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Exponentiated Generalized Exponential Gompertz Distribution

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Abstract:

- In this paper, a new generalization of the Gompertz distribution is introduced. This distribution is widely used across diverse fields like reliability analysis and data fitting within various data science domains, spanning disciplines such as computer science, marketing, biology, and insurance. Some properties of this new distribution are studied. The exact form of the density function of this distribution is obtained. In addition, the new distribution parameters are estimated using several methods and these methods are compared by using the Monte Carlo simulation. Finally, the performance of the proposed model is shown to demonstrate its advantages.

Keywords:

- *Gompertz distribution; Exponentiated generalized exponential distribution; Parameter estimation; Simulation study.*

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- 62Exx, 97K80.

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1. INTRODUCTION

Gompertz distribution was first introduced by Gompertz (1825) and plays an important role in analyzing lifetime data and modeling data in some areas such as engineering, actuarial, environmental, medical sciences, biological studies, demography, economics, finance, and insurance. The Gompertz distribution has an increasing hazard rate. A study on Gompertz distribution was done by Pollard and Valkovics (1992) in 1992. Marshall and Olkin (2007) considered a negative Gompertz distribution. This distribution has been studied by many authors. For more details refer to references Chen (1997), Franses (1994), Garg et al. (1970), Minimol and Thomas (2014), Lenart (2014), Lenart and Missov (2016), Rao et al. (1992) Wu and Lee (1999), Wu et al. (2003), and Wu et al. (2004).

Although classical distributions are simple and flexible in practical situations, classical distributions do not always provide adequate fits to real data. Hence, to increase the flexibility of existing statistical distributions, for modeling data sets, different generalizations of the classical distributions have been proposed in the statistical literature recently. A new generalization of Gompertz distribution which includes exponential (E), generalized exponential (GE), and Gompertz (G) distributions as special cases were proposed by El-Gohary et al. (2013) that is called generalized Gompertz (GG) distribution. Jafari et al. (2014) introduced Beta-Gompertz (BG) distribution as another generalization of the Gompertz distribution. The transmuted Gompertz (TG) distribution was introduced by Abdul-Moniem and Seham (2015). El-Damcese et al. (2015) also proposed another generalization of Gompertz distribution for modeling lifetime, which is known as odd generalized exponential Gompertz (OGEG) distribution. Yari et al. (2020) studied a new generalization called Marshall Olkin Gompertz Makeham (MOGM) distribution. Karimi Ezmareh and Yari (2022a) introduced Kumaraswamy-G Generalized Gompertz distribution with application to lifetime data. Inference and prediction for modified Weibull distribution based on doubly censored samples presented by Karimi Ezmareh and Yari (2022b).

One of the most important issues in statistical analysis is finding the suitable distribution for data set modeling. Knowing the appropriate distribution in the modeling of data sets leads to an accurate analysis of the data. Consequently, the main purpose of this paper is to introduce a new generalization of Gompertz distribution called as exponentiated generalized exponential Gompertz (EGEG) distribution.

The rest of the paper is organized as follows: in Section 2, the new model and its sub-models are introduced. In Section 3, some of the statistical properties of the introduced distribution are calculated. Section 4 estimates the unknown parameters of this distribution using the maximum likelihood (MLE), least square (LS), and Bayes methods, and related simulation studies are presented. In Section 5, the criteria for evaluating the appropriate model are presented. Finally, Section 6 illustrates the application of the new model by means of a real data set.

2. NEW MODEL

Before introducing the new model, the exponentiated generalized exponential family and Gompertz distribution are defined.

Definition 2.1. Let T be an exponential random variable with probability density function (pdf) $f(t) = \lambda e^{-\lambda t}, t, \lambda > 0$ and X be a continuous random variable with pdf F, where f and F are pdf and cumulative distribution function (cdf) respectively. Then the cdf of the exponentiated generalized exponential EGE-X family of distributions is written as follows

$$G(x) = \int_0^{-\log[1-(1-\bar{F}^d(x))^c]} \lambda e^{-\lambda t} dt = 1 - \{1 - [1 - (1 - F(x))^d]^c\}^\lambda, \quad \lambda, x, c, d > 0, \tag{2.1}$$

where $\bar{F}(x) = 1 - F(x)$ (Nasiru et al., 2017).

Definition 2.2. A random variable X has a Gompertz distribution with positive parameters γ and β , i.e., $X \sim Go(\gamma, \beta)$, if its cdf and pdf are given as follows

$$F_{Go}(x) = 1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}, \quad x > 0, \tag{2.2}$$

$$f_{Go}(x) = \beta e^{\gamma x} e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}, \quad x > 0. \tag{2.3}$$

Now by replacing expression (2.2) into expression (2.1), the cdf of the new EGEG distribution for $x, \beta, \gamma > 0$ is obtained by

$$G(x) = 1 - \left[1 - \left(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)}\right)^c\right]^\lambda, \ d, c, \lambda > 0. \tag{2.4}$$

The pdf of the EGEG distribution is as follows

$$g(x) = \beta\lambda c d e^{\gamma x} e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)} \left(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)}\right)^{c-1} \left[1 - \left(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)}\right)^c\right]^{\lambda-1}. \tag{2.5}$$

The pdf and cdf plots of the EGEG distribution for different parameter values are shown in Figure 2.

2.1. Expansion for the density function

In the following Lemma, the EGEG pdf expansion is obtained.

Lemma 2.1. *The EGEG distribution pdf is a linear combination of Gompertz distribution with different parameters as*

$$g(x) = \lambda c d \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_{ij.\lambda} f_{Go}(x; \gamma, \beta'), \tag{2.6}$$

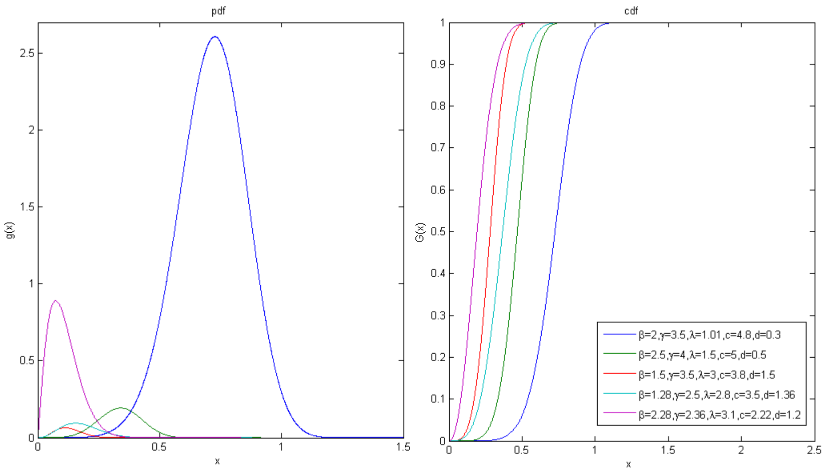

Figure 1: The pdf and cdf plots of the EGEG distribution for some parameter values

where $\beta' = \beta d(j+1)$, Γ *represents the gamma function, and*

$$\omega_{ij.\lambda} = \binom{\lambda-1}{i}\binom{ci+c-1}{j}\frac{(-1)^{i+j}}{d(j+1)}, \tag{2.7}$$

where, for k *a non-negative integer and* r *a real number*

$$\binom{r}{k} = \frac{r(r-1)...(r-k+1)}{k!} = \frac{(r)_k}{k!},$$

where $()_k$ *reprsents the Pochhammer symbol.*

Proof: For a real non-integer $\eta > 0$ and $|z| < 1$

$$(1-z)^{\eta-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\eta)}{i!\Gamma(\eta-i)} z^i. \tag{2.8}$$

By using expansion (2.8) and the fact that $0 < e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)} < 1$ implies that

$$\begin{aligned}
g(x) &= \beta\lambda c d e^{\gamma x} A_{d,\beta,\gamma}\big(1-A_{d,\beta,\gamma}\big)^{c-1}\big[1-\big(1-A_{d,\beta,\gamma}\big)^{c}\big]^{\lambda-1}, \\
&= \beta\lambda c d e^{\gamma x} A_{d,\beta,\gamma}\big(1-A_{d,\beta,\gamma}\big)^{c-1}\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\lambda)}{i!\Gamma(\lambda-i)}\big(1-A_{d,\beta,\gamma}\big)^{ci}, \\
&= \beta\lambda c d e^{\gamma x} A_{d,\beta,\gamma}\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\lambda)}{i!\Gamma(\lambda-i)}\big(1-A_{d,\beta,\gamma}\big)^{ci+c-1}, \\
&= \beta\lambda c d e^{\gamma x} A_{d,\beta,\gamma}\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\lambda)}{i!\Gamma(\lambda-i)}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(ci+c)}{j!\Gamma(ci+c-j)}e^{-\frac{\beta d j}{\gamma}(e^{\gamma x}-1)}, \\
&= \lambda c d\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(\lambda)\Gamma(ci+c)}{i!j!\Gamma(\lambda-i)\Gamma(ci+c-j)d(j+1)} \times \underbrace{\beta d(j+1)e^{\gamma x}e^{-\frac{\beta d(j+1)}{\gamma}(e^{\gamma x}-1)}}_{f_{Go}(x;\gamma,\beta d(j+1))}, \\
&= \lambda c d\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij.\lambda} f_{Go}(x;\gamma,\beta'),
\end{aligned}$$

where $A_{d,\beta,\gamma} = \exp\big[-\frac{\beta d}{\gamma}\big(\exp(\gamma x)-1\big)\big]$, and f_{Go} is the pdf of Gompetrz distribution. □

2.2. Sub-models

1) If we set $\lambda = c = d = 1$, Gompertz distribution is obtained with two parameters of β and γ.

2) If we set $\lambda = 1$, exponentiated generalized Gompertz distribution is obtained with four parameters of β, γ, c and d.

3. SOME STATISTICAL PROPERTIES

Now, some of the statistical properties of the EGEG distribution are studied.

3.1. Survival and hazard rate function

The survival function (S) and hazard rate function (H) of the EGEG distribution, for non-negative x and $\beta, \gamma, d, c, \lambda > 0$, are defined as follows, respectively,

$$S(x) = \left[1 - \left(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)}\right)^c\right]^\lambda, \quad \beta, \gamma, d, c, \lambda > 0 \tag{3.1}$$

$$H(x) = \frac{\beta\lambda c d e^{\gamma x} e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)}\left(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)}\right)^{c-1}}{1 - \left(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x}-1)}\right)^c}, \quad \beta, \gamma, d, c, \lambda > 0\cdot \tag{3.2}$$

3.2. Moments

The moments play a very important role in statistical analysis. They are used to study the properties of distributions such as tendency, dispersion, skewness, and kurtosis. Hence, the r^{th} moment of the EGEG distribution is presented in Theorem 3.1.

Theorem 3.1. *The r^{th} non-central moment of the EGEG distribution is given by*

$$\mu'_r = \lambda c d \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij.\lambda} \frac{r!}{\gamma^r} e^{\frac{\beta'}{\gamma}} E_1^{r-1}\left(\frac{\beta'}{\gamma}\right), \quad \beta', \gamma, d, c, \lambda > 0, \tag{3.3}$$

where $\omega_{ij.\lambda}$ is defined by (2.7), $\beta' = \beta d(j+1)$ and

$$E_1^{r-1}\left(\frac{\beta'}{\gamma}\right) = \frac{1}{(r-1)!}\int_1^\infty (\log x)^{r-1} x^{-1} e^{-\frac{\beta'}{\gamma}x} dx. \tag{3.4}$$

Proof: It is known that if $X \sim Go(\gamma, \beta)$, then $E(X^r) = \frac{r!}{\gamma^r} e^{\frac{\beta}{\gamma}} E_1^{r-1}(\frac{\beta}{\gamma})$. Thus, based on expression (2.6), we have

$$\begin{aligned}\mu'_r = E(X^r) &= \int_0^\infty x^r g(x)dx,\\ &= \int_0^\infty x^r \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \omega_{ij.\lambda} f_{Go}(x;\gamma,\beta')dx,\\ &= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \omega_{ij.\lambda} \int_0^\infty x^r f_{Go}(x;\gamma,\beta')dx,\\ &= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \omega_{ij.\lambda} \frac{r!}{\gamma^r} e^{\frac{\beta'}{\gamma}} E_1^{r-1}(\frac{\beta'}{\gamma}).\end{aligned}$$

□

3.3. Quantile function

The quantile function plays an important role in specifying a distribution, generating a sample from a distribution, and calculating quartiles, skewness, and kurtosis.

Lemma 3.1. *The EGEG distribution quantile function is given by*

$$Q_x(\tau) = \frac{1}{\gamma} \log\left\{1 - \frac{\gamma}{\beta d} \log\left[1 - \left(1 - (1-\tau)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]\right\}, \tag{3.5}$$

where $0 < \tau < 1$.

Proof: In order to obtain the quantile function, it is sufficient to solve the following equation

$$G(x_\tau) = P(X \le x_\tau) = \tau. \tag{3.6}$$

Thus

$$1 - \left\{1 - \left[1 - \left(e^{-\frac{\beta}{\gamma}(e^{\gamma x_\tau}-1)}\right)^d\right]^c\right\}^\lambda = \tau. \tag{3.7}$$

Let $x_\tau = Q_x(\tau)$. By solving Equation (3.7) for $Q_x(\tau)$, yields

$$Q_x(\tau) = \frac{1}{\gamma} \log\left\{1 - \frac{\gamma}{\beta d} \log\left[1 - \left(1 - (1-\tau)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]\right\}.$$

□

When $\tau = 0.25, 0.5$ and 0.75, the first quartile (Q_1), median and third quartile (Q_3) of the EGEG distribution are obtained, respectively.

Skewness is a measure indicating the presence or absence of symmetry and kurtosis describes the degree to which a probability distribution is peaked and flat. These two measures are often given by $\gamma_1 = \frac{\mu_3}{\sigma^3}$ and $\gamma_2 = \frac{\mu_4}{\sigma^4}$ respectively, where $\mu_r = E(X-\mu)^r$. When the third and fourth moments do not exist, these two criteria can be approximated based on quantile function by Galton (1883) and Moors (1988) as follows

$$Skewness = \frac{Q(\frac{6}{8}) - 2Q(\frac{4}{8}) + Q(\frac{2}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}, \tag{3.8}$$

$$Kurtosis = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) + Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}. \tag{3.9}$$

Since the exact value of this new distribution moments can not be calculated, using relations (3.8) and (3.9), the skewness and kurtosis of the EGEG distribution can be obtained.

The median, quartiles, skewness and kurtosis values for $\beta = 1.1, \gamma = 2, c = 4, d = 6$ and for different values of observation x and λ are listed in Table 1.

Table 1: The median, Q_1, Q_3, skewness, and kurtosis for different values of x and λ.

Statistics	$x_1 = 1.5$	$x_2 = 3$	$x_3 = 2$	$x_4 = 5$	$x_5 = 8$
	$\lambda_1 = 1.2$	$\lambda_2 = 2$	$\lambda_3 = 3.2$	$\lambda_4 = 4.5$	$\lambda_5 = 6$
Median	0.2059	0.1694	0.1429	0.1225	0.1152
Q_1	0.1481	0.1240	0.1061	0.0919	0.0868
Q_3	0.2737	0.2214	0.1842	0.1561	0.1462
Skewness	1.7193	1.8075	1.8862	1.9570	1.9845
Kurtosis	1.2250	1.2247	1.2235	1.2220	1.2214

3.4. Order statistics

Consider n a positive integer, a random sample $(X_1, ..., X_n)$ from a EGEG distribution and $X_{(p)}$ the p^{th} order statistics. The pdf of $X_{(p)}$ is given by

$$g_{X(p)}(x) = \frac{1}{B(p, n-p+1)}[G(x)]^{p-1}[1 - G(x)]^{n-p}g(x), \tag{3.10}$$

where G and g are the pdf and cdf of the EGEG distribution and B is the beta function, defined for positive υ_1 and υ_2 by $(B(\upsilon_1, \upsilon_2) = \frac{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}{\Gamma(\upsilon_1+\upsilon_2)})$.

Theorem 3.2. *The p^{th} order statistics of the EGEG distribution can be expressed by*

$$g_{X(p)}(x) = \frac{\lambda c d}{B(p, n-p+1)} \sum_{i=0}^{n-p} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \omega_{ijkl} f_G(x; \gamma, \beta''), \tag{3.11}$$

where $\beta'' = \beta_{d(l+1)}$ and

$$\omega_{ijkl} = \frac{(-1)^{i+j+k+l}\Gamma(n-p+1)\Gamma(i+p)\Gamma(\lambda j+\lambda)\Gamma(ck+c)}{i!j!k!l!\Gamma(n-p-i+1)\Gamma(i+p-j)\Gamma(\lambda j+\lambda-k)\Gamma(ck+c-l)d(l+1)}. \tag{3.12}$$

Proof: At first, by using the binomial extension and the fact that $0 < G(x) < 1$ for $x > 0$, $[1 - G(x)]^{n-p}$ can be written as follows

$$[1 - G(x)]^{n-p} = \sum_{i=0}^{n-p} (-1)^i \binom{n-p}{i} [G(x)]^i. \tag{3.13}$$

Substituting expression (3.13) into expression (3.10), we get

$$g_{X(p)}(x) = \frac{1}{B(p, n-p+1)} \sum_{i=0}^{n-p} (-1)^i \binom{n-p}{i} [G(x)]^{i+p-1} g(x). \tag{3.14}$$

By replacing expressions (2.4) and (2.5) into expression (3.14), yields

$$\begin{aligned} g_{X(p)}(x) = {} & \frac{1}{B(p, n-p+1)} \sum_{i=0}^{n-p} (-1)^i \binom{n-p}{i} \left\{1 - \left[1 - \left(1 - A_{d,\beta,\gamma}\right)^c\right]^\lambda\right\}^{i+p-1} \\ & \times \beta\lambda c d A_{d,\beta,\gamma} \left(1 - A_{d,\beta,\gamma}\right)^{c-1} \left[1 - \left(1 - A_{d,\beta,\gamma}\right)^c\right]^{\lambda-1}. \end{aligned} \tag{3.15}$$

By using expansion (2.8), we obtain

$$\left\{1 - \left[1 - \left(1 - A_{d,\beta,\gamma}\right)^c\right]^\lambda\right\}^{i+p-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(i+p)}{j! \Gamma(i+p-j)} \left[1 - \left(1 - A_{d,\beta,\gamma}\right)^c\right]^{\lambda j}, \tag{3.16}$$

$$\left[1 - \left(1 - A_{d,\beta,\gamma}\right)^c\right]^{\lambda j + \lambda - 1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\lambda j + \lambda)}{k! \Gamma(\lambda j + \lambda - k)} \left(1 - A_{d,\beta,\gamma}\right)^{ck}, \tag{3.17}$$

$$\left(1 - A_{d,\beta,\gamma}\right)^{ck+c-1} = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(ck+c)}{l! \Gamma(ck+c-l)} e^{-\frac{\beta d l}{\gamma}(e^{\gamma x} - 1)}. \tag{3.18}$$

Finally, by substituting expressions (3.16), (3.17), and (3.18) into expression (3.15), the p^{th} order statistics pdf is obtained as follows

$$\begin{aligned} g_{X(p)}(x) = {} & \frac{\lambda c d}{B(p, n-p+1)} \sum_{i=0}^{n-p} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \beta d(l+1) e^{\gamma x} e^{-\frac{\beta d(l+1)}{\gamma}(e^{\gamma x} - 1)} \\ & \times \frac{(-1)^{i+j+k+l} \Gamma(n-p+1) \Gamma(i+p) \Gamma(\lambda j + \lambda) \Gamma(ck+c)}{i! j! k! l! \Gamma(n-p-i+1) \Gamma(i+p-j) \Gamma(\lambda j + \lambda - k) \Gamma(ck+c-l) d(l+1)}, \\ = {} & \frac{\lambda c d}{B(p, n-p+1)} \sum_{i=0}^{n-p} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f_G(x; \gamma, \beta'') \omega_{ijkl}. \end{aligned}$$

□

4. PARAMETERS ESTIMATION

In this section, the parameters of EGEG distribution are estimated using three methods maximum likelihood (MLE), least square (LS), and Bayes.

4.1. MLE method

Let $X_1, X_2, ..., X_n$ be a random sample that has EGEG distribution with positive parameters $\Theta = (\beta, \gamma, \lambda, c, d)$, then the likelihood function is as follows

$$l(\Theta|x) = (\beta\lambda cd)^n e^{\gamma\sum_{i=1}^n x_i} \prod_{i=1}^n e^{-\frac{\beta d}{\gamma}(e^{\gamma x_i}-1)}\big(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x_i}-1)}\big)^{c-1}$$
$$\times \big[1 - \big(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x_i}-1)}\big)^c\big]^{\lambda-1}, \quad x > 0. \tag{4.1}$$

The log-likelihood function is given by

$$L(\Theta \mid x) = n\log\beta + n\log\lambda + n\log c + n\log d + \frac{\beta nd}{\gamma} + \gamma\sum_{i=1}^n x_i - \frac{\beta d}{\gamma}\sum_{i=1}^n e^{\gamma x_i}$$
$$+ (c-1)\sum_{i=1}^n \log\big(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x_i}-1)}\big) + (\lambda-1)\sum_{i=1}^n \log\big[1 - \big(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x_i}-1)}\big)^c\big].$$

To calculate the MLE estimators of the EGEG distribution parameters, it is enough to maximize the log-likelihood function concerning the parameters. For this purpose, the partial derivative of the log-likelihood function for each of the parameters is obtained and is set equal to zero. Partial derivatives to parameters are

$$\frac{\partial L(\Theta \mid x)}{\partial\beta} = \frac{n}{\beta} + \frac{nd}{\gamma} - \frac{d}{\gamma}\sum_{i=1}^n e^{\gamma x_i} + \frac{(c-1)d}{\gamma}\sum_{i=1}^n \frac{(e^{\gamma x_i}-1)z_i}{1-z_i}$$
$$- \frac{(\lambda-1)cd}{\gamma}\sum_{i=1}^n \frac{(e^{\gamma x_i}-1)z_i(1-z_i)^{c-1}}{1-(1-z_i)^c} = 0,$$
$$\frac{\partial L(\Theta \mid x)}{\partial\gamma} = \sum_{i=1}^n x_i - \frac{\beta nd}{\gamma^2} + \frac{\beta d}{\gamma}\sum_{i=1}^n e^{\gamma x_i}(\frac{1}{\gamma} - x_i) - d(c-1)\sum_{i=1}^n \frac{Az_i}{1-z_i}$$
$$+ (\lambda-1)cd\sum_{i=1}^n \frac{Az_i(1-z_i)^{c-1}}{1-(1-z_i)^c} = 0,$$
$$\frac{\partial L(\Theta \mid x)}{\partial d} = \frac{n}{d} + \frac{\beta n}{\gamma} - \frac{\beta}{\gamma}\sum_{i=1}^n e^{\gamma x_i} + \frac{(c-1)\beta}{\gamma}\sum_{i=1}^n \frac{(e^{\gamma x_i}-1)z_i}{1-z_i}$$
$$- \frac{(\lambda-1)c\beta}{\gamma}\sum_{i=1}^n \frac{(e^{\gamma x_i}-1)z_i(1-z_i)^{c-1}}{1-(1-z_i)^c} = 0,$$
$$\frac{\partial L(\Theta \mid x)}{\partial c} = \frac{n}{c} + \sum_{i=1}^n \log(1-z_i) - (\lambda-1)\sum_{i=1}^n \frac{(1-z_i)^c\log(1-z_i)}{1-(1-z_i)^c} = 0,$$
$$\frac{\partial L(\Theta \mid x)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \log\big[1-(1-z_i)^c\big] = 0,$$

where

$$z_i = e^{-\frac{\beta d}{\gamma}(e^{\gamma x_i}-1)}, \tag{4.2}$$
$$A = \frac{\beta}{\gamma^2}(e^{\gamma x_i}-1) - \frac{\beta}{\gamma}x_i e^{\gamma x_i}. \tag{4.3}$$

The parameter estimators can not be calculated by analytical methods. Consequently, the Monte Carlo simulation is used to obtain estimators.

4.1.1. Asymptotic confidence bounds

For large samples, the MLEs $(\hat{\beta}, \hat{\gamma}, \hat{\lambda}, \hat{c}, \hat{d})$ have approximately multivariate normal distribution with mean $(\beta, \gamma, \lambda, c, d)$ and the covariance matrix $I^{-1}(\Theta)$, where $\Theta = (\beta, \gamma, \lambda, c, d)$ and $I^{-1}(\Theta)$ is the inverse of the observed information matrix which is defined as follows

$$I_{ij}(\Theta) = \frac{\partial^2 L(\Theta \mid x)}{\partial\theta_i \partial\theta_j} \tag{4.4}$$

$$\left(I^{-1}(\Theta \mid x)\right)_{ij} = Cov(\hat{\theta}_i, \hat{\theta}_j), \quad i, j = 1, 2, 3, 4, 5, \tag{4.5}$$

where $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (\beta, \gamma, \lambda, c, d)$. Thus, we have

$$\begin{aligned}
\frac{\partial^2 L(\Theta \mid x)}{\partial\beta^2} &= -\frac{n}{\beta^2} - \frac{(c-1)d^2}{\gamma^2}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)^2 z_i}{(1-z_i)^2} - \frac{(\lambda-1)cd^2}{\gamma^2}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)^2 z_i(1-z_i)^{c-1}C}{[1-(1-z_i)^c]^2},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial\gamma^2} &= \frac{2\beta n d}{\gamma^3} + d\sum_{i=1}^{n} B - d(c-1)\sum_{i=1}^{n}\frac{Bz_i(1-z_i)+dA^2 z_i(2-z_i)}{(1-z_i)^2}\\
&\quad + cd(\lambda-1)\sum_{i=1}^{n}\frac{1}{[1-(1-z_i)^c]^2}\times\Big\{\big[Bz_i(1-z_i)^{c-1}+dA^2 z_i(1-z_i)^{c-1}\\
&\quad -(c-1)dA^2 z_i^2(1-z_i)^{c-2}\big]\big[1-(1-z_i)^c\big] - cdA^2 z_i^2(1-z_i)^{2c-2}\Big\},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial d^2} &= -\frac{n}{d^2} - \frac{(c-1)\beta^2}{\gamma^2}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)^2 z_i}{(1-z_i)^2} - \frac{(\lambda-1)c\beta^2}{\gamma^2}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)^2 z_i(1-z_i)^{c-1}C}{[1-(1-z_i)^c]^2},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial c^2} &= -\frac{n}{c^2} - (\lambda-1)\sum_{i=1}^{n}\frac{[\log(1-z_i)]^2(1-z_i)^c}{\left(1-(1-z_i)^c\right)^2},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial\lambda^2} &= -\frac{n}{\lambda^2},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial\beta\partial\lambda} &= -\frac{cd}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)z_i(1-z_i)^{c-1}}{1-(1-z_i)^c},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial\gamma\partial\lambda} &= cd\sum_{i=1}^{n}\frac{Az_i(1-z_i)^{c-1}}{1-(1-z_i)^c},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial d\partial\lambda} &= -\frac{c\beta}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)z_i[1-z_i]^{c-1}}{1-(1-z_i)^c},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial c\partial\lambda} &= -\sum_{i=1}^{n}\frac{(1-z_i)^c\log(1-z_i)}{1-(1-z_i)^c},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial\beta\partial c} &= \frac{d}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)z_i}{1-z_i} - \frac{(\lambda-1)d}{\gamma}\sum_{i=1}^{n}(e^{\gamma x_i}-1)z_i(1-z_i)^{c-1}\Big\{\frac{1}{1-(1-z_i)^c} + \frac{c\log(1-z_i)}{[1-(1-z_i)^c]^2}\Big\},\\
\frac{\partial^2 L(\Theta \mid x)}{\partial\gamma\partial c} &= -d\sum_{i=1}^{n}\frac{Az_i}{1-z_i} + cd(\lambda-1)\sum_{i=1}^{n}\frac{Az_i(1-z_i)^{c-1}\log(1-z_i)}{[1-(1-z_i)^c]^2} + d(\lambda-1)\sum_{i=1}^{n}\frac{Az_i(1-z_i)^{c-1}}{1-(1-z_i)^c},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 L(\Theta \mid x)}{\partial d \partial c} &= \frac{\beta}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)z_i}{1-z_i} - \frac{(\lambda-1)\beta c}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)z_i(1-z_i)^{c-1}\log(1-z_i)}{[1-(1-z_i)^c]^2} \\
&\quad - \frac{(\lambda-1)\beta}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)z_i(1-z_i)^{c-1}}{1-(1-z_i)^c}, \\
\frac{\partial^2 L(\Theta \mid x)}{\partial \gamma \partial d} &= -\frac{\beta n}{\gamma^2} + \frac{\beta}{\gamma^2}\sum_{i=1}^{n} e^{\gamma x_i} - \frac{\beta}{\gamma}\sum_{i=1}^{n} x_i e^{\gamma x_i} - (c-1)\sum_{i=1}^{n}\frac{Az_i}{1-z_i} \\
&\quad + \frac{(c-1)d\beta}{\gamma}\sum_{i=1}^{n}\frac{A(e^{\gamma x_i}-1)z_i}{(1-z_i)^2} + c(\lambda-1)\sum_{i=1}^{n}\frac{Az_i(1-z_i)^{c-1}}{1-(1-z_i)^c} \\
&\quad - \frac{(\lambda-1)cd\beta}{\gamma}\sum_{i=1}^{n}\frac{A(e^{\gamma x_i}-1)z_i(1-z_i)^{c-1}}{[1-(1-z_i)^c]^2}\left(\frac{z_i}{1-z_i}-1\right)[1-(1-z_i)^c], \\
\frac{\partial^2 L(\Theta \mid x)}{\partial \gamma \partial \beta} &= -\frac{n}{\gamma^2} + \frac{d}{\gamma^2}\sum_{i=1}^{n} e^{\gamma x_i} - \frac{d}{\gamma}\sum_{i=1}^{n} x_i e^{\gamma x_i} - d(c-1)\sum_{i=1}^{n}\frac{\frac{A}{\beta}z_i(1-z_i) - \frac{d}{\gamma}(e^{\gamma x_i}-1)Az_i}{(1-z_i)^2} \\
&\quad + cd(\lambda-1)\sum_{i=1}^{n}\frac{[\frac{A}{\beta}z_i(1-z_i)^{c-1} - \frac{d}{\gamma}(e^{\gamma x_i}-1)Az_i(1-z_i)^{c-1}][1-(1-z_i)^c]}{[1-(1-z_i)^c]^2} \\
&\quad + \frac{cd^2(\lambda-1)}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)Az_i^2(1-z_i)^{c-2}[(c-1)+(1-z_i)^c]}{[1-(1-z_i)^c]^2},
\end{aligned}
$$

where z_i and A are given in expressions (4.2) and (4.3), respectively. In addition

$$
\begin{aligned}
B &= -\frac{2\beta}{\gamma^3}(e^{\gamma x_i}-1) + \frac{2\beta}{\gamma^2}x_i e^{\gamma x_i} - \frac{\beta}{\gamma}x_i^2 e^{\gamma x_i}, \\
C &= (1-z_i)^c + z_i(1-z_i)^{c-1} + \frac{(c-1)z_i}{1-z_i} - 1.
\end{aligned}
$$

The approximate $(1-\alpha)100\%$ confidence intervals for parameters $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (\beta, \gamma, \lambda, c, d)$, are

$$
\hat{\theta}_i \pm \xi_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta}_i)}, \quad i = 1, 2, 3, 4, 5,
$$

where $\xi_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ percentile of the standard normal distribution.

4.2. LS method

To estimate the new model parameters, it is enough to minimize the following function with respect to parameters

$$
LS(\Theta, \mathbf{x}) = \sum_{i=1}^{n}\left(G(x_{(i)}) - u_i\right)^2, \quad x_{(i)} > 0, \tag{4.6}
$$

where $x_{(i)}$ is i^{th} order sample, $\Theta = (\beta, \gamma, \lambda, c, d)$, and $u_i = \frac{i}{n+1}$. By putting expression (2.4) into expression (4.6), the LS function is given by

$$
LS(\Theta, \mathbf{x}) = \sum_{i=1}^{n}\left(1 - \left[1 - \left(1 - e^{-\frac{\beta d}{\gamma}(e^{\gamma x_{(i)}}-1)}\right)^c\right]^\lambda - u_i\right)^2. \tag{4.7}
$$

Now, partial derivatives of the function (4.7) relative to $\Theta = (\beta, \gamma, \lambda, c, d)$ parameters are obtained and then are set equal to zero.

$$\frac{\partial LS}{\partial \beta} = \frac{2\lambda cd}{\gamma} \sum_{i=1}^{n} (e^{\gamma x_{(i)}} - 1) z_{(i)} (1 - z_{(i)})^{c-1} \left[1 - (1 - z_{(i)})^{c}\right]^{\lambda - 1} B_i = 0, \tag{4.8}$$

$$\frac{\partial LS}{\partial \gamma} = -2\lambda cd \sum_{i=1}^{n} z_{(i)} (1 - z_{(i)})^{c-1} A_i B_i = 0, \tag{4.9}$$

$$\frac{\partial LS}{\partial d} = \frac{2\lambda c\beta}{\gamma} \sum_{i=1}^{n} z_{(i)} \log(z_{(i)}) (1 - z_{(i)})^{c-1} \left[1 - (1 - z_{(i)})^{c}\right]^{\lambda - 1} B_i = 0, \tag{4.10}$$

$$\frac{\partial LS}{\partial c} = 2\lambda \sum_{i=1}^{n} (1 - z_{(i)})^{c} \log(1 - z_{(i)}) \left[1 - (1 - z_{(i)})^{c}\right]^{\lambda - 1} B_i = 0, \tag{4.11}$$

$$\frac{\partial LS}{\partial \lambda} = -2 \sum_{i=1}^{n} \left[1 - (1 - z_{(i)})^{c}\right]^{\lambda} \log\left(1 - (1 - z_{(i)})^{c}\right) B_i = 0, \tag{4.12}$$

where $z_{(i)} = e^{\frac{\beta d}{\gamma}(e^{\gamma x_{(i)}} - 1)}$ and

$$A_i = \frac{\beta}{\gamma^2}(e^{\gamma x_{(i)}} - 1) - \frac{\beta}{\gamma} x_{(i)} e^{\gamma x_{(i)}},$$
$$B_i = 1 - \left[1 - \left(1 - z_{(i)}\right)^{c}\right]^{\lambda} - u_i.$$

The LS estimators are obtained by solving Equations (4.8), (4.9), (4.10), (4.11), and (4.12).

4.3. Bayes method

Let $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (\beta, \gamma, \lambda, c, d)$ be the vector of parameters of the EGEG distribution and suppose that these have independent Uniform prior distributions, with pdf given by

$$\pi(\theta_i) = \frac{1}{b_i - a_i}, \quad a_i \le \theta_i \le b_i, \quad i = 1, ..., 5. \tag{4.13}$$

The joint posterior pdf is defined as

$$g(\Theta \mid \mathbf{x}) = \frac{l(\mathbf{x} \mid \Theta)\pi(\beta)\pi(\gamma)\pi(\lambda)\pi(c)\pi(d)}{\int_{a_5}^{b_5}\int_{a_4}^{b_4}\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} L(\mathbf{x} \mid \Theta)\pi(\beta)\pi(\gamma)\pi(\lambda)\pi(c)\pi(d) d\Theta}. \tag{4.14}$$

Since the denominator in equation (4.14) contains a five-fold integral, direct calculation is challenging. Therefore, we employ the "important sampling" method (Rubinstein and Kroese, 2016). The following section outlines this technique.

4.3.1. Important sampling technique

The algorithm of the important sampling technique has the following steps:

Algorithm 1 Important sampling

1: A large sample of N size, from the prior distributions of the parameters is generated. This sample is represented by $\Theta_1 = (\beta_1, \gamma_1, \lambda_1, c_1, d_1), ..., \Theta_N = (\beta_N, \gamma_N, \lambda_N, c_N, d_N)$.
2: For $i = 1, 2, ..., N$, the values of $f(\mathbf{x} \mid \Theta_i)$ are calculated using the $\mathbf{x}$ observation vector.
3: For $i = 1, 2, ..., N$, the values of $C_i = \frac{f(\mathbf{x}|\Theta_i)}{\sum_{i=1}^{N} f(\mathbf{x}|\Theta_i)}$ are calculated.
4: Each function in form $E\big(h(\Theta \mid \mathbf{x})\big)$ can be estimated with the sum of $\sum_{i=1}^{N} C_i h(\Theta_i)$.

In the Bayes method, usually, the mean of the posterior distribution is considered as the parameter estimators. The relations $\sum_{i=1}^{N} C_i\beta_i$, $\sum_{i=1}^{N} C_i\gamma_i$, $\sum_{i=1}^{N} C_i\lambda_i$, $\sum_{i=1}^{N} C_ic_i$ and $\sum_{i=1}^{N} C_id_i$ in step 4, are calculated as the $\beta, \gamma, \lambda, c$ and d estimators, respectively.

4.4. Simulation study

For estimating the EGEG distribution parameters using the MLE, LS, and Bayes methods, we encountered equations that were not solvable by analytical methods. Consequently, these methods are compared using Monte Carlo simulation and with the help of suitable numerical methods. Suppose the sample size $n = 20, 40, 60, 80, 100, 120$ and the number of repetitions $m = 1000$ are considered. In the first step, a random sample of this distribution using the inverse transform method is generated with parameters $\beta = 5, \gamma = 3, d = 1.5, c = 2.5$, and $\lambda = 1.28$. The inverse transform method is as follows

1. Generate U from Uniform(0,1),
2. Return $X = G^{-1}(U)$,

where $X = \frac{1}{\gamma} \log\left\{1 - \frac{\gamma}{\beta d} \log\left[1 - \left(1 - (1-U)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]\right\}$. Then, the parameter estimators are obtained using the Monte Carlo simulation. Here, criteria such as variance, bias, and mean square error (MSE) are used to compare estimators. The formulas for the bias and the MSE are respectively

$$Bias(\hat{\Theta}) = E(\hat{\Theta}) - \Theta, \tag{4.15}$$

$$MSE(\hat{\Theta}) = var(\hat{\Theta}) - [Bias(\hat{\Theta})]^2. \tag{4.16}$$

The best estimator is the one with the lowest variance and MSE and its bias close to zero. The results of simulation studies are summarized in Figures 2, 3, 4, 5, and 6.

Based on these Figures, for large samples, the MLE method is the most appropriate method of estimation. The criteria bias, variance, and MSE confirm these results.

5. GOODNESS OF FIT TESTS AND MODEL SELECTION CRITERIA

In this section, several criteria have been used to compare different distributions in terms of the ability to fit into real data. The most widely used criteria are: root mean

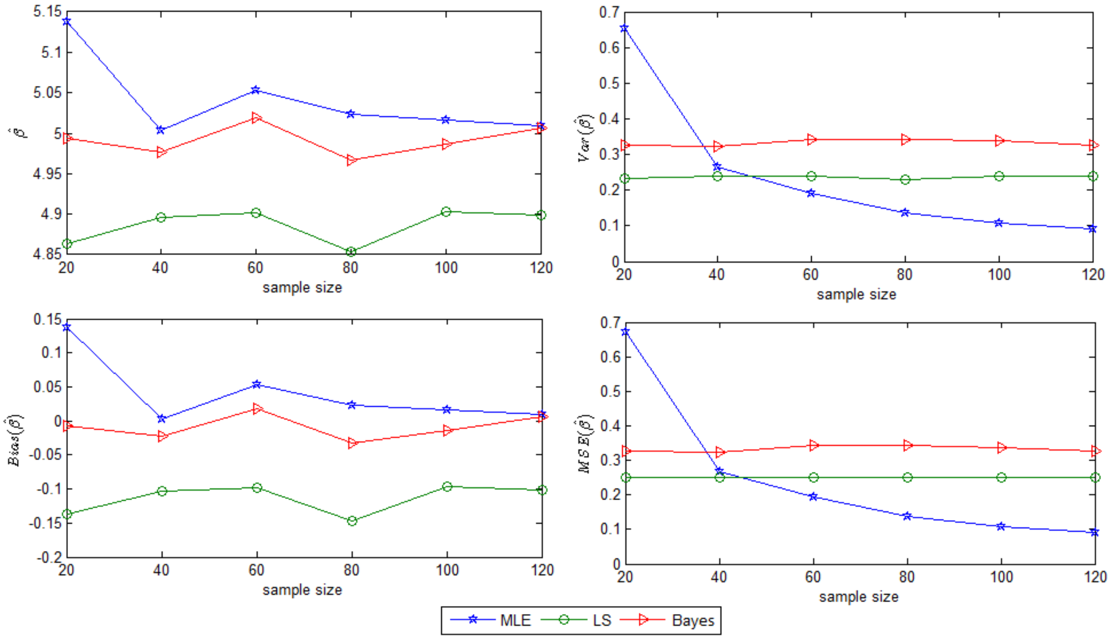

Figure 2: Comparison of the MLE, LS, and Bayes estimators for β.

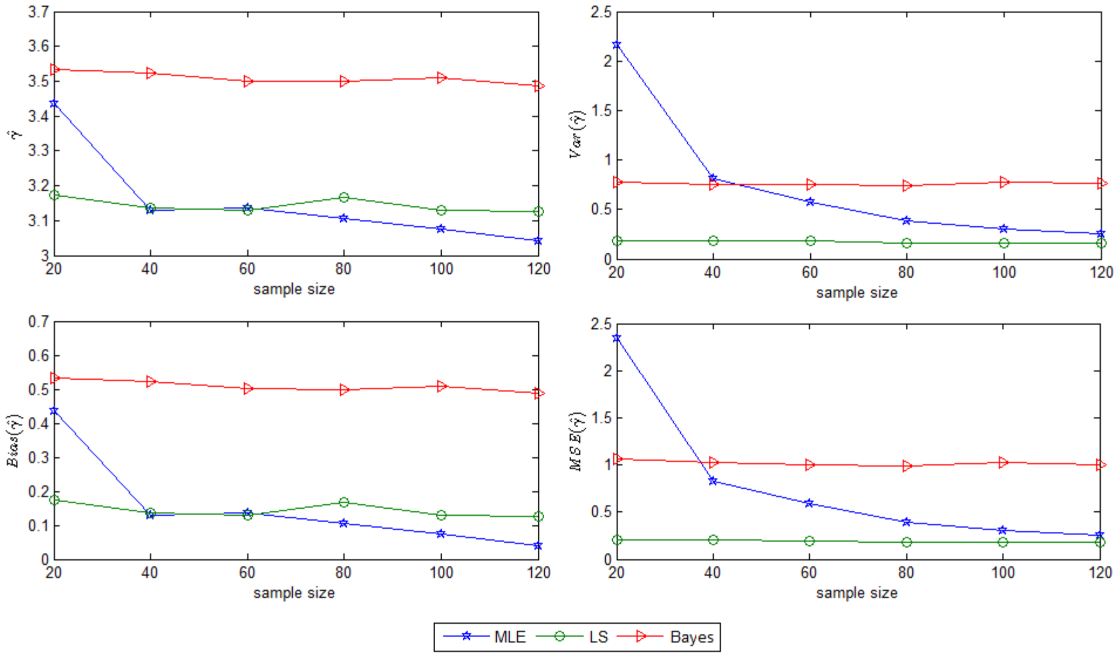

Figure 3: Comparison of the MLE, LS, and Bayes estimators for γ.

square errors (RMSE), coefficient of determination (R^2), Kolmogorov-Simirnov test (KS), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criteria (CAIC) and log-likelihood function (L). The formulas for calculating these criteria are listed in Table 2, where q is the number of parameters, n is the sample size,

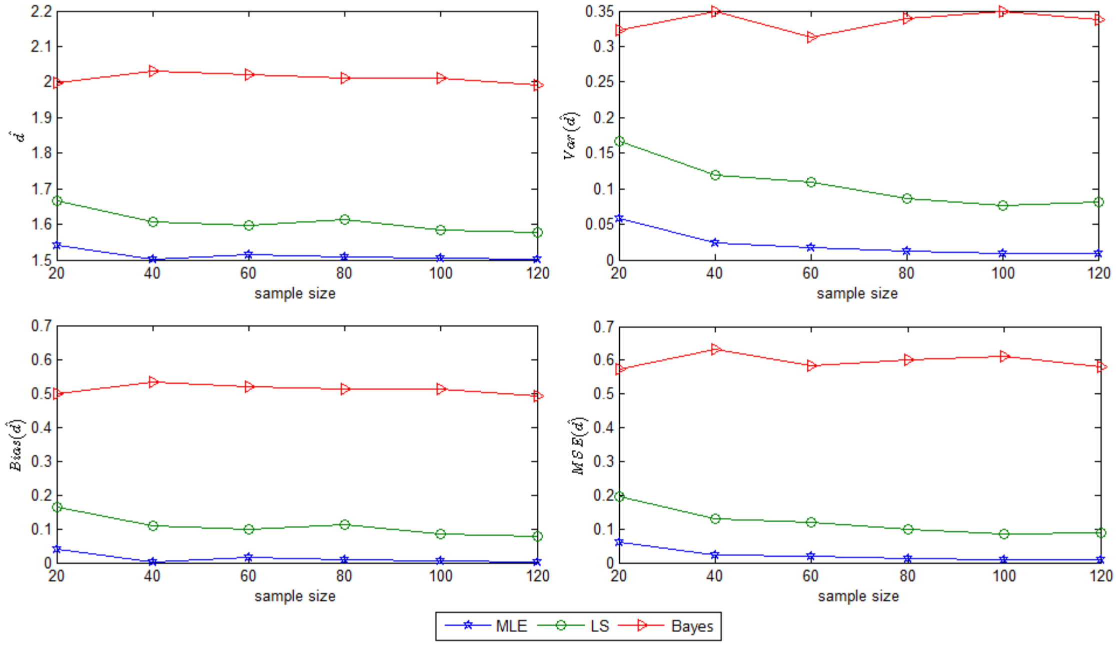

Figure 4: Comparison of the MLE, LS, and Bayes estimators for d.

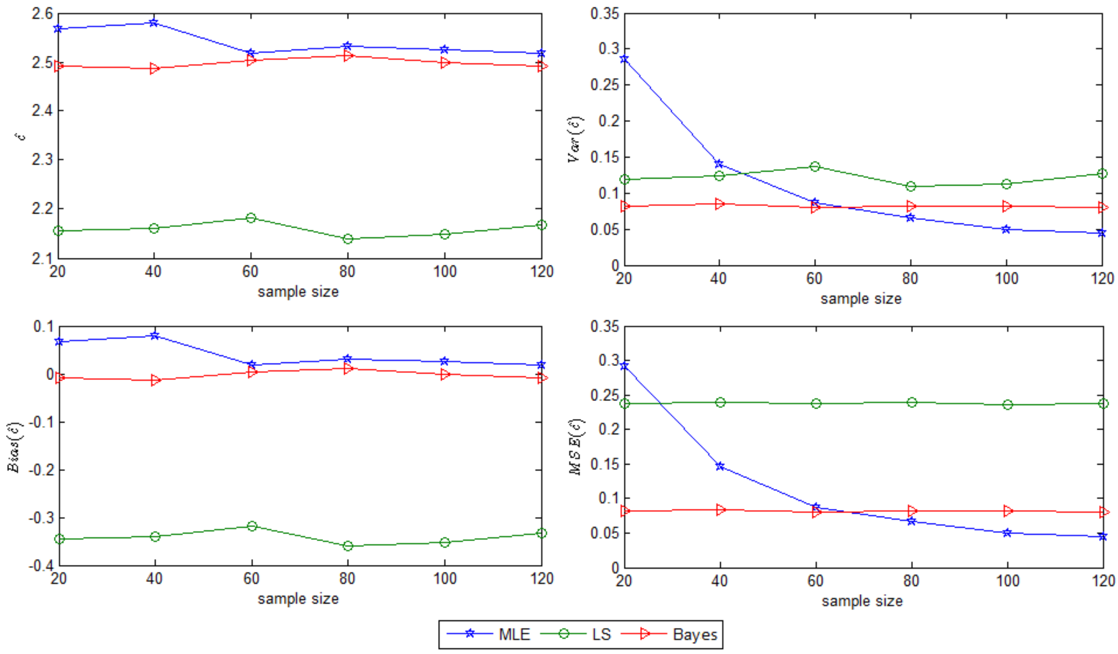

Figure 5: Comparison of the MLE, LS, and Bayes estimators for c.

and $x_{(i)}$ is the i^{th} ascending order observation.

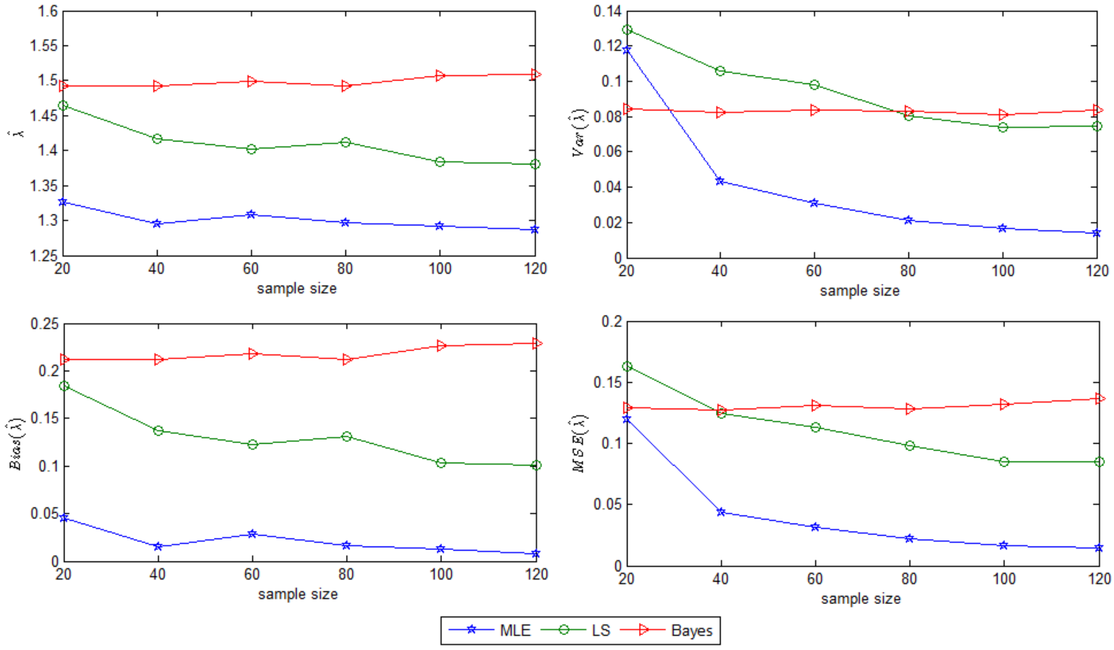

Figure 6: Comparison of the MLE, LS, and Bayes estimators for λ.

Table 2: The formulas of criteria for model evaluation.

Criteria	Formulas
KS	$KS = max_{1\leq i\leq n}\{\lvert\frac{i}{n} - F(x_{(i)})\rvert, \lvert F(x_{(i)}) - \frac{i-1}{n}\rvert\}$
RMSE	$RMSE = \left[\frac{\sum_{i=1}^{n}\left(\hat{F}(x_{(i)} - \frac{i}{n+1})\right)^2}{n}\right]^{\frac{1}{2}}$
R^2	$R^2 = 1 - \frac{\sum_{i=1}^{n}\left(\hat{F}(x_{(i)} - \frac{i}{n+1})\right)^2}{\sum_{i=1}^{n}\left(\hat{F}(x_{(i)} - \bar{\hat{F}}(x_{(i)}))\right)^2}$
AIC	$AIC = -2L + 2q$
BIC	$BIC = -2L + q\log(n)$
CAIC	$CAIC = -2L + \frac{2qn}{n-q-1}$

6. APPLICATION

This section presents the new model application on real data. The used data are the windmill data from Kotb and Raqab (2017). This data set has been listed in Table 3. The EGEG distribution performance is compared with odd generalized Gompertz (OGG), generalized Gompertz (GG), and Gamma (Ga) distributions. The pdf of these distributions, defined for positive parameters, are listed in Table 4. Table 5 shows the MLE estimators of unknown parameters for the selected distributions. In Table 6, the performance of the proposed model is evaluated against the selected distributions. Figures 7, 8, and 9 show the empirical density and cumulative distribution, histogram and theoretical densities, and Empirical and theoretical CDFs on windmill data, respectively.

Table 3: Windmill data

0.123	0.5	0.558	0.653	1.057	1.137	1.144	1.194	1.501	1.562
1.582	1.737	1.800	1.822	1.866	1.930	2.088	2.112	2.166	2.179
2.236	2.294	2.303	2.310	2.386					

Table 4: PDF of the fitted models

Distribution	PDF, g(x)
EGEG	$\beta\lambda cd\big(e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\big)^{d}\big[1-\big(e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\big)^{d}\big]^{c-1}\big\{1-\big[1-\big(e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\big)^{d}\big]^{c}\big\}^{\lambda-1}I_{[0,\infty)}(x)$
OGG	$ab\beta e^{\gamma x}e^{\frac{\beta}{\gamma}(e^{\gamma x}-1)}e^{-a[e^{\frac{\beta}{\gamma}(e^{\gamma x}-1)}-1]}\big\{1-e^{-a[e^{\frac{\beta}{\gamma}(e^{\gamma x}-1)}-1]}\big\}^{b-1}I_{[0,\infty)}(x)$
GG	$b\beta e^{\gamma x}e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\big[1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\big]^{b-1}I_{[0,\infty)}(x)$
Ga	$\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}I_{[0,\infty)}(x)$

Table 5: MLE

Distribution	EGEG	OGG	GG	Ga
MLE	$\hat{\beta}=2.218e-05$	$\hat{\beta}=1.290e-06$	$\hat{\beta}=2.992e-06$	$\hat{\alpha}=3.574639$
	$\hat{\gamma}=6.826e+00$	$\hat{\gamma}=6.612e+00$	$\hat{\gamma}=6.068e+00$	$\hat{\beta}=2.220997$
	$\hat{d}=6.961e-02$	$\hat{a}=1.195e-01$	$\hat{b}=1.109e-01$	
	$\hat{c}=1.949e-01$	$\hat{b}=1.464e-01$		
	$\hat{\lambda}=8.224e-01$			

6.1. Result

According to Table 6 and Figures 8, and 9, the superiority of the EGEG distribution for modeling real data is shown. This is evident from its highest L, Pvalue, and R^2, and its lowest AIC, BIC, CAIC, and RMSE.

7. CONCLUSION

This study proposed a new family of distributions. This family contains several commonly used distributions in statistical analysis as submodels. Some important statistical properties of the EGEG distribution, such as quantile function, moments, and order statistics were obtained. The distribution parameters were estimated using three methods MLE, LS, and Bayes. Simulation studies were presented to compare estimation methods. Finally, an application of this new distribution was presented to prove its suitability and it turns out that this distribution fits better on windmill data than OGG, GG, and Ga distributions.

Table 6: Goodness-of-fit statistics for windmill data

Distribution	L	AIC	BIC	CAIC	KS	Pvalue	RMSE	R^2
EGEG	−19.70936	49.41872	55.5131	52.5766	0.111	0.8847	0.0465931	0.9768793
OGG	−31.76159	71.52318	76.39868	73.52318	0.2061	0.2078	0.08630715	0.851494
GG	−24.79431	55.58862	59.24525	56.73148	0.2468	0.07936	0.1459928	0.5862012
Ga	−28.95077	61.90153	64.33929	62.44699	7.6624	$2.22e-16$	0.09863256	0.8650929

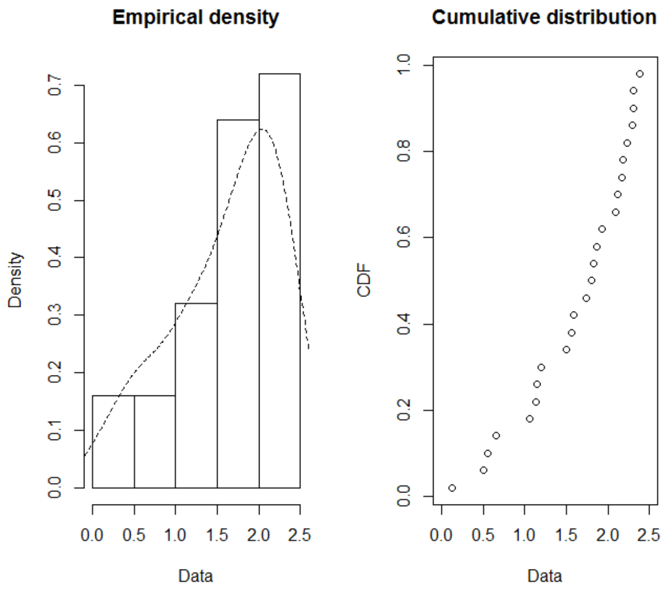

Figure 7: Empirical density and cumulative distribution for windmill data.

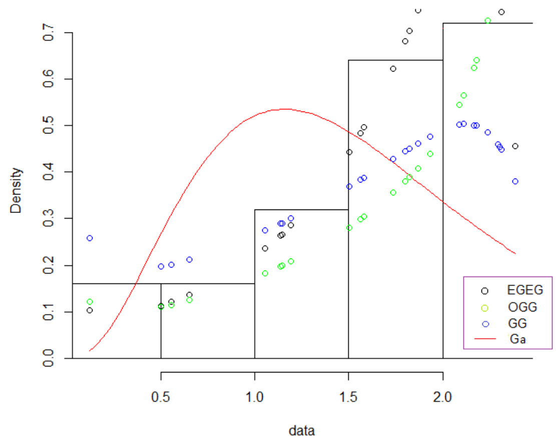

Figure 8: Histogram and theoretical densities.

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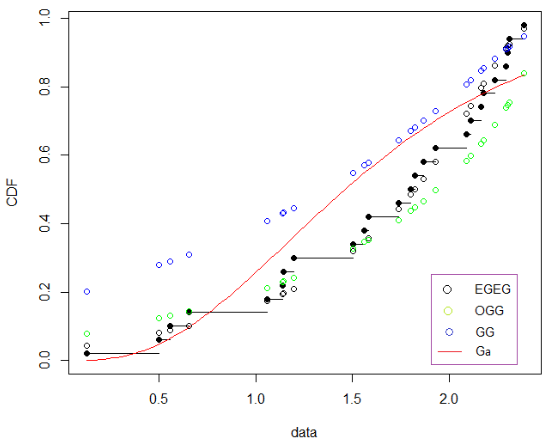

Figure 9: Empirical and theoretical CDFs.

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