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Supplementary Material for "The Extended Chen-Poisson Lifetime Distribution"

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1. PROOF OF PROPOSITION 3.3

This section provides the Proof of Proposition 3.3 regarding the monotonicity study of the probability density function (pdf) of the ECP distribution.

Proof of Proposition 3.3:

The first derivative of the pdf (3.4) of the ECP distribution is given by

$$f'(t;\lambda,\gamma,\phi) = \frac{f(t;\lambda,\gamma,\phi)}{t}\left\{\gamma - 1 - \gamma t^{\gamma}\left[-1 + \lambda e^{t^{\gamma}}\left(1 - \phi e^{\lambda(1-e^{t^{\gamma}})}\right)\right]\right\}, \quad t > 0,$$

where $\lambda, \gamma > 0$ and $\phi \in \mathbb{R}\backslash\{0\}$. The sign of $f'(t;\lambda,\gamma,\phi)$ is the sign of the expression in curly brackets and $f'(t;\lambda,\gamma,\phi)$ is zero when that expression is zero. Consider the change of variable $u = e^{t^{\gamma}}$ and rewrite the expression in curly brackets as $g(u;\lambda,\gamma,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$,

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where $r_1(u;\lambda,\phi) = \log(u)\big[-1+\lambda u(1-\phi e^{\lambda(1-u)})\big]$ for $u>1$. The monotonicity study of the pdf is done separately for $\phi<0$ (distribution of the minimum) and $\phi>0$ (distribution of the maximum).

1. For $\phi<0$ (distribution of the minimum):
If $\phi<0$, then $1<1-\phi e^{\lambda(1-u)}<1-\phi$ since $0<e^{\lambda(1-u)}<1$, for $u>1$. Hence, $1-\phi e^{\lambda(1-u)}$ is never zero when $\phi<0$.

Case 1.1. ($\phi<0$, $0<\gamma\le 1$ and $\lambda\ge 1$)
If $\phi<0$, $0<\gamma\le 1$ and $\lambda\ge 1$, then $g(u;\lambda,\gamma,\phi)<0$ and, therefore, $f(t;\lambda,\gamma,\phi)$ is monotonically decreasing.

Case 1.2. ($\phi<0$, $\gamma=1$ and $0<\lambda<1$)
Here, from $g(u;\lambda,\gamma,\phi)=0$ it is straightforward to see that $\lambda u(1-\phi e^{\lambda(1-u)})$ takes values less or greater than 1 and so there exists at least one solution of this equation. Let u_0 be one of these solutions, that is, $\lambda u_0(1-\phi e^{\lambda(1-u_0)})=1$. Given that $1-\phi e^{\lambda(1-u_0)}>1$, then $\lambda u_0 = 1/(1-\phi e^{\lambda(1-u_0)})<1$ and so $u_0<1/\lambda$. Accordingly, u_0 belongs to the interval $(1,1/\lambda)$, with $0<\lambda<1$. Let $r_2(u;\lambda,\phi)=-1+\lambda u(1-\phi e^{\lambda(1-u)})$, so its first derivative is $r_2'(u;\lambda,\phi)=\lambda[1-\phi(1-u\lambda)e^{\lambda(1-u)}]$. Evaluating $r_2'(u;\lambda,\phi)$ at u_0, it follows that $r_2'(u_0;\lambda,\phi)>0$, for $u_0\in(1,1/\lambda)$. Hence, $r_2(u;\lambda,\phi)$ is monotonically increasing and u_0 is the only solution, if it exists. Thus, $g(u;\lambda,\gamma,\phi)=-\log(u)r_2(u;\lambda,\phi)$ also has just a single zero at u_0, when it exists, because $\log(u)>0$ for $u>1$. However, since $\lim_{u\to 1} r_2(u;\lambda,\phi)=-1+(1-\phi)\lambda$, u_0 only exists if $-1+(1-\phi)\lambda<0$. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 1.2.1. ($\phi<0$, $\gamma=1$ and $0<\lambda<(1-\phi)^{-1}$)
If $0<\lambda<(1-\phi)^{-1}$, then $r_2(u;\lambda,\phi)$ has a zero at u_0 and, since $r_2(u;\lambda,\phi)$ is monotonically increasing, it follows that $g(u;\lambda,\gamma,\phi)=-\log(u)r_2(u;\lambda,\phi)$ is positive until u_0, it has a zero at u_0 and it is negative thereafter. Therefore, $f(t;\lambda,\gamma,\phi)$ is unimodal. In this sub-case, the mode is equal to $\log(u_0)^{1/\gamma}$, where u_0 is the root of the non-linear equation $\lambda u(1-\phi e^{\lambda(1-u)})=1$, for $\gamma=1$ and $0<\lambda<(1-\phi)^{-1}$.

Sub-case 1.2.2. ($\phi<0$, $\gamma=1$ and $(1-\phi)^{-1}\le\lambda<1$)
If $(1-\phi)^{-1}\le\lambda<1$, then $r_2(u;\lambda,\phi)$ is always greater than or equal to 0 and, consequently, $g(u;\lambda,\gamma,\phi)$ is always less than or equal to 0. Therefore, $f(t;\lambda,\gamma,\phi)$ is monotonically decreasing.

Case 1.3. ($\phi<0$, $0<\gamma<1$ and $0<\lambda<1$)
Here, from $g(u;\lambda,\gamma,\phi)=0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_1 be one of these solutions, that is, $r_1(u_1;\lambda,\phi)=(\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma<0$ for $0<\gamma<1$, then $r_1(u_1;\lambda,\phi)<0$, which implies that $\lambda u_1(1-\phi e^{\lambda(1-u_1)})<1$ since $\log(u_1)>0$ for $u_1>1$. Moreover, it is known that $1-\phi e^{\lambda(1-u_1)}>1$ implies that $u_1<1/\lambda$. Accordingly, u_1 belongs to the interval $(1,1/\lambda)$, with $0<\lambda<1$. The first derivative of $r_1(u;\lambda,\phi)$ is given by $r_1'(u;\lambda,\phi)=u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1+\log(u)\big)-\big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$. Evaluating $r_1'(u;\lambda,\phi)$ at $u_1\in(1,1/\lambda)$, it follows that it can take both negative and positive values. In addition, as $\lim_{u\to 1} r_1'(u;\lambda,\phi)=-1+(1-\phi)\lambda$, it is clear that $r_1'(u;\lambda,\phi)$ can be initially negative or positive. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 1.3.1. ($\phi < 0$, $0 < \gamma < 1$ and $0 < \lambda < (1-\phi)^{-1}$)
If $0 < \lambda < (1-\phi)^{-1}$, then $r_1'(u;\lambda,\phi)$ is initially negative and, furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$ it is seen that $r_1'(u;\lambda,\phi) = 0$ has at least one root. Let u_2 be one of these solutions, that is, $r_1'(u_2;\lambda,\phi) = 0$, but now u_2 belongs to the interval $(1, 1/\lambda)$ with $0 < \lambda < (1-\phi)^{-1}$. Thus, it follows that

$$\begin{cases} r_1'(u;\lambda,\phi) < 0, & \text{if} \quad u < u_2 \\ r_1'(u;\lambda,\phi) = 0, & \text{if} \quad u = u_2 \\ r_1'(u;\lambda,\phi) > 0, & \text{if} \quad u > u_2 \end{cases}.$$

This means that $r_1(u;\lambda,\phi)$ decreases until u_2 and then increases. So, u_2 is the only solution of the non-linear equation $r_1'(u;\lambda,\phi) = 0$. Note that if u_2 minimizes $r_1(u;\lambda,\phi)$, then u_2 maximizes $g(u;\lambda,\gamma,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ for $0 < \gamma < 1$ and $0 < \lambda < (1-\phi)^{-1}$. Therefore, $g(u;\lambda,\gamma,\phi)$ is unimodal. Since $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 < 0$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$, then $g(u;\lambda,\gamma,\phi)$ is initially negative for $0 < \gamma < 1$ and goes to $-\infty$ as $u \to \infty$. As $g(u;\lambda,\gamma,\phi)$ is unimodal, then two situations can occur depending on whether the maximum of this function is negative or positive. If the maximum of $g(u;\lambda,\gamma,\phi)$ is less than or equal to 0, that is, $g(u_2;\lambda,\gamma,\phi) \leq 0$, then $f(u;\lambda,\gamma,\phi)$ is monotonically decreasing. On the other hand, if the maximum of $g(u;\lambda,\gamma,\phi)$ is greater than 0, that is, $g(u_2;\lambda,\gamma,\phi) > 0$, then $g(u;\lambda,\gamma,\phi)$ will have two zeros because it is unimodal. This means that, in this situation, $g(u;\lambda,\gamma,\phi) = 0$ has two solutions, say $u_{1,1}$ and $u_{1,2}$, which can only be obtained using numerical methods. Remembering that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 < 0$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$, it follows that $f(u;\lambda,\gamma,\phi)$ is decreasing-increasing-decreasing (DID).

Sub-case 1.3.2. ($\phi < 0$, $0 < \gamma < 1$ and $(1-\phi)^{-1} < \lambda < 1$)
If $(1-\phi)^{-1} \leq \lambda < 1$, then $r_1'(u;\lambda,\phi) \geq 0$. Therefore, $r_1(u;\lambda,\phi)$ is monotonically increasing and, consequently, $g(u;\lambda,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ is monotonically decreasing. Moreover, it is known that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 < 0$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$. So $g(u;\lambda,\gamma,\phi) < 0$ for $0 < \gamma < 1$ and $(1-\phi)^{-1} \leq \lambda < 1$. This shows that $f(u;\lambda,\gamma,\phi)$ is monotonically decreasing.

Case 1.4. ($\phi < 0$, $\gamma > 1$ and $\lambda \geq 1$)
Here, from $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that once again $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_3 be one of these solutions, that is, $r_1(u_3;\lambda,\phi) = (\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma < 1$ for $\gamma > 1$, then $r_1(u;\lambda,\phi) < 1$, for $u > 1$. Since $\log(u_3) > 0$, for $u_3 > 1$, from $r_1(u;\lambda,\phi) = \log(u)\big[-1 + \lambda u(1 - \phi \mathrm{e}^{\lambda(1-u)})\big]$ it is not possible to obtain the upper bound of the interval to which u_3 belongs. However, it can be seen that $r_1(u;\lambda,\phi) > 0$, for $u > 1$ and $\lambda \geq 1$. Then, $r_1(u;\lambda,\phi)$ is never zero and it belongs to the interval $(0,1)$. Thus, in this case, $g(u;\lambda,\gamma,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ can be seen as a straight line with slope $-\gamma$ and vertical intercept $\gamma - 1$. Consequently, $g(u;\lambda,\gamma,\phi)$ is monotonically decreasing and u_3 is the only solution. In fact, since $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$ for $\gamma > 1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$, it turns out that $g(u;\lambda,\gamma,\phi)$ is initially positive and will eventually become negative as $u \to \infty$. Therefore, it follows that $f(t;\lambda,\gamma,\phi)$ is unimodal. In this case, the mode is equal to $\log(u_3)^{1/\gamma}$, where u_3 is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $\gamma > 1$ and $\lambda \geq 1$.

Case 1.5. ($\phi < 0$, $\gamma > 1$ and $0 < \lambda < 1$)
From $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_4 be one

of these solutions, that is, $r_1(u_4;\lambda,\phi)=(\gamma-1)/\gamma$. As in the previous case, it is not possible to obtain the upper bound of the interval to which u_4 belongs. An added difficulty is that now $r_1(u;\lambda,\phi)=\log(u)r_2(u;\lambda,\phi)$, where $r_2(u;\lambda,\phi)=-1+\lambda u(1-\phi\mathrm{e}^{\lambda(1-u)})$, can be negative, positive or even zero since $\log(u)>0$ and $r_2(u;\lambda,\phi)>-1$, for $u>1$ and $0<\lambda<1$. From Case 1.3., it is known that $r_1'(u;\lambda,\phi)=u^{-1}\big[u\lambda(1-\phi\mathrm{e}^{\lambda(1-u)})\big(1+\log(u)\big)-\big(1-\phi\lambda^2u^2\mathrm{e}^{\lambda(1-u)}\log(u)\big)\big]$ and $\lim_{u\to1} r_1'(u;\lambda,\phi)=-1+(1-\phi)\lambda$. Therefore, this case will be separated into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 1.5.1. ($\phi<0$, $0<\gamma<1$ and $0<\lambda<(1-\phi)^{-1}$)
If $0<\lambda<(1-\phi)^{-1}$, then $r_1'(u;\lambda,\phi)$ is initially negative and, given that $\lim_{u\to\infty} r_1'(u;\lambda,\phi)=\infty$, it will eventually become positive as $u\to\infty$. From Sub-case 1.3.1., it follows that $r_1(u;\lambda,\phi)$ decreases until a given point and increases thereafter. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Sub-case 1.3.1, it is seen that $\lim_{u\to1} g(u;\lambda,\gamma,\phi)=\gamma-1>0$, for $\gamma>1$. This means that, although $g(u;\lambda,\gamma,\phi)$ is unimodal, it is initially positive and, consequently, it has only one root which will be denoted by $u_{4,1}$. Thus, $f(t;\lambda,\gamma,\phi)$ is unimodal. In this case, the mode is equal to $\log(u_{4,1})^{1/\gamma}$, where $u_{4,1}$ is the root of the non-linear equation $r_1(u;\lambda,\phi)=(\gamma-1)/\gamma$, for $0<\gamma<1$ and $0<\lambda<(1-\phi)^{-1}$.

Sub-case 1.5.2. ($\phi<0$, $0<\gamma<1$ and $(1-\phi)^{-1}\le\lambda<1$)
If $(1-\phi)^{-1}\le\lambda<1$, then $r_1'(u;\lambda,\phi)\ge0$. As in Sub-case 1.3.2., it follows that $r_1(u;\lambda,\phi)$ is monotonically increasing and, consequently, $g(u;\lambda,\gamma,\phi)$ is monotonically decreasing. However, in contrast to Sub-case 1.3.1, it is seen that $\lim_{u\to1} g(u;\lambda,\gamma,\phi)=\gamma-1>0$, for $\gamma>1$. Thus, $g(u;\lambda,\gamma,\phi)$ is initially positive and, because it is monotonically decreasing, it has one root which will be denoted by $u_{4,2}$. Therefore, $f(u;\lambda,\gamma,\phi)$ is also unimodal and the mode is equal to $\log(u_{4,2})^{1/\gamma}$, where $u_{4,2}$ is the root of the non-linear equation $r_1(u;\lambda,\phi)=(\gamma-1)/\gamma$, for $0<\gamma<1$ and $(1-\phi)^{-1}\le\lambda<1$.

2. For $\phi>0$ (distribution of the maximum):
If $\phi>0$, then $1-\phi<1-\phi\mathrm{e}^{\lambda(1-u)}<1$ since $0<\mathrm{e}^{\lambda(1-u)}<1$, for $u>1$. Hence, $1-\phi\mathrm{e}^{\lambda(1-u)}$ can be negative, positive or even zero when $\phi>0$. Note that the first derivative of $1-\phi\mathrm{e}^{\lambda(1-u)}$ is given by $\lambda\phi\mathrm{e}^{\lambda(1-u)}$, which is always positive for $u>1$ and $\lambda,\phi>0$. Then, $1-\phi\mathrm{e}^{\lambda(1-u)}$ is monotonically increasing and is zero at $u^*=1+\lambda^{-1}\log(\phi)$.

Case 2.1. ($\phi>0$, $0<\gamma<1$ and $\lambda>1$)
Here, from $g(u;\lambda,\gamma,\phi)=0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_5 be one of these solutions, that is, $r_1(u_5;\lambda,\phi)=(\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma<0$ for $0<\gamma<1$, then $r_1(u_5;\lambda,\phi)<0$, which implies that $\lambda u_5(1-\phi\mathrm{e}^{\lambda(1-u_5)})<1$ since $\log(u_5)>0$ for $u_5>1$. Then $\lambda(1-\phi\mathrm{e}^{\lambda(1-u_5)})<1$ and so $u_5<1-\lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]$, for $\lambda>1$ and $\phi>0$. Accordingly, u_5 belongs to the interval $\big(1,1-\lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]\big)$. It is noteworthy that u_5 only exists if $1-\lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]>1$, which implies that $\phi>1-\lambda^{-1}$ for $\lambda>1$. The first derivative of $r_1(u;\lambda,\phi)$ is given by $r_1'(u;\lambda,\phi)=u^{-1}\big[u\lambda(1-\phi\mathrm{e}^{\lambda(1-u)})\big(1+\log(u)\big)-\big(1-\phi\lambda^2u^2\mathrm{e}^{\lambda(1-u)}\log(u)\big)\big]$. Evaluating $r_1'(u;\lambda,\phi)$ at u_5, it follows that it can take both negative and positive values. In addition, since $\lim_{u\to1} r_1'(u;\lambda,\phi)=-1+(1-\phi)\lambda$ it is clear that $r_1'(u;\lambda,\phi)$ can be initially negative or positive. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 2.1.1. ($0 < \phi \leq 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$)
If $0 < \phi \leq 1 - \lambda^{-1}$ with $\lambda > 1$, then $r_1'(u; \lambda, \phi) \geq 0$. Therefore, $r_1(u; \lambda, \phi)$ is monotonically increasing and, consequently, $g(u; \lambda, \phi) = \gamma - 1 - \gamma r_1(u; \lambda, \phi)$ is monotonically decreasing. Moreover, $\lim_{u \to 1} g(u; \lambda, \gamma, \phi) = \gamma - 1 < 0$ for $0 < \gamma < 1$ and $\lim_{u \to \infty} g(u; \lambda, \gamma, \phi) = -\infty$. Therefore, $g(u; \lambda, \gamma, \phi) < 0$ for $0 < \phi \leq 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$. This shows that $f(u; \lambda, \gamma, \phi)$ is monotonically decreasing.

Sub-case 2.1.2. ($\phi > 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$)
If $\phi > 1 - \lambda^{-1}$ for $\lambda > 1$, then $r_1'(u; \lambda, \phi)$ is initially negative and, furthermore, as $\lim_{u \to \infty} r_1'(u; \lambda, \phi) = \infty$ it is seen that $r_1'(u; \lambda, \phi)$ has at least one root. Let u_6 be one of these solutions, that is, $r_1'(u_6; \lambda, \phi) = 0$, where u_6 belongs to the interval $\big(1, 1 - \lambda^{-1} \log[\phi^{-1}(1 - \lambda^{-1})]\big)$ with $\phi > 1 - \lambda^{-1}$ and $\lambda > 1$. Thus, it follows that

$$\begin{cases} r_1'(u; \lambda, \phi) < 0, \text{ if } & u < u_6 \\ r_1'(u; \lambda, \phi) = 0, \text{ if } & u = u_6 \\ r_1'(u; \lambda, \phi) > 0, \text{ if } & u > u_6 \end{cases}.$$

This means that $r_1(u; \lambda, \phi)$ decreases until u_6 and then increases. So, u_6 is the only solution of the non-linear equation $r_1'(u; \lambda, \phi) = 0$. Note that if u_6 minimizes $r_1(u; \lambda, \phi)$, then u_6 maximizes $g(u; \lambda, \gamma, \phi) = \gamma - 1 - \gamma r_1(u; \lambda, \phi)$ for $\phi > 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$. Therefore, $g(u; \lambda, \gamma, \phi)$ is unimodal. Since $\lim_{u \to 1} g(u; \lambda, \gamma, \phi) = \gamma - 1 < 0$ and $\lim_{u \to \infty} g(u; \lambda, \gamma, \phi) = -\infty$, then $g(u; \lambda, \gamma, \phi)$ is initially negative for $0 < \gamma < 1$ and goes to $-\infty$ as $u \to \infty$. As $g(u; \lambda, \gamma, \phi)$ is unimodal, then two situations can occur depending on whether the maximum of this function is negative or positive. If the maximum of $g(u; \lambda, \gamma, \phi)$ is less than or equal to 0, that is, $g(u_6; \lambda, \gamma, \phi) \leq 0$, then $f(u; \lambda, \gamma, \phi)$ is monotonically decreasing. On the other hand, if the maximum of $g(u; \lambda, \gamma, \phi)$ is greater than 0, that is, $g(u_6; \lambda, \gamma, \phi) > 0$, then $g(u; \lambda, \gamma, \phi)$ will have two zeros because it is unimodal. This means that, in this situation, $g(u; \lambda, \gamma, \phi) = 0$ has two solutions, say $u_{5,1}$ and $u_{5,2}$, which can only be obtained using numerical methods. Thus, it follows that $f(u; \lambda, \gamma, \phi)$ is DID.

Case 2.2. ($\phi > 0$, $\gamma = 1$ and $\lambda > 1$)
Here, from $g(u; \lambda, \gamma, \phi) = 0$ it is straightforward to see that $\lambda u(1 - \phi e^{\lambda(1-u)})$ takes values less or greater than 1 and so there exists at least one solution of this equation. Let u_7 be one of these solutions, that is, $\lambda u_7(1 - \phi e^{\lambda(1-u_7)}) = 1$. Given that $u_7 > 1$, then $\lambda(1 - \phi e^{\lambda(1-u_7)}) < 1$ and so $u_7 < 1 - \lambda^{-1} \log[\phi^{-1}(1 - \lambda^{-1})]$, for $\lambda > 1$ and $\phi > 0$. Accordingly, u_7 belongs to the interval $\big(1, 1 - \lambda^{-1} \log[\phi^{-1}(1 - \lambda^{-1})]\big)$ and u_7 only exists if $\phi > 1 - \lambda^{-1}$ for $\lambda > 1$. Evaluating $r_2'(u; \lambda, \phi) = \lambda[1 - \phi(1 - u\lambda)e^{\lambda(1-u)}]$ at u_7, it follows that $r_2'(u_7; \lambda, \phi) > \lambda$. Hence, $r_2(u; \lambda, \phi)$ is monotonically increasing and u_7 is the only solution when it exists. Thus, for $\gamma = 1$, $g(u; \lambda, \gamma, \phi) = -\log(u) r_2(u; \lambda, \phi)$ also has a single zero at u_7, when it exists, because $\log(u) > 0$ for $u > 1$. However, since $\lim_{u \to 1} r_2(u; \lambda, \phi) = -1 + (1 - \phi)\lambda$, once again it is seen that u_7 only exists if $\phi > 1 - \lambda^{-1}$ for $\lambda > 1$. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 2.2.1. ($0 < \phi \leq 1 - \lambda^{-1}$, $\gamma = 1$ and $\lambda > 1$)
If $0 < \phi \leq 1 - \lambda^{-1}$ and $\gamma = 1$, then $r_2(u; \lambda, \phi)$ is always greater than or equal to 0 and, consequently, $g(u; \lambda, \gamma, \phi)$ is always less than or equal to 0. Therefore, $f(t; \lambda, \gamma, \phi)$ is monotonically decreasing.

Sub-case 2.2.2. ($\phi > 1 - \lambda^{-1}$, $\gamma = 1$ and $\lambda > 1$)
If $\phi > 1 - \lambda^{-1}$ and $\lambda > 1$, then $r_2(u; \lambda, \phi)$ has a zero at u_7 and, since $r_2(u; \lambda, \phi)$ is

monotonically increasing, it follows that $g(u;\lambda,\gamma,\phi) = -\log(u)r_2(u;\lambda,\phi)$ is positive until u_7, it has a zero at u_7 and it is negative thereafter. In this sub-case, the mode is equal to $\log(u_7)^{1/\gamma}$, where u_7 is the root of the non-linear equation $\lambda u(1-\phi e^{\lambda(1-u)}) = 1$, for $\phi > 1-\lambda^{-1}$, $\gamma = 1$ and $\lambda > 1$.

Case 2.3. ($\phi > 0$, $\gamma > 1$ and $\lambda > 1$)
From $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_8 be one of these solutions, that is, $r_1(u_8;\lambda,\phi) = (\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma < 1$ for $\gamma > 1$, then $r_1(u;\lambda,\phi) < 1$, for $u > 1$. Since $\log(u_8) > 0$, for $u_8 > 1$, from $r_1(u;\lambda,\phi)$ it is not possible to obtain the upper bound of the interval to which u_8 belongs. An added difficulty is that $r_1(u;\lambda,\phi) = \log(u)r_2(u;\lambda,\phi)$ can be negative, positive or even zero since $1-\phi < 1-\phi e^{\lambda(1-u)} < 1$, which implies that $-1+\lambda u(1-\phi) < r_2(u;\lambda,\phi) < -1+\lambda u$, for $u > 1$, $\lambda > 1$ and $\phi > 0$. In addition, it is known that $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1+\log(u)\big) - \big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$ and $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1+(1-\phi)\lambda$. Therefore, this case will be separated into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 2.3.1. ($0 < \phi \le 1-\lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$)
If $0 < \phi \le 1-\lambda^{-1}$ with $\lambda > 1$, then $r_1'(u;\lambda,\phi) \ge 0$. It follows that $r_1(u;\lambda,\phi)$ is monotonically increasing and, consequently, $g(u;\lambda,\gamma,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ is monotonically decreasing. However, in contrast to Sub-case 2.1.1, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$, for $\gamma > 1$. Thus, $g(u;\lambda,\gamma,\phi)$ is initially positive and, because it is monotonically decreasing with $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$, it has one root which will be denoted by $u_{8,1}$. Therefore, $f(u;\lambda,\gamma,\phi)$ is unimodal and the mode is equal to $\log(u_{8,1})^{1/\gamma}$, where $u_{8,1}$ is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $0 < \phi \le 1-\lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$.

Sub-case 2.3.2. ($\phi > 1-\lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$)
If $\phi > 1-\lambda^{-1}$ with $\lambda > 1$, then $r_1'(u;\lambda,\phi)$ is initially negative and, given that $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$, it will eventually become positive as $u \to \infty$. From Sub-case 2.1.2, it follows that $r_1(u;\lambda,\phi)$ decreases until a given point and increases thereafter. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Sub-case 2.1.2, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$, for $\gamma > 1$. This means that, although $g(u;\lambda,\gamma,\phi)$ is unimodal, it is initially positive and, consequently, it has only one root which will be denoted by $u_{8,2}$. Thus, $f(t;\lambda,\gamma,\phi)$ is also unimodal. In this case, the mode is equal to $\log(u_{8,2})^{1/\gamma}$, where $u_{8,2}$ is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $\phi > 1-\lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$.

Case 2.4. ($\phi > 0$, $0 < \gamma < 1$ and $0 < \lambda \le 1$)
From $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so this equation has at least one solution. Let u_9 be one of these solutions, that is, $r_1(u_9;\lambda,\phi) = (\gamma-1)/\gamma$. As in Case 2.1, given that $(\gamma-1)/\gamma < 0$ for $0 < \gamma < 1$ then $r_1(u_9;\lambda,\phi) < 0$, which implies that $\lambda u_9(1-\phi e^{\lambda(1-u_9)}) < 1$ since $\log(u_9) > 0$ for $u_9 > 1$. Then $\lambda(1-\phi e^{\lambda(1-u_9)}) < 1$ and so $e^{\lambda(1-u_9)} > \phi^{-1}(1-\lambda^{-1})$. However, in contrast to Case 2.1, from $r_1(u;\lambda,\phi)$ it is not possible to obtain the upper bound of the interval to which u_9 belongs for $0 < \lambda \le 1$. Nonetheless, knowing that $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1+\log(u)\big) - \big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$ and since $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1+(1-\phi)\lambda < 0$, it is clear that $r_1'(u;\lambda,\phi)$ is always initially negative for $\phi > 0$ and $0 < \lambda \le 1$. Furthermore, as

$\lim_{u\to\infty} r_1'(u;\lambda,\phi)=\infty$, it is seen that $r_1'(u;\lambda,\phi)$ has at least one root. Let u_{10} be one of these solutions, that is, $r_1'(u_{10};\lambda,\phi)=0$, where $u_{10}>1$. As in Sub-case 2.1.2, it can be seen that $r_1(u;\lambda,\phi)$ decreases until u_{10} and increases thereafter. So, u_{10} is the only solution of the non-linear equation $r_1'(u;\lambda,\phi)=0$. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. Note that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi)=\gamma-1<0$ for $0<\gamma<1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi)=-\infty$. Consequently, as described in Sub-case 2.1.2, $f(u;\lambda,\gamma,\phi)$ can take two different shapes depending on whether the maximum of $g(u;\lambda,\gamma,\phi)$, that is, $g(u_{10};\lambda,\gamma,\phi)$, is negative or positive. If the maximum of $g(u;\lambda,\gamma,\phi)$ is less than or equal to 0, then $f(u;\lambda,\gamma,\phi)$ is monotonically decreasing. On the other hand, if the maximum of $g(u;\lambda,\gamma,\phi)$ is greater than 0, then $f(u;\lambda,\gamma,\phi)$ is DID because $g(u;\lambda,\gamma,\phi)$ has two zeros and it is unimodal. Thus, in this second situation, $g(u;\lambda,\gamma,\phi)=0$ has two solutions, say $u_{9,1}$ and $u_{9,2}$, which can only be obtained using numerical methods.

Case 2.5. ($\phi>0$, $\gamma=1$ and $0<\lambda\leq 1$)
From $g(u;\lambda,\gamma,\phi)=0$ it is straightforward to see that $\lambda u(1-\phi e^{\lambda(1-u)})$ takes values less or greater than 1 and so there exists at least one solution of this equation. Let u_{11} be one of these solutions, that is, $\lambda u_{11}(1-\phi e^{\lambda(1-u_{11})})=1$. Given that $u_{11}>1$, then $\lambda(1-\phi e^{\lambda(1-u_{11})})<1$ and so $e^{\lambda(1-u_{11})}>\phi^{-1}(1-\lambda^{-1})$. In contrast to Case 2.2, it is not possible to obtain the upper bound of the interval to which u_{11} belongs for $0<\lambda\leq 1$. Nonetheless, knowing that $r_1'(u;\lambda,\phi)=u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1+\log(u)\big)-\big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$ and $\lim_{u\to 1} r_1'(u;\lambda,\phi)=-1+(1-\phi)\lambda<0$, it is clear that $r_1'(u;\lambda,\phi)$ is always initially negative for $\phi>0$ and $0<\lambda\leq 1$. Furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi)=\infty$ it is seen that $r_1'(u;\lambda,\phi)$ has at least one root. Let u_{12} be one of these solutions, that is, $r_1'(u_{12};\lambda,\phi)=0$, where $u_{12}>1$. As in Case 2.4, it can be seen that $r_1(u;\lambda,\phi)$ decreases until u_{12} and increases thereafter. So, u_{12} is the only solution of the non-linear equation $r_1'(u;\lambda,\phi)=0$. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Case 2.4, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi)=\gamma-1=0$ for $\gamma=1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi)=-\infty$. This means that, in this situation, $g(u;\lambda,\gamma,\phi)=0$ has two solutions, say $u_{11,1}$ and $u_{11,2}$, which can only be obtained using numerical methods. Although $g(u;\lambda,\gamma,\phi)$ has two zeros, it is initially equal to 0 and, because it is unimodal, it crosses the horizontal axis only once, more precisely at $u_{11,2}$, with $u_{11,2}>u_{11,1}$. Therefore, $f(t;\lambda,\gamma,\phi)$ is unimodal and the mode is equal to $\log(u_{11,2})^{1/\gamma}$, where $u_{11,2}$ is the root of the non-linear equation $\lambda u(1-\phi e^{\lambda(1-u)})=1$, for $\gamma=1$ and $0<\lambda\leq 1$.

Case 2.6.($\phi>0$, $\gamma>1$ and $0<\lambda\leq 1$)
From $g(u;\lambda,\gamma,\phi)=0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_{13} be one of these solutions, that is, $r_1(u_{13};\lambda,\phi)=(\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma<1$ for $\gamma>1$, then $r_1(u;\lambda,\phi)<1$, for $u>1$. However, as in Case 2.3, from $r_1(u;\lambda,\phi)$ it is not possible to obtain the upper bound of the interval to which u_{13} belongs, since $\log(u_{13})>0$ for $u_{13}>1$. An added difficulty is that $r_1(u;\lambda,\phi)=\log(u)r_2(u;\lambda,\phi)$, where $r_2(u;\lambda,\phi)=-1+\lambda u(1-\phi e^{\lambda(1-u)})$, can be negative, positive or even zero since $1-\phi<1-\phi e^{\lambda(1-u)}<1$, which implies that $-1+\lambda u(1-\phi)<r_2(u;\lambda,\phi)<-1+\lambda u$, for $u>1$, $0<\lambda\leq 1$ and $\phi>0$. Nonetheless, knowing that $r_1'(u;\lambda,\phi)=u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1+\log(u)\big)-\big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$ and $\lim_{u\to 1} r_1'(u;\lambda,\phi)=-1+(1-\phi)\lambda<0$, it is clear that $r_1'(u;\lambda,\phi)$ is always initially negative for $\phi>0$ and $0<\lambda\leq 1$. Furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi)=\infty$ it is seen that $r_1'(u;\lambda,\phi)$ has at least one root. Let u_{14} be one of these solutions, that is, $r_1'(u_{14};\lambda,\phi)=0$, where $u_{14}>1$. As in Cases 2.4 and 2.5, it can be seen that $r_1(u;\lambda,\phi)$ decreases until u_{14} and increases thereafter. Therefore, u_{14} is the only solution of the non-linear equation $r_1'(u;\lambda,\phi)=0$. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Cases 2.4 and 2.5, it is seen that

$\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$ for $\gamma > 1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$. This means that, although $g(u;\lambda,\gamma,\phi)$ is unimodal, it is initially positive and, consequently, it has only one root, denoted by u_{13}. Thus, $f(t;\lambda,\gamma,\phi)$ is unimodal. In this case, the mode is equal to $\log(u_{13})^{1/\gamma}$, where u_{13} is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $\gamma > 1$ and $0 < \lambda \leq 1$. □

2. ELEMENTS OF THE OBSERVED INFORMATION MATRIX

The elements of the observed information, $\boldsymbol{I}(\lambda,\gamma,\phi)$, are given by

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \lambda^2} &= -\frac{m}{\lambda^2} - \phi^2 \sum_{i=1}^{n} (1-\delta_i) \frac{\left(\mathrm{e}^{t_i^\gamma}-1\right)^2 \mathrm{e}^{2\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)+\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\left(\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right)^2} \\
&\quad -\phi \sum_{i=1}^{n} \left(\delta_i \mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right) \frac{\left(\mathrm{e}^{t_i^\gamma}-1\right)^2 \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}{\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \lambda \partial \gamma} &= -\sum_{i=1}^{n} \delta_i t_i^\gamma \mathrm{e}^{t_i^\gamma} \log(t_i) - \lambda\phi \sum_{i=1}^{n} \delta_i \frac{t_i^\gamma \log(t_i) \mathrm{e}^{2t_i^\gamma+\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)+\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1} \\
&\quad +\lambda\phi^2 \sum_{i=1}^{n} (1-\delta_i) \frac{t_i^\gamma \left(1-\mathrm{e}^{t_i^\gamma}\right) \log(t_i) \mathrm{e}^{t_i^\gamma+2\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)+\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\left(\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right)^2} \\
&\quad +\phi \sum_{i=1}^{n} \frac{\left(\lambda \mathrm{e}^{t_i^\gamma} + (\lambda+1)\left(\delta_i \mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right)\right) t_i^\gamma \log(t_i) \mathrm{e}^{t_i^\gamma+\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}{\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \lambda \partial \phi} &= -\sum_{i=1}^{n} \left(\delta_i \mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right) \frac{\left(1-\mathrm{e}^{t_i^\gamma}\right) \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}{\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1} \\
&\quad -\phi \sum_{i=1}^{n} (1-\delta_i) \frac{\left(1-\mathrm{e}^{t_i^\gamma}\right) \mathrm{e}^{2\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)+\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\left(\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right)^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \gamma^2} &= -\frac{m}{\gamma^2} + \sum_{i=1}^{n} \delta_i t_i^\gamma \log(t_i)^2 \left(1-\lambda\left(t_i^\gamma+1\right)\mathrm{e}^{t_i^\gamma}\right) \\
&\quad -\lambda^2\phi^2 \sum_{i=1}^{n} (1-\delta_i) \frac{t_i^{2\gamma} \log(t_i)^2 \; \mathrm{e}^{2t_i^\gamma+2\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)+\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\left(\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right)^2} \\
&\quad +\lambda\phi \sum_{i=1}^{n} \left(\delta_i \mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1\right) \frac{t_i^\gamma \log(t_i)^2 \left(1+t_i^\gamma\left(1-\lambda \mathrm{e}^{t_i^\gamma}\right)\right) \mathrm{e}^{t_i^\gamma+\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}{\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}-1},
\end{aligned}$$

$$\frac{\partial^2 \ell}{\partial\gamma\partial\phi} = \lambda\sum_{i=1}^{n}\left(\delta_i \mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}} - 1\right)\frac{t_i^\gamma \log(t_i)\mathrm{e}^{t_i^\gamma+\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}{\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}} - 1}$$
$$+\lambda\phi\sum_{i=1}^{n}(1-\delta_i)\frac{t_i^\gamma \log(t_i)\mathrm{e}^{t_i^\gamma+2\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)+\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\left(\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}} - 1\right)^2},$$

$$\frac{\partial^2 \ell}{\partial\phi^2} = -\frac{m}{\phi^2} + \frac{n\mathrm{e}^{\phi}}{\left(\mathrm{e}^{\phi}-1\right)^2} - \sum_{i=1}^{n}(1-\delta_i)\frac{\mathrm{e}^{2\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)+\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\left(\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}} - 1\right)^2},$$

where $m = \sum_{i=1}^{n}\delta_i$ is the observed number of events.

3. SOME PROGRAMS DEVELOPED IN R SOFTWARE

This section provides the R programming codes to reproduce the results of the simulation study discussed in Section 3.4.

```
#========================================================
# function to calculate the expected value of a variable
# with extended Chen-Poisson distribution
#========================================================
Echenpois <- function(lambda, gamma, phi) {
  if ((!is.numeric(lambda)) || (!is.numeric(gamma))
      || (!is.numeric(phi)))
    stop("non-numeric argument")
  if ((min(lambda) <= 0) || (min(gamma) <= 0) ||
      (min(phi) == 0))
    stop("Invalid arguments")
  func <- function(y) {(phi*exp(-phi*y)*
  ((log(1-lambda^(-1)*log(y)))^(1/gamma)))/
  (1-exp(-phi))}
  integral<-integrate(Vectorize(func),
                      lower = 0, upper = 1)
  arr<-array(c(integral$value,integral$abs.error),
             dim=c(1,2))
  dimnames(arr)<-list("",c("estimate ",
                          " integral abs. error <"))
  return(arr)
}


#========================================================
# function to generate pseudo-random data from an extended
# Chen-Poisson distribution, considering random censoring
#========================================================
# lambda, gamma, phi: parameter values;
# n: sample size; p: percentage of censoring

rchenpoi <- function(lambda, gamma, phi, n, p) {
```

```
  temp <- matrix(0, nrow=n, ncol=1);
  t.event <- matrix(0, nrow=n, ncol=1);
  cens <- matrix(0, nrow=n, ncol=1)

  u<-runif(n,0,1) # for time-to-events
  t.event <- (log(1-(log(1-(log((exp(phi)-1)*u+1))/
                          phi))/lambda))^(1/gamma)

  # determine maux associated to percentage of censoring p
  if(p==0){temp<-t.event ;cens<-rep(1,n)}
  if(p!=0){maux <- Echenpois(lambda=lambda, gamma=gamma,
                             phi=phi)[1]/p

  # for random censoring
  cax<-runif(n,0,maux)
  for (i in 1:n) {
    if (t.event[i]<=cax[i]) {
      temp[i]<-t.event[i] ;cens[i]<-1}
    if (t.event[i]>cax[i])  {
      temp[i]<-cax[i]     ;cens[i]<-0}
  }}
  return(list(temp=temp, cens=cens))
}

#==========================================================
# log-likelihood function of the extended Chen-Poisson
# distribution
#==========================================================
# param: vector of parameter; cens: censoring vector;
# temp: times vector; n: sample size
# Note: In order to ensure that the estimate of phi is:
# positive, then consider exp(param[3])
# negative, then consider log(1/(1+exp(param[3])))

param = numeric(0)
fvero <- function(param, cens, temp, n) {
  vetsoma = 0

  p1 <- exp(param[1])  # lambda
  p2 <- exp(param[2])  # gamma
  p3 <- exp(param[3])  # phi

  vetsoma = lapply(1:n, function(z) {
    aux <- (-log(p3/(1-exp(-p3)))-cens[z]*(p1+log(p1*p2))-
    (p2-1)*cens[z]*log(temp[z])-cens[z]*(temp[z]^p2)+
    p1*cens[z]*exp(temp[z]^p2)-(1-cens[z])*
    log((1-exp(-p3*exp(p1*(1-exp(temp[z]^p2)))))/p3)+
    p3*cens[z]*exp(p1*(1-exp(temp[z]^p2)))); sum(aux)})
  llike <- sum(unlist(vetsoma))
  return(llike)
}

#==========================================================
# function to calculate the observed information matrix
#==========================================================
# param: vector of parameter; cens: censoring vector;
# temp: times vector; n: sample size
```

```
hess <- function(param, cens, temp, n) {
  aux11=0; aux12=0; aux13=0; aux22=0; aux23=0; aux33=0

  p1 <- param[1]  # lambda
  p2 <- param[2]  # gamma
  p3 <- param[3]  # phi

  # second derivative with respect to lambda
  aux11 <- lapply(1:n, function(z) {
    hessiL = (((cens[z]*(-1 + exp(exp(p1 -
    exp((temp[z])^p2)*p1)*p3))^2)/(p1^2) +
    ((-1 + exp((temp[z])^p2))^2*p3^2)/
    exp(2*(-1 + exp((temp[z])^p2))*p1) +
    cens[z]*exp(p1 - 2*exp((temp[z])^p2)*p1 +
    exp(p1 - exp((temp[z])^p2)*p1)*p3)*
    (-1 + exp((temp[z])^p2))^2*p3*
    (-exp(exp((temp[z])^p2)*p1) +
    exp(exp((temp[z])^p2)*p1 +
    exp(p1 - exp((temp[z])^p2)*p1)*p3) - exp(p1)*p3) +
    exp(p1 - exp((temp[z])^p2)*p1)*
    (-1 + exp((temp[z])^p2))^2*
    (-1 + exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))*p3*
    (-1 + exp(p1 - exp((temp[z])^p2)*p1)*p3))/
    ((-1 + exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))^2));
  sum(hessiL)})

  a11 <- sum(unlist(aux11))

  # second derivative of lambda with respect to gamma
  aux12 <- lapply(1:n, function(z) {
    hessiLG = ((1/(-1+(exp(exp(p1-exp((temp[z])^p2)*p1)*
    p3)))^2)*(exp((temp[z])^p2-2*exp((temp[z])^p2)*p1)*
    (temp[z])^p2*(exp(p1)*p3*(exp((temp[z])^p2+
    exp((temp[z])^p2)*p1)*p1-exp((temp[z])^p2 +
    exp((temp[z])^p2)*p1+exp(p1- exp((temp[z])^p2)*p1)*
    p3)*p1-exp(exp((temp[z])^p2)*p1)*(1 + p1) +
    exp(exp((temp[z])^p2)*p1+exp(p1 - exp((temp[z])^p2)*
    p1)*p3)*(1 + p1) - exp(p1 + exp(p1 -
    exp((temp[z])^p2)*p1)*p3)*p1*p3 + exp((temp[z])^p2+
    p1 + exp(p1 - exp((temp[z])^p2)*p1)*p3)*p1*p3) +
    cens[z]*(exp(2*exp((temp[z])^p2)*p1) -
    2*exp(2*exp((temp[z])^p2)*p1 + exp(p1 -
    exp((temp[z])^p2)*p1)*p3)+exp(2*exp((temp[z])^p2)*p1+
    2*exp(p1-exp((temp[z])^p2)*p1)*p3)-exp((temp[z])^p2+
    p1+exp((temp[z])^p2)*p1+exp(p1-exp((temp[z])^p2)*p1)*
    p3)*p1*p3 + exp((temp[z])^p2+p1+exp((temp[z])^p2)*p1 +
    2*exp(p1 - exp((temp[z])^p2)*p1)*p3)*p1*p3 + exp(p1 +
    exp((temp[z])^p2)*p1+exp(p1-exp((temp[z])^p2)*p1)*p3)*
    (1 + p1)*p3 - exp(p1 + exp((temp[z])^p2)*p1 +
    2*exp(p1 - exp((temp[z])^p2)*p1)*p3)*(1 + p1)*p3 +
    exp(2*p1+exp(p1-exp((temp[z])^p2)*p1)*p3)*p1*p3^2 -
    exp((temp[z])^p2+2*p1+exp(p1-exp((temp[z])^p2)*p1)*
    p3)*p1*p3^2))*log(temp[z])));
  sum(hessiLG)})

  a12 <- sum(unlist(aux12))

  # second derivative of lambda with respect to phi
```

```
aux13 <- lapply(1:n, function(z) {hessiLP = ((exp(p1 -
  2*exp((temp[z])^p2)*p1)*(1 - exp((temp[z])^p2))*
  (exp(exp((temp[z])^p2)*p1) - (1 + cens[z])*
  exp(exp((temp[z])^p2)*p1 + exp(p1 - exp((temp[z])^p2)*
  p1)*p3) + cens[z]*exp(exp((temp[z])^p2)*p1+2*exp(p1 -
  exp((temp[z])^p2)*p1)*p3) - (-1 + cens[z])*exp(p1 +
  exp(p1 - exp((temp[z])^p2)*p1)*p3)*p3))/((-1 +
  exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))^2));
sum(hessiLP)})

a13 <- sum(unlist(aux13))

# second derivative with respect to gamma
aux22 <- lapply(1:n, function(z) {
  hessiG = (cens[z]/(p2^2) + (1/((-1 + exp(exp(p1 -
  exp((temp[z])^p2)*p1)*p3))^2))*(((temp[z])^p2*
  (exp((temp[z])^p2+p1)*p1*p3*((-exp(exp((temp[z])^p2)*
  p1))*(1 + (temp[z])^p2) + exp(exp((temp[z])^p2)*p1 +
  exp(p1-exp((temp[z])^p2)*p1)*p3)*(1+(temp[z])^p2) +
  exp((temp[z])^p2 + exp((temp[z])^p2)*p1)*(temp[z])^p2
  *p1 - exp((temp[z])^p2+exp((temp[z])^p2)*p1+exp(p1 -
  exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*p1 +
  exp((temp[z])^p2+p1+exp(p1-exp((temp[z])^p2)*p1)*p3)*
  (temp[z])^p2*p1*p3)+cens[z]*(-exp(2*exp((temp[z])^p2)*
  p1) + 2*exp(2*exp((temp[z])^p2)*p1 + exp(p1 -
  exp((temp[z])^p2)*p1)*p3)-exp(2*exp((temp[z])^p2)*p1+
  2*exp(p1-exp((temp[z])^p2)*p1)*p3)+exp((temp[z])^p2+
  2*exp((temp[z])^p2)*p1)*(1+(temp[z])^p2)*p1-
  2*exp((temp[z])^p2 + 2*exp((temp[z])^p2)*p1 +
  exp(p1-exp((temp[z])^p2)*p1)*p3)*(1+(temp[z])^p2)*p1+
  exp((temp[z])^p2+2*exp((temp[z])^p2)*p1+2*exp(p1 -
  exp((temp[z])^p2)*p1)*p3)*(1 + (temp[z])^p2)*p1 +
  exp((temp[z])^p2+p1+exp((temp[z])^p2)*p1+exp(p1 -
  exp((temp[z])^p2)*p1)*p3)*(1 + (temp[z])^p2)*p1*p3 -
  exp((temp[z])^p2 + p1 + exp((temp[z])^p2)*p1 +
  2*exp(p1-exp((temp[z])^p2)*p1)*p3)*(1+(temp[z])^p2)*
  p1*p3 - exp(2*(temp[z])^p2+p1+exp((temp[z])^p2)*p1 +
  exp(p1 - exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*
  p1^2*p3+exp(2*(temp[z])^p2+p1+exp((temp[z])^p2)*p1 +
  2*exp(p1 - exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*
  p1^2*p3 - exp(2*(temp[z])^p2 + 2*p1 + exp(p1 -
  exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*p1^2*p3^2))*
  log((temp[z]))^2)/exp(2*exp((temp[z])^p2)*p1)));
sum(hessiG)})

a22 <- sum(unlist(aux22))

# second derivative of gamma with respect to phi
aux23 <- lapply(1:n, function(z) {
  hessiGP = ((exp((temp[z])^p2+p1-2*exp((temp[z])^p2)*
  p1)*(temp[z])^p2*p1*(-exp(exp((temp[z])^p2)*p1)+(1+
  cens[z])*exp(exp((temp[z])^p2)*p1 + exp(p1 -
  exp((temp[z])^p2)*p1)*p3)-cens[z]*
  exp(exp((temp[z])^p2)*p1+2*exp(p1-exp((temp[z])^p2)*
  p1)*p3)+(-1+cens[z])*exp(p1+exp(p1-exp((temp[z])^p2)*
  p1)*p3)*p3)*log(temp[z]))/((-1+exp(exp(p1-
  exp((temp[z])^p2)*p1)*p3))^2));
sum(hessiGP)})
```

```
  a23 <- sum(unlist(aux23))

  # second derivative with respect to phi
  aux33 <- lapply(1:n, function(z) {
    hessiP = (-(exp(p3)/((-1+exp(p3))^2))-((-1+cens[z])*
    exp(2*p1-2*exp((temp[z])^p2)*p1+exp(p1-
    exp((temp[z])^p2)*p1)*p3))/
    ((-1 + exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))^2) +
    cens[z]/(p3^2));
  sum(hessiP)})

  a33 <- sum(unlist(aux33))

  matrix(c(a11, a12, a13, a12, a22, a23, a13, a23, a33),
         nrow=3, byrow=T)
}

#==========================================================
# Set parameter values for the simulations scenarios
# n = 20, 50, 100, 500, 1000 ; p = 0, 0.1, 0.3
# lambda 0.2; gamma= 1.5; phi= 3    (hf is increasing)
# lambda 3;   gamma= 0.3; phi= 20   (hf is unimodal)
# lambda 1.3; gamma= 0.2; phi= -2   (hf is decreasing)
# lambda 0.6; gamma= 0.6; phi= -3.5 (hf is bathtub-shaped)
#==========================================================
# installing and loading library MASS to use ginv()
install.packages("MASS"); library(MASS)

# sample size
n <- c(20,50,100,500,1000)

# lambda, gamma and phi parameter values
lambda <- c(0.2,3)  # c(1.3,0.6)
gamma <-  c(1.5,0.3) # c(0.2,0.6)
phi <-  c(3,20)  #c(-2,-3.5)

# vector of initial values for parameters (see below)
# Note: If in the log-likelihood function was considered:
# exp(param[3]), then put log(phi)
# log(1/(1+exp(param[3]))), then put log(1-exp(phi))-phi
condinit.l = log(lambda)
condinit.g = log(gamma)
condinit.p = log(phi)

# percentage of censoring
p <- c(0,0.1,0.3)

# number of simulations
simul = 1000

#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
# Program for the simulation study
#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
# initializing table
table1 <- data.frame()

for (m in 1:length(p)){
```

```
for (a in 1:length(lambda)){
for (x in 1:length(n)){
set.seed(2143)
result=data.frame(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0);
names(result) <- c("lambda","Varlambda","LI","LS","gamma",
            "Vargamma","LI","LS","phi","Varphi","LI","LS")
s=1

options(warn=-1) #Note: warnings are disabled because we
    #have already dealt the problems in the simulations.

while (s <= simul) {
# generate the data
data = rchenpoi(lambda=lambda[a], gamma=gamma[a],
                phi=phi[a], n=n[x], p=p[m])

# fit model
# par=initial values for each lambda, gamma and phi
otim <- optim(par=c(condinit.l[a],condinit.g[a],
              condinit.p[a]), method="BFGS", fn=fvero,
              cens=data$cens, temp=data$temp, n=n[x],
              control=list(reltol=1e-5))

# compute the observed information matrix
# Note: If in the log-likelihood function was considered:
# exp(param[3]), then put exp(otim$par)
# log(1/(1+exp(param[3]))), then put c(exp(otim$par[1]),
# exp(otim$par[2]),log(1/(1+exp(otim$par[3]))))
Inf.Fisher <- hess(exp(otim$par), cens=data$cens,
                   temp=data$temp, n=n[x])

if (is.nan(sum(Inf.Fisher))) {  }
else {# compute the variance from the information matrix
  aux <- ginv(Inf.Fisher)
  vetvar <- diag(aux);
  if (is.nan(sqrt(vetvar[1]))||is.nan(sqrt(vetvar[2]))||
      is.nan(sqrt(vetvar[3]))) {  }
  else {
# compute the 95% CI of the parameters estimates
# Note: If in the log-likelihood function was considered:
# exp(param[3]), then here put
# matrix(c(exp(otim$par)- 1.96*sqrt(vetvar),
# exp(otim$par)+1.96*sqrt(vetvar)), ncol=2, byrow=F)
#
# log(1/(1+exp(param[3]))), then here put
# matrix(c(exp(otim$par[1])-1.96*sqrt(vetvar[1]),
# exp(otim$par[1])+1.96*sqrt(vetvar[1]),
# exp(otim$par[2])-1.96*sqrt(vetvar[2]),
# exp(otim$par[2])+1.96*sqrt(vetvar[2]),
# log(1/(1+exp(otim$par[3])))-1.96*sqrt(vetvar[3]),
# log(1/(1+exp(otim$par[3])))+1.96*sqrt(vetvar[3])),
# ncol=2, byrow=T)

     IC <- matrix(c(exp(otim$par)-1.96*sqrt(vetvar),
       exp(otim$par)+1.96*sqrt(vetvar)), ncol=2, byrow=F)

  # get the results for parameter lambda
  result[s,1] = exp(otim$par[1]); result[s,2] <- vetvar[1]
```

```
  result[s,3] <- IC[1,1]; result[s,4] <- IC[1,2]

  # get the results for parameter gamma
  result[s,5] = exp(otim$par[2]); result[s,6] <- vetvar[2]
  result[s,7] <- IC[2,1]; result[s,8] <- IC[2,2]

  # get the results for parameter phi
  # Note: If in the log-likelihood function was considered:
  # exp(param[3]), then here put exp(otim$par[3])
  # log(1/(1+exp(param[3]))), then here put
  # log(1/(1+exp(otim$par[3])))
  result[s,9] = exp(otim$par[3]); result[s,10] <- vetvar[3]
  result[s,11] <- IC[3,1]; result[s,12] <- IC[3,2]

  s=s+1}}}
  options(warn=0) # warnings turned on

  L1 <- length(which(result[,3] > lambda[a]))/simul
  U1 <- length(which(result[,4] < lambda[a]))/simul
  L2 <- length(which(result[,7] > gamma[a]))/simul
  U2 <- length(which(result[,8] < gamma[a]))/simul
  L3 <- length(which(result[,11] > phi[a]))/simul
  U3 <- length(which(result[,12] < phi[a]))/simul

  table1 <- rbind(table1,c(p[m]*100,lambda[a],gamma[a],
   phi[a],n[x],mean(result[,1]), mean(result[,5]),
   mean(result[,9]), mean(sqrt(result[,2])),
   mean(sqrt(result[,6])), mean(sqrt(result[,10])),
   sum(result[,1]-lambda[a])/simul,
   sum(result[,5]-gamma[a])/simul,
   sum(result[,9]-phi[a])/simul,
   sum((result[,1]-lambda[a])^2)/simul,
   sum((result[,5]-gamma[a])^2)/simul,
   sum((result[,9]-phi[a])^2)/simul,(1-(L1+U1))*100,
   (1-(L2+U2))*100, (1-(L3+U3))*100))

}}}

colnames(table1) <- c("% Cens","lambda","gamma","phi","n",
      "avg(l)","avg(g)","avg(p)","sd(l)","sd(g)","sd(p)",
      "bias(l)","bias(g)","bias(p)","mse(l)","mse(g)",
      "mse(p)","CP(l)","CP(g)","CP(p)")

#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
# show results for Table 1:
#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
table1
```