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The Extended Chen–Poisson Lifetime Distribution

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Abstract:

- A three-parameter lifetime distribution is proposed, named extended Chen–Poisson distribution, by compounding the Chen and zero-truncated Poisson distributions. The new distribution belongs to the unified Poisson family, where both distributions of the minimum and maximum are merged into one. Several properties of the distribution are studied. The proposed distribution is quite flexible since it accommodates different complex hazard shapes. Inference is based on the maximum likelihood method in the presence of a right-censoring mechanism. A simulation study is performed to evaluate the properties of the parameters estimators. Two real lifetime data sets are analysed for purposes of comparison with other generalizations of the Chen distribution, as well as with other members of the unified Poisson family. The obtained results allow to highlight the potential of the new distribution.

Keywords:

- *Chen distribution; compounding Poisson; maximum likelihood estimation; survival analysis; unified Poisson family.*

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1. INTRODUCTION

In recent years, several researchers have proposed many generalizations of classical distributions by adding further parameters. Generally, the aim behind such generalized distributions is to improve goodness-of-fit. For instance, the choice for modelling a monotonic hazard function (hf) usually falls on the exponential, Weibull, gamma or others generalized exponential distributions. However, for complex phenomena in survival and reliability studies, the hazard behaviour is almost certainly not monotonic. Therefore, in a situation of non-monotonic hf, such as bathtub-shaped or unimodal, the aforementioned distributions are unreasonable or even unrealistic. These limitations have naturally increased the interest in developing new extensions or generalizations of the more traditional distributions.

In the current literature, the methods for generating new distributions can be divided into two main approaches. The first one consists in the introduction of shape parameter(s) in the baseline distribution to explore tail properties. Some well-known techniques are: Lehman alternatives (also known as exponentiated), Marshall–Olkin, Kumaraswamy, transmuted, among others. The second approach concerns compounding a baseline continuous lifetime distribution with a discrete distribution, namely Poisson, geometric, negative-binomial or logarithmic. One of the reasons for developing compounding distributions is that the lifetime of a system constituted by Z (discrete random variable) components can be characterized by the distribution of the minimum or maximum of the lifetimes of its components (non-negative continuous random variables), depending on whether they form a series or a parallel system, respectively. A detailed and comprehensive survey of the existing methods are presented in Tahir and Cordeiro [30], which also proposed some new distributions.

An interesting two-parameter lifetime distribution that exhibits an increasing or a bathtub-shaped hf was proposed by Chen [11]. Some merits of this distribution are related with the exact confidence intervals and exact joint confidence region for the parameters. Over the years, several generalizations of this distribution have been developed. One of the first extensions, named XTG distribution, was introduced by Xie *et al.* [34] by adding the lacking scale parameter. Although the resulting model provided a better fit to the analysed data, the variety of shapes of the hf was not enriched. Other researchers have proposed models with an increased number of alternative hazard shapes. The family of distributions given by Lehman alternatives was considered by Chaubey and Zang [10] and Sarhan and Apaloo [28], who obtained the exponentiated Chen and exponentiated XTG distributions, respectively. Nadarajah *et al.* [23] derived general properties of the Kumaraswamy family of distributions and illustrated the new results obtaining the Kumaraswamy versions of the Chen and XTG distributions. The Marshall–Olkin technique was applied by Alawadhi *et al.* [2] in order to develop the Marshall–Olkin Chen distribution. The Chen-geometric and Marshall–Olkin Chen distributions can be seen as similar models with the same number of parameters, but the parameter space of the former model takes a more limited range of values. Cordeiro *et al.* [13] proposed a new family of lifetime distributions compounding a given class of generalized Weibull distributions with the geometric distribution. Since the Chen and XTG distributions were shown to be members of such class of models, these authors described the Chen-geometric and XTG-geometric distributions as particular cases. Another compounding distribution was proposed by Pappas *et al.* [24], who studied the Chen-logarithmic distribution and also extended the parameter space of the logarithmic distribution to $\mathbb{R}^+\backslash\{0\}$.

The transmuted Chen distribution has already been developed and was reported in Tahir and Cordeiro [30]. For other recent generalized versions of the Chen distribution, the reader is referred to [3, 7, 31].

In the light of the above context, the aim of this paper is to propose a new flexible generalization of the Chen distribution [11] by compounding it with the zero-truncated Poisson (ZTP) distribution. The remainder of the paper is organized as follows. In Section 2, a brief review on the unified Poisson family of distributions discussed by Ramos *et al.* [26] is presented. Section 3 begins with the definition of the new lifetime distribution, followed by the study of its properties, including the shapes of the probability density function (pdf) and hf in Subsection 3.1, as well as the quantiles, moment generating function and mean residual life function in Subsection 3.2. In Subsection 3.3, the maximum likelihood (ML) method is applied in the presence of a right-censoring mechanism and the estimators performance is evaluated by a simulation study in Subsection 3.4. In Subsections 3.5 and 3.6, the usefulness of the new distribution is illustrated in two real data applications with uncensored and censored observations. Some final remarks are presented in Section 4.

2. THE UNIFIED POISSON FAMILY OF DISTRIBUTIONS: A BRIEF REVIEW

The new distribution arises on competitive and complementary risks (CCR) scenarios, wherein it is only possible to observe the minimum/maximum lifetime among all causes instead of observing the lifetime associated with a particular cause [5]. In these settings, a difficulty emerges if the causes are latent in the sense that there is no information about which cause was responsible for the occurrence of the event. On many situations, it is impossible to specify the true cause, even by an expert, because it is somehow masked. For instance, in the biomedical sciences the interest is often to study the time until death, which can occur due to several competing causes such as respiratory infection, cardiac arrest, stroke, cancer, diabetes, among others. This triggers a competitive risks problem (time-to-event of a series system) due to the fact that it is only possible to observe the minimum lifetime among all causes. In an opposite example, suppose that the death of a patient with a given infection is due to multiple organ failures such as in lungs, kidneys and liver. This is now a complementary risks problem (time-to-event of a parallel system) since only the maximum lifetime among all causes is observed. As mentioned by Basu and Klein [6], since a complementary risks problem is the dual of a competitive risks problem, in general it is sufficient to establish the results in terms of the distribution of the minimum or the maximum, although there are some situations where the distribution of the maximum is simpler to handle analytically.

Recently, Ramos *et al.* [26] showed that both distributions of the minimum and the maximum can be unified in a simple form using a latent variable with ZTP distribution. Let $X_1, ..., X_Z$ be the times to event associated with each cause and Z a random variable with ZTP distribution, with probability mass function $P(Z = z; \phi) = \phi^z \big(z!(\mathrm{e}^{\phi} - 1)\big)^{-1}$, $z \in \mathbb{N}$, $\phi \in \mathbb{R}^+$. Assume that the random variables X's and Z are independent and that $X_1, ..., X_Z$ are independent and identically distributed according to a continuous lifetime distribution with a generic baseline cumulative distribution function (cdf) $F_0(x; \boldsymbol{\theta})$, indexed by the parameters vector $\boldsymbol{\theta}$.

Defining $Y = \min\{X_1, ..., X_Z\}$ in a competitive risks problem, the conditional cdf of Y given that $Z = z$ is

$$F(y|z;\boldsymbol{\theta}) = 1 - P(Y > y|Z = z;\boldsymbol{\theta}) = 1 - \left[1 - F_0(y;\boldsymbol{\theta})\right]^z, \qquad y > 0.$$

Then, the marginal cdf of Y is

$$F(y;\boldsymbol{\theta},\phi) = \sum_{z=1}^{\infty} \frac{\phi^z}{z!(\mathrm{e}^{\phi}-1)}\left(1 - \left[1 - F_0(y;\boldsymbol{\theta})\right]^z\right) = \frac{1 - \mathrm{e}^{-\phi F_0(y;\boldsymbol{\theta})}}{1 - \mathrm{e}^{-\phi}}, \quad \phi > 0. \tag{2.1}$$

On the other hand, defining $T = \max\{X_1, ..., X_Z\}$ in a complementary risks problem, the conditional cdf of T given that $Z = z$ is

$$F(t|z;\boldsymbol{\theta}) = P(T \leq t|Z = z;\boldsymbol{\theta}) = \left[F_0(t;\boldsymbol{\theta})\right]^z, \qquad t > 0.$$

Consequently, the marginal cdf of T is

$$F(t;\boldsymbol{\theta},\phi) = \sum_{z=1}^{\infty} \frac{\phi^z}{z!(\mathrm{e}^{\phi}-1)}\left[F_0(t;\boldsymbol{\theta})\right]^z = \frac{1 - \mathrm{e}^{\phi F_0(t;\boldsymbol{\theta})}}{1 - \mathrm{e}^{\phi}}, \qquad \phi > 0. \tag{2.2}$$

Thus, the distribution obtained from (2.2) belongs to the same family of distributions presented in (2.1) if it is assumed that ϕ takes negative values. So, when the latent variable has a ZTP distribution, the distributions of the minimum and the maximum can be merged into one, giving rise to the unified Poisson family of distributions.

Thereafter, assume that T has a distribution from the unified Poisson family, wherein the parameter space is extended to $\mathbb{R}\backslash\{0\}$. Since the cdf of T is still defined by (2.2), the parameter ϕ of this family of models has a particular interpretation in CCR problems. When $\phi < 0$ ($\phi > 0$), T represents the minimum (maximum) lifetime among all causes.

A large number of compounded ZTP distributions has already been proposed considering separately the minimum or maximum, as reviewed by Tahir and Cordeiro [30]. Following the unified approach, some of these distributions can be merged or even extended. For instance, Ramos *et al.* [26] considered the extended Weibull–Poisson (EWP) distribution [16, 19] (that was initially derived only by taking the minimum) and showed that the exponential-Poisson [18] and Poisson-exponential [9] distributions (that were derived by taking the minimum and maximum, respectively) can be unified into a single distribution, named extended exponential-Poisson (EEP) distribution.

3. A NEW LIFETIME DISTRIBUTION

Let X be a random variable following a Chen distribution [11] with cdf and hf given by

$$F_0(x;\lambda,\gamma) = 1 - \mathrm{e}^{\lambda\left(1-\mathrm{e}^{x^{\gamma}}\right)}, \qquad x > 0, \tag{3.1}$$

and

$$h_0(x;\lambda,\gamma) = \lambda\gamma x^{\gamma-1}\mathrm{e}^{x^{\gamma}}, \qquad x > 0, \tag{3.2}$$

respectively, where $\lambda, \gamma > 0$. Since $h_0'(x;\lambda,\gamma) = [\gamma(x^\gamma + 1) - 1]h_0(x;\lambda,\gamma)x^{-1}$, only the parameter γ affects the shape of the hf, which is: i) bathtub-shaped for $\gamma < 1$ (decreasing for $0 < x \leq (1/\gamma - 1)^{1/\gamma}$ and increasing for $x > (1/\gamma - 1)^{1/\gamma}$); and ii) monotonically increasing for $\gamma \geq 1$.

By substituting (3.1) in the unified Poisson family of distributions (2.2), a new generalization of the Chen distribution arises with cdf given by

$$F(t;\lambda,\gamma,\phi) = \frac{1 - \mathrm{e}^{\phi\left[1-\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t^\gamma}\right)}\right]}}{1 - \mathrm{e}^{\phi}}, \qquad t > 0, \tag{3.3}$$

where $\lambda, \gamma > 0$ and $\phi \in \mathbb{R}\backslash\{0\}$ are the parameters of the distribution. The corresponding pdf is

$$f(t;\lambda,\gamma,\phi) = \frac{\lambda\gamma\phi t^{\gamma-1}}{1 - \mathrm{e}^{-\phi}} \mathrm{e}^{t^\gamma + \lambda\left(1-\mathrm{e}^{t^\gamma}\right) - \phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t^\gamma}\right)}}, \qquad t > 0. \tag{3.4}$$

Hereafter, the distribution of T will be referred to as extended Chen–Poisson (ECP) distribution, which is a customary name for distributions belonging to the unified Poisson family. In fact, this distribution unifies both the minimum ($\phi < 0$) and the maximum ($\phi > 0$) distributions, which correspond to the Chen–Poisson and Poisson-Chen distributions, respectively.

The survival function (sf) and hf of the ECP distribution are defined, respectively, as follows

$$S(t;\lambda,\gamma,\phi) = \frac{1 - \mathrm{e}^{-\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t^\gamma}\right)}}}{1 - \mathrm{e}^{-\phi}}, \qquad t > 0,$$

and

$$h(t;\lambda,\gamma,\phi) = \frac{\lambda\gamma\phi t^{\gamma-1}\mathrm{e}^{t^\gamma + \lambda\left(1-\mathrm{e}^{t^\gamma}\right)}}{\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t^\gamma}\right)}} - 1}, \qquad t > 0. \tag{3.5}$$

3.1. Shapes of the probability density function and hazard function

The pdf (3.4) and hf (3.5) for some combinations of parameters values are depicted in Figures 1 and 2, respectively. It is challenging to study analytically the theoretical behaviour of these functions due to their complex expressions. In addition, the monotonicity study is hampered by the fact that all three parameters, λ, γ and ϕ, affect both the density and hazard shapes.

Based on the analytical analysis of the pdf, and as illustrated on the graphical representation in Figure 1, the density shape can be: (a) monotonic decreasing; (b)–(c) unimodal; or (d) decreasing-increasing-decreasing (DID). In what concerns the hazard shape, Figure 2 suggests that it can be: (a) monotonic increasing; (b) monotonic decreasing; (c) unimodal; (d) bathtub; (e) increasing-decreasing-increasing (IDI); or (f) decreasing-increasing-decreasing-increasing (DIDI). Accordingly, the ECP distribution is shown to be quite flexible. Nonetheless, some care is needed as the monotonicity study of the hf should not be solely based on graphical analysis. Since $\lim_{t\to\infty} h(t;\lambda,\gamma,\phi) = \infty$, for all $\lambda, \gamma > 0$ and $\phi \in \mathbb{R}\backslash\{0\}$, the hf is ultimately increasing, so a pure monotonic decreasing or unimodal shape is impossible.

However, it was verified that when γ takes values close to zero the hf takes a long time to increase. In such cases it is usual to admit that, from the practical point of view, the hf has a generally decreasing right tail.

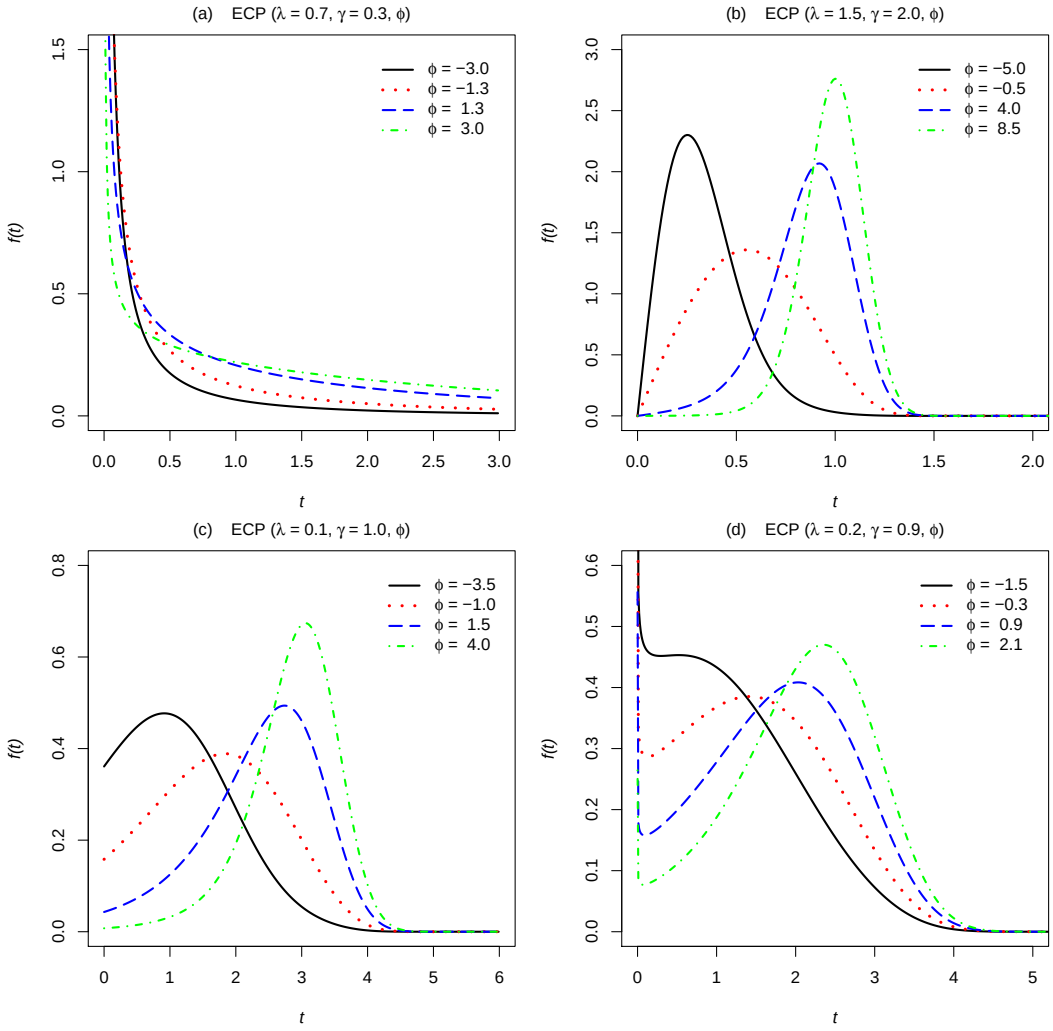

Figure 1: Probability density functions of the ECP distribution for different combinations of parameters values.

Proposition 3.1. *The Chen distribution is a limiting case of the ECP distribution, since when ϕ approaches 0 it follows that*

$$\lim_{\phi \to 0} h(t; \lambda, \gamma, \phi) = \lambda \gamma t^{\gamma - 1} \mathrm{e}^{t^{\gamma}},$$

which is the hf (3.2) of the Chen distribution.

Proposition 3.2. *The limiting behaviour of the pdf (3.4) and hf (3.5) of the ECP distribution is*

$$\textbf{(i)} \quad \lim_{t \to 0^+} f(t; \lambda, \gamma, \phi) = \lim_{t \to 0^+} h(t; \lambda, \gamma, \phi) = \begin{cases} \infty, & 0 < \gamma < 1, \\ \dfrac{\lambda \phi}{\mathrm{e}^{\phi} - 1}, & \gamma = 1, \\ 0, & \gamma > 1, \end{cases}$$

$\forall \lambda > 0$ *and* $\phi \in \mathbb{R} \backslash \{0\}$;

$$\textbf{(ii)} \quad \lim_{t \to \infty} f(t; \lambda, \gamma, \phi) = 0 \text{ and } \lim_{t \to \infty} h(t; \lambda, \gamma, \phi) = \infty, \ \forall \lambda, \gamma > 0 \text{ and } \phi \in \mathbb{R} \backslash \{0\}.$$

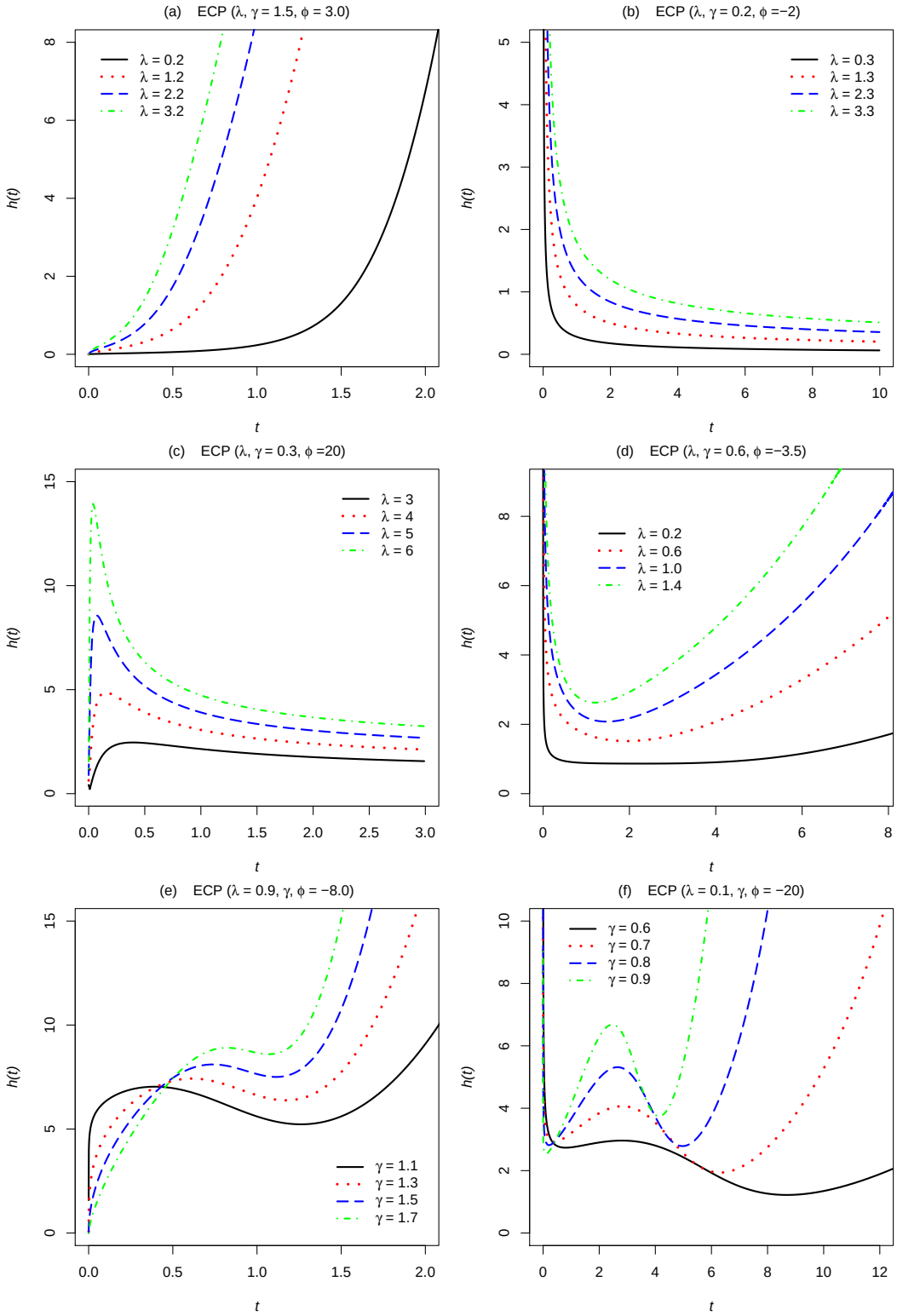

Figure 2: Hazard functions of the ECP distribution for different combinations of parameters values.

Proposition 3.3. *The theoretical behaviour of the pdf (3.4) of the ECP distribution may be characterized separately for the minimum ($\phi < 0$) and maximum ($\phi > 0$) distributions, as summarized in the following statements.*

(**i**) *Distribution of the minimum:*
- *For $\phi < 0$, $0 < \gamma \leq 1$ and $\lambda \geq (1-\phi)^{-1}$, the pdf is monotonically decreasing;*
- *For $\phi < 0$, $\gamma = 1$ and $0 < \lambda < (1-\phi)^{-1}$, the pdf is unimodal;*
- *For $\phi < 0$, $0 < \gamma < 1$ and $0 < \lambda < (1-\phi)^{-1}$, the pdf is monotonically decreasing or DID;*
- *For $\phi < 0$, $\gamma > 1$ and $\lambda > 0$, the pdf is unimodal;*

(**ii**) *Distribution of the maximum:*
- *For $0 < \phi \leq 1 - \lambda^{-1}$, $0 < \gamma \leq 1$ and $\lambda > 1$, the pdf is monotonically decreasing;*
- *For $\phi > 1 - \lambda^{-1}$, $\gamma = 1$ and $\lambda > 1$, the pdf is unimodal;*
- *For $\phi > 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$, the pdf is monotonically decreasing or DID;*
- *For $\phi > 0$, $\gamma > 1$ and $\lambda > 1$, the pdf is unimodal;*
- *For $\phi > 0$, $\gamma \geq 1$ and $0 < \lambda \leq 1$, the pdf is unimodal;*
- *For $\phi > 0$, $0 < \gamma < 1$ and $0 < \lambda \leq 1$, the pdf is monotonically decreasing or DID.*

The proofs of Propositions 3.1 and 3.2 are straightforward and, therefore, are omitted. The proof of Proposition 3.3 is given in supplementary material file.

3.2. Quantiles, moments and mean residual life function

Some of the most important characteristics of a distribution, such as dispersion, skewness and kurtosis, can be studied through its quantiles and moments. By inverting the cdf (3.3), the quantile function of the ECP distribution is given by

$$Q(u;\lambda,\gamma,\phi) = \left\{ \log\left[1 - \lambda^{-1}\log\left(1 - \phi^{-1}\log\left((\mathrm{e}^{\phi}-1)u+1\right)\right)\right]\right\}^{1/\gamma}, \tag{3.6}$$

for $0 < u < 1$. This expression can be used for simulating pseudo-random values of $T \sim \mathrm{ECP}(\lambda,\gamma,\phi)$, considering that

$$T = \left\{ \log\left[1 - \lambda^{-1}\log\left(1 - \phi^{-1}\log\left((\mathrm{e}^{\phi}-1)U+1\right)\right)\right]\right\}^{1/\gamma}, \tag{3.7}$$

where U is a uniformly distributed random variable on $(0,1)$ interval.

The moment generating function of T can be defined as

$$M_T(w) = E(\mathrm{e}^{wT}) = \phi(1-\mathrm{e}^{-\phi})^{-1}\int_0^1 \exp\left\{w\left[\log\left(1-\lambda^{-1}\log(v)\right)\right]^{1/\gamma} - \phi v\right\}dv,$$

by making the change of variable $v = \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t^\gamma}\right)}$. Then, the r-th raw moment of T is given by

$$E(T^r) = \phi(1-\mathrm{e}^{-\phi})^{-1}\int_0^1 \mathrm{e}^{-\phi v}\left[\log\left(1-\lambda^{-1}\log(v)\right)\right]^{r/\gamma} dv, \quad r = 1, 2, \ldots .$$

In particular, the mean and variance of ECP distribution are, respectively, given by

$$E(T) = \phi(1-\mathrm{e}^{-\phi})^{-1}\int_0^1 \mathrm{e}^{-\phi v}\left[\log\left(1-\lambda^{-1}\log(v)\right)\right]^{1/\gamma} dv,$$

and

$$\mathrm{Var}(T) = \phi(1-\mathrm{e}^{-\phi})^{-1}\int_0^1 \mathrm{e}^{-\phi v}\left[\log\left(1-\lambda^{-1}\log(v)\right)\right]^{2/\gamma} dv - [E(T)]^2.$$

The mean residual life function, as well as the hf, plays an important role in survival analysis for characterizing lifetime. While the latter represents the instantaneous event rate, the former summarizes the entire residual lifetime. The mean residual life function, $\mathrm{mrl}(t;\lambda,\gamma,\phi) = E(T-t|T \geq t)$, of the ECP distribution is given by

$$\mathrm{mrl}(t;\lambda,\gamma,\phi) = \phi(1-\mathrm{e}^{-\phi A})^{-1}\int_0^A \mathrm{e}^{-\phi v}\left[\log\left(1-\lambda^{-1}\log(v)\right)\right]^{1/\gamma} dv - t,$$

with $A = \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t^\gamma}\right)}$.

The moments have no closed-form expressions and so they can only be obtained using numerical integration. Therefore, the classical measures of skewness and kurtosis based on moments are intractable. In this case, quantile-based measures are often considered, namely the Bowley skewness and Moors kurtosis that are given, respectively, by $B = [Q(3/4) - 2Q(1/2) + Q(1/4)]/[Q(3/4) - Q(1/4)]$ and $M = [Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)]/[Q(3/4) - Q(1/4)]$, where $Q(\cdot)$ comes from (3.6). These measures exist even for distributions without finite moments and are less sensitive to outliers.

3.3. Statistical inference

For statistical inference, the ML method is usually preferred due to the attractive properties of the resulting estimators, such as consistency, asymptotic efficiency, invariance property and asymptotic normality. Therefore, the ML method to estimate the three unknown parameters of the ECP distribution for the general case of right-censored time-to-event data is presented.

Let $\tilde{T}_i = \min\{T_i, C_i\}$, $i = 1, \ldots, n$, where T_i is the lifetime of i-th subject, following a ECP distribution, and C_i is the censoring time, assumed to have a distribution that does not depend on the parameters of T_i. Moreover, it is assumed that T_i and C_i are independent. So, the censoring mechanism is non-informative. The censoring indicator is defined as $\delta_i = I(T_i \leq C_i)$, taking the value 1 if T_i is a time-to-event and 0 if it is right-censored. Considering a random sample of n pairs, $(t_1, \delta_1), \ldots, (t_n, \delta_n)$, the log-likelihood function $\ell = \log L(\lambda, \gamma, \phi)$

is given by

$$\begin{aligned}\ell &= \sum_{i=1}^{n}\Big\{\delta_i \log f(t_i;\lambda,\gamma,\phi) + (1-\delta_i)\log S(t_i;\lambda,\gamma,\phi)\Big\}\\ &= n\log\left(\frac{\phi}{1-\mathrm{e}^{-\phi}}\right) + m\big(\lambda + \log(\lambda\gamma)\big) + (\gamma-1)\sum_{i=1}^{n}\delta_i\log(t_i) + \sum_{i=1}^{n}\delta_i t_i^{\gamma}\\ &\quad - \lambda\sum_{i=1}^{n}\delta_i \mathrm{e}^{t_i^{\gamma}} + \sum_{i=1}^{n}(1-\delta_i)\log\left(\frac{1-\mathrm{e}^{-\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{\phi}\right) - \phi\sum_{i=1}^{n}\delta_i\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)},\end{aligned} \tag{3.8}$$

where $m=\sum_{i=1}^{n}\delta_i$ is the observed number of events. Some care must be taken when $\phi<0$, since the values of $\log(\phi)$ cannot be computed. This problem is easily overcome by considering the fact that $\log\big(\phi/(1-\mathrm{e}^{-\phi})\big)\in\mathbb{R}$, $\forall\phi\in\mathbb{R}\backslash\{0\}$, and $\log\big((1-\exp\{-\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}\})/\phi\big)\in\mathbb{R}$, $\forall\lambda,\gamma>0$ and $\phi\in\mathbb{R}\backslash\{0\}$.

The first-order partial derivatives of the log-likelihood function with respect to each of the three parameters are

$$\frac{\partial\ell}{\partial\lambda} = m\left(1+\frac{1}{\lambda}\right) - \sum_{i=1}^{n}\delta_i\mathrm{e}^{t_i^{\gamma}} - \sum_{i=1}^{n}(1-\delta_i)\frac{\phi\big(1-\mathrm{e}^{t_i^{\gamma}}\big)\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}{1-\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}} - \phi\sum_{i=1}^{n}\delta_i\big(1-\mathrm{e}^{t_i^{\gamma}}\big)\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)},$$

$$\begin{aligned}\frac{\partial\ell}{\partial\gamma} &= \frac{m}{\gamma} + \sum_{i=1}^{n}\delta_i\log\big(t_i\big) + \sum_{i=1}^{n}\delta_i t_i^{\gamma}\log\big(t_i\big) - \lambda\sum_{i=1}^{n}\delta_i t_i^{\gamma}\log\big(t_i\big)\mathrm{e}^{t_i^{\gamma}}\\ &\quad + \lambda\phi\sum_{i=1}^{n}(1-\delta_i)\frac{t_i^{\gamma}\log\big(t_i\big)\mathrm{e}^{t_i^{\gamma}+\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}{1-\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}} + \lambda\phi\sum_{i=1}^{n}\delta_i t_i^{\gamma}\log\big(t_i\big)\mathrm{e}^{t_i^{\gamma}+\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)},\end{aligned}$$

$$\frac{\partial\ell}{\partial\phi} = n\left(\frac{1}{\phi}+\frac{1}{1-\mathrm{e}^{\phi}}\right) - \frac{1}{\phi}\sum_{i=1}^{n}(1-\delta_i)\frac{1+\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)} - \mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{1-\mathrm{e}^{\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}} - \sum_{i=1}^{n}\delta_i\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}.$$

The ML estimates are determined by setting these partial derivatives equal to zero, obtaining a nonlinear system of equations that can only be solved using a numerical optimization method such as Newton–Raphson or Broyden–Fletcher–Goldfarb–Shanno (BFGS).

Under mild regularity conditions, the ML estimators of λ, γ and ϕ have an asymptotic multivariate normal distribution given by

$$(\hat{\lambda},\hat{\gamma},\hat{\phi}) \overset{a}{\sim} N\Big[(\lambda,\gamma,\phi),\ \boldsymbol{I}^{-1}(\lambda,\gamma,\phi)\Big], \quad \text{as} \quad n\longrightarrow\infty,$$

where the observed information matrix, $\boldsymbol{I}(\lambda,\gamma,\phi)$, is defined as

$$\boldsymbol{I}(\lambda,\gamma,\phi) = -\begin{bmatrix} \dfrac{\partial^2\ell}{\partial\lambda^2}, & \dfrac{\partial^2\ell}{\partial\lambda\partial\gamma}, & \dfrac{\partial^2\ell}{\partial\lambda\partial\phi} \\ \dfrac{\partial^2\ell}{\partial\gamma\partial\lambda}, & \dfrac{\partial^2\ell}{\partial\gamma^2}, & \dfrac{\partial^2\ell}{\partial\gamma\partial\phi} \\ \dfrac{\partial^2\ell}{\partial\phi\partial\lambda}, & \dfrac{\partial^2\ell}{\partial\phi\partial\gamma}, & \dfrac{\partial^2\ell}{\partial\phi^2} \end{bmatrix}.$$

The mathematical expressions of the elements of $\boldsymbol{I}(\lambda,\gamma,\phi)$ are given in supplementary material file.

For interval estimation and hypothesis testing, let $\widehat{\text{Var}}(\hat{\lambda})$, $\widehat{\text{Var}}(\hat{\gamma})$ and $\widehat{\text{Var}}(\hat{\phi})$ denote the estimates of the main diagonal elements of the inverse of the observed information matrix, evaluated at the ML estimates of the parameters. The large-sample $(1-\alpha)100\%$ confidence intervals (CI) for λ, γ and ϕ are

$$\hat{\lambda} \pm z_{\alpha/2}\sqrt{\widehat{\text{Var}}(\hat{\lambda})}, \qquad \hat{\gamma} \pm z_{\alpha/2}\sqrt{\widehat{\text{Var}}(\hat{\gamma})} \qquad \text{and} \qquad \hat{\phi} \pm z_{\alpha/2}\sqrt{\widehat{\text{Var}}(\hat{\phi})},$$

respectively, where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution.

For computational implementation, the `optim` function available in `R` [25] statistical software (version 4.1.0) was used for direct maximization of the log-likelihood function (3.8).

3.4. Simulation study

In order to investigate the performance of ML estimators of the three parameters of the ECP distribution and to evaluate the accuracy of the resulting estimates, a simulation study was conducted through `R` [25] statistical software. In such simulation, the following steps were followed:

1. Specification of the parameters values $(\lambda, \gamma, \phi) =$ (0.2, 1.5, 3.0), (1.3, 0.2, -2.0), (3.0, 0.3, 20.0) and (0.6, 0.6, -3.5). These sets of parameters values were selected in order to yield increasing, decreasing, unimodal and bathtub shapes of the hazard function, respectively, as shown in Figure 2.
2. Specification of the sample size $n =$ 20, 50, 100, 500 and 1000.
3. Generation of a pseudo-random sample from (3.7), in the presence of random censoring (that has the types I and II of censoring mechanisms as special cases). Here, it is assumed that the event times follow an ECP distribution and the censoring times are uniformly distributed. The percentage of pseudo-random censoring is specified as 0%, 10% and 30%, following the procedures discussed in [27].
4. Computation of the ML estimates of the three parameters using the BFGS method and evaluation of the elements of the inverse of the observed information matrix at the ML estimates.
5. Repetition of the steps 1 to 4, $N = 1\,000$ times.
6. Calculation of the average of the N ML estimates and their standard errors.
7. Calculation of the bias, mean squared error (MSE) and coverage probability (CP) of the 95% CI for each parameter. The bias and MSE associated with the ML estimates of the parameter ϑ are, respectively, given by

$$\text{Bias}_{\vartheta} = \frac{1}{N}\sum_{l=1}^{N}\left(\hat{\vartheta}_l - \vartheta\right) \qquad \text{and} \qquad \text{MSE}_{\vartheta} = \frac{1}{N}\sum_{l=1}^{N}\left(\hat{\vartheta}_l - \vartheta\right)^2,$$

where $\hat{\vartheta}_l$ is the ML estimate obtained from the l-th sample, $l = 1, ..., N$, and $\boldsymbol{\vartheta} = (\lambda, \gamma, \phi)'$. The CP is the proportion of the N generated 95% CIs that include the real value of the parameter.

Table 1: The averages of the 1000 ML estimates for λ, γ and ϕ, their standard errors, bias, mean square errors (MSE) and coverage probabilities (CP) of the 95% CI. (Continues.)

Censoring	(λ, γ, ϕ)	n	Average			Standard error			Bias			MSE			CP		
			$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$
0%	(0.2, 1.5, 3.0)	20	0.268	1.580	6.593	0.240	0.394	7.865	0.068	0.080	3.593	0.093	0.156	588.379	80.7%	93.7%	99.1%
		50	0.215	1.551	3.260	0.140	0.278	2.727	0.015	0.051	0.260	0.021	0.071	6.965	88.5%	93.0%	98.9%
		100	0.204	1.538	3.021	0.102	0.209	1.831	0.004	0.038	0.021	0.010	0.040	2.814	90.1%	93.2%	98.0%
		500	0.196	1.515	2.940	0.045	0.087	0.721	−0.004	0.015	−0.060	0.002	0.007	0.487	96.2%	96.7%	97.9%
		1000	0.197	1.509	2.963	0.031	0.058	0.482	−0.003	0.009	−0.037	0.001	0.003	0.235	95.4%	96.7%	97.1%
	(1.3, 0.2, −2.0)	20	1.863	0.213	−1.591	1.311	0.057	3.013	0.563	0.013	0.409	1.461	0.002	1.685	99.6%	97.4%	100.0%
		50	1.507	0.203	−1.878	0.833	0.036	2.336	0.207	0.003	0.122	0.374	0.001	1.674	98.3%	97.8%	100.0%
		100	1.416	0.201	−1.988	0.666	0.026	1.953	0.116	0.001	0.012	0.229	0.000	1.502	97.2%	96.9%	99.8%
		500	1.337	0.199	−1.983	0.345	0.011	1.018	0.037	−0.001	0.017	0.079	0.000	0.701	95.6%	97.2%	97.9%
		1000	1.315	0.200	−2.000	0.252	0.008	0.725	0.015	0.000	0.000	0.036	0.000	0.287	97.0%	97.1%	97.7%
	(3.0, 0.3, 20.0)	20	3.213	0.432	58.535	0.823	0.215	29.647	0.213	0.132	38.535	0.872	0.130	12950.676	79.0%	79.6%	63.7%
		50	3.174	0.321	46.065	0.525	0.103	23.152	0.174	0.021	26.065	0.373	0.018	6438.042	82.9%	84.9%	74.1%
		100	3.150	0.304	35.482	0.388	0.068	18.428	0.150	0.004	15.482	0.216	0.006	2836.665	88.1%	89.9%	81.1%
		500	3.014	0.302	21.291	0.174	0.031	6.630	0.014	0.002	1.291	0.030	0.001	48.590	95.9%	95.8%	93.4%
		1000	3.005	0.302	20.479	0.121	0.022	4.310	0.005	0.002	0.479	0.014	0.000	18.554	95.4%	94.8%	92.9%
	(0.6, 0.6, −3.5)	20	0.987	0.634	−3.947	0.743	0.151	6.246	0.387	0.034	−0.447	0.526	0.020	34.523	93.1%	97.1%	99.7%
		50	0.882	0.604	−2.990	0.528	0.095	3.130	0.282	0.004	0.510	0.252	0.006	9.711	96.2%	97.0%	96.3%
		100	0.803	0.597	−3.101	0.431	0.066	2.539	0.203	−0.003	0.399	0.172	0.003	5.334	94.3%	96.4%	92.8%
		500	0.673	0.595	−3.400	0.233	0.028	1.450	0.073	−0.005	0.100	0.060	0.001	2.121	90.7%	96.5%	90.7%
		1000	0.644	0.597	−3.452	0.168	0.019	1.046	0.044	−0.003	0.048	0.034	0.000	1.255	91.1%	96.8%	91.4%

Table 1: (Continued.) The averages of the 1000 ML estimates for λ, γ and ϕ, their standard errors, bias, mean square errors (MSE) and coverage probabilities (CP) of the 95% CI. (Continues.)

Censoring	(λ, γ, ϕ)	n	Average			Standard error			Bias			MSE			CP		
			$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$
10%	(0.2, 1.5, 3.0)	20	0.282	1.575	7.146	0.274	0.421	10.866	0.082	0.075	4.146	0.110	0.168	502.776	82.6%	94.3%	99.4%
		50	0.219	1.556	3.340	0.153	0.302	2.990	0.019	0.056	0.340	0.025	0.080	8.137	88.3%	93.2%	99.3%
		100	0.200	1.549	2.965	0.103	0.210	1.826	0.000	0.049	−0.035	0.011	0.044	3.187	88.4%	92.3%	97.6%
		500	0.197	1.516	2.944	0.047	0.091	0.749	−0.003	0.016	−0.056	0.002	0.009	0.563	94.8%	95.9%	97.0%
		1000	0.197	1.511	2.947	0.033	0.062	0.508	−0.003	0.011	−0.053	0.001	0.004	0.255	95.7%	96.6%	97.3%
	(1.3, 0.2, −2.0)	20	1.810	0.214	−1.825	1.495	0.058	3.612	0.510	0.014	0.175	2.091	0.003	1.811	100.0%	95.7%	100.0%
		50	1.456	0.202	−2.055	1.042	0.036	2.929	0.156	0.002	−0.055	0.449	0.001	1.734	99.3%	98.0%	100.0%
		100	1.387	0.200	−2.058	0.845	0.027	2.428	0.087	0.000	−0.058	0.252	0.000	1.474	99.1%	97.1%	100.0%
		500	1.325	0.199	−2.051	0.472	0.012	1.357	0.025	−0.001	−0.051	0.111	0.000	0.889	96.6%	97.6%	99.1%
		1000	1.297	0.199	−2.061	0.387	0.008	1.080	−0.003	−0.001	−0.061	0.056	0.000	0.473	95.8%	97.8%	97.8%
	(3.0, 0.3, 20.0)	20	3.198	0.470	54.671	0.868	0.246	30.938	0.198	0.170	34.671	0.882	0.176	11453.983	79.0%	78.5%	62.5%
		50	3.224	0.324	51.708	0.534	0.110	24.354	0.224	0.024	31.708	0.431	0.026	8374.052	80.1%	81.5%	70.9%
		100	3.149	0.305	37.308	0.393	0.074	18.146	0.149	0.005	17.308	0.231	0.008	3257.225	86.7%	88.1%	78.5%
		500	3.029	0.301	21.980	0.182	0.034	7.398	0.029	0.001	1.980	0.036	0.001	71.255	96.1%	95.2%	92.8%
		1000	3.006	0.301	20.639	0.126	0.024	4.663	0.006	0.001	0.639	0.015	0.001	21.348	95.8%	95.5%	92.8%
	(0.6, 0.6, −3.5)	20	0.902	0.641	−4.820	0.853	0.157	8.715	0.302	0.041	−1.320	0.519	0.024	38.210	91.6%	96.0%	99.7%
		50	0.801	0.607	−3.962	0.596	0.096	4.970	0.201	0.007	−0.462	0.242	0.007	19.043	94.1%	96.4%	98.5%
		100	0.759	0.601	−3.582	0.470	0.068	3.349	0.159	0.001	−0.082	0.155	0.004	9.607	95.3%	96.7%	95.8%
		500	0.655	0.596	−3.586	0.282	0.031	1.849	0.055	−0.004	−0.086	0.067	0.001	2.374	90.9%	96.0%	91.0%
		1000	0.647	0.595	−3.517	0.230	0.022	1.431	0.047	−0.005	−0.017	0.049	0.000	1.808	89.4%	96.3%	89.7%

Table 1: (Continued.) The averages of the 1000 ML estimates for λ, γ and ϕ, their standard errors, bias, mean square errors (MSE) and coverage probabilities (CP) of the 95% CI.

Censoring	(λ, γ, ϕ)	n	Average			Standard error			Bias			MSE			CP		
			$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$
30%	(0.2, 1.5, 3.0)	20	0.309	1.594	11.608	0.307	0.465	13.178	0.109	0.094	8.608	0.173	0.223	2317.641	78.8%	92.2%	97.9%
		50	0.223	1.569	3.432	0.164	0.326	3.411	0.023	0.069	0.432	0.032	0.101	12.296	85.5%	93.6%	99.7%
		100	0.201	1.554	2.975	0.120	0.253	2.153	0.001	0.054	−0.025	0.013	0.053	3.814	89.4%	94.4%	98.8%
		500	0.196	1.520	2.922	0.055	0.111	0.873	−0.004	0.020	−0.078	0.003	0.012	0.700	94.7%	95.2%	96.2%
		1000	0.197	1.511	2.947	0.038	0.072	0.565	−0.003	0.011	−0.053	0.001	0.005	0.304	95.6%	96.1%	97.1%
	(1.3 0.2 −2.0)	20	1.771	0.215	−1.883	1.703	0.058	4.049	0.471	0.015	0.117	2.086	0.003	1.617	99.9%	95.7%	100.0%
		50	1.464	0.203	−2.015	1.249	0.037	3.321	0.164	0.003	−0.015	0.455	0.001	1.461	99.3%	97.5%	100.0%
		100	1.394	0.201	−2.045	1.041	0.028	2.871	0.094	0.001	−0.045	0.256	0.000	1.312	98.7%	97.2%	99.9%
		500	1.353	0.199	−1.979	0.592	0.012	1.595	0.053	−0.001	0.021	0.119	0.000	0.752	97.5%	97.4%	99.1%
		1000	1.314	0.199	−2.019	0.486	0.008	1.290	0.014	−0.001	−0.019	0.065	0.000	0.456	97.6%	97.6%	98.7%
	(3.0, 0.3, 20.0)	20	3.332	0.551	55.909	1.006	0.327	28.306	0.332	0.251	35.909	2.243	0.302	14102.353	80.7%	77.9%	57.8%
		50	3.266	0.342	65.858	0.562	0.141	24.938	0.266	0.042	45.858	0.504	0.049	15920.982	77.7%	77.2%	67.4%
		100	3.218	0.303	52.297	0.396	0.088	18.083	0.218	0.003	32.297	0.324	0.014	9011.748	79.6%	80.6%	71.9%
		500	3.046	0.299	23.555	0.200	0.042	9.380	0.046	−0.001	3.555	0.048	0.002	166.505	94.9%	94.9%	91.4%
		1000	3.016	0.300	21.256	0.139	0.030	5.855	0.016	0.000	1.256	0.018	0.001	37.586	96.6%	95.4%	94.1%
	(0.6 0.6 −3.5)	20	0.730	0.653	−6.067	1.076	0.170	13.444	0.130	0.053	−2.567	0.404	0.033	40.916	93.6%	95.8%	99.8%
		50	0.646	0.614	−5.323	0.857	0.105	9.228	0.046	0.014	−1.823	0.174	0.010	25.074	92.1%	96.6%	99.9%
		100	0.606	0.605	−5.033	0.703	0.076	7.275	0.006	0.005	−1.533	0.101	0.005	18.459	91.7%	96.4%	99.9%
		500	0.595	0.598	−4.091	0.496	0.039	3.805	−0.005	−0.002	−0.591	0.046	0.001	4.743	95.5%	96.6%	99.1%
		1000	0.588	0.595	−3.941	0.391	0.028	2.617	−0.012	−0.005	−0.441	0.037	0.001	2.665	92.6%	95.8%	99.3%

The results obtained from the simulation study are presented in Table 1. For samples generated with 0% of censoring, it is observed that the averages of the ML estimates of λ, γ and ϕ tend to the true value of the parameter as the sample size increases, as well as their standard errors tend to zero. Both the bias and MSE are smaller for larger sample sizes, reflecting that the ML estimators are asymptotically unbiased. Besides, the CP tends to be closer to the nominal level of 95%. However, it appears that ϕ has higher values for bias and MSE in comparison to the remaining parameters. This aspect is more visible for the set of parameters values corresponding to a unimodal hazard shape, but then it vanishes for large sample sizes and does not compromise the estimation of λ and γ.

In general, these results suggest that the estimation of parameters was performed consistently. Similar results were obtained for samples generated with 10% and 30%, despite the bias and MSE of all three parameters having slightly higher values. Although it is not shown here, the results were similar to the ones obtained for other choices of parameter values.

The programming codes of the simulation study, developed in `R`, are available in supplementary material file. Further research may be carried out to assess and explore other potential estimation procedures for the parameters of the ECP distribution, such as least-square estimators, minimum distance estimators, percentile based estimators, among others (see, for example, Dey *et al.* [14]).

3.5. Application to uncensored data: guinea pigs

In this section, the ECP distribution is applied to the (uncensored) guinea pigs data set reported by Bjerkedal [8]. The data represent the survival times, in days, of 72 guinea pigs infected with virulent tubercle bacilli. Dey *et al.* [15] analysed a transformed version of the original data (divided by 100), which is also considered in this work. Moreover, the adequacy of the ECP distribution is assessed in comparison with some other generalizations of the Chen distribution. Those models are listed in Table 2.

Table 2: List of distributions fitted to the guinea pigs data.

j-th Model, [ref.]	Probability density function, $f(t)$, $t > 0$
1 Chen, [11]	$\lambda_1\gamma_1 t^{\gamma_1-1}\mathrm{e}^{t^{\gamma_1}+\lambda_1(1-\mathrm{e}^{t^{\gamma_1}})}$, $\lambda_1, \gamma_1 > 0$
2 XTG, [34]	$\lambda_2\gamma_2(t/\phi_2)^{\gamma_2-1}\mathrm{e}^{(t/\phi_2)^{\gamma_2}+\lambda_2\phi_2(1-\mathrm{e}^{(t/\phi_2)^{\gamma_2}})}$, $\lambda_2, \gamma_2, \phi_2 > 0$
3 ECP	$\dfrac{\lambda_3\gamma_3\phi_3 t^{\gamma_3-1}}{1-\mathrm{e}^{-\phi_3}}\mathrm{e}^{t^{\gamma_3}+\lambda_3(1-\mathrm{e}^{t^{\gamma_3}})-\phi_3\mathrm{e}^{\lambda_3(1-\mathrm{e}^{t^{\gamma_3}})}}$, $\lambda_3, \gamma_3 > 0$, $\phi_3 \in \mathbb{R}\backslash\{0\}$
4 Chen-logarithmic, [24]	$\dfrac{\lambda_4\gamma_4(\phi_4-1)t^{\gamma_4-1}}{\left[1-(1-\phi_4)\mathrm{e}^{\lambda_4(1-\mathrm{e}^{t^{\gamma_4}})}\right]\log\phi_4}\mathrm{e}^{t^{\gamma_4}+\lambda_4(1-\mathrm{e}^{t^{\gamma_4}})}$, $\lambda_4, \gamma_4, \phi_4 > 0$
5 Exponentiated Chen, [10]	$\lambda_5\gamma_5\phi_5 t^{\gamma_5-1}\left[1-\mathrm{e}^{\lambda_5(1-\mathrm{e}^{t^{\gamma_5}})}\right]^{\phi_5-1}\mathrm{e}^{t^{\gamma_5}+\lambda_5(1-\mathrm{e}^{t^{\gamma_5}})}$, $\lambda_5, \gamma_5, \phi_5 > 0$
6 Marshall–Olkin Chen, [2]	$\dfrac{\lambda_6\gamma_6\phi_6 t^{\gamma_6-1}}{\left[1-(1-\phi_6)\mathrm{e}^{\lambda_6(1-\mathrm{e}^{t^{\gamma_6}})}\right]^2}\mathrm{e}^{t^{\gamma_6}+\lambda_6(1-\mathrm{e}^{t^{\gamma_6}})}$, $\lambda_6, \gamma_6, \phi_6 > 0$
7 Transmuted Chen, [30]	$\dfrac{\lambda_7\gamma_7 t^{\gamma_7-1}}{\left[1-\phi_7+2\phi_7\mathrm{e}^{\lambda_7(1-\mathrm{e}^{t^{\gamma_7}})}\right]^{-1}}\mathrm{e}^{t^{\gamma_7}+\lambda_7(1-\mathrm{e}^{t^{\gamma_7}})}$, $\lambda_7, \gamma_7 > 0$, $\phi_7 \in (-1,\ 1)$
8 Kumaraswamy Chen, [23]	$\dfrac{\lambda_8\gamma_8\phi_8\psi_8 t^{\gamma_8-1}(1-\mathrm{e}^{\lambda_8(1-\mathrm{e}^{t^{\gamma_8}})})^{\phi_8-1}}{\left[1-\left[1-\mathrm{e}^{\lambda_8(1-\mathrm{e}^{t^{\gamma_8}})}\right]^{\phi_8}\right]^{1-\psi_8}}\mathrm{e}^{t^{\gamma_8}+\lambda_8(1-\mathrm{e}^{t^{\gamma_8}})}$, $\lambda_8, \gamma_8, \phi_8, \psi_8 > 0$

The `AdequacyModel` [21] package was used for fitting models to the guinea pigs data. The ML estimates, their corresponding standard errors and −log-likelihood values of the fitted models are shown in Table 3. The `AdequacyModel` package also provides some useful statistics to assess the adequacy of the fitted models [22], such as the Cramér–von Mises (CM), Anderson–Darling (AD), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC) and in addition performs the Kolmogorov–Smirnov (KS) test. The obtained values are compiled in Table 4.

Table 3: ML estimates, standard errors and −log-likelihood values for the guinea pigs data.

Model	ML estimates				Standard error				$-\hat{\ell}$
	$\hat{\lambda}_j$	$\hat{\gamma}_j$	$\hat{\phi}_j$	$\hat{\psi}_j$	$\hat{\lambda}_j$	$\hat{\gamma}_j$	$\hat{\phi}_j$	$\hat{\psi}_j$	
Chen	0.208	0.759	—	—	0.034	0.043	—	—	104.241
XTG	0.391	0.322	0.010	—	0.165	0.023	0.005	—	100.839
ECP	1.225	0.407	12.094	—	0.256	0.061	5.158	—	93.537
Chen-logarithmic	0.208	0.758	1.008	—	0.131	0.094	1.395	—	104.241
Exponentiated Chen	0.995	0.444	7.209	—	0.306	0.080	4.095	—	94.186
Marshall–Olkin Chen	0.003	1.131	0.016	—	0.001	0.043	0.006	—	97.975
Transmuted Chen	0.117	0.809	0.753	—	0.025	0.045	0.203	—	102.617
Kumaraswamy Chen	0.896	0.339	9.229	2.364	0.391	0.324	11.159	6.413	94.108

Table 4: Goodness-of-fit statistics for the guinea pigs data.

Model	CM	AD	KS (p-value)	AIC	CAIC	BIC	HQIC
Chen	0.367	2.130	0.165 (0.040)	212.482	212.656	217.036	214.295
XTG	0.304	1.775	0.131 (0.172)	207.678	208.031	214.508	210.397
ECP	**0.085**	**0.514**	**0.082 (0.719)**	**193.075**	**193.428**	**199.905**	**195.794**
Chen-logarithmic	0.367	2.130	0.165 (0.040)	214.482	214.835	221.312	217.201
Exponentiated Chen	0.094	0.585	0.090 (0.601)	194.372	194.725	201.202	197.091
Marshall–Olkin Chen	0.199	1.153	0.137 (0.134)	201.652	202.005	208.482	204.371
Transmuted Chen	0.336	1.950	0.158 (0.055)	211.235	211.588	218.065	213.954
Kumaraswamy Chen	0.092	0.570	0.090 (0.610)	196.217	196.814	205.323	199.842

Bold values correspond to the best model.

The ECP distribution stands out as the best model among the fitted models, since its values of goodness-of-fit measures are the smaller ones and it has the highest p-value from the KS test. Interestingly, Dey *et al.* [15] showed that the alpha power transformed inverse Lindley (APTIL) distribution provides a better fit to the guinea pigs data, when compared to the fits of the inverse Lindley, generalized inverse Lindley, exponentiated generalized inverse Lindley, exponentiated inverse Lindley and inverse Weibull distributions. Nevertheless, the reported values of the AIC, BIC and KS statistic associated to the fit of the APTIL distribution are 234.817, 239.370 and 0.146, respectively, which are much higher than those obtained for the ECP distribution.

Additionally, the adequacy of the ECP distribution to model the guinea pigs data was informally evaluated through the two plots positioned on the upper panel of Figure 3, where plot (a) displays the empirical and model-based estimates of the sf; and plot (b) exhibits the histogram and model-based estimates of the pdf. In order to avoid a graphical overload, only the estimates of the Chen and ECP distributions are depicted. In both plots, the curves corresponding to the ECP distribution show close agreement, corroborating the fact that this distribution provides an adequate superior fitting to the survival times of guinea pigs with tuberculous infection.

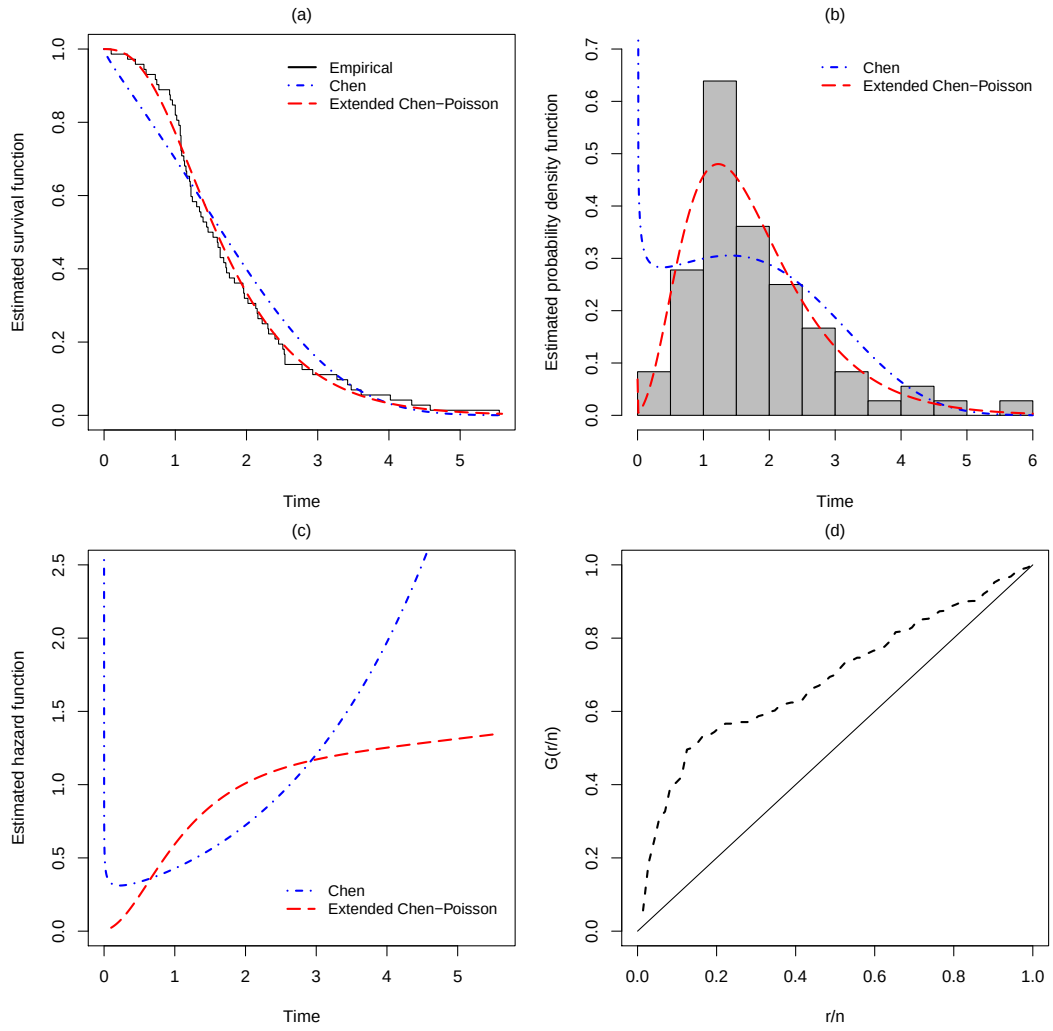

Figure 3: (a) Empirical and estimated survival functions of the Chen and ECP distributions; (b) Histogram and estimated probability density functions; (c) Estimated hazard functions; (d) Empirical scaled TTT-transform for the guinea pigs data.

The hf estimates of the referred distributions are shown in Figure 3 (c). With the purpose of identifying the hazard shape, a graphical method based on the total time on test (TTT) transform suggested by Aarset [1] was considered. The TTT plot is obtained by plotting the empirical scaled TTT-transform given by $G(r/n) = \left[\sum_{i=1}^{r} T_{i:n} + (n-r)T_{r:n}\right] / \left[\sum_{i=1}^{n} T_{i:n}\right]$ versus r/n, where $r = 1, ..., n$ and $T_{i:n}$ are the order statistics of the sample. It has been shown that the hf is increasing or decreasing if the TTT plot is concave or convex, respectively.

Although this is a sufficient but not a necessary condition, this graphical method is commonly used as a rough indicative of the hazard shape. Figure 3 (d) shows that the TTT plot is concave for the considered data, suggesting an increasing hf, which in theory would be properly accommodated by both distributions. However, the ML estimate of γ_1 of the Chen distribution is less than 1 (see Table 3), indicating that its hf is bathtub-shaped, as confirmed by Figure 3 (c). Hence, this distribution provides a poor fit. In fact, based on the p-value of the KS test (see Table 4), at significance level of 5%, there is evidence that the Chen distribution is not adequate for modelling this data. In contrast, the ECP distribution was able to capture an increasing hazard shape, reinforcing that it provides a good fit to the guinea pigs data.

Under the unified approach of Ramos *et al.* [26], it is possible to find whether the ECP distribution comes from the distribution of the minimum or maximum. Since the ML estimate of ϕ_3 is a positive value (see Table 3), the resulting distribution comes from the maximum of Chen distributions, that is, if T_i, $i = 1, ..., 72$, are the guinea pigs lifetimes, then $T_i = \max\{X_{i,1}, ..., X_{i,Z}\}$, where $X_{i,z}$, $z = 1, ..., Z$, follows a Chen distribution and Z is a non-observable random variable following a ZTP distribution.

3.6. Application to censored data: Rotterdam breast cancer

In this section, the ECP distribution is applied to the Rotterdam breast cancer data set reported by Sauerbrei *et al.* [29]. The data represent the relapse-free survival from 2982 patients with primary breast cancer whose records were included in the Rotterdam tumour bank. Here, the survival times (in years) since tumour removal until death from the disease is analysed. The maximum follow-up time is 19.283 years, the median (estimated by the reverse Kaplan–Meier method) is 9.273 years and the percentage of censoring is 57.3%. The Rotterdam data is also available in the `survival` [32] package.

The adequacy of the ECP distribution is assessed in comparison with some other members of the unified Poisson family [26], in particular with the EEP, EWP, generalized extended exponential-Poisson (GE2P) and extended exponenciated Weibull–Poisson (E2WP) distributions. Those models are listed in Table 5. Note that the E2WP distribution was proposed only by taking the maximum ($\phi_7 > 0$) [20], but we consider $\phi_7 \in \mathbb{R}\backslash\{0\}$ because it belongs to the unified Poisson family. Besides the Chen distribution being a limiting case of the ECP distribution (when $\phi_4 \to 0$), the Weibull distribution is a limiting case of the EWP (when $\phi_5 \to 0$) and E2WP (when $\psi_7 = 1$ and $\phi_7 \to 0$) distributions. For this reason, the Weibull distribution was also fitted to the Rotterdam data.

Given that in this application there are censored observations, the `maxLik` [33] package was conveniently used to maximize the log-likelihood function for censored data associated to each model, using the BFGS method. Table 6 compiles the ML estimates, their corresponding standard errors and $-$log-likelihood values. Here, it is verified that almost all fitted models come from the distribution of the maximum, except the GE2P distribution that comes from the distribution of the minimum ($\hat{\phi}_6 < 0$). Since the current application is not a CCR problem, the sign of $\hat{\phi}_j$ is not relevant. The observed values of the AIC, CAIC, BIC and HQIC statistics were also calculated in order to informally assess the adequacy of the fitted models,

as presented in Table 7. From these results it is seen that, although the ECP distribution has the smaller values of those criteria, the EWP distribution provides a similar fit. Thus, both ECP and EWP distributions are the best models among the fitted models to analyse the Rotterdam data.

Table 5: List of distributions fitted to the Rotterdam breast cancer data.

j-th Model, [ref.]	Probability density function, $f(t)$, $t > 0$
1 Chen, [11]	$\lambda_1\gamma_1 t^{\gamma_1-1}\mathrm{e}^{t^{\gamma_1}+\lambda_1(1-\mathrm{e}^{t^{\gamma_1}})}$, $\lambda_1, \gamma_1 > 0$
2 Weibull	$\lambda_2\gamma_2 t^{\gamma_2-1}\mathrm{e}^{-\lambda_2 t^{\gamma_2}}$, $\lambda_2, \gamma_2 > 0$
3 EEP, [18, 9]	$\dfrac{\lambda_3\phi_3}{1-\mathrm{e}^{-\phi_3}}\mathrm{e}^{-\lambda_3 t-\phi_3\mathrm{e}^{-\lambda_3 t}}$, $\lambda_3 > 0$, $\phi_3 \in \mathbb{R}\backslash\{0\}$
4 ECP	$\dfrac{\lambda_4\gamma_4\phi_4 t^{\gamma_4-1}}{1-\mathrm{e}^{-\phi_4}}\mathrm{e}^{t^{\gamma_4}+\lambda_4(1-\mathrm{e}^{t^{\gamma_4}})-\phi_4\mathrm{e}^{\lambda_4(1-\mathrm{e}^{t^{\gamma_4}})}}$, $\lambda_4, \gamma_4 > 0$, $\phi_4 \in \mathbb{R}\backslash\{0\}$
5 EWP, [16, 19, 26]	$\dfrac{\lambda_5\gamma_5\phi_5 t^{\gamma_5-1}}{1-\mathrm{e}^{-\phi_5}}\mathrm{e}^{-\lambda_5 t^{\gamma_5}-\phi_5\mathrm{e}^{-\lambda_5 t^{\gamma_5}}}$, $\lambda_5, \gamma_5 > 0$, $\phi_5 \in \mathbb{R}\backslash\{0\}$
6 GE2P, [4, 26]	$\dfrac{\lambda_6\gamma_6\phi_6}{1-\mathrm{e}^{-\phi_6}}\left(\dfrac{\mathrm{e}^{-\phi_6\mathrm{e}^{-\lambda_6 t}}-\mathrm{e}^{-\phi_6}}{1-\mathrm{e}^{-\phi_6}}\right)^{\gamma_6-1}\mathrm{e}^{-\lambda_6 t-\phi_6\mathrm{e}^{-\lambda_6 t}}$, $\lambda_6, \gamma_6 > 0, \phi_6 \in \mathbb{R}\backslash\{0\}$
7 E2WP, [20]	$\dfrac{\lambda_7^{\gamma_7}\gamma_7\phi_7\psi_7 t^{\gamma_7-1}\left[1-\mathrm{e}^{-(\lambda_7 t)^{\gamma_7}}\right]^{\psi_7-1}}{\left(\mathrm{e}^{\phi_7}-1\right)\mathrm{e}^{(\lambda_7 t)^{\gamma_7}-\phi_7\left[1-\mathrm{e}^{-(\lambda_7 t)^{\gamma_7}}\right]^{\psi_7}}}$, $\lambda_7, \gamma_7, \psi_7 > 0, \phi_7 \in \mathbb{R}\backslash\{0\}$

Table 6: ML estimates, standard errors and $-$log-likelihood values for the Rotterdam breast cancer data.

Model	ML estimates				Standard error				$-\hat{\ell}$
	$\hat{\lambda}_j$	$\hat{\gamma}_j$	$\hat{\phi}_j$	$\hat{\psi}_j$	$\hat{\lambda}_j$	$\hat{\gamma}_j$	$\hat{\phi}_j$	$\hat{\psi}_j$	
Chen	0.034	0.469	—	—	0.002	0.007	—	—	4913.724
Weibull	0.035	1.254	—	—	0.003	0.031	—	—	4817.114
EEP	0.101	—	1.479	—	0.006	—	0.194	—	4839.735
ECP	1.792	0.108	83.000	—	0.017	0.002	1.407	—	4780.796
EWP	0.227	2.330	39.353	—	0.005	0.035	1.047	—	4780.882
GE2P	1.609	0.046	−2.233	—	0.065	0.015	0.993	—	4797.261
E2WP	14.908	0.257	0.378	25.107	0.903	0.013	0.840	1.332	4780.297

Table 7: Goodness-of-fit statistics for the Rotterdam breast cancer data.

Model	AIC	CAIC	BIC	HQIC
Chen	9831.448	9831.452	9843.449	9835.766
Weibull	9638.228	9638.232	9650.229	9642.546
EEP	9683.471	9683.475	9695.472	9687.789
ECP	**9567.591**	**9567.599**	**9585.592**	**9574.068**
EWP	**9567.765**	**9567.773**	**9585.766**	**9574.242**
GE2P	9600.522	9600.530	9618.523	9606.999
E2WP	9569.527	9569.540	9593.528	9578.163

Bold values correspond to the best models.

In addition, the overall goodness-of-fit of the ECP distribution was informally evaluated through the two plots positioned on the upper panel of Figure 4, where plot (a) displays the estimates of the sf based on the Kaplan–Meier estimator and on the Chen and ECP distributions; and plot (b) exhibits the Cox–Snell residuals of the ECP distribution. The residuals are defined as $\hat{r}_i = \widehat{H}(t_i; \hat{\lambda}, \hat{\gamma}, \hat{\phi})$, $i = 1, ..., n$, where $\widehat{H}(t_i; \hat{\lambda}, \hat{\gamma}, \hat{\phi})$ is the estimated cumulative hazard function (chf) of the fitted model. When the model is adequate, the residuals behave approximately as a sample from a population with unit exponential distribution [12]. This assumption is informally checked through the graphical representation of $\big(\hat{r}_i, \widehat{H}_{NA}(\hat{r}_i)\big)$, where $\widehat{H}_{NA}(\hat{r}_i)$ is the Nelson-Aalen estimate of the chf of the residuals. There is a good fit when this representation yields a straight line through the origin with slope 1. In both (a) and (b) plots, the curves corresponding to the ECP distribution show general agreement, even though there are a few poorly fitted observations on the upper tail. This is acceptable since the 90-th quantile of the follow-up time (estimated by the reverse Kaplan–Meier method) is equal to 13.227 years, from which the model begins to provide a poor fit to the data.

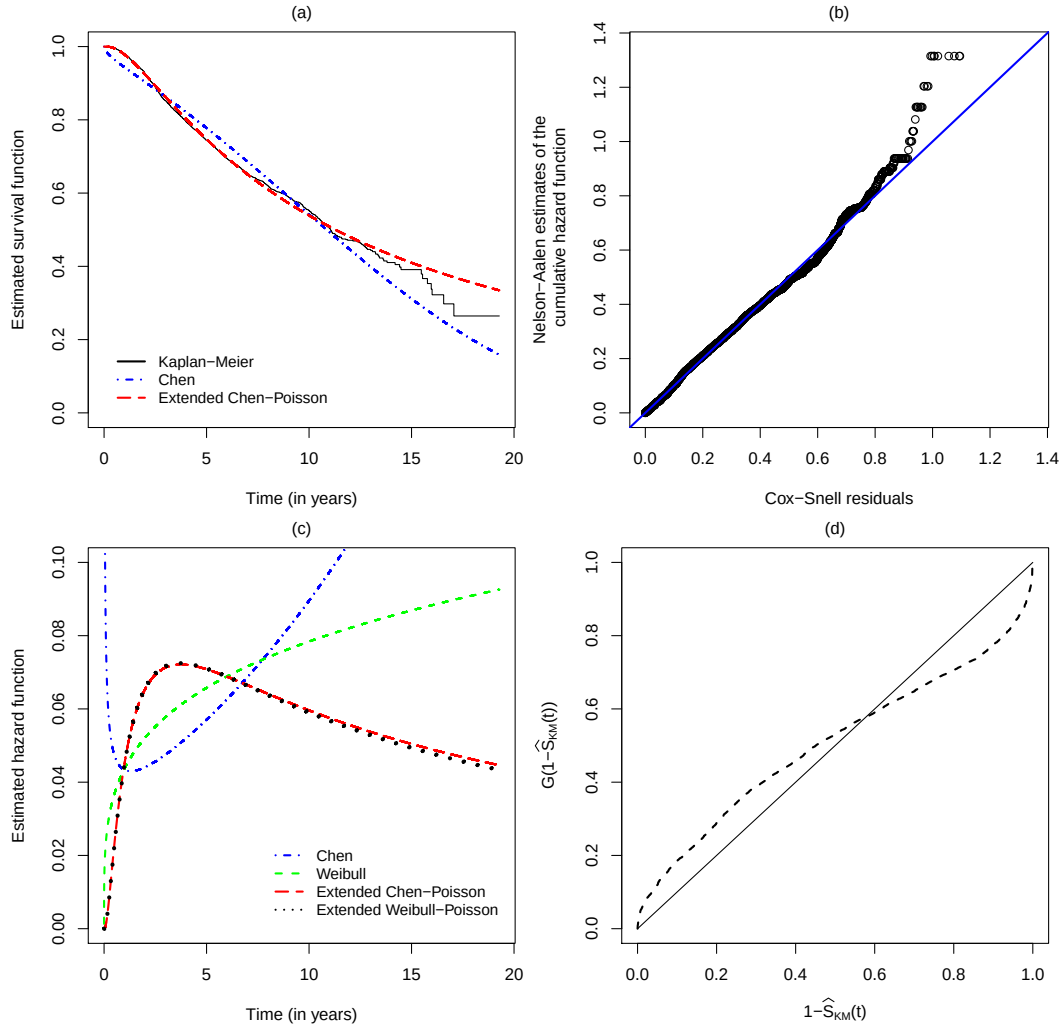

Figure 4: (a) Estimated survival functions based on the Kaplan–Meier estimator and on the Chen and ECP distributions; (b) Cox–Snell residuals of the ECP distribution; (c) Estimated hazard functions of the Chen, Weibull, ECP and EWP distributions; (d) Empirical scaled TTT-transform based on the Kaplan–Meier estimator for the Rotterdam breast cancer data.

The hf estimates of the Chen, Weibull, ECP and EWP distributions are depicted in Figure 4 (c). With the purpose of identifying the hazard shape, the TTT plot is once again considered. However, the existence of censored observations must be taken into account. As mentioned by Klefsjö [17], a natural generalization of the empirical scaled TTT-transform, $G(r/n)$, to accommodate right censored data consists in replacing the empirical cdf, r/n, by the estimator of the cdf based on the Kaplan–Meier estimator, $1-\widehat{S}_{\mathrm{KM}}(t)$. Figure 4 (d) shows that, in this case, the TTT plot is initially concave and then becomes convex, suggesting an unimodal hf. Both ECP and EWP distributions were able to capture an unimodal hazard shape, providing quite similar estimates. Therefore, in addition to both models being suitable for modelling the Rotterdam data, the proposed distribution is an adequate parametric alternative to the EWP distribution.

4. CONCLUDING REMARKS

In this paper, we introduce a new three-parameter lifetime distribution, named ECP distribution. The proposed distribution is a generalization of the Chen distribution [11] and arises from the unified Poisson compounding approach of Ramos *et al.* [26], where both distributions of the minimum and maximum are merged into one when it is assumed that the latent variable follows a ZTP distribution. Under this approach, the obtained distribution allows a practical interpretation in CCR settings. It was verified that if the parameter from the ZTP distribution takes a negative (or positive) value, then the random variable with ECP distribution represents the minimum (or maximum) lifetime among all unobservable causes. Several features of the new distribution are deduced, including the explicit expressions for the sf, pdf, hf, quantile function, moment generating function (particularly, for the mean and variance) and mean residual life function. The ECP distribution can take a richer variety of flexible hazard shapes regarding to the baseline distribution. In fact, the main advantage of the ECP distribution is that its hf can be monotonic increasing, monotonic decreasing, unimodal, bathtub, IDI or DIDI.

The estimation of the parameters is done by the ML method, considering a right-censoring mechanism. The results of the simulation study showed the effectiveness of the ML method, in which the bias and MSE of the parameters estimates are close to zero as the sample size increases. Additionally, two real data applications were presented with the following purposes:

- **i**) to assess the adequacy of the ECP distribution for modelling uncensored (guinea pigs) and censored (Rotterdam breast cancer) data;
- **ii**) to compare the proposed distribution with other generalizations of the Chen distribution, as well as with other members of the unified Poisson family.

In both applications, the ECP distribution clearly revealed to be a suitable parametric alternative for modelling the data, when compared with the competing models. It is noteworthy that some of the considered models have quite flexible hfs (such as the Marshal-Olkin Chen and E2WP distributions) but, for the analysed data sets, none was better than the ECP distribution. This fact emphasizes the potential and flexibility of the proposed model.

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