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Ordering Properties of the Smallest and Largest Order Statistics from Exponentiated Location-Scale Models Under Random Shocks

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Abstract:

- In this paper, we discuss stochastic comparisons of lifetimes of series and parallel systems when the components are exponentiated location-scale models under random shocks. The results established here are developed in two directions. First, the comparisons are carried out with respect to usual stochastic ordering by using the concept of vector majorization for series and parallel systems. Next, when the matrix of parameters changes to another matrix of parameters in the sense of multivariate chain majorization, we study the usual stochastic order of the smallest order statistics when each component receives a random shock.

Keywords:

- *exponentiated location-scale family; matrix majorization; random shock; usual stochastic order; vector majorization.*

AMS Subject Classification:

- 60E15, 60K10.

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1. INTRODUCTION

In actuarial science, it is often of interest to compare stochastically extreme claim amounts from heterogeneous portfolios. In this regard, in the present work, we compare the extreme order statistics arising from two heterogeneous portfolios in the sense of the usual stochastic ordering. It is assumed that the portfolios belong to the general exponentiated location-scale model. The general exponentiated location-scale model includes several important statistical models such as generalized exponential distribution, generalized Weibull distribution, generalized Pareto distribution and many more. The exponentiated location-scale model has three types of parameters: location, scale and shape(or skewness) parameters. Location parameter is useful in modeling the insurance related data, since an insurance company suffers a claim from policyholder after a certain period of time from the date of beginning of the policy. Also, most of the data dealing with health care costs and economy are skewed. In finance, an investigator may often have small gains, but occasionally may have a few large losses. In this case, the data will invariably be negatively skewed. If, howeve, we have a reverse situation, the data in this case will be positively skewed. Thus, fitting skewed models to these types of data in finance and some other fields is a very important issue. in order to fit a skewed data, we require a model having a skewness parameter. The general exponentiated location-scale model possesses this kind of flexibility. For this reason, such a general model is useful for fitting various kinds of data sets. In practical situations, the extreme claim amounts play an important role as these provide useful information for determining annual premium. In actuarial science, it is an important issue in expressing preferences between random future gains or losses. In this direction, comparisons of claim amounts in two heterogeneous portfolio of risks based on different stochastic ordering such as usual become very useful. Order statistics have received a great amount of atention from various authors. It plays an important role in several areas of probability and statistics such as reliability theory, queueing theory, and survival analysis. Let $X_{1:n} \leq ... \leq X_{n:n}$ denote the order statistics corresponding to the random variables $X_1, ..., X_n$, where $X_{1:n}$ and $X_{n:n}$ corresponds to the sample minimum and sample maximum, respectively. The results of stochastic comparisons of the order statistics with independent and dependent sampling units can be seen in Dykstra *et al.* (1997) [9], Zhao and Balakrishnan (2011) [30], Li and Li (2015) [25], Torrado (2015) [28], Torrado and Kochar (2015) [29], Kochar and Torrado (2015) [18], Kundu *et al.* (2016) [24], Kundu and Chowdhury (2016 [19], 2018 [20], 2019 [21], 2020 [22]), Chowdhury and Kundu (2017 [4], 2018 [5]), Hazra *et al.* (2017) [14], Fang and Zhang (2013) [12], Fang and Xu (2019) [13], Das *et al.* (2020) [8], Kundu *et al.* (2022) [23], Chowdhury *et al.* (2022) [6], and the references there in for a variety of parametric models. The assumption in the papers lies in the fact that each of the order statistics $X_{1:n}, X_{2:n}, ..., X_{n:n}$ occurs with certainty and the comparison is carried out on the minimums or the maximums of the order statistics. Now, it may so happen that the order statistics experience random shocks which may or may not result in its occurrence and it is of interest to compare two such systems stochastically with respect to vector or matrix majorization. A random variable X is said to follow the exponentiated location-scale model if it's cumulative distribution function is given by

$$F_X(x; \lambda, \theta, \alpha) = \left[F\left(\frac{x-\lambda}{\theta}\right)\right]^{\alpha}, \qquad x > \lambda, \tag{1.1}$$

where $\lambda \in R$, $\alpha > 0$, $\theta > 0$ and F is the baseline distribution function. Here, λ, θ and α are respectively known as the location, scale, and shape parameters. We write $X \sim ELS(\lambda, \theta, \alpha)$

if X has the distribution function given by (1.1). The probability density function of the exponentiated location-scale model with (1.1) is denoted by f_X. In particular, when $\alpha = 1$, the model given in (1.1) reduces to the location-scale family of distributions. Further, when $\alpha = 1$ and $\lambda = 0$, (1.1) reduces to the scale family and when $\alpha = 1$ and $\theta = 1$, (1.1) becomes location family. The model in (1.1) coincides with the exponentiated-scale family when the location parameter λ is equal to 0. Thus (1.1) is a general family of distribution with great flexibility.

Let us assume series and parallel systems consist of n independent components in working conditions. Each component of the system receives a shock which may cause the component to fail. Let the random variable X_i denote lifetime of the i-th component in the system which experiences a random shock at binging. Also, suppose that I_{p_i} denotes independent Bernoulli random variables, independent of X_i's, with $E(I_{p_i}) = p_i$, where p_i will be called shock parameter hereafter. Then, the random shock impacts the i-th component ($I_{p_i} = 1$) with probability p_i or doesn't impact the i-th component ($I_{p_i} = 0$) with probability $1 - p_i$. Hence, the random variable $Y_i = I_{p_i} X_i$ corresponds to the lifetime of i-th component in a system under shock. Fang and Balakrishnan (2018) [10] have compared two such systems with generalized Birnbaum–Saunders components. Similar comparisons are carried out in the context of the insurance where largest or smallest claim amounts in a portfolio of risks are compared stochastically. One may refer to Barmalzan *et al.* (2017) [3], and Balakrishnan *et al.* (2018) [2].

The survival function of $Y_{1:n} = \min\{Y_1, ..., Y_n\}$ is given by

$$\bar{F}_{Y_{1:n}}(x; \underset{\sim}{p}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \underset{\sim}{\alpha}) = \prod_{i=1}^{n} p_i \left[1 - F^{\alpha_i}\left(\frac{x - \lambda_i}{\theta_i}\right)\right], \qquad x > \max\{\lambda_i,\ i = 1, ..., n\}, \tag{1.2}$$

and the cumulative distribution function of $Y_{n:n} = \max\{Y_1, ..., Y_n\}$ is given by

$$F_{Y_{n:n}}(x; \underset{\sim}{p}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \underset{\sim}{\alpha}) = \prod_{i=1}^{n} \left[1 - p_i \left[1 - F^{\alpha_i}\left(\frac{x - \lambda_i}{\theta_i}\right)\right]\right], \qquad x > \max\{\lambda_i,\ i = 1, ..., n\}, \tag{1.3}$$

where $\underset{\sim}{x} = (x_1, ..., x_n) \in I^n$ be any real vector and I^n denote a n-dimensional Euclidean space where $I \subseteq R$. Hereafter, we assume that $Y^*_{1:n}(Y^*_{n:n})$ denotes similarly the smallest (largest) order statistic arising from $Y^*_i = I_{p^*_i} X^*_i$, $i = 1, ..., n$, where $X^*_1, ..., X^*_n$ are independent non-negative random variables with $X^*_i \sim ELS(\lambda^*_i, \theta^*_i, \alpha^*_i)$, $i = 1, ..., n$ and $I_{p^*_1}, ..., I_{p^*_n}$ are independent Bernoulli random variables, independent of X^*_i's, with $E(I_{p^*_i}) = p^*_i$, $i = 1, ..., n$. Let

$$\begin{aligned}
P_n &= \left\{(\underset{\sim}{x}, \underset{\sim}{y}; n) :\ x_i > 0,\ y_j > 0 \text{ and } (x_i - x_j)(y_i - y_j) \leqslant 0,\ i, j = 1, ..., n\right\}, \\
S_n &= \left\{(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}; n) :\ x_i, y_j, z_k > 0 \text{ and } x_i \leqslant (\geqslant) x_j,\ y_i \geqslant (\leqslant) y_j,\ z_i \geqslant (\leqslant) z_j\right\}, \\
N_n &= \left\{(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}; n) :\ z_i \geq 1,\ x_i > 0,\ y_i > 0,\ x_i \leq (\geq) x_j,\ y_i \leq (\geq) y_j,\ z_i \leq (\geq) z_j\right\}, \\
N^*_n &= \left\{(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}; n) :\ z_i > 0,\ x_i > 0,\ y_i > 0,\ x_i \leq (\geq) x_j,\ y_i \leq (\geq) y_j,\ z_i \leq (\geq) z_j\right\}, \\
U_n &= \left\{(\underset{\sim}{w}, \underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}; n) :\ w_i, x_j, y_k, z_l > 0,\ w_i \leqslant (\geqslant) w_j,\ x_i \geqslant (\leqslant) x_j,\ y_i \geqslant (\leqslant) y_j,\ z_i \geqslant (\leqslant) z_j\right\}.
\end{aligned}$$

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and fundamental lemmas. In Section 3, we establish some ordering properties for the smallest and largest order statistics of the ELS model with associated random shocks. In Section 4, some special cases of our main results are added. Section 5 provides applications of the established results. Finally, in Section 6, we include some concluding.

2. PRELIMINARIES

In this section we provide some preliminary definitions and lemmas which will be useful in the sequel. To compare lifetimes of series and parallel systems, stochastic orders have been extensively used in the literature. Below, we present a few of them. Throughout the paper, we use the notations $R=(-\infty,+\infty)$, $R_+=(0,+\infty)$ and $a \overset{\text{sign}}{=} b$ means that a and b have the same sign. Let X be non-negative random variable with distribution function F, and density function f. The survival function, hazard rate, and reversed hazard rate are $\bar{F}=1-F$, $r_X=\frac{f}{\bar{F}}$, and $\tilde{r}_X=\frac{f}{F}$, respectively.

Definition 2.1. Let X and Y be two absolutely continuous random variables with respective supports (l_X,u_X) and (l_Y,u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then, X is said to be smaller than Y in usual stochastic (st) order, denoted as $X \leqslant_{st} Y$, if $\bar{F}_X(t) \leqslant \bar{F}_Y(t)$ for all $t \in (-\infty,+\infty)$.

Let $\underset{\sim}{x}=(x_1,...,x_n)\in I^n$ and $\underset{\sim}{y}=(y_1,...,y_n)\in I^n$ be any two real vectors with $x_{(1)},...,x_{(n)}$ being the increasing arrangement of the components of the vector $\underset{\sim}{x}$. The following definitions may be found in Marshall *et al.* (2011) [27].

Definition 2.2.

(i) The vector $\underset{\sim}{x}$ is said to majorize the vector $\underset{\sim}{y}$ (written as $\underset{\sim}{x} \overset{m}{\geqslant} \underset{\sim}{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leqslant \sum_{i=1}^{j} y_{(i)},\ j=1,...,n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)};$$

(ii) The vector $\underset{\sim}{x}$ is said to weakly supermajorize the vector $\underset{\sim}{y}$ (written as $\underset{\sim}{x} \overset{w}{\geqslant} \underset{\sim}{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leqslant \sum_{i=1}^{j} y_{(i)} \qquad \text{for} \quad j=1,...,n;$$

(iii) The vector $\underset{\sim}{x}$ is said to weakly submajorize the vector $\underset{\sim}{y}$ (written as $\underset{\sim}{x} \geqslant_w \underset{\sim}{y}$) if

$$\sum_{i=j}^{n} x_{(i)} \geqslant \sum_{i=j}^{n} y_{(i)} \qquad \text{for} \quad j=1,...,n;$$

(iv) The vector $\underset{\sim}{x}$ is said to be p-larger than the vector $\underset{\sim}{y}$ (written as $\underset{\sim}{x} \overset{p}{\geqslant} \underset{\sim}{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leqslant \prod_{i=1}^{j} y_{(i)} \qquad \text{for} \quad j=1,...,n;$$

(v) The vector $\underset{\sim}{x}$ is said to reciprocally majorize the vector $\underset{\sim}{y}$ (written as $\underset{\sim}{x} \overset{rm}{\geqslant} \underset{\sim}{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \geqslant \sum_{i=1}^{j} \frac{1}{y_{(i)}} \qquad \text{for} \quad j = 1, ..., n.$$

It is not difficult to show that $\underset{\sim}{x} \overset{m}{\geqslant} \underset{\sim}{y} \Rightarrow \underset{\sim}{x} \overset{w}{\geqslant} \underset{\sim}{y} \Rightarrow \underset{\sim}{x} \overset{p}{\geqslant} \underset{\sim}{y} \Rightarrow \underset{\sim}{x} \overset{rm}{\geqslant} \underset{\sim}{y}$.

Definition 2.3. A function $\psi\colon I^n \to R$ is said to be schur-convex (schur-concave) on I^n if

$$\underset{\sim}{x} \overset{m}{\geqslant} \underset{\sim}{y} \quad \text{implies} \quad \psi(\underset{\sim}{x}) \geqslant (\leqslant) \psi(\underset{\sim}{y}) \quad \text{for all} \quad \underset{\sim}{x}, \underset{\sim}{y} \in I^n.$$

The following definitions related to matrix majorization may be found in Marshall *et al.* (2011) [27].

Definition 2.4.

(i) A square matrix Π_n, of order n, is said to be a permutation matrix if each row and column has a single entry 1, and all other entries as zero;

(ii) A square matrix $P = (p_{ij})$, of order n, is said to be doubly stochastic if $p_{ij} \geqslant 0$, for all $i, j = 1, ..., n$, $\sum_{i=1}^{n} p_{ij} = 1$, $j = 1, ..., n$ and $\sum_{j=1}^{n} p_{ij} = 1$, $i = 1, ..., n$;

(iii) A square matrix T_w, is said to be T-transform matrix if it has form $T_w = wI + (1 - w)\Pi$; where $0 \leqslant w \leqslant 1$, I is the identity matrix and Π is the permutation matrix. Let $T_{w_1} = w_1 I + (1 - w_1)\Pi_1$ and $T_{w_2} = w_2 I + (1 - w_2)\Pi_2$ be two transform matrices, where $0 \leq w_1, w_2 \leq 1$ and Π_1 and Π_2 are two permutation matrices that interchange two coordinates. Then, we say T_{w_1} and T_{w_2} have the same structure if $\Pi_1 = \Pi_2$, where Π_1 and Π_2 are permutation matrices with the same dimension, otherwise they are different structures.

Definition 2.5. Consider the $m \times n$ matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ with rows $a_1, ..., a_m$ and $b_1, ..., b_n$, respectively.

(i) A is said to be larger than B in chain majorization, denoted by $A \gg B$, if there exists a finite set of $n \times n$ T-transform matrices $T_1, ..., T_k$ such that $B = AT_1 \cdots T_k$;

(ii) A is said to majorize B, denoted by $A > B$, if $A = BP$, where the $n \times n$ matrix P is doubly stochastic. Since a product of T-transforms is doubly stochastic, it follows that $A \gg B \Rightarrow A > B$;

(iii) A is said to be larger than the matrix B in row majorization, denoted by $A \overset{row}{>} B$, if $a_i \overset{m}{\geqslant} b_i$ for $i = 1, ..., m$. It is clear that $A > B \Rightarrow A \overset{row}{>} B$;

(iv) A is said to be larger than the matrix B in row weakly majorization, denoted by $A \overset{w}{>} B$, if $a_i \overset{w}{\geqslant} b_i$ for $i = 1, ..., m$. It is clear that $A \overset{row}{>} B \Rightarrow A \overset{w}{>} B$. Thus it can be written that $A \gg B \Rightarrow A > B \Rightarrow A \overset{row}{>} B \Rightarrow A \overset{w}{>} B$.

Also, we introduce the following notations.

(i) $D_+ = \{(x_1, ..., x_n) : x_1 \geqslant ... \geqslant x_n > 0\}$;

(ii) $\varepsilon_+ = \{(x_1, ..., x_n) : 0 < x_1 \leqslant ... \leqslant x_n\}$;

(iii) $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}) = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix}$.

The following two Lemmas are used to prove the two Theorems 2.17 and 2.18.

Lemma 2.1. *A differentiable function* $\Phi : R_+^8 \to R_+$ *satisfies*

$$\Phi(A) \geqslant (\leqslant)\Phi(B) \quad \text{for all } A, B \text{ such that } A \in U_2, \ A \gg B \tag{2.1}$$

if and only if

(i) $\Phi(A) = \Phi(A\Pi)$ *for all permutation matrice* Π *and for all* $A \in U_2$, *and*

(ii) $\sum_{i=1}^{4} (a_{ik} - a_{ij})[\Phi_{ik}(A) - \Phi_{ij}(A)] \geqslant (\leqslant) 0$ *for all* $j, k = 1, 2$ *and for all* $A \in U_2$, *where* $\Phi_{ij}(A) = \frac{\partial \Phi(A)}{\partial a_{ij}}$.

Lemma 2.2. *Let the function* $\gamma : R_+^4 \to R_+$ *be differentiable and the functin* $\Phi_n : R_+^{4n} \to R_+$ *be defined as*

$$\Phi_n(A) = \prod_{i=1}^{n} \gamma(a_{1i}, a_{2i}, a_{3i}, a_{4i}).$$

Assume that Φ_2 *satisfies (2.1). Then for* $A \in U_n$ *and* $B = AT_w$, *we have* $\Phi_n(A) \geqslant (\leqslant)\Phi_n(B)$, *where* T_w *is the T-transform matrix.*

Proof: The proofs of Lemmas 2.1–2.2 are similar to those of Theorems 2 and 3 of Balakrishnan *et al.* (2015) [1], and Marshall and Olkin (1997) [26]. □

3. MAIN RESULTS

In this section we establish some ordering properties for the smallest and largest order statistics of the ELS model with associated random shocks. We now consider the following assumption.

Assumption 3.1. Suppose $X_1, ..., X_n$ are independent non-negative random variables with $X_i \sim ELS(\lambda_i, \theta_i, \alpha_i)$, and $I_{p_1}, ..., I_{p_n}$ are independent Bernoulli random variables, independent of X_i's, with $E(I_{p_i}) = p_i, i = 1, ..., n$. Further, suppose $X_1^*, ..., X_n^*$ are independent non-negative random variables with $X_i^* \sim ELS(\lambda_i^*, \theta_i^*, \alpha_i^*)$, and $I_{p_1^*}, ..., I_{p_n^*}$ are independent Bernoulli random variables, independent of X_i^*'s, with $E(I_{p_i^*}) = p_i^*, i = 1, ..., n$.

Theorem 3.1 shows that usual stochastic ordering holds between two parallel systems of heterogeneous components under random shocks for fixed θ and α.

Theorem 3.1. *Let Assumption 3.1 hold and $h : [0,1] \to R$ be a differentiable, increasing and strictly convex function. Also, $\lambda_i = \lambda_i^*$, $\theta_i = \theta_i^* = \theta$ and $\alpha_i = \alpha_i^* = \alpha$, where $i = 1, ..., n$. Then, $h(\underset{\sim}{p}) \geqslant_w h(\underset{\sim}{p}^*)$ implies $Y_{n:n} \geqslant_{st} Y_{n:n}^*$, provided $\underset{\sim}{\lambda} \in D_+$, $h(\underset{\sim}{p}) \in D_+$, and $h(\underset{\sim}{p}) = (h(p_1), ..., h(p_n))$.*

Proof: The cumulative distribution function of $Y_{n:n}$ is given by

$$F_{Y_{n:n}}(x) = \prod_{i=1}^{n} \left[1 - h^{-1}(u_i)\left[1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right]\right],$$

where $h(p_i) = u_i$. Let us define $\psi_1(\underset{\sim}{u}) = F_{Y_{n:n}}(x)$. Differentiating $\psi_1(\underset{\sim}{u})$, partially, with respect to u_i, we get

$$\text{(3.1)} \qquad \frac{\partial \psi_1(\underset{\sim}{u})}{\partial u_i} = -\frac{\frac{dh^{-1}(u_i)}{du_i}\left(1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right)}{1 - h^{-1}(u_i)\left(1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right)}\psi_1(\underset{\sim}{u}) \leqslant 0,$$

so, $\psi_1(\underset{\sim}{u})$ is decreasing in each u_i. Again, it can be shown that

$$\text{(3.2)} \quad \frac{\partial \psi_1(\underset{\sim}{u})}{\partial u_i} - \frac{\partial \psi_1(\underset{\sim}{u})}{\partial u_j} \overset{\text{sign}}{=} \frac{\frac{dh^{-1}(u_j)}{du_j}\left(1 - F^{\alpha}\left(\frac{x-\lambda_j}{\theta}\right)\right)}{1 - h^{-1}(u_j)\left(1 - F^{\alpha}\left(\frac{x-\lambda_j}{\theta}\right)\right)} - \frac{\frac{dh^{-1}(u_i)}{du_i}\left(1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right)}{1 - h^{-1}(u_i)\left(1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right)}$$

Now,

$$\frac{\partial}{\partial u}\left(\frac{1}{1 - h^{-1}(u)(1 - F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}\right) = \frac{dh^{-1}(u)}{du}\left(1 - F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right) \geqslant 0,$$

implying that $\frac{1}{1-h^{-1}(u)\left(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}$ is increasing in u. Thus, as $\underset{\sim}{\lambda} \in D_+$, $h(\underset{\sim}{p}) \in D_+$, for $i \leqslant j$ taking $\lambda_i \geqslant \lambda_j$ and $u_i \geqslant u_j$ and noticing that $h^{-1}(u)$ is increasing in u, it can be written that

$$\frac{1 - F^{\alpha}\left(\frac{x-\lambda_j}{\theta}\right)}{1 - h^{-1}(u_j)\left(1 - F^{\alpha}\left(\frac{x-\lambda_j}{\theta}\right)\right)} \leqslant \frac{1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)}{1 - h^{-1}(u_i)\left(1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right)},$$

Again, if $h(u)$ is convex in u, then $u_i \geqslant u_j$ gives $\frac{dh^{-1}(u_i)}{du_i} \geqslant \frac{dh^{-1}(u_j)}{du_j}$ which yields

$$\frac{\frac{dh^{-1}(u_j)}{du_j}\left(1 - F^{\alpha}\left(\frac{x-\lambda_j}{\theta}\right)\right)}{1 - h^{-1}(u_j)\left(1 - F^{\alpha}\left(\frac{x-\lambda_j}{\theta}\right)\right)} \leqslant \frac{\frac{dh^{-1}(u_i)}{du_i}\left(1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right)}{1 - h^{-1}(u_i)\left(1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta}\right)\right)}.$$

Substituting the above result in (3.2), we get $\frac{\partial \psi_1(\underset{\sim}{u})}{\partial u_i} - \frac{\partial \psi_1(\underset{\sim}{u})}{\partial u_j} \leqslant 0$. Thus, by Lemma 3.1 of Kundu *et al.* (2016) [24], $\psi_1(\underset{\sim}{u})$ is Schur concave in $\underset{\sim}{u}$. Thus the result is proved by Theorem A.8 of Marshall *et al.* (2011) [27]. □

Theorem 3.2 shows that the majorized shape parameter vector leads to smaller systems lifetime in the sense of the usual stochastic ordering when the location and scale parameter vectors are constant and shock parameter vectors are heterogeneous.

Theorem 3.2. *Let Assumption 3.1 hold and $h:[0,1]\to R$ be a differentiable and decreasing function. Also, $\lambda_i=\lambda_i^*=\lambda$, $\theta_i=\theta_i^*=\theta$, and $\alpha_i^*=\beta_i$, where $i=1,...,n$. Then, $\underset{\sim}{\alpha}\overset{w}{\geqslant}\underset{\sim}{\beta}$ implies $Y_{n:n}\leqslant_{st}Y_{n:n}^*$, provided $\underset{\sim}{\alpha},\underset{\sim}{\beta},h(\underset{\sim}{p})\in\varepsilon_+$.*

Proof: The cumulative distribution function of $Y_{n:n}$ is given by

$$F_{Y_{n:n}}(x)=\prod_{i=1}^{n}\left[1-h^{-1}(u_i)\left[1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right]\right],$$

where $h(p_i)=u_i$. Differentiating $F_{Y_{n:n}}(x)$, partially, with respect to α_i, we get

$$\frac{\partial F_{Y_{n:n}}(x)}{\partial\alpha_i}=\frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\ln[F\left(\frac{x-\lambda}{\theta}\right)]}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right)}F_{Y_{n:n}}(x)\leqslant 0,$$

so, $F_{Y_{n:n}}(x)$ is decreasing in each α_i. Again, it can be shown that
(3.3)

$$\frac{\partial F_{Y_{n:n}}(x)}{\partial\alpha_i}-\frac{\partial F_{Y_{n:n}}(x)}{\partial\alpha_j}\overset{\text{sign}}{=}\frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\ln[F\left(\frac{x-\lambda}{\theta}\right)]}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right)}-\frac{h^{-1}(u_j)F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)\ln[F\left(\frac{x-\lambda}{\theta}\right)]}{1-h^{-1}(u_j)\left(1-F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)\right)}.$$

Now,

$$\frac{\partial}{\partial\alpha}\left(\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u)(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}\right)\overset{\text{sign}}{=}(1-h^{-1}(u))F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]\leqslant 0,$$

implying that $\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u)\left(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}$ is decreasing in α. Again, as $h(u)$ is decreasing in u, then

$$\frac{\partial}{\partial u}\left(\frac{h^{-1}(u)}{1-h^{-1}(u)(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}\right)\overset{\text{sign}}{=}\frac{\partial h^{-1}(u)}{\partial u}\leqslant 0,$$

implying that $\frac{h^{-1}(u)}{1-h^{-1}(u)(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}$ is decreasing in u. Thus, as $\underset{\sim}{\alpha}\in\varepsilon_+$, $h(\underset{\sim}{p})\in\varepsilon_+$, for $i\leqslant j$ taking $\alpha_i\leqslant\alpha_j$ and $u_i\leqslant u_j$ and noticing that $h^{-1}(u)$ is decreasing in u, it can be written that

$$\begin{aligned}\frac{h^{-1}(u_j)F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u_j)\left(1-F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)\right)}&\leqslant\frac{h^{-1}(u_j)F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u_j)\left(1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right)}\\&\leqslant\frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right)},\end{aligned}$$

which implies

$$\begin{aligned}\frac{h^{-1}(u_j)F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)\ln[F\left(\frac{x-\lambda}{\theta}\right)]}{1-h^{-1}(u_j)\left(1-F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)\right)}&\geqslant\frac{h^{-1}(u_j)F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\ln[F\left(\frac{x-\lambda}{\theta}\right)]}{1-h^{-1}(u_j)\left(1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right)}\\&\geqslant\frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\ln[F\left(\frac{x-\lambda}{\theta}\right)]}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right)}.\end{aligned}$$

Hence, substituting the above results in (3.3), we get $\frac{\partial F_{Y_{n:n}}(x)}{\partial \alpha_i} - \frac{\partial F_{Y_{n:n}}(x)}{\partial \alpha_j} \leqslant 0$. Thus, by Lemma 3.3 of Kundu *et al.* (2016) [24], it can be proved that $F_{Y_{n:n}}(x)$ is Schur-convex in $\underset{\sim}{\alpha}$. Thus the result is proved by Theorem A.8 of Marshall *et al.* (2011) [27]. □

Theorem 3.3. *Let Assumption 3.1 hold and $h : [0, 1] \to R$ be a differentiable and decreasing function. Also, $\lambda_i = \lambda_i^* = \lambda$, $\theta_i = \theta_i^* = \theta$, and $\alpha_i^* = \beta_i$, where $i = 1, ..., n$. Then, $\underset{\sim}{\alpha}^{-1} \geqslant_w \underset{\sim}{\beta}^{-1}$ implies $Y_{n:n} \leqslant_{st} Y_{n:n}^*$, provided, $\underset{\sim}{\alpha}, \underset{\sim}{\beta}, h(\underset{\sim}{p}) \in \varepsilon_+$.*

Proof: The cumulative distribution function of $Y_{n:n}$ can be expressed as the function of a_i, where $a_i = \frac{1}{\alpha_i}$, $i = 1, ..., n$. We denote it by $\psi_2(\underset{\sim}{a})$, where $\underset{\sim}{a} = (a_1, ..., a_n)$, and

$$\psi_2(\underset{\sim}{a}) = \prod_{i=1}^{n} \left[1 - h^{-1}(u_i)\left[1 - F^{\frac{1}{a_i}}\left(\frac{x-\lambda}{\theta}\right)\right]\right],$$

where $h(p_i) = u_i$. It can be shown that the partial derivative of $\psi_2(\underset{\sim}{a})$, with respect to a_i increasing in $i = 1, ..., n$. Thus, by Lemma 3.1 of Kundu *et al.* (2016) [24], it can be proved that $\psi_2(\underset{\sim}{a})$ is Schur-convex in $\underset{\sim}{a}$. Thus the result is proved by Theorem A.8 of Marshall *et al.* (2011) [27]. □

Theorem 3.4 shows that $Y_{n:n}$ is smaller than $Y_{n:n}^*$ with respect to the usual stochastic ordering when a vector of scale parameters is p-larger than that of another vector of the scale parameters with some additional conditions when the location and shape parameter vectors are constant and shock parameter vectors are heterogeneous. Similar result also hold under reciprocally majorized based conditions among the scale parameters.

Theorem 3.4. *Let Assumption 3.1 hold and $h : [0, 1] \to R$ be a differentiable and decreasing function. Also, $\lambda_i = \lambda_i^* = \lambda$, $\theta_i^* = \delta_i$, and $\alpha_i = \alpha_i^* = \alpha$, where $i = 1, ..., n$. Then,*

- **(i)** *$\underset{\sim}{\theta} \overset{p}{\geqslant} \underset{\sim}{\delta}$ implies $Y_{n:n} \leqslant_{st} Y_{n:n}^*$, provided $\underset{\sim}{\theta}, \underset{\sim}{\delta}, h(\underset{\sim}{p}) \in D_+$,, and $u\tilde{r}(u)$ is increasing in u;*
- **(ii)** *$\underset{\sim}{\theta} \overset{rm}{\geqslant} \underset{\sim}{\delta}$ implies $Y_{n:n} \leqslant_{st} Y_{n:n}^*$, provided $\underset{\sim}{\theta}, \underset{\sim}{\delta}, h(\underset{\sim}{p}) \in \varepsilon_+$, and $\tilde{r}(u)$ is increasing in u.*

Proof:

(**i**): The cumulative distribution function of $Y_{n:n}$ can be expressed as the function of a_i, where $a_i = \ln \theta_i$, $i = 1, ..., n$. We denote it by $\psi_3(\underset{\sim}{a})$, where $\underset{\sim}{a} = (a_1, ..., a_n)$, and

$$\psi_3(\underset{\sim}{a}) = \prod_{i=1}^{n} \left[1 - h^{-1}(u_i)\Big(1 - F^{\alpha}(e^{-a_i}(x-\lambda))\Big)\right],$$

where $h(p_i) = u_i$. Differentiating $\psi_3(\underset{\sim}{a})$, partially, with respect to a_i, we get

$$\frac{\partial \psi_3(\underset{\sim}{a})}{\partial a_i} = -\frac{\alpha h^{-1}(u_i) e^{-a_i}(x-\lambda)\tilde{r}(e^{-a_i}(x-\lambda))F^{\alpha}(e^{-a_i}(x-\lambda))}{1 - h^{-1}(u_i)\left(1 - F^{\alpha}(e^{-a_i}(x-\lambda))\right)} \psi_3(\underset{\sim}{a}) \leqslant 0,$$

so, $\psi_3(\underset{\sim}{a})$ is decreasing in each a_i. Again, it can be shown that

$$\frac{\partial\psi_3(\underset{\sim}{a})}{\partial a_i}-\frac{\partial\psi_3(\underset{\sim}{a})}{\partial a_j}\overset{\text{sign}}{=}\frac{\alpha h^{-1}(u_j)e^{-a_j}(x-\lambda)\tilde{r}(e^{-a_j}(x-\lambda))F^{\alpha}(e^{-a_j}(x-\lambda))}{1-h^{-1}(u_j)\left(1-F^{\alpha}(e^{-a_j}(x-\lambda))\right)} - \frac{\alpha h^{-1}(u_i)e^{-a_i}(x-\lambda)\tilde{r}(e^{-a_i}(x-\lambda))F^{\alpha}(e^{-a_i}(x-\lambda))}{1-h^{-1}(u_i)\left(1-F^{\alpha}(e^{-a_i}(x-\lambda))\right)}. \tag{3.4}$$

Now,

$$\frac{\partial}{\partial a}\left(\frac{F^{\alpha}(e^{-a}(x-\lambda))}{1-h^{-1}(u)\left(1-F^{\alpha}(e^{-a}(x-\lambda))\right)}\right)\overset{\text{sign}}{=}-\alpha e^{-a}(x-\lambda)(1-h^{-1}(u))\times\tilde{r}(e^{-a}(x-\lambda))\times F^{\alpha}(e^{-a}(x-\lambda))\leqslant 0,$$

implying that $\frac{F^{\alpha}(e^{-a}(x-\lambda))}{1-h^{-1}(u)\,(1-F^{\alpha}(e^{-a}(x-\lambda)))}$ is decreasing in a. Again, as $h(u)$ is decreasing in u, then Fang, L. and Balakrishnan, N. (2018) [10]. Ordering properties of the small25 est order statistics from generalized Birnbaum–Saunders models with associated 26 random shocks, Metrika, 81, 1, 19-35.

$$\frac{\partial}{\partial u}\left(\frac{h^{-1}(u)}{1-h^{-1}(u)\left(1-F^{a}(e^{-a}(x-\lambda))\right)}\right)\overset{\text{sign}}{=}\frac{\partial h^{-1}(u)}{\partial u}\leqslant 0,$$

implying that $\frac{h^{-1}(u)}{1-h^{-1}(u)\,(1-F^{a}(e^{-a}(x-\lambda)))}$ is decreasing in u. Thus, as $\underset{\sim}{\theta}, h(\underset{\sim}{p})\in D_+$, for $i\leqslant j$ taking $a_i\geqslant a_j$ and $u_i\geqslant u_j$ and noticing that $h^{-1}(u)$ is decreasing in u, it can be written that

$$\frac{h^{-1}(u_j)F^{\alpha}(e^{-a_j}(x-\lambda))}{1-h^{-1}(u_j)\left(1-F^{\alpha}(e^{-a_j}(x-\lambda))\right)}\geqslant\frac{h^{-1}(u_j)F^{\alpha}(e^{-a_i}(x-\lambda))}{1-h^{-1}(u_j)\left(1-F^{\alpha}(e^{-a_i}(x-\lambda))\right)}\geqslant\frac{h^{-1}(u_i)F^{\alpha}(e^{-a_i}(x-\lambda))}{1-h^{-1}(u_i)\left(1-F^{\alpha}(e^{-a_i}(x-\lambda))\right)},$$

As $u\tilde{r}(u)$ is increasing in u, then

$$\frac{\alpha h^{-1}(u_j)e^{-a_j}(x-\lambda)\tilde{r}(e^{-a_j}(x-\lambda))F^{\alpha}(e^{-a_j}(x-\lambda))}{1-h^{-1}(u_j)\left(1-F^{\alpha}(e^{-a_j}(x-\lambda))\right)}\geqslant\frac{\alpha h^{-1}(u_i)e^{-\alpha_i}(x-\lambda)\tilde{r}(e^{-a_i}(x-\lambda))F^{\alpha}(e^{-a_i}(x-\lambda))}{1-h^{-1}(u_i)\left(1-F^{\alpha}(e^{-a_i}(x-\lambda))\right)}.$$

Hence, from (3.4), we get $\frac{\partial\psi_3(\underset{\sim}{a})}{\partial a_i}-\frac{\partial\psi_3(\underset{\sim}{a})}{\partial a_j}\geqslant 0$. Thus, by Lemma 3.1 of Kundu *et al.* (2016) [24], it can be proved that $\psi_3(\underset{\sim}{a})$ is Schur-convex in $\underset{\sim}{a}$. Thus the result is proved by Lemma 3.1 of Khaledi *et al.* (2002) [16].

(**ii**): The cumulative distribution function of $Y_{n:n}$ can be expressed as the function of $b_i=\frac{1}{\theta_i}$, $i=1,...,n$. We denote it by $\psi_4(\underset{\sim}{b})$, where $\underset{\sim}{b}=(b_1,...,b_n)$:

$$\psi_4(\underset{\sim}{b})=\prod_{i=1}^{n}\left[1-h^{-1}(u_i)\left(1-F^{\alpha}(b_i(x-\lambda))\right)\right],$$

where $h(p_i)=u_i$. Differentiating $\psi_4(\underset{\sim}{b})$, partially, with respect to b_i, we see that $\psi_4(\underset{\sim}{b})$ is increasing in each $i=1,...,n$. Thus, by Lemma 3.1 of Kundu *et al.* (2016) [24], it can be proved that $\psi_4(\underset{\sim}{b})$ is Schur-convex in $\underset{\sim}{b}$. Thus the result is proved by Lemma 4.1 of Hazra *et al.* (2017) [14]. □

Theorem 3.5 shows that the majorized shape parameter vector leads to smaller systems lifetime in the sense of the usual stochastic ordering when the location and scale and shock parameter vectors are heterogeneous.

Theorem 3.5. *Let Assumption 3.1 hold and $h : [0,1] \to R$ be a differentiable and decreasing function. Also, $\lambda_i = \lambda_i^*$, $\theta_i = \theta_i^*$, and $\alpha_i^* = \beta_i$, where $i = 1, ..., n$. Then, $\underset{\sim}{\alpha} \overset{w}{\geqslant} \underset{\sim}{\beta}$ implies $Y_{n:n} \leqslant_{st} Y_{n:n}^*$, provided $\underset{\sim}{\alpha}, \underset{\sim}{\beta}, h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \in \varepsilon_+$.*

Proof: The cumulative distribution function of $Y_{n:n}$ is given by

$$F_{Y_{n:n}}(x) = \prod_{i=1}^{n} \left[1 - h^{-1}(u_i)\left[1 - F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)\right]\right],$$

where $h(p_i) = u_i$. Differentiating $F_{Y_{n:n}}(x)$, partially, with respect to α_i, we get

$$\frac{\partial F_{Y_{n:n}}(x)}{\partial \alpha_i} = \frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)\ln\left[F\left(\frac{x-\lambda_i}{\theta_i}\right)\right]}{1 - h^{-1}(u_i)\left(1 - F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)\right)} F_{Y_{n:n}}(x) \leqslant 0,$$

so, $F_{Y_{n:n}}(x)$ is decreasing in each α_i. Again, it can be shown that

$$\frac{\partial F_{Y_{n:n}}(x)}{\partial \alpha_i} - \frac{\partial F_{Y_{n:n}}(x)}{\partial \alpha_j} \overset{\text{sign}}{=} \frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)\ln\left[F\left(\frac{x-\lambda_i}{\theta_i}\right)\right]}{1 - h^{-1}(u_i)\left(1 - F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)\right)} - \frac{h^{-1}(u_j)F^{\alpha_j}\left(\frac{x-\lambda_j}{\theta_j}\right)\ln\left[F\left(\frac{x-\lambda_j}{\theta_j}\right)\right]}{1 - h^{-1}(u_j)\left(1 - F^{\alpha_j}\left(\frac{x-\lambda_j}{\theta_j}\right)\right)}.$$

Now,

$$\frac{\partial}{\partial \alpha}\left(\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1 - h^{-1}(u)\left(1 - F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}\right) \overset{\text{sign}}{=} (1 - h^{-1}(u))F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right] \leqslant 0,$$

implying that $\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u)\left(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}$ is decreasing in α. Again, as $h(u)$ is decreasing in u, then,

$$\frac{\partial}{\partial u}\left(\frac{h^{-1}(u)}{1 - h^{-1}(u)\left(1 - F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}\right) \overset{\text{sign}}{=} \frac{\partial h^{-1}(u)}{\partial u} \leqslant 0,$$

implying that $\frac{h^{-1}(u)}{1-h^{-1}(u)\left(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}$ is decreasing in u. Again,

$$\frac{\partial}{\partial \lambda}\left(\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1 - h^{-1}(u)\left(1 - F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}\right) \overset{\text{sign}}{=} -(\frac{\alpha}{\theta})\tilde{r}\left(\frac{x-\lambda}{\theta}\right)F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\left(1 - h^{-1}(u)\right) \leq 0,$$

implying that $\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u)\left(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}$ is decreasing in λ. Also,

$$\frac{\partial}{\partial \theta}\left(\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1 - h^{-1}(u)\left(1 - F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}\right) \overset{\text{sign}}{=} -(\frac{\alpha}{\theta^2})(x-\lambda)\tilde{r}\left(\frac{x-\lambda}{\theta}\right)F^{\alpha}\left(\frac{x-\lambda}{\theta}\right) \times \left(1 - h^{-1}(u)\right) \leq 0,$$

implying that $\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1-h^{-1}(u)\left(1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\right)}$ is decreasing in θ. Thus, as $\underset{\sim}{\alpha}, \underset{\sim}{\beta}, h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \in \varepsilon_+$, for $i \leq j$ taking $\alpha_i \leq \alpha_j$, $u_i \leq u_j$, $\lambda_i \leq \lambda_j$, $\theta_i \leq \theta_j$ and noticing that $h^{-1}(u)$ is decreasing in u, it can be written that

$$\begin{aligned}
\frac{h^{-1}(u_j)F^{\alpha_j}\left(\frac{x-\lambda_j}{\theta_j}\right)}{1-h^{-1}(u_j)\left(1-F^{\alpha_j}\left(\frac{x-\lambda_j}{\theta_j}\right)\right)} &\leqslant \frac{h^{-1}(u_j)F^{\alpha_i}\left(\frac{x-\lambda_j}{\theta_j}\right)}{1-h^{-1}(u_j)\left(1-F^{\alpha_i}\left(\frac{x-\lambda_j}{\theta_j}\right)\right)} \\
&\leqslant \frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda_j}{\theta_j}\right)}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda_j}{\theta_j}\right)\right)} \\
&\leqslant \frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_j}\right)}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_j}\right)\right)} \\
&\leqslant \frac{h^{-1}(u_i)F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)\right)},
\end{aligned}$$

which implies

$$\frac{h^{-1}(u_j)F^{\alpha_j}\left(\frac{x-\lambda_j}{\theta_j}\right)}{1-h^{-1}(u_j)\left(1-F^{\alpha_j}\left(\frac{x-\lambda_j}{\theta_j}\right)\right)} \leqslant \frac{h^{-1}(u_i)F^{\alpha_i}(\frac{x-\lambda_i}{\theta_i})}{1-h^{-1}(u_i)\left(1-F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)\right)}.$$

Hence substituting the above results, we get $\frac{\partial F_{Y_{n:n}}(x)}{\partial\alpha_i} - \frac{\partial F_{Y_{n:n}}(x)}{\partial\alpha_j} \leq 0$. Thus by Lemma 3.3 of Kundu *et al.* (2016) [24], it can be proved that $F_{Y_{n:n}}(x)$ is Schur-convex in $\underset{\sim}{\alpha}$. thus the result is proved by Theorem A.8 of Marshall *et al.* (2011) [27]. □

Theorem 3.6. *Let Assumption 3.1 hold and $h : [0, 1] \to R$ be a differentiable function. Also, $\lambda_i = \lambda_i^* = \lambda$, $\theta_i = \theta_i^* = \theta$, $\alpha_i^* = \beta_i$, where $i = 1, ..., n$. Then,*

(i) $\underset{\sim}{\alpha} \overset{w}{\geqslant} \underset{\sim}{\beta}$ *implies* $Y_{1:n} \leqslant_{st} Y^*_{1:n}$, *provided* $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in \varepsilon_+$, $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$;

(ii) $\underset{\sim}{\alpha}^{-1} \geqslant_w \underset{\sim}{\beta}^{-1}$ *implies* $Y_{1:n} \leqslant_{st} Y^*_{1:n}$, *provided* $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in \varepsilon_+$, $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$.

Proof:

(**i**): The survival function of $Y_{1:n}$ is given by

$$\bar{F}_{Y_{1:n}}(x) = \prod_{i=1}^{n} p_i\left[1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right],$$

where $h(p_i) = u_i$. To prove the required result, it sufficient to show that $\bar{F}_{Y_{1:n}}(x) \leq \bar{F}_{Y^*_{1:n}}(x)$, which is equivalent to proving that $\prod_{i=1}^{n}\left[1-F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right] \leq \prod_{i=1}^{n}\left[1-F^{\beta_i}\left(\frac{x-\lambda}{\theta}\right)\right]$, since $\prod_{i=1}^{n} p_i \leq$

$\prod_{i=1}^{n} p_i^*$. Let $\phi(\underset{\sim}{\alpha}) = \prod_{i=1}^{n}\left[1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right]$. Differentiating $\phi(\underset{\sim}{\alpha})$, partially, with respect to α_i, we get

$$\frac{\partial \phi(\underset{\sim}{\alpha})}{\partial \alpha_i} = -\frac{F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)}\phi(\underset{\sim}{\alpha}) \geqslant 0,$$

so, $\phi(\underset{\sim}{\alpha})$ is increasing in each α_i. Again, it can be shown that

$$\frac{\partial \phi(\underset{\sim}{\alpha})}{\partial \alpha_i} - \frac{\partial \phi(\underset{\sim}{\alpha})}{\partial \alpha_j} \overset{\text{sign}}{=} \frac{F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)} - \frac{F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)} \tag{3.5}$$

Now,

$$\frac{\partial}{\partial \alpha}\left(\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1 - F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}\right) \overset{\text{sign}}{=} F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right] \leqslant 0,$$

implying that $\frac{F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}{1-F^{\alpha}\left(\frac{x-\lambda}{\theta}\right)}$ is decreasing in α. Thus, as $\underset{\sim}{\alpha} \in \varepsilon_+$, for $i \leqslant j$ taking $\alpha_i \leqslant \alpha_j$, it can be written that

$$\frac{F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)}{1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)} \geqslant \frac{F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)}{1 - F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)},$$

which implies

$$\frac{F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)} \leqslant \frac{F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{\alpha_j}\left(\frac{x-\lambda}{\theta}\right)}.$$

Hence, from (3.5), we get $\frac{\partial \phi(\underset{\sim}{\alpha})}{\partial \alpha_i} - \frac{\partial \phi(\underset{\sim}{\alpha})}{\partial \alpha_j} \geqslant 0$. Thus, by Lemma 3.3 of Kundu *et al.* (2016) [24], it can be proved that $\phi(\underset{\sim}{\alpha})$ is Schur-concave in $\underset{\sim}{\alpha}$. Thus the result is proved by Theorem A.8 of Marshall *et al.* (2011) [27].

(ii): The survival function of $Y_{1:n}$ can be expressed as the function of $c_i = \frac{1}{\alpha_i}$, $i = 1, ..., n$. We denote it by $\psi_5(\underset{\sim}{c})$, where $\underset{\sim}{c} = (c_1, ..., c_n)$, and

$$\psi_5(\underset{\sim}{\alpha}) = \prod_{i=1}^{n} p_i\left[1 - F^{\frac{1}{c_i}}\left(\frac{x-\lambda}{\theta}\right)\right],$$

where $h(p_i) = u_i$. To prove the required result, it is sufficient to show that $\psi_5(\underset{\sim}{\alpha}) \leq \psi_5(\underset{\sim}{\beta})$, which is equivalent to proving that $\prod_{i=1}^{n}\left[1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right] \leq \prod_{i=1}^{n}\left[1 - F^{\beta_i}\left(\frac{x-\lambda}{\theta}\right)\right]$, since $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$. Let $\phi(\underset{\sim}{c}) = \prod_{i=1}^{n}\left[1 - F^{\frac{1}{c_i}}\left(\frac{x-\lambda}{\theta}\right)\right]$. It can be shown that the partial derivative of $\phi(\underset{\sim}{c})$, with respect to c_i, is decreasing in each c_i. Thus, by Lemma 3.1 of Kundu *et al.* (2016) [24], it can be proved that $\phi(\underset{\sim}{c})$ is Schur-concave in $\underset{\sim}{c}$. Thus the result is proved by Theorem A.8 of Marshall *et al.* (2011) [27]. □

Theorem 3.7. *Let Assumption 3.1 hold and $h : [0, 1] \to R$ be a differentiable function. Also, $\lambda_i = \lambda_i^* = \lambda$, $\theta_i = \theta_i^* = \theta$, and $\alpha_i^* = \beta_i$, where $i = 1, ..., n$. Then, $\underset{\sim}{\alpha} \overset{p}{\geqslant} \underset{\sim}{\beta}$ implies $Y_{1:n} \leqslant_{st} Y_{1:n}^*$, provided $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in D_+$, $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$.*

Proof: The survival function of $Y_{1:n}$ can be expressed as the function of d_i, where $d_i = \ln\alpha_i$, $i = 1, ..., n$. We denote it by $\psi_6(\underset{\sim}{d})$, where $\underset{\sim}{d} = (d_1, ..., d_n)$:

$$\psi_6(\underset{\sim}{d}) = \prod_{i=1}^{n} p_i \left[1 - F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)\right],$$

where $h(p_i) = u_i$. To prove the required result, it is sufficient to show that $\psi_6(\underset{\sim}{\alpha}) \le \psi_6(\underset{\sim}{\beta})$, which is equivalent to proving that $\prod_{i=1}^{n} p_i\left[1 - F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\right] \le \prod_{i=1}^{n} p_i\left[1 - F^{\beta_i}\left(\frac{x-\lambda}{\theta}\right)\right]$, since $\prod\limits_{i=1}^{n} p_i \le \prod\limits_{i=1}^{n} p_i^*$. Let $\phi(\underset{\sim}{d}) = \prod_{i=1}^{n}\left[1 - F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)\right]$ Differentiating $\psi_6(\underset{\sim}{d})$, partially, with respect to d_i, we get

$$\frac{\partial\phi(\underset{\sim}{d})}{\partial d_i} = -\frac{e^{d_i}F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)}\phi(\underset{\sim}{d}) \geqslant 0$$

so, $\phi(\underset{\sim}{d})$ is increasing in each d_i. Again, it can be shown that

$$\frac{\partial\phi(\underset{\sim}{d})}{\partial d_i} - \frac{\partial\phi(\underset{\sim}{d})}{\partial d_j} \overset{\text{sign}}{=} \frac{e^{d_j}F^{e^{d_j}}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{e^{d_j}}\left(\frac{x-\lambda}{\theta}\right)} - \frac{e^{d_i}F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)}. \tag{3.6}$$

Now,

$$\frac{\partial}{\partial d}\left(\frac{F^{e^d}\left(\frac{x-\lambda}{\theta}\right)}{1 - F^{e^d}\left(\frac{x-\lambda}{\theta}\right)}\right) \overset{\text{sign}}{=} e^d F^{e^d}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right] \leqslant 0,$$

implying that $\frac{F^{e^d}\left(\frac{x-\lambda}{\theta}\right)}{1-F^{e^d}\left(\frac{x-\lambda}{\theta}\right)}$ is decreasing in d. Thus, as $\underset{\sim}{d} \in D_+$, for $i \leqslant j$ taking $d_i \geqslant d_j$, it can be written that

$$\frac{F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)}{1 - F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)} \leqslant \frac{F^{e^{d_j}}\left(\frac{x-\lambda}{\theta}\right)}{1 - F^{e^{d_j}}\left(\frac{x-\lambda}{\theta}\right)},$$

which implies

$$\frac{e^{d_i}F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{e^{d_i}}\left(\frac{x-\lambda}{\theta}\right)} \geqslant \frac{e^{d_j}F^{e^{d_j}}\left(\frac{x-\lambda}{\theta}\right)\ln\left[F\left(\frac{x-\lambda}{\theta}\right)\right]}{1 - F^{e^{d_j}}\left(\frac{x-\lambda}{\theta}\right)}.$$

Hence, from (3.6), we get $\frac{\partial\phi(\underset{\sim}{d})}{\partial d_i} - \frac{\partial\phi(\underset{\sim}{d})}{\partial d_j} \leqslant 0$. Thus, by Lemma 3.1 of Kundu *et al.* (2016) [24], it can be proved that $\phi(\underset{\sim}{d})$ is Schur-concave in $\underset{\sim}{d}$. Thus the result is proved by Lemma 3.1 of Khaledi *et al.* (2002) [16]. □

Theorem 3.8. *Let Assumption 3.1 hold and $h : [0,1] \to R$ be a differentiable function. Also, $\lambda_i = \lambda_i^* = \lambda$, $\theta_i^* = \delta_i$, and $\alpha_i = \alpha_i^* = \alpha$, where $i = 1, ..., n$. Then, $\underset{\sim}{\theta}^{-1} \geqslant_w \underset{\sim}{\delta}^{-1}$ implies $Y_{1:n} \leqslant_{st} Y_{1:n}^*$, provided $\underset{\sim}{\theta}, \underset{\sim}{\delta} \in \varepsilon_+$, $\prod\limits_{i=1}^{n} p_i \le \prod\limits_{i=1}^{n} p_i^*$, and $\tilde{r}(u)$ is increasing in u.*

Proof: The survival function of $Y_{1:n}$ can be expressed as the function of e_i, where $e_i = \frac{1}{\theta_i}$, $i = 1, ..., n$. We denote it by $\psi_7(\underset{\sim}{e})$, where $\underset{\sim}{e} = (e_1, ..., e_n)$, and

$$\psi_7(\underset{\sim}{e}) = \prod_{i=1}^{n} p_i\left[1 - F^{\alpha}\big(e_i(x-\lambda)\big)\right],$$

where $h(p_i) = u_i$. To prove the required result, it is sufficient to show that $\psi_7(\underset{\sim}{\theta}^{-1}) \leq \psi_7(\underset{\sim}{\delta}^{-1})$, which is equivalent to proving that $\prod_{i=1}^n \left[1 - F^\alpha\left(\frac{1}{\theta_i}(x-\lambda)\right)\right] \leq \prod_{i=1}^n \left[1 - F^\alpha\left(\frac{1}{\delta_i}(x-\lambda)\right)\right]$, since $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$. Let $\phi(\underset{\sim}{e}) = \prod_{i=1}^n \left[1 - F^\alpha\left(e_i(x-\lambda)\right)\right]$. Differentiating $\phi(\underset{\sim}{e})$, partially, with respect to e_i, we get

$$\frac{\partial \phi(\underset{\sim}{e})}{\partial e_i} = -\frac{\alpha(x-\lambda)\tilde{r}(e_i(x-\lambda))F^\alpha(e_i(x-\lambda))}{1 - F^\alpha(e_i(x-\lambda))}\phi(\underset{\sim}{e}) \leqslant 0,$$

so, $\psi_7(\underset{\sim}{e})$ is decreasing in each e_i. Again, it can be shown that

$$\frac{\partial \phi(\underset{\sim}{e})}{\partial e_i} - \frac{\partial \phi(\underset{\sim}{e})}{\partial e_j} \overset{\text{sign}}{=} \frac{\alpha(x-\lambda)\tilde{r}(e_j(x-\lambda))F^\alpha(e_j(x-\lambda))}{1 - F^\alpha(e_j(x-\lambda))} - \frac{\alpha(x-\lambda)\tilde{r}(e_i(x-\lambda))F^\alpha(e_i(x-\lambda))}{1 - F^\alpha(e_i(x-\lambda))}. \tag{3.7}$$

Now,

$$\frac{\partial}{\partial e}\left(\frac{F^\alpha(e(x-\lambda))}{1 - F^\alpha(e(x-\lambda))}\right) \overset{\text{sign}}{=} \alpha(x-\lambda)\tilde{r}(e(x-\lambda))F^\alpha(e(x-\lambda)) \geqslant 0,$$

implying that $\frac{F^\alpha(e(x-\lambda))}{1-F^\alpha(e(x-\lambda))}$ is inscreasing in e. Thus, us $\underset{\sim}{\theta} \in \varepsilon_+$, for $i \leqslant j$ taking $e_i \geqslant e_j$, it can be written that

$$\frac{F^\alpha(e_i(x-\lambda))}{1 - F^\alpha(e_i(x-\lambda))} \geqslant \frac{F^\alpha(e_j(x-\lambda))}{1 - F^\alpha(e_j(x-\lambda))}.$$

As $\tilde{r}(u)$ is increasing in u, then

$$\frac{\alpha(x-\lambda)\tilde{r}(e_i(x-\lambda))F^\alpha(e_i(x-\lambda))}{1 - F^\alpha(e_i(x-\lambda))} \geqslant \frac{\alpha(x-\lambda)\tilde{r}(e_j(x-\lambda))F^\alpha(e_j(x-\lambda))}{1 - F^\alpha(e_j(x-\lambda))}.$$

Hence, from (3.7), we get $\frac{\partial \phi(\underset{\sim}{e})}{\partial e_i} - \frac{\partial \phi(\underset{\sim}{e})}{\partial e_j} \leqslant 0$. Thus, by Lemma 3.1 of Kundu *et al.* (2016) [24], it can be proved that $\phi(\underset{\sim}{e})$ is Schur-concave in $\underset{\sim}{e}$. Thus the result is proved by Theorem A.8 Marshall *et al.* (2011) [27]. □

Theorem 3.8 shows that if both the location and shock parameter vectors i.e. the matrix of location and shock parameters of one system majorizes the other when the scale and shape parameter vectors remain constant do not lead to better system reliability.

Theorem 3.9. *For $n = 2$, let Assumption 3.1 hold. Further, let $h : [0,1] \to R_+$ be a differentiable and strictly increasing concave function. Then, for $i = 1,2$, if $\theta_i = \theta_i^* = \alpha_i = \alpha_i^* = \theta$, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}) \in P_2$, we have that*

$$\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \end{pmatrix},$$

implies $Y_{1:2}^* \geqslant_{st} Y_{1:2}$, provided $\tilde{r}(u)$ is increasing in u.

Proof: With $u_1 = h(p_1)$, $u_2 = h(p_2)$, we have $p_1 = h^{-1}(u_1), p_2 = h^{-1}(u_2)$, where h^{-1} denotes the inverse of the function h. From (1.2), the survival function of $Y_{1:2}$ is

$$\bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta) = \prod_{i=1}^{2} h^{-1}(u_i)\left[1 - F^{\theta}\left(\frac{x-\lambda_i}{\theta}\right)\right], \qquad x > \max\{\lambda_i,\ i = 1, ..., n\}.$$

Note that the function $\bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)$ is permutation invariant in (u_i, λ_i), and so condition (i) of Theorem 2 of Balakrishnan *et al.* (2015) [1] is satisfied. Next, we have to show that condition (ii) of Theorem 2 of Balakrishnan *et al.* (2015) [1] also holds. The assumption $(\underset{\sim}{u}, \underset{\sim}{\lambda}) \in P_2$ implies that $(u_1 - u_2)(\lambda_1 - \lambda_2) \leqslant 0$. This implies that $u_1 \geqslant u_2$ and $\lambda_1 \leqslant \lambda_2$ or $u_1 \leqslant u_2$ and $\lambda_1 \geqslant \lambda_2$. We proof only for the case when $u_1 \geqslant u_2$ and $\lambda_1 \leqslant \lambda_2$. The proof for the other case is similar. The partial derivatives of $\bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)$ with respect to u_i and λ_i are

$$\begin{aligned}
\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)}{\partial u_i} &= \frac{\frac{\partial h^{-1}(u_i)}{\partial u_i}}{h^{-1}(u_i)} \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta), \\
\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)}{\partial \lambda_i} &= -\frac{\tilde{r}\left(\frac{x-\lambda_i}{\theta}\right) F^{\theta}\left(\frac{x-\lambda_i}{\theta}\right)}{1 - F^{\theta}\left(\frac{x-\lambda_i}{\theta}\right)} \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta).
\end{aligned}$$

For fixed $x > \max\{\lambda_i,\ i = 1, ..., n\}$, let us define the function $\varphi(\underset{\sim}{u}, \underset{\sim}{\lambda})$ as follows:

$$\begin{aligned}
\varphi(\underset{\sim}{u}, \underset{\sim}{\lambda}) &= (u_1 - u_2)\left(\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)}{\partial u_1} - \frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)}{\partial u_2}\right) \\
&\quad + (\lambda_1 - \lambda_2)\left(\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)}{\partial \lambda_1} - \frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta)}{\partial \lambda_2}\right) \\
&= (u_1 - u_2)\left(\frac{\frac{\partial h^{-1}(u_1)}{\partial u_1}}{h^{-1}(u_1)} - \frac{\frac{\partial h^{-1}(u_2)}{\partial u_2}}{h^{-1}(u_2)}\right) \times \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta) \\
&\quad + (\lambda_1 - \lambda_2)\left(\frac{\tilde{r}\left(\frac{x-\lambda_1}{\theta}\right) F^{\theta}\left(\frac{x-\lambda_1}{\theta}\right)}{1 - F^{\theta}\left(\frac{x-\lambda_1}{\theta}\right)} - \frac{\tilde{r}\left(\frac{x-\lambda_2}{\theta}\right) F^{\theta}\left(\frac{x-\lambda_2}{\theta}\right)}{1 - F^{\theta}\left(\frac{x-\lambda_2}{\theta}\right)}\right) \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \theta, \theta). \qquad (3.8)
\end{aligned}$$

Since h is strictly increasing and concave, then for $u_1 \geqslant u_2$ and $\lambda_1 \leqslant \lambda_2$, we have

$$\frac{\frac{\partial h^{-1}(u_1)}{\partial u_1}}{h^{-1}(u_1)} \leq \frac{\frac{\partial h^{-1}(u_2)}{\partial u_2}}{h^{-1}(u_2)}. \tag{3.9}$$

Furthermore, $\frac{t^{\theta}}{1-t^{\theta}}$ is increasing in t for $\theta > 0$. For the reversed hazard rate function $\tilde{r}$ that is increasing, we have

$$\tilde{r}(\frac{x-\lambda_1}{\theta}) \geq \tilde{r}\left(\frac{x-\lambda_2}{\theta}\right) \tag{3.10}$$

and

$$\frac{\tilde{r}\left(\frac{x-\lambda_1}{\theta}\right) F^{\theta}\left(\frac{x-\lambda_1}{\theta}\right)}{1 - F^{\theta}\left(\frac{x-\lambda_1}{\theta}\right)} \geqslant \frac{\tilde{r}\left(\frac{x-\lambda_2}{\theta}\right) F^{\theta}\left(\frac{x-\lambda_2}{\theta}\right)}{1 - F^{\theta}\left(\frac{x-\lambda_2}{\theta}\right)}. \tag{3.11}$$

Combining (3.9) and (3.10) and (3.11), we see that $\varphi(\underset{\sim}{u}, \underset{\sim}{\lambda}) \leqslant 0$. Condition (ii) Theorem 2 of Balakrishnan *et al.* (2015) [1] is satisfied, which completes the proof. □

Counterexample 3.1. Let the baseline distribution function be $F(t) = e^{\frac{-1}{t}}$, $t > 0$. Take $h(p) = -\ln p$. Here, the baseline reversed hazard rate function is decreasing and $h(p)$ is decreasing and convex. Thus, the assumptions of the Theorem 3.9 are violated. Let us take $\theta_1 = \theta_2 = \theta_1^* = \theta_2^* = \alpha_1 = \alpha_2 = \alpha_1^* = \alpha_2^* = 1.5$, $(\lambda_1, \lambda_2) = (0.3, 0.9)$, $(\lambda_1^*, \lambda_2^*) = (0.54, 0.66)$, $(p_1, p_2) = (e^{-0.4}, e^{-0.5})$ and $(p_1^*, p_2^*) = (e^{-0.44}, e^{-0.46})$. Consider the T-transform matrix $T_{0.6} = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}$. It can be shown that

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \end{pmatrix} T_{0.6},$$

which implies $\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \end{pmatrix}$. Under this set up, $\bar{F}_{Y_{1:2}}(2) = 0.2597610428$, $\bar{F}_{Y_{1:2}^*}(2) = 0.2599036428$, $\bar{F}_{Y_{1:2}}(5) = 0.06532417018$, $\bar{F}_{Y_{1:2}^*}(5) = 0.06516286182$, which readily shows that $Y_{1:2}^* \not\geqslant_{st} Y_{1:2}$.

The following theorem extends Theorem 3.8 when two sets of n-independent observations from *ELS* distribution. The generalization is the direct result of the Theorem 3.8 and Lemma 5 of Balakrishnan *et al.* (2018) [2]. So, the proof is omitted.

Theorem 3.10. *Let Assumption 3.1 hold and $h : [0, 1] \to R_+$ be a differentiable and strictly increasing concave function. Further, let T_w be a T-transform matrix. Then, for $i = 1, ..., n$, if $\theta_i = \theta_i^* = \alpha_i = \alpha_i^* = \theta$, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}) \in P_n$, we have that*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} T_w,$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$, provided $\tilde{r}(u)$ is increasing in u.

Theorem 3.11. *Let Assumption 3.1 hold. Further, let $T_{w_1}, ..., T_{w_k}$ have the same structure. Suppose $h : [0, 1] \to R_+$ is a differentiable and strictly increasing concave function. Then, for $i = 1, ..., n$, if $\theta_i = \theta_i^* = \alpha_i = \alpha_i^* = \theta$, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}) \in P_n$, we have that*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} T_{w_1} \cdots T_{w_k},$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$, provided $\tilde{r}(u)$ is increasing in u.

Proof: Since a finite product of T-transform matrices with the same structure is also a T-transform matrix, so, the desired result is obtained from Theorem 3.9. □

Our next Theorem shows that the result in Theorem 3.10 holds for T-transform matrices with different structure.

Theorem 3.12. *Let Assumption 3.1 hold. Further, let $T_{w_1}, ..., T_{w_k}, k > 2$, have different structures. Suppose $h : [0, 1] \to R_+$ is a differentiable and strictly increasing concave function. Then, for $i = 1, ..., n$, if $\theta_i = \theta_i^* = \alpha_i = \alpha_i^* = \theta$, $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}) \in P_n$ and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda})\, T_{w_1} \cdots T_{w_k} \in P_n$, we have that*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} T_{w_1} \cdots T_{w_k},$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$, provided $\tilde{r}(u)$ is increasing in u.

Proof:

$$\begin{pmatrix} h(p_1)^{(i)} & \cdots & h(p_n)^{(i)} \\ \lambda_1^{(i)} & \cdots & \lambda_n^{(i)} \end{pmatrix} = \begin{pmatrix} h(p_1) & \cdots & h(p_n) \\ \lambda_1 & \cdots & \lambda_n \end{pmatrix} T_{w_1} \cdots T_{w_i}, \qquad \text{for} \quad i = 1, ..., k.$$

Assume $V_1^{(i)}, ..., V_n^{(i)}$, $i = 1, ..., k$, are independent sets of random variables with $V_j^{(i)} \sim ELS(\lambda_j^{(i)}, \theta_j, \alpha_j)$ where $\theta_j = \theta_j^* = \alpha_j = \alpha_j^* = \theta$, $j = 1, ..., n$ and $i = 1, ..., k$. From the assumption of the theorem, it follows that

$$\begin{pmatrix} h(p_1)^{(i)} & \cdots & h(p_n)^{(i)} \\ \lambda_1^{(i)} & \cdots & \lambda_n^{(i)} \end{pmatrix} \in P_n, \qquad \text{for} \quad i = 1, ..., k.$$

From these observations and the results of Theorem 3.9, it then follows that

$$Y_{1:n} \leqslant_{st} V_{1:n}^{(1)} \leqslant_{st} \cdots \leqslant_{st} V_{1:n}^{(k-2)} \leqslant_{st} V_{1:n}^{(k-1)} \leqslant_{st} Y_{1:n}^*,$$

which completes the proof of the theorem. □

The following example illustrates the result established in Theorem 3.11.

Example 3.1. Suppose X_1, X_2 and X_3 are independent non-negative random variables with $X_i \sim ELS(\lambda_i, \theta_i, \alpha_i)$, and I_{p_1}, I_{p_2} and I_{p_3} are independent Bernoulli random variables, independent of $X_i^{,}s$, with $E(I_{p_i}) = p_i, i = 1, 2, 3$. Further, suppose X_1^*, X_2^* and X_3^* are independent non-negative random variables with $X_i^* \sim ELS(\lambda_i^*, \theta_i^*, \alpha_i^*)$, and $I_{p_1^*}, I_{p_2^*}$ and $I_{p_3^*}$ are independent Bernoulli random variables, independent of $X_i^{*,}s$, with $E(I_{p_i^*}) = p_i^*$, $i = 1, 2, 3$. Consider a baseline distribution with distribution function $F(x) = 1 - e^{-x}$, $x > 0$. Consider the T-transform matrices as follows:

$$T_{0.7} = \begin{pmatrix} 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \\ 0.3 & 0 & 0.7 \end{pmatrix}, \quad T_{0.6} = \begin{pmatrix} 0.4 & 0.6 & 0 \\ 0.6 & 0.4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{0.4} = \begin{pmatrix} 0.6 & 0 & 0.4 \\ 0 & 1 & 0 \\ 0.4 & 0 & 0.6 \end{pmatrix}.$$

Suppose $h(p) = \frac{p}{1+p}$. Then, for $\theta_i = \theta_i^* = \alpha_i = \alpha_i^* = 1$, $i = 1, 2, 3$, let $(\lambda_1, \lambda_2, \lambda_3) = (2, 3, 4)$, $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (3.06, 2.76, 3.17)$, $(p_1, p_2, p_3) = (0.4, 0.3, 0.2)$ and $(p_1^*, p_2^*, p_3^*) = (0.285, 0.317, 0.273)$. It is easy to observe that $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}) \in P_3$, $(h(\underset{\sim}{p}), \underset{\sim}{\lambda})T_{0.7} \in P_3$ and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda})T_{0.7}T_{0.6} \in P_3$ and $(h(\underset{\sim}{p}^*), \underset{\sim}{\lambda}^*) = (h(\underset{\sim}{p}), \underset{\sim}{\lambda})T_{0.7}T_{0.6}T_{0.4}$. So, from Theorem 3.11, we have $Y_{1:3}^* \geqslant_{st} Y_{1:3}$.

Theorem 3.13. *Let Assumption 3.1 hold for $n = 2$. Suppose $h : [0, 1] \to R_+$ is differentiable and strictly increasing concave function. Further, let $\tilde{r}(u)$ and $u\tilde{r}(u)$ increasing in u. Then, if $\alpha_i = \alpha_i^* = \alpha$, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}) \in S_2$, we have that*

$$\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \theta_1 & \theta_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \\ \theta_1^* & \theta_2^* \end{pmatrix}$$

implies $Y_{1:2}^* \geqslant_{st} Y_{1:2}$.

Proof: With $u_1 = h(p_1), u_2 = h(p_2)$, we have $p_1 = h^{-1}(u_1)$, $p_2 = h^{-1}(u_2)$, where h^{-1} denotes the inverse of the function h. From (1.2), the survival function of $Y_{1:2}$ is

$$\bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha) = \prod_{i=1}^{2} h^{-1}(u_i)\left[1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta_i}\right)\right], \qquad x > \max\{\lambda_i,\, i = 1, ..., n\}.$$

Note that the function $\bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)$ is permutation invariant in $(u_i, \lambda_i, \theta_i)$ and therefore condition (i) of Lemma 6 of Balakrishnan *et al.* (2018) [2] is satisfied. Next, we have to show that Condition (ii) of Lemma 6 of Balakrishnan *et al.* (2018) [2] also holds. The assumption $(\underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}) \in S_2$ implies that $u_1 \leqslant (\geqslant) u_2$ and $\lambda_1 \geqslant (\leqslant)\lambda_2$ and $\theta_1 \geqslant (\leqslant)\theta_2$. We proof only for the case when $u_1 \leqslant u_2$ and $\lambda_1 \geqslant \lambda_2$ and $\theta_1 \geqslant \theta_2$. The proof for the other case is similar. The partial derivatives of $\bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)$ with respect to u_i and λ_i and θ_i are

$$\begin{aligned}
\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial u_i} &= \frac{\frac{\partial h^{-1}(u_i)}{\partial u_i}}{h^{-1}(u_i)} \times \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha),\\
\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial \lambda_i} &= \frac{\alpha}{\theta_i} \times \frac{\tilde{r}\left(\frac{x-\lambda_i}{\theta_i}\right) F^{\alpha}\left(\frac{x-\lambda_i}{\theta_i}\right)}{1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta_i}\right)} \times \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha),\\
\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial \theta_i} &= \frac{\alpha}{\theta_i} \times \frac{\left(\frac{x-\lambda_i}{\theta_i}\right)\tilde{r}\left(\frac{x-\lambda_i}{\theta_i}\right) F^{\alpha}\left(\frac{x-\lambda_i}{\theta_i}\right)}{1 - F^{\alpha}\left(\frac{x-\lambda_i}{\theta_i}\right)} \times \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha).
\end{aligned}$$

For fixed $x > \max\{\lambda_i,\, i = 1, ..., n\}$, let us define the function $\varphi(\underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta})$ as follows:

$$\begin{aligned}
\varphi(\underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}) &= (u_1 - u_2)\left(\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial u_1} - \frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial u_2}\right)\\
&\quad + (\lambda_1 - \lambda_2)\left(\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial \lambda_1} - \frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial \lambda_2}\right)\\
&\quad + (\theta_1 - \theta_2)\left(\frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial \theta_1} - \frac{\partial \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)}{\partial \theta_2}\right)\\
&= (u_1 - u_2)\left(\frac{\frac{\partial h^{-1}(u_1)}{\partial u_1}}{h^{-1}(u_1)} - \frac{\frac{\partial h^{-1}(u_2)}{\partial u_2}}{h^{-1}(u_2)}\right) \times \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)\\
&\quad + (\lambda_1 - \lambda_2)\left(\frac{1}{\theta_1}\tilde{r}\left(\frac{x-\lambda_1}{\theta_1}\right) l_1 - \frac{1}{\theta_2}\tilde{r}\left(\frac{x-\lambda_2}{\theta_2}\right) l_2\right) \times \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha)\\
&\quad + (\theta_1 - \theta_2)\left(\frac{1}{\theta_1}\left(\frac{x-\lambda_1}{\theta_1}\right)\tilde{r}\left(\frac{x-\lambda_1}{\theta_1}\right) l_1 - \frac{1}{\theta_2}\left(\frac{x-\lambda_2}{\theta_2}\right)\tilde{r}\left(\frac{x-\lambda_2}{\theta_2}\right) l_2\right)\\
&\quad \times \bar{F}_{Y_{1:2}}(x; \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \alpha),
\end{aligned} \tag{3.12}$$

where $l_i = l\Big(\alpha, F^{\alpha}\big(\frac{x-\lambda_i}{\theta_i}\big)\Big) = \frac{\alpha F^{\alpha}\left(\frac{x-\lambda_i}{\theta_i}\right)}{1-F^{\alpha}\left(\frac{x-\lambda_i}{\theta_i}\right)}$, $i = 1, 2$, is defined in Lemma 2.8 of Torrado (2015) [28]. Since h is strictly increasing and concave function, then for $u_1 \leqslant u_2$ and $\lambda_1 \geqslant \lambda_2$ and $\theta_1 \geqslant \theta_2$, we have

$$\frac{\frac{\partial h^{-1}(u_1)}{\partial u_1}}{h^{-1}(u_1)} \geqslant \frac{\frac{\partial h^{-1}(u_2)}{\partial u_2}}{h^{-1}(u_2)}. \tag{3.13}$$

$$\frac{1}{\theta_1}\tilde{r}\left(\frac{x-\lambda_1}{\theta_1}\right)l\left(\alpha, F^{\alpha}\left(\frac{x-\lambda_1}{\theta_1}\right)\right) \leqslant \frac{1}{\theta_2}\tilde{r}\left(\frac{x-\lambda_2}{\theta_2}\right)l\left(\alpha, F^{\alpha}\left(\frac{x-\lambda_2}{\theta_2}\right)\right) \tag{3.14}$$

and

$$\frac{1}{\theta_1}\left(\frac{x-\lambda_1}{\theta_1}\right)\tilde{r}\left(\frac{x-\lambda_1}{\theta_1}\right)l\left(\alpha, F^{\alpha}\left(\frac{x-\lambda_1}{\theta_1}\right)\right) \leqslant \frac{1}{\theta_2}\left(\frac{x-\lambda_2}{\theta_2}\right)\tilde{r}\left(\frac{x-\lambda_2}{\theta_2}\right)l\left(\alpha, F^{\alpha}\left(\frac{x-\lambda_2}{\theta_2}\right)\right) \tag{3.15}$$

combining (3.13), (3.14) and (3.15) in (3.12), we see that $\varphi(\underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\theta}) \leqslant 0$. So condition (ii) of Lemma 6 of Balakrishnan *et al.* (2018) [2] is satisfied, which completes the proof. □

Counterexample 3.2. Let the baseline distribution function be $F(t) = e^{\frac{-1}{t}}$, $t > 0$. Here, $\tilde{r}(t)$ and $t\tilde{r}(t)$ are decreasing. Take $h(p) = p$. Thus, the assumption of Theorem 3.13 are violated. Let us take $\alpha_1 = \alpha_2 = \alpha_1^* = \alpha_2^* = 2.2$, $(\theta_1, \theta_2) = (5.2, 2.7)$, $(\theta_1^*, \theta_2^*) = (4.2, 3.7)$, $(\lambda_1, \lambda_2) = (2.2, 2.5)$, $(\lambda_1^*, \lambda_2^*) = (2.32, 2.38)$, $(p_1, p_2) = (p_1^*, p_2^*) = (0.2, 0.2)$. Consider the T-transform matrix $T_{0.6} = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}$. It can be shown that

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \\ \theta_1^* & \theta_2^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \theta_1 & \theta_2 \end{pmatrix} T_{0.6},$$

which implies $\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \theta_1 & \theta_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \\ \theta_1^* & \theta_2^* \end{pmatrix}$. Under this set up, $\bar{F}_{Y_{1:2}}(2.8) = 0.9999999923$, $\bar{F}_{Y^*_{1:2}}(2.8) = 0.9999999918$, $\bar{F}_{Y_{1:2}}(5) = 0.8918294672$, $\bar{F}_{Y^*_{1:2}} = 0.9248659543$, which readily shows that $Y^*_{1:2} \not\geqslant_{st} Y_{1:2}$.

The following theorem extends Theorem 3.12 when two sets of n-independent observations are from ELS distribution. The generalization is the direct of the Theorem 3.12 and Lemma 7 of Balakrishnan *et al.* (2018) [2]. So, the proof is omitted.

Theorem 3.14. *Let Assumption 3.1 hold. Suppose $h : [0, 1] \to R_+$ is differentiable and strictly increasing concave function. Further, let $\tilde{r}(u)$ and $u\tilde{r}(u)$ are increasing in u. Then, for $i = 1, ..., n$ and T-transform matrix T_w, if $\alpha_i = \alpha_i^* = \alpha$, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}) \in S_n$, we have that*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \\ \theta_1^* & \theta_2^* & \cdots & \theta_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \theta_1 & \theta_2 & \cdots & \theta_n \end{pmatrix} T_w$$

implies $Y^*_{1:n} \geqslant_{st} Y_{1:n}$.

Theorem 3.15. *Let Assumption 3.1 hold and $T_{w_1}, ..., T_{w_k}$ be T-transform matrices with same structures. Suppose, $h : [0, 1] \to R_+$ is differentiable and strictly increasing concave* *function. Further, let $\tilde{r}(u)$, and $u\tilde{r}(u)$ are strictly increasing in u,. Then, if $\alpha_i = \alpha_i^* = \alpha$,, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}) \in S_n$, we have that*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \\ \theta_1^* & \theta_2^* & \cdots & \theta_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \theta_1 & \theta_2 & \cdots & \theta_n \end{pmatrix} T_{w_1} \cdots T_{w_k}$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$.

Proof: Since, a finite product of T-transform matrices with the same structure is also a T-transform matrix, so, the desired result can be obtained by repeating the result of Theorem 3.13. □

Theorem 3.16. *Let Assumption 3.1 hold and $T_{w_1}, ..., T_{w_k}$, $k > 2$ be T-transform matrices with different structures. Suppose, $h : [0,1] \to R_+$ is differentiable and strictly increasing concave function. Further, let $\tilde{r}(u)$ and $u\tilde{r}(u)$ are increasing in u. Then, if $\alpha_i = \alpha_i^* = \alpha$, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}) \in S_n$ and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta})\, T_{w_1} \cdots T_{w_i} \in S_n, i = 1, ..., k-1$, we have*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \\ \theta_1^* & \theta_2^* & \cdots & \theta_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \theta_1 & \theta_2 & \cdots & \theta_n \end{pmatrix} T_{w_1} \cdots T_{w_k}$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$.

Proof:

$$\begin{pmatrix} h^{(i)}(p_1) & \dots & h^{(i)}(p_n) \\ \lambda_1^{(i)} & \cdots & \lambda_n^{(i)} \\ \theta_1^{(i)} & \cdots & \theta_n^{(i)} \end{pmatrix} = \begin{pmatrix} h(p_1) & \cdots & h(p_n) \\ \lambda_1 & \cdots & \lambda_n \\ \theta_1 & \cdots & \theta_n \end{pmatrix} T_{w_1} ... T_{w_i}, \qquad \text{for} \quad i = 1, ..., k.$$

Assume $V_1^{(i)}, ..., V_n^{(i)}$, $i = 1, ..., k$, are independent sets of random variables with $V_j^{(i)} \sim ELS(\lambda_j^{(i)}, \theta_j^{(i)}, \alpha_j)$ where $\alpha_i = \alpha_i^* = \alpha$, $j = 1, ..., n$ and $i = 1, ..., k$. From the assumption of the theorem, it follows that

$$\begin{pmatrix} h(p_1)^{(i)} & \dots & h(p_n)^{(i)} \\ \lambda_1^{(i)} & \cdots & \lambda_n^{(i)} \\ \theta_1^{(i)} & \cdots & \theta_n^{(i)} \end{pmatrix} \in S_n.$$

Using the results of Theorem 3.13, it then follows that

$$Y_{1:n} \leqslant_{st} V_{1:n}^{(1)} \leqslant_{st} \cdots \leqslant_{st} V_{1:n}^{(k-2)} \leqslant_{st} V_{1:n}^{(k-1)} \leqslant_{st} Y_{1:n}^*,$$

which completes the proof of the theorem. □

Theorem 3.17. *Let Assumption 3.1 hold for $n = 2$. Suppose $h : [0, 1] \to R_+$ is a differentiable and strictly increasing concave function. Further, assume that $\tilde{r}(u)$ and $u\tilde{r}(u)$ are increasing in u, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \underset{\sim}{\alpha}) \in U_2$,. Then,*

$$\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \theta_1 & \theta_2 \\ \alpha_1 & \alpha_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \\ \theta_1^* & \theta_2^* \\ \alpha_1^* & \alpha_2^* \end{pmatrix}$$

implies $Y_{1:2}^* \geqslant_{st} Y_{1:2}$.

Proof: With the help of Lemma 2.1, the proof follows from arguments similar to those in the proof of Theorem 3.13. It is omitted for brevity. □

We present a counterexample to show that the comparison result may not hold if the assumptions are not satisfied.

Counterexample 3.3. Let the baseline distribution function be $F(t)=1-\exp(1-t^{0.5})$, $t \geq 1$. Here $\tilde{r}(t)$ and $t\tilde{r}(t)$ are decreasing. Take $h(p) = e^p$, where $h(p)$ is convex. Thus, the assumption of Theorem 3.17 are not violated. Let us set $(\alpha_1, \alpha_2) = (0.2, 0.5)$, $(\alpha_1^*, \alpha_2^*) = (0.44, 0.26)$, $(\lambda_1, \lambda_2) = (1, 1.5)$, $(\lambda_1^*, \lambda_2^*) = (1.4, 1.1)$, $(\theta_1, \theta_2) = (4, 2)$, $(\theta_1^*, \theta_2^*) = (2.4, 3.6)$, $(p_1, p_2) = (\ln(4), \ln(5))$, $(p_1^*, p_2^*) = (\ln(4.8), \ln(4.2))$,. Consider a T-transform matrix $T_{0.2} = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$. Then, it can be shown that

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \\ \theta_1^* & \theta_2^* \\ \alpha_1^* & \alpha_2^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \theta_1 & \theta_2 \\ \alpha_1 & \alpha_2 \end{pmatrix} T_{0.2},$$

which implies $\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \theta_1 & \theta_2 \\ \alpha_1 & \alpha_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \lambda_1^* & \lambda_2^* \\ \theta_1^* & \theta_2^* \\ \alpha_1^* & \alpha_2^* \end{pmatrix}$. Finally $\bar{F}_{Y_{1:2}}(5) - \bar{F}_{Y_{1:2}^*}(5) = 0.4133022299$, $\bar{F}_{Y_{1:2}}(7.5) - \bar{F}_{Y_{1:2}^*}(7.5) = -0.0324688838$, which readily shows that $Y_{1:2}^* \ngeqslant_{st} Y_{1:2}$.

In the following theorem, we present a generalization of Theorem 3.17 to the case of n independent variables.

Theorem 3.18. *Let Assumption 3.1 hold. Further, let T_w be a T-transform matrix. Suppose $h : [0, 1] \to R_+$ is a differentiable strictly increasing concave function. Further, assume that $\tilde{r}(u)$ and $u\tilde{r}(u)$ are increasing in u, and let $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \underset{\sim}{\alpha}) \in U_n$. Then,*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \\ \theta_1^* & \theta_2^* & \cdots & \theta_n^* \\ \alpha_1^* & \alpha_2^* & \cdots & \alpha_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \theta_1 & \theta_2 & \cdots & \theta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} T_w$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$.

Theorem 3.19. *Let $T_{w_1}, ..., T_{w_k}$ be T-transform matrices with same structure. Let Assumption 3.1 hold and $h : [0,1] \to R_+$ be a differentiable and strictly increasing concave function. Further, assume that $\tilde{r}(u)$ and $u\tilde{r}(u)$ are strictly increasing in u, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \underset{\sim}{\alpha}) \in U_n$. Then,*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \\ \theta_1^* & \theta_2^* & \cdots & \theta_n^* \\ \alpha_1^* & \alpha_2^* & \cdots & \alpha_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \theta_1 & \theta_2 & \cdots & \theta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} T_{w_1} \cdots T_{w_k}$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$.

The following theorem presents a generalization to the case of a finite number of T-transform matrices with different structures.

Theorem 3.20. *Let Assumption 3.1 hold. Further, let $T_{w_1}, ..., T_{w_k}$, $k > 2$ be T-transform matrices, with different structures. Suppose $h : [0,1] \to R_+$ is a differentiable and strictly increasing concave function. Further, assume that $\tilde{r}(u)$ and $u\tilde{r}(u)$ are increasing in u, and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \underset{\sim}{\alpha}) \in U_n$ and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}, \underset{\sim}{\alpha})\, T_{w_1} \cdots T_{w_i} \in U_n, i = 1, ..., k-1$. Then,*

$$\begin{pmatrix} h(p_1^*) & h(p_2^*) & \cdots & h(p_n^*) \\ \lambda_1^* & \lambda_2^* & \cdots & \lambda_n^* \\ \theta_1^* & \theta_2^* & \cdots & \theta_n^* \\ \alpha_1^* & \alpha_2^* & \cdots & \alpha_n^* \end{pmatrix} = \begin{pmatrix} h(p_1) & h(p_2) & \cdots & h(p_n) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \theta_1 & \theta_2 & \cdots & \theta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} T_{w_1} \cdots T_{w_k}$$

implies $Y_{1:n}^* \geqslant_{st} Y_{1:n}$.

4. SOME SPECIAL CASES

In this section, we present some special cases of the results obtained in the previous sections. We consider two special distributions - generalized gamma, half - normal distributions. For these distributions, we present some comparisons results using the general results established earlier. In terms of hazard rate function, the following theorems can be proved like the theorems above, which are proved by reversed hazard rate function. To prevent recurrence, you can refer to Das *et al.* (2021) [7].

4.1. Generalized gamma distribution

In this subsection, we consider the generalized gamma distribution with density function

$$f(t) \propto t^{a-1} e^{-t^b}, \qquad a, b, t > 0.$$

It is easy to check that the hazard rate function of this distribution is increasing for $a, b \geq 1$ and decreasing for $0 < a, b \leq 1$(see Hazra *et al.*, 2018 [15]). We now consider the baseline distribution function to the generalized gamma distribution.

Theorem 4.1. *For a baseline distribution function $F(.)$, let $X_1, ..., X_n$ $(X_1^*, ..., X_n^*)$ be non-negative independent random variables with $X_i \sim ELS(\lambda_i, \theta_i, \alpha_i)[X_i^* \sim ELS(\mu_i, \delta_i, \beta_i)]$, $i = 1, ..., n$. Further, let $I_{p_1}, ..., I_{p_n}[I_{p_1^*}, ..., I_{p_n^*}]$ be a set of independent Bernoulli random variables, independent of $X_i[X_i^*]$'s with $E(I_{p_i}) = p_i[E(I_{p_i^*}) = p_i^*]$, $i = 1, ..., n$. Further, let $h : [0, 1] \to R_+$ be a differentiable, increasing and convex function. Then, for $i = 1, 2$, if $\theta_i = \delta_i = \delta$ and $\alpha_i = \beta_i = \alpha \geq 1$, $r(x)$ is increasing. Suppose $h(p) = e^p \ln(1 + p)$. Then, for $a, b \geq 1$, we have $\begin{pmatrix} h(p_1) & h(p_2) \\ \theta_1 & \theta_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \delta_1 & \delta_2 \end{pmatrix}$ implies $Y_{1:2}^* \geq_{st} Y_{1:2}$.*

Theorem 4.2. *For a baseline distribution function $F(.)$, let $X_1, ..., X_n$ $(X_1^*, ..., X_n^*)$ be non-negative independent random variables with $X_i \sim ELS(\lambda_i, \theta_i, \alpha_i)[X_i^* \sim ELS(\mu_i, \delta_i, \beta_i)]$, $i = 1, ..., n$. Further, let $I_{p_1}, ..., I_{p_n}[I_{p_1^*}, ..., I_{p_n^*}]$ be a set of independent Bernoulli random variables, independent of $X_i[X_i^*]$'s with $E(I_{p_i}) = p_i[E(I_{p_i^*}) = p_i^*]$, $i = 1, ..., n$. Further, let $h : [0, 1] \to R_+$ be a differentiable, increasing and convex function. Further, let the baseline hazard rate $r(.)$ be increasing. If $\alpha_i = \beta_i = \alpha \geq 1$ and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\theta}) \in N_2$. Suppose $h(p) = p^2$. Then, for $a, b \geq 1$, we have $\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \theta_1 & \theta_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \mu_1 & \mu_2 \\ \delta_1 & \delta_2 \end{pmatrix}$ implies $Y_{1:2}^* \geq_{st} Y_{1:2}$.*

4.2. Half-normal distribution

Consider the probability distribution function of a half-normal distribution given by

$$f(t) \propto e^{\frac{-t^2}{2}}, \qquad t > 0.$$

The hazard rate function of the above half-normal distribution is increasing (see Hazra *et al.* (2018) [15]). The distribution function of the half-normal distribution is now taken as the baseline distribution function.

Theorem 4.3. *For a baseline distribution function $F(.)$, let $X_1, ..., X_n$ $(X_1^*, ..., X_n^*)$ be non-negative independent random variables with $X_i \sim ELS(\lambda_i, \theta_i, \alpha_i)$ $(X_i^* \sim ELS(\mu_i, \delta_i, \beta_i))$, $i = 1, ..., n$. Further, let $I_{p_1}, ..., I_{p_n}[I_{p_1^*}, ..., I_{p_n^*}]$ be a set of independent Bernoulli random variables, independent of $X_i[X_i^*]$'s with $E(I_{p_i}) = p_i[E(I_{p_i^*}) = p_i^*]$, $i = 1, ..., n$. Further, let $h : [0, 1] \to R_+$ be a differentiable, increasing and convex function. Further, let the baseline hazard rate $r(.)$ be increasing. If $\theta_i = \delta_i = \theta$ and $(h(\underset{\sim}{p}), \underset{\sim}{\lambda}, \underset{\sim}{\alpha}) \in N_2^*$. Let $h(p) = -p \ln(1 - p)$. Then $\begin{pmatrix} h(p_1) & h(p_2) \\ \lambda_1 & \lambda_2 \\ \alpha_1 & \alpha_2 \end{pmatrix} \gg \begin{pmatrix} h(p_1^*) & h(p_2^*) \\ \mu_1 & \mu_2 \\ \beta_1 & \beta_2 \end{pmatrix}$ implies $Y_{1:2}^* \geq_{st} Y_{1:2}$.*

5. APPLICATIONS

In this section, we discuss application of few of our established results in insurance and auction theory. Suppose $X_1, ..., X_n$ are independent exponentiated location-scale random variables with $X_i \sim ELS(\lambda_i, \mu_i, \alpha_i)$, for $i = 1, ..., n$, and $I_{p_1}, ..., I_{p_n}$ are independent Bernoulli random variables, independent of the X_i's, with $E(I_{p_i}) = p_i$. Let $Y_i = I_{p_i} X_i$, for $i = 1, ..., n$.

Suppose X_i denotes the total of random claims that can be made in an insurance period and I_{p_i} denotes a Bernoulli random variables associated with X_i defined as follows: $I_{p_i} = 1$ when ever the ith policyholder makes random claim X_i and $I_{p_i} = 0$ whenever he/she does not make make a claim. In setting, $Y_i = I_{p_i} X_i$ corresponds to the claim amount in a portfolio of risks. The problem of comparison of number of claims and aggregate claim amounts with respect to some well-known stochastic orders is of interest on both theoretical and practical view points. Under some conditions Theorems 3.2, 3.3, 3.4 respectively conclude that $Y_{n:n}$ in the weakly supermajorized order, weakly submajorized order, p-larger and reciprocally majorized order is stochastically smaller. There are many real-life applications of the ordering results. We discuss applications of few of our established results in auction theory. Auction theory has been an interest topic to various scientists because of its usefulness for sale of variety of items or purchasing services. For more details in auction theory, we refer to (Klemperer, (2004) [17]). In real world,among all types of auctions, the sealed-bid private-value auction is of theoretical interest. Also, this type of auction has been used extensively. In this case, bidders hand in their bids to the auctioneer simultaneously and can neither observe their rival bids nor revise their own bids.The bidders having the highest bid wins. The bidders with the lowest bid wins in the reverse auction. Consequently, the bidder pays his own bid in the sealed-bid first-price auction (FPA). Few of our established results could be useful for some new light in the auction theory. Let the bids follow exponentiated locatio-scale model. Then, under some conditions, Theorems 3.2, 3.3, 3.4 respectively conclude that the final price in the FPA with more heterogeneous shape parameters in the weakly super majorized order, reciprocal of the shape parameters in the weakly submajorized, scale parameters in the p-larger and reciprocally majorized orders is stochastically smaller.

6. CONCLUDING

In this paper, when the matrix of parameters changes to another matrix of parameters with respect to multivariate chain majorization, we study the usual stochastic order of the smallest order statistics when each component receives a random shock. Under certain conditions, by using the concept of vector majorization and related orders, we have also discussed stochastic comparison between series and parallel systems in the sense of the usual stochastic order under random shock. We have then applied the results for some special cases of the exponentiated location-scale model with possibly different scale, location and shape parameters to illustrate the established results.

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