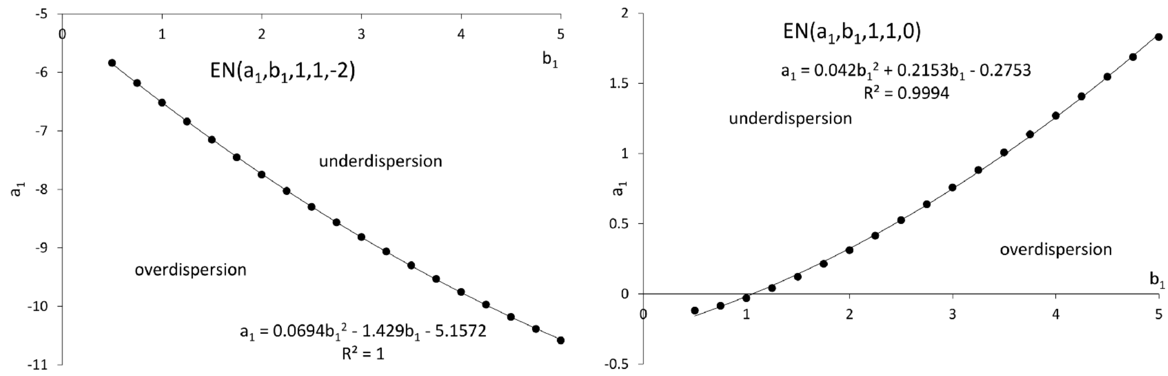

Figure 2: Dispersion regions for the $EN\,(a_1, b_1, 1, 1, -2)$ and $EN\,(a_1, b_1, 1, 1, 0)$.

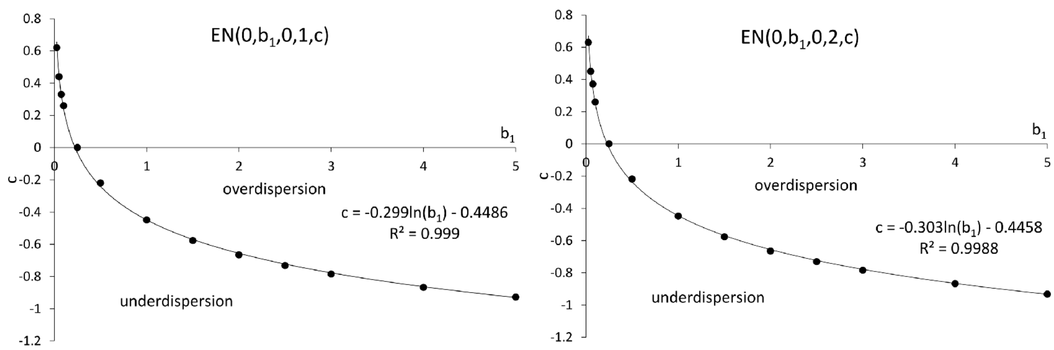

Figure 3: Dispersion regions for the $EN\,(0, b_1, 0, 1, c)$ and $EN\,(0, b_1, 0, 2, c)$.

The relationship between a_1 and b_1 in the $EN\,(a_1, b_1, a_2 \neq 0, b_2, c)$ remains quadratic for $a_2 = 1$, $b_2 = 1, c = -2$ and $a_2 = 1$, $b_2 = 1, c = 0$ (see Figure 2). The relationship between b_1 and c in the $EN\,(0, b_1, 0, b_2, c)$ remains logarithmic for $b_2 = 1, 2$ (see Figure 3).

3. Estimation procedures

We observe in Tables 1–2 that the estimates approach true values and RMSEs decrease as the sample size increases implying the consistency of the estimates. Biases and RMSEs are the smallest for $\widehat{b}_1$ (see Table 1). RMSEs are the smallest for $\widehat{b}_1$ (see Table 2). The smallest biases are for maximum likelihood estimate (MLE) related to the $EN\,(1,1,0,2,-1)$.

Table 1: Biases and RMSEs of the MLEs (denoted as 1), OLSEs (denoted as 2), WLSEs (denoted as 3) for the $EN\,(0,b_1,1,1,1)$).

b_1	EP	n	Bias					RMSE				
			$\widehat{a}_1$	$\widehat{b}_1$	$\widehat{a}_2$	$\widehat{b}_2$	$\widehat{c}$	$\widehat{a}_1$	$\widehat{b}_1$	$\widehat{a}_2$	$\widehat{b}_2$	$\widehat{c}$
0.5	1	50	0.09	0.02	-0.42	-0.16	0.09	0.33	0.19	2.32	3.41	1.30
	2		0.14	0.07	-0.25	0.80	-0.06	0.62	0.32	2.67	3.40	0.81
	3		0.11	0.06	-0.12	1.20	-0.08	0.50	0.25	4.85	5.94	0.81
	1	500	0.03	0.01	-0.24	0.01	-0.06	0.17	0.09	1.09	0.66	0.90
	2		0.02	0.01	-0.06	0.18	-0.01	0.12	0.07	0.47	0.70	0.34
	3		0.01	0.01	-0.10	0.08	-0.03	0.10	0.06	0.45	0.48	0.38
	1	1e3	0.02	0.01	-0.18	0.03	-0.06	0.13	0.07	0.84	0.53	0.72
	2		0.02	0.01	-0.03	0.10	0.00	0.09	0.05	0.32	0.48	0.26
	3		0.01	0.00	-0.05	0.03	-0.02	0.08	0.04	0.35	0.32	0.32
1	1	50	0.22	0.03	-0.73	0.18	-0.07	1.10	0.56	2.51	4.12	1.51
	2		0.35	0.21	-0.61	0.79	-0.10	1.26	0.74	2.70	3.05	0.85
	3		0.37	0.20	-0.77	0.94	-0.10	1.27	0.70	2.17	2.46	0.90
	1	500	0.08	0.02	-0.21	0.04	-0.05	0.51	0.25	0.88	0.54	0.95
	2		0.07	0.04	-0.18	0.19	-0.03	0.30	0.19	0.55	0.71	0.36
	3		0.03	0.02	-0.13	0.09	-0.04	0.23	0.14	0.48	0.46	0.39
	1	1e3	0.05	0.02	-0.13	0.02	-0.02	0.40	0.19	0.69	0.42	0.76
	2		0.04	0.02	-0.09	0.10	-0.02	0.19	0.12	0.36	0.45	0.27
	3		0.02	0.01	-0.05	0.03	-0.01	0.18	0.10	0.32	0.29	0.33
1.5	1	50	0.38	0.06	-0.77	0.14	-0.08	2.09	1.00	2.73	1.68	1.64
	2		0.57	0.42	-1.00	0.80	-0.19	1.97	1.32	2.61	2.17	0.98
	3		0.64	0.41	-1.15	0.91	-0.19	2.21	1.28	2.86	2.40	1.04
	1	500	0.13	0.04	-0.15	0.03	0.00	0.91	0.45	0.69	0.43	0.94
	2		0.07	0.06	-0.17	0.14	-0.04	0.42	0.28	0.56	0.58	0.36
	3		0.04	0.02	-0.08	0.05	-0.02	0.40	0.23	0.38	0.34	0.41
	1	1e3	0.09	0.03	-0.09	0.02	0.00	0.72	0.35	0.51	0.33	0.75
	2		0.04	0.03	-0.08	0.07	-0.02	0.28	0.19	0.32	0.35	0.27
	3		0.02	0.01	-0.05	0.03	-0.01	0.33	0.18	0.27	0.23	0.35

Table 2: Biases and RMSEs of the MLEs (denoted as 1), OLSEs (denoted as 2), WLSEs (denoted as 3) for the $EN(1,1,0,2,c)$.

c	EP	n	Bias					RMSE				
			$\widehat{a}_1$	$\widehat{b}_1$	$\widehat{a}_2$	$\widehat{b}_2$	$\widehat{c}$	$\widehat{a}_1$	$\widehat{b}_1$	$\widehat{a}_2$	$\widehat{b}_2$	$\widehat{c}$
-1	1	50	0.51	0.07	-0.51	0.15	0.25	2.59	1.26	3.98	2.57	1.74
	2		0.64	0.47	-0.80	0.57	0.13	1.91	1.74	2.54	1.93	1.09
	3		0.89	0.51	-1.18	0.83	0.19	2.31	1.55	3.19	2.43	1.26
	1	500	0.13	0.03	-0.06	0.02	0.09	0.74	0.39	1.20	0.68	0.98
	2		0.32	0.14	-0.32	0.23	0.13	0.98	0.57	1.20	0.87	0.62
	3		0.25	0.11	-0.23	0.17	0.12	0.86	0.48	1.05	0.75	0.60
	1	1e3	0.07	0.01	-0.04	0.00	0.04	0.45	0.25	0.93	0.45	0.84
	2		0.24	0.11	-0.21	0.17	0.12	0.75	0.44	0.89	0.67	0.52
	3		0.13	0.06	-0.12	0.09	0.06	0.55	0.33	0.67	0.47	0.42
1	1	50	0.02	-0.07	-0.51	0.98	0.14	1.19	0.55	17.91	41.07	1.76
	2		0.23	0.13	-0.64	1.40	-0.12	1.38	0.70	4.54	9.23	1.11
	3		0.24	0.11	-0.62	1.49	0.00	1.29	0.63	5.11	12.31	1.12
	1	500	0.06	0.00	-0.20	0.30	0.11	0.57	0.25	1.86	2.54	1.05
	2		0.04	0.01	-0.12	0.28	0.03	0.44	0.22	0.76	1.19	0.57
	3		0.05	0.01	-0.14	0.30	0.04	0.51	0.23	0.90	1.37	0.74
	1	1e3	0.05	0.01	-0.09	0.12	0.07	0.44	0.20	1.30	1.38	0.90
	2		0.03	0.01	-0.07	0.19	0.04	0.38	0.18	0.55	0.92	0.53
	3		0.05	0.01	-0.09	0.20	0.04	0.42	0.19	0.72	0.99	0.67
2	1	50	0.01	-0.04	-0.75	2.18	-0.02	1.43	0.56	18.65	57.32	1.80
	2		0.19	0.10	-0.70	2.89	-0.28	1.32	0.59	5.81	28.47	1.24
	3		0.20	0.10	-0.63	2.93	-0.20	1.35	0.57	6.66	25.50	1.28
	1	500	0.00	-0.02	-0.26	0.90	0.09	0.52	0.22	2.17	9.60	1.03
	2		0.02	0.01	-0.11	0.74	-0.01	0.35	0.16	1.05	3.88	0.53
	3		0.00	0.00	-0.10	0.49	-0.01	0.36	0.16	0.84	2.43	0.63
	1	1e3	0.01	-0.01	-0.20	0.64	0.06	0.45	0.19	1.59	3.74	0.92
	2		0.01	0.00	-0.05	0.39	0.01	0.28	0.13	0.61	2.40	0.47
	3		0.01	0.00	-0.08	0.29	-0.01	0.31	0.13	0.68	1.34	0.60

5.1. Example 2

The second real data present Intercountry Life-Cycle Savings Data. The data consist of 50 observations of the variable "real per-capita disposable income" and are available in the R software with code LifeCycleSavings[4]. Figures 4–5 present histograms, estimated PDFs and CDFs of the analyzed models sorted by AIC values. The information criteria are used for model comparisons, while the GoFTs and the bootstrap method described below are used for model fitting.

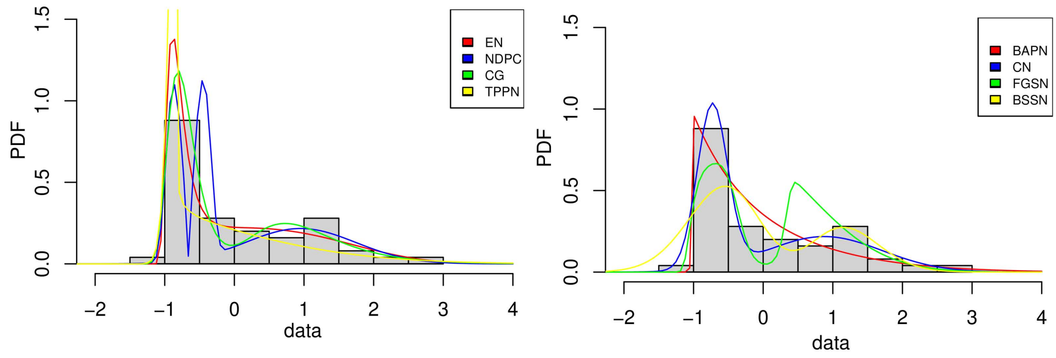

Figure 4: Estimated PDF of analyzed distributions. Example 2.

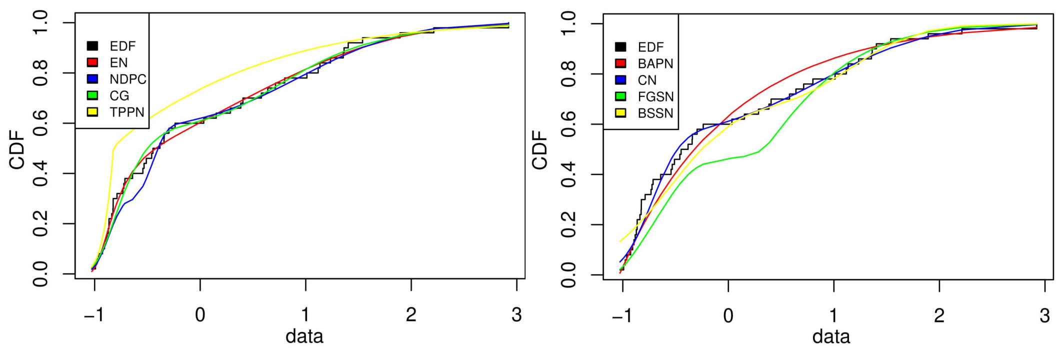

Figure 5: Estimated CDF of analyzed distributions. Example 2.

As shown in Table 3 the EN model is the best in terms of the AIC, BIC and HQIC values. The AIC ranking is not the same as the BIC and HQIC rankings, but it is similar. The EN model has the highest p-values for all analyzed tests. (see Table 4). The p-value ranking for the KS test is similar to the p-value rankings for the AD and CvM tests. The p-value rankings for the AD i CvM tests are the same. The information criteria ranking is not the same as the p-value ranking. It is noteworthy that the rankings are similar for some models (e.g. the CG and NDPC) and the biggest difference in the ranking is for the TPPN model.

Concluding, the EN model fits better than the other models analyzed in this case.

Table 3: Results of estimation. Information criteria. Example 2.

Model		$\widehat{\Theta}$	95%CI	-l	AIC	BIC	HQIC
EN	$\widehat{a}_1$	-0.917	$[-0.972, -0.751]$	48.172	106.343	115.903	109.984
	$\widehat{b}_1$	0.150	$[0.078, 0.301]$				
	$\widehat{a}_2$	-4.546	$[-16.854, 1.677]$				
	$\widehat{b}_2$	4.670	$[0.770, 9.426]$				
	$\widehat{c}$	-2.383	$[-5.903, 0.361]$				
NDPC	$\widehat{a}_1$	0.951	$[0.563, 1.358]$	47.409	106.819	118.291	111.187
	$\widehat{b}_1$	0.791	$[0.468, 1.025]$				
	$\widehat{a}_2$	-0.660	$[-0.701, -0.615]$				
	$\widehat{b}_2$	0.259	$[0.213, 0.306]$				
	$\widehat{c}$	1.869	$[1.428, 3.050]$				
	$\widehat{\omega}$	0.430	$[0.275, 0.575]$				
CG	$\widehat{a}_1$	0.735	$[0.152, 1.101]$	49.406	108.812	118.372	112.452
	$\widehat{b}_1$	0.587	$[0.298, 0.856]$				
	$\widehat{a}_2$	-0.809	$[-0.889, -0.714]$				
	$\widehat{b}_2$	0.188	$[0.109, 0.250]$				
	$\widehat{\omega}$	0.394	$[0.221, 0.568]$				
TPPN	$\widehat{\theta}_1$	-0.824	$[-0.843, -0.818]$	50.566	109.132	116.780	112.045
	$\widehat{\sigma}_1$	0.094	$[0.062, 0.124]$				
	$\widehat{\sigma}_2$	1.432	$[1.017, 1.865]$				
	$\widehat{c}$	0.834	$[0.686, 1.042]$				
BAPN	$\widehat{\alpha}$	-1.573	$[-4.328, 0.989]$	51.419	110.839	118.487	113.751
	$\widehat{\beta}$	150.545	$[150.544, 188.933]$				
	$\widehat{\theta}$	-1.031	$[-1.050, -0.956]$				
	$\widehat{\sigma}$	1.893	$[1.151, 2.966]$				
CN	$\widehat{a}_1$	-0.730	$[-0.822, -0.623]$	52.142	114.284	123.844	117.925
	$\widehat{b}_1$	0.217	$[0.128, 0.302]$				
	$\widehat{a}_2$	0.877	$[0.454, 1.338]$				
	$\widehat{b}_2$	0.831	$[0.461, 1.062]$				
	$\widehat{\omega}$	0.546	$[0.384, 0.718]$				
FGSN	$\widehat{a}$	-0.363	$[-0.433, -0.256]$	54.2611	116.522	124.170	119.435
	$\widehat{b}$	1.054	$[0.841, 1.247]$				
	$\widehat{\alpha}_0$	-6.004	$[-14.343, -3.358]$				
	$\widehat{\alpha}_1$	14.504	$[5.886, 41.609]$				
BSSN	$\widehat{\theta}_1$	0.237	$[0.050, 0.430]$	62.748	133.497	141.145	136.409
	$\widehat{\theta}_2$	1.188	$[0.884, 1.735]$				
	$\widehat{c}$	0.399	$[0.166, 0.679]$				
	$\widehat{d}$	0.128	$[-0.042, 0.450]$				

Table 4: The KS, AD and CvM tests. Example 2.

Model	**KS test**		**AD test**		**CvM test**	
	TS	p-value	TS	p-value	TS	p-value
EN	0.0594	0.9584	0.132	0.9996	0.0203	0.9973
CG	0.0955	0.6084	0.2752	0.9559	0.0466	0.9009
NDPC	0.1042	0.5101	0.4617	0.7864	4.0426	0.6916
CN	0.1098	0.4412	0.4519	0.7965	0.0642	0.7911
BSSN	0.1127	0.4127	1.2907	0.2373	0.1762	0.3215
BAPN	0.1151	0.3975	0.8480	0.4514	15.5652	0.4027
FGSN	0.1776	0.0552	2.6937	0.0408	0.5042	0.0405
TPPN	0.2204	0.0089	3.8934	0.0103	0.8077	0.0061