Supplementary material for "The extended Chen-Poisson lifetime distribution"

Authors: Ivo Sousa-Ferreira ✉*, Ana Maria Abreu and Cristina Rocha

1. PROOF OF PROPOSITION 3.3

This section provides the Proof of Proposition 3.3 regarding the monotonicity study of the probability density function (pdf) of the ECP distribution.

Proof of Proposition 3.3: The first derivative of the pdf (3.4) of the ECP distribution is given by

$$f'(t;\lambda,\gamma,\phi)=\frac{f(t;\lambda,\gamma,\phi)}{t}\left\{\gamma-1-\gamma t^{\gamma}\Big[-1+\lambda \mathrm{e}^{t^{\gamma}}\big(1-\phi \mathrm{e}^{\lambda(1-\mathrm{e}^{t^{\gamma}})}\big)\Big]\right\},\quad t>0,$$

where $\lambda,\gamma>0$ and $\phi\in\mathbb{R}\backslash\{0\}$. The sign of $f'(t;\lambda,\gamma,\phi)$ is the sign of the expression in curly brackets and $f'(t;\lambda,\gamma,\phi)$ is zero when that expression is zero. Consider the change of variable $u=\mathrm{e}^{t^{\gamma}}$ and rewrite the expression in curly brackets as $g(u;\lambda,\gamma,\phi)=\gamma-1-\gamma r_1(u;\lambda,\phi)$, where $r_1(u;\lambda,\phi)=\log(u)\big[-1+\lambda u(1-\phi \mathrm{e}^{\lambda(1-u)})\big]$ for $u>1$. The monotonicity study of the pdf is done separately for $\phi<0$ (distribution of the minimum) and $\phi>0$ (distribution of the maximum).

1. <u>For $\phi<0$ (distribution of the minimum):</u>
If $\phi<0$, then $1<1-\phi \mathrm{e}^{\lambda(1-u)}<1-\phi$ since $0<\mathrm{e}^{\lambda(1-u)}<1$, for $u>1$. Hence, $1-\phi \mathrm{e}^{\lambda(1-u)}$ is never zero when $\phi<0$.

Case 1.1. ($\phi<0$, $0<\gamma\leq 1$ and $\lambda\geq 1$)
If $\phi<0$, $0<\gamma\leq 1$ and $\lambda\geq 1$, then $g(u;\lambda,\gamma,\phi)<0$ and, therefore, $f(t;\lambda,\gamma,\phi)$ is monotonically decreasing.

Case 1.2. ($\phi<0$, $\gamma=1$ and $0<\lambda<1$)
Here, from $g(u;\lambda,\gamma,\phi)=0$ it is straightforward to see that $\lambda u(1-\phi \mathrm{e}^{\lambda(1-u)})$ takes values less or greater than 1 and so there exists at least one solution of this equation. Let u_0 be one of these solutions, that is, $\lambda u_0(1-\phi \mathrm{e}^{\lambda(1-u_0)})=1$. Given that $1-\phi \mathrm{e}^{\lambda(1-u_0)}>1$, then $\lambda u_0=1/(1-\phi \mathrm{e}^{\lambda(1-u_0)})<1$ and so $u_0<1/\lambda$. Accordingly, u_0 belongs to the interval $(1,1/\lambda)$, with $0<\lambda<1$. Let $r_2(u;\lambda,\phi)=-1+\lambda u(1-\phi \mathrm{e}^{\lambda(1-u)})$, so its first derivative is $r_2'(u;\lambda,\phi)=\lambda[1-\phi(1-u\lambda)\mathrm{e}^{\lambda(1-u)}]$. Evaluating $r_2'(u;\lambda,\phi)$ at u_0, it follows that $r_2'(u_0;\lambda,\phi)>0$, for $u_0\in(1,1/\lambda)$. Hence, $r_2(u;\lambda,\phi)$ is monotonically increasing and u_0 is the only solution, if it exists. Thus, $g(u;\lambda,\gamma,\phi)=-\log(u)r_2(u;\lambda,\phi)$ also has just a single zero at u_0,

*Corresponding author

when it exists, because $\log(u) > 0$ for $u > 1$. However, since $\lim_{u\to 1} r_2(u;\lambda,\phi) = -1 + (1-\phi)\lambda$, u_0 only exists if $-1 + (1-\phi)\lambda < 0$. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 1.2.1. ($\phi < 0$, $\gamma = 1$ and $0 < \lambda < (1-\phi)^{-1}$)
If $0 < \lambda < (1-\phi)^{-1}$, then $r_2(u;\lambda,\phi)$ has a zero at u_0 and, since $r_2(u;\lambda,\phi)$ is monotonically increasing, it follows that $g(u;\lambda,\gamma,\phi) = -\log(u) r_2(u;\lambda,\phi)$ is positive until u_0, it has a zero at u_0 and it is negative thereafter. Therefore, $f(t;\lambda,\gamma,\phi)$ is unimodal. In this sub-case, the mode is equal to $\log(u_0)^{1/\gamma}$, where u_0 is the root of the non-linear equation $\lambda u(1-\phi e^{\lambda(1-u)}) = 1$, for $\gamma = 1$ and $0 < \lambda < (1-\phi)^{-1}$.

Sub-case 1.2.2. ($\phi < 0$, $\gamma = 1$ and $(1-\phi)^{-1} \le \lambda < 1$)
If $(1-\phi)^{-1} \le \lambda < 1$, then $r_2(u;\lambda,\phi)$ is always greater than or equal to 0 and, consequently, $g(u;\lambda,\gamma,\phi)$ is always less than or equal to 0. Therefore, $f(t;\lambda,\gamma,\phi)$ is monotonically decreasing.

Case 1.3. ($\phi < 0$, $0 < \gamma < 1$ and $0 < \lambda < 1$)
Here, from $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_1 be one of these solutions, that is, $r_1(u_1;\lambda,\phi) = (\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma < 0$ for $0 < \gamma < 1$, then $r_1(u_1;\lambda,\phi) < 0$, which implies that $\lambda u_1(1-\phi e^{\lambda(1-u_1)}) < 1$ since $\log(u_1) > 0$ for $u_1 > 1$. Moreover, it is known that $1-\phi e^{\lambda(1-u_1)} > 1$ implies that $u_1 < 1/\lambda$. Accordingly, u_1 belongs to the interval $(1, 1/\lambda)$, with $0 < \lambda < 1$. The first derivative of $r_1(u;\lambda,\phi)$ is given by $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1+\log(u)\big) - \big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$. Evaluating $r_1'(u;\lambda,\phi)$ at $u_1 \in (1, 1/\lambda)$, it follows that it can take both negative and positive values. In addition, as $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1 + (1-\phi)\lambda$, it is clear that $r_1'(u;\lambda,\phi)$ can be initially negative or positive. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 1.3.1. ($\phi < 0$, $0 < \gamma < 1$ and $0 < \lambda < (1-\phi)^{-1}$)
If $0 < \lambda < (1-\phi)^{-1}$, then $r_1'(u;\lambda,\phi)$ is initially negative and, furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$ it is seen that $r_1'(u;\lambda,\phi) = 0$ has at least one root. Let u_2 be one of these solutions, that is, $r_1'(u_2;\lambda,\phi) = 0$, but now u_2 belongs to the interval $(1, 1/\lambda)$ with $0 < \lambda < (1-\phi)^{-1}$. Thus, it follows that

$$\begin{cases} r_1'(u;\lambda,\phi) < 0, \text{ if } & u < u_2 \\ r_1'(u;\lambda,\phi) = 0, \text{ if } & u = u_2 \\ r_1'(u;\lambda,\phi) > 0, \text{ if } & u > u_2 \end{cases}.$$

This means that $r_1(u;\lambda,\phi)$ decreases until u_2 and then increases. So, u_2 is the only solution of the non-linear equation $r_1'(u;\lambda,\phi) = 0$. Note that if u_2 minimizes $r_1(u;\lambda,\phi)$, then u_2 maximizes $g(u;\lambda,\gamma,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ for $0 < \gamma < 1$ and $0 < \lambda < (1-\phi)^{-1}$. Therefore, $g(u;\lambda,\gamma,\phi)$ is unimodal. Since $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 < 0$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$, then

$g(u; \lambda, \gamma, \phi)$ is initially negative for $0 < \gamma < 1$ and goes to $-\infty$ as $u \to \infty$. As $g(u; \lambda, \gamma, \phi)$ is unimodal, then two situations can occur depending on whether the maximum of this function is negative or positive. If the maximum of $g(u; \lambda, \gamma, \phi)$ is less than or equal to 0, that is, $g(u_2; \lambda, \gamma, \phi) \leq 0$, then $f(u; \lambda, \gamma, \phi)$ is monotonically decreasing. On the other hand, if the maximum of $g(u; \lambda, \gamma, \phi)$ is greater than 0, that is, $g(u_2; \lambda, \gamma, \phi) > 0$, then $g(u; \lambda, \gamma, \phi)$ will have two zeros because it is unimodal. This means that, in this situation, $g(u; \lambda, \gamma, \phi) = 0$ has two solutions, say $u_{1,1}$ and $u_{1,2}$, which can only be obtained using numerical methods. Remembering that $\lim_{u\to 1} g(u; \lambda, \gamma, \phi) = \gamma - 1 < 0$ and $\lim_{u\to\infty} g(u; \lambda, \gamma, \phi) = -\infty$, it follows that $f(u; \lambda, \gamma, \phi)$ is decreasing-increasing-decreasing (DID).

Sub-case 1.3.2. ($\phi < 0$, $0 < \gamma < 1$ and $(1 - \phi)^{-1} < \lambda < 1$)
If $(1 - \phi)^{-1} \leq \lambda < 1$, then $r_1'(u; \lambda, \phi) \geq 0$. Therefore, $r_1(u; \lambda, \phi)$ is monotonically increasing and, consequently, $g(u; \lambda, \phi) = \gamma - 1 - \gamma r_1(u; \lambda, \phi)$ is monotonically decreasing. Moreover, it is known that $\lim_{u\to 1} g(u; \lambda, \gamma, \phi) = \gamma - 1 < 0$ and $\lim_{u\to\infty} g(u; \lambda, \gamma, \phi) = -\infty$. So $g(u; \lambda, \gamma, \phi) < 0$ for $0 < \gamma < 1$ and $(1 - \phi)^{-1} \leq \lambda < 1$. This shows that $f(u; \lambda, \gamma, \phi)$ is monotonically decreasing.

Case 1.4. ($\phi < 0$, $\gamma > 1$ and $\lambda \geq 1$)
Here, from $g(u; \lambda, \gamma, \phi) = 0$ it is straightforward to see that once again $r_1(u; \lambda, \phi)$ takes values less or greater than $(\gamma - 1)/\gamma$ and so there exists at least one solution of this equation. Let u_3 be one of these solutions, that is, $r_1(u_3; \lambda, \phi) = (\gamma - 1)/\gamma$. Given that $(\gamma - 1)/\gamma < 1$ for $\gamma > 1$, then $r_1(u; \lambda, \phi) < 1$, for $u > 1$. Since $\log(u_3) > 0$, for $u_3 > 1$, from $r_1(u; \lambda, \phi) = \log(u)\big[-1 + \lambda u(1 - \phi e^{\lambda(1-u)})\big]$ it is not possible to obtain the upper bound of the interval to which u_3 belongs. However, it can be seen that $r_1(u; \lambda, \phi) > 0$, for $u > 1$ and $\lambda \geq 1$. Then, $r_1(u; \lambda, \phi)$ is never zero and it belongs to the interval $(0, 1)$. Thus, in this case, $g(u; \lambda, \gamma, \phi) = \gamma - 1 - \gamma r_1(u; \lambda, \phi)$ can be seen as a straight line with slope $-\gamma$ and vertical intercept $\gamma - 1$. Consequently, $g(u; \lambda, \gamma, \phi)$ is monotonically decreasing and u_3 is the only solution. In fact, since $\lim_{u\to 1} g(u; \lambda, \gamma, \phi) = \gamma - 1 > 0$ for $\gamma > 1$ and $\lim_{u\to\infty} g(u; \lambda, \gamma, \phi) = -\infty$, it turns out that $g(u; \lambda, \gamma, \phi)$ is initially positive and will eventually become negative as $u \to \infty$. Therefore, it follows that $f(t; \lambda, \gamma, \phi)$ is unimodal. In this case, the mode is equal to $\log(u_3)^{1/\gamma}$, where u_3 is the root of the non-linear equation $r_1(u; \lambda, \phi) = (\gamma - 1)/\gamma$, for $\gamma > 1$ and $\lambda \geq 1$.

Case 1.5. ($\phi < 0$, $\gamma > 1$ and $0 < \lambda < 1$)
From $g(u; \lambda, \gamma, \phi) = 0$ it is straightforward to see that $r_1(u; \lambda, \phi)$ takes values less or greater than $(\gamma - 1)/\gamma$ and so there exists at least one solution of this equation. Let u_4 be one of these solutions, that is, $r_1(u_4; \lambda, \phi) = (\gamma - 1)/\gamma$. As in the previous case, it is not possible to obtain the upper bound of the interval to which u_4 belongs. An added difficulty is that now $r_1(u; \lambda, \phi) = \log(u) r_2(u; \lambda, \phi)$, where $r_2(u; \lambda, \phi) = -1 + \lambda u(1 - \phi e^{\lambda(1-u)})$, can be negative, positive or even zero since $\log(u) > 0$ and $r_2(u; \lambda, \phi) > -1$, for $u > 1$ and $0 < \lambda < 1$. From Case 1.3., it is known that $r_1'(u; \lambda, \phi) = u^{-1}\big[u\lambda(1 - \phi e^{\lambda(1-u)})\big(1 + \log(u)\big) - \big(1 - \phi\lambda^2 u^2 e^{\lambda(1-u)} \log(u)\big)\big]$ and $\lim_{u\to 1} r_1'(u; \lambda, \phi) = -1 + (1 - \phi)\lambda$. Therefore, this case

will be separated into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 1.5.1. ($\phi < 0$, $0 < \gamma < 1$ and $0 < \lambda < (1-\phi)^{-1}$)
If $0 < \lambda < (1-\phi)^{-1}$, then $r_1'(u;\lambda,\phi)$ is initially negative and, given that $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$, it will eventually become positive as $u \to \infty$. From Sub-case 1.3.1., it follows that $r_1(u;\lambda,\phi)$ decreases until a given point and increases thereafter. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Sub-case 1.3.1, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$, for $\gamma > 1$. This means that, although $g(u;\lambda,\gamma,\phi)$ is unimodal, it is initially positive and, consequently, it has only one root which will be denoted by $u_{4,1}$. Thus, $f(t;\lambda,\gamma,\phi)$ is unimodal. In this case, the mode is equal to $\log(u_{4,1})^{1/\gamma}$, where $u_{4,1}$ is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $0 < \gamma < 1$ and $0 < \lambda < (1-\phi)^{-1}$.

Sub-case 1.5.2. ($\phi < 0$, $0 < \gamma < 1$ and $(1-\phi)^{-1} \le \lambda < 1$)
If $(1-\phi)^{-1} \le \lambda < 1$, then $r_1'(u;\lambda,\phi) \ge 0$. As in Sub-case 1.3.2., it follows that $r_1(u;\lambda,\phi)$ is monotonically increasing and, consequently, $g(u;\lambda,\gamma,\phi)$ is monotonically decreasing. However, in contrast to Sub-case 1.3.1, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$, for $\gamma > 1$. Thus, $g(u;\lambda,\gamma,\phi)$ is initially positive and, because it is monotonically decreasing, it has one root which will be denoted by $u_{4,2}$. Therefore, $f(u;\lambda,\gamma,\phi)$ is also unimodal and the mode is equal to $\log(u_{4,2})^{1/\gamma}$, where $u_{4,2}$ is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $0 < \gamma < 1$ and $(1-\phi)^{-1} \le \lambda < 1$.

2. For $\phi > 0$ (distribution of the maximum):
If $\phi > 0$, then $1 - \phi < 1 - \phi e^{\lambda(1-u)} < 1$ since $0 < e^{\lambda(1-u)} < 1$, for $u > 1$. Hence, $1 - \phi e^{\lambda(1-u)}$ can be negative, positive or even zero when $\phi > 0$. Note that the first derivative of $1 - \phi e^{\lambda(1-u)}$ is given by $\lambda\phi e^{\lambda(1-u)}$, which is always positive for $u > 1$ and $\lambda, \phi > 0$. Then, $1 - \phi e^{\lambda(1-u)}$ is monotonically increasing and is zero at $u^* = 1 + \lambda^{-1}\log(\phi)$.

Case 2.1. ($\phi > 0$, $0 < \gamma < 1$ and $\lambda > 1$)
Here, from $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_5 be one of these solutions, that is, $r_1(u_5;\lambda,\phi) = (\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma < 0$ for $0 < \gamma < 1$, then $r_1(u_5;\lambda,\phi) < 0$, which implies that $\lambda u_5(1 - \phi e^{\lambda(1-u_5)}) < 1$ since $\log(u_5) > 0$ for $u_5 > 1$. Then $\lambda(1 - \phi e^{\lambda(1-u_5)}) < 1$ and so $u_5 < 1 - \lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]$, for $\lambda > 1$ and $\phi > 0$. Accordingly, u_5 belongs to the interval $\big(1, 1 - \lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]\big)$. It is noteworthy that u_5 only exists if $1 - \lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})] > 1$, which implies that $\phi > 1 - \lambda^{-1}$ for $\lambda > 1$. The first derivative of $r_1(u;\lambda,\phi)$ is given by $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1 - \phi e^{\lambda(1-u)})\big(1+\log(u)\big) - \big(1 - \phi\lambda^2 u^2 e^{\lambda(1-u)}\log(u)\big)\big]$. Evaluating $r_1'(u;\lambda,\phi)$ at u_5, it follows that it can take both negative and positive values. In addition, since $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1 + (1-\phi)\lambda$ it is clear that $r_1'(u;\lambda,\phi)$ can be initially negative or positive. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 2.1.1. ($0 < \phi \le 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$)
If $0 < \phi \le 1 - \lambda^{-1}$ with $\lambda > 1$, then $r_1'(u;\lambda,\phi) \ge 0$. Therefore, $r_1(u;\lambda,\phi)$ is monotonically increasing and, consequently, $g(u;\lambda,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ is monotonically decreasing. Moreover, $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 < 0$ for $0 < \gamma < 1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$. Therefore, $g(u;\lambda,\gamma,\phi) < 0$ for $0 < \phi \le 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$. This shows that $f(u;\lambda,\gamma,\phi)$ is monotonically decreasing.

Sub-case 2.1.2. ($\phi > 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$)
If $\phi > 1 - \lambda^{-1}$ for $\lambda > 1$, then $r_1'(u;\lambda,\phi)$ is initially negative and, furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$ it is seen that $r_1'(u;\lambda,\phi)$ has at least one root. Let u_6 be one of these solutions, that is, $r_1'(u_6;\lambda,\phi) = 0$, where u_6 belongs to the interval $\left(1, 1 - \lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]\right)$ with $\phi > 1 - \lambda^{-1}$ and $\lambda > 1$. Thus, it follows that

$$\begin{cases} r_1'(u;\lambda,\phi) < 0, & \text{if} \quad u < u_6 \\ r_1'(u;\lambda,\phi) = 0, & \text{if} \quad u = u_6 \\ r_1'(u;\lambda,\phi) > 0, & \text{if} \quad u > u_6 \end{cases}.$$

This means that $r_1(u;\lambda,\phi)$ decreases until u_6 and then increases. So, u_6 is the only solution of the non-linear equation $r_1'(u;\lambda,\phi) = 0$. Note that if u_6 minimizes $r_1(u;\lambda,\phi)$, then u_6 maximizes $g(u;\lambda,\gamma,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ for $\phi > 1 - \lambda^{-1}$, $0 < \gamma < 1$ and $\lambda > 1$. Therefore, $g(u;\lambda,\gamma,\phi)$ is unimodal. Since $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 < 0$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$, then $g(u;\lambda,\gamma,\phi)$ is initially negative for $0 < \gamma < 1$ and goes to $-\infty$ as $u \to \infty$. As $g(u;\lambda,\gamma,\phi)$ is unimodal, then two situations can occur depending on whether the maximum of this function is negative or positive. If the maximum of $g(u;\lambda,\gamma,\phi)$ is less than or equal to 0, that is, $g(u_6;\lambda,\gamma,\phi) \le 0$, then $f(u;\lambda,\gamma,\phi)$ is monotonically decreasing. On the other hand, if the maximum of $g(u;\lambda,\gamma,\phi)$ is greater than 0, that is, $g(u_6;\lambda,\gamma,\phi) > 0$, then $g(u;\lambda,\gamma,\phi)$ will have two zeros because it is unimodal. This means that, in this situation, $g(u;\lambda,\gamma,\phi) = 0$ has two solutions, say $u_{5,1}$ and $u_{5,2}$, which can only be obtained using numerical methods. Thus, it follows that $f(u;\lambda,\gamma,\phi)$ is DID.

Case 2.2. ($\phi > 0$, $\gamma = 1$ and $\lambda > 1$)
Here, from $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $\lambda u(1-\phi e^{\lambda(1-u)})$ takes values less or greater than 1 and so there exists at least one solution of this equation. Let u_7 be one of these solutions, that is, $\lambda u_7(1-\phi e^{\lambda(1-u_7)}) = 1$. Given that $u_7 > 1$, then $\lambda(1-\phi e^{\lambda(1-u_7)}) < 1$ and so $u_7 < 1 - \lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]$, for $\lambda > 1$ and $\phi > 0$. Accordingly, u_7 belongs to the interval $\left(1, 1 - \lambda^{-1}\log[\phi^{-1}(1-\lambda^{-1})]\right)$ and u_7 only exists if $\phi > 1 - \lambda^{-1}$ for $\lambda > 1$. Evaluating $r_2'(u;\lambda,\phi) = \lambda[1 - \phi(1 - u\lambda)e^{\lambda(1-u)}]$ at u_7, it follows that $r_2'(u_7;\lambda,\phi) > \lambda$. Hence, $r_2(u;\lambda,\phi)$ is monotonically increasing and u_7 is the only solution when it exists. Thus, for $\gamma = 1$, $g(u;\lambda,\gamma,\phi) = -\log(u) r_2(u;\lambda,\phi)$ also has a single zero at u_7, when it exists, because $\log(u) > 0$ for $u > 1$. However, since $\lim_{u\to 1} r_2(u;\lambda,\phi) = -1 + (1-\phi)\lambda$, once again it is seen that u_7 only exists if $\phi > 1 - \lambda^{-1}$ for $\lambda > 1$. Therefore, it is necessary to split this case into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 2.2.1. ($0 < \phi \leq 1 - \lambda^{-1}$, $\gamma = 1$ and $\lambda > 1$)
If $0 < \phi \leq 1 - \lambda^{-1}$ and $\gamma = 1$, then $r_2(u;\lambda,\phi)$ is always greater than or equal to 0 and, consequently, $g(u;\lambda,\gamma,\phi)$ is always less than or equal to 0. Therefore, $f(t;\lambda,\gamma,\phi)$ is monotonically decreasing.

Sub-case 2.2.2. ($\phi > 1 - \lambda^{-1}$, $\gamma = 1$ and $\lambda > 1$)
If $\phi > 1 - \lambda^{-1}$ and $\lambda > 1$, then $r_2(u;\lambda,\phi)$ has a zero at u_7 and, since $r_2(u;\lambda,\phi)$ is monotonically increasing, it follows that $g(u;\lambda,\gamma,\phi) = -\log(u) r_2(u;\lambda,\phi)$ is positive until u_7, it has a zero at u_7 and it is negative thereafter. In this sub-case, the mode is equal to $\log(u_7)^{1/\gamma}$, where u_7 is the root of the non-linear equation $\lambda u(1-\phi \mathrm{e}^{\lambda(1-u)}) = 1$, for $\phi > 1-\lambda^{-1}$, $\gamma = 1$ and $\lambda > 1$.

Case 2.3. ($\phi > 0$, $\gamma > 1$ and $\lambda > 1$)
From $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_8 be one of these solutions, that is, $r_1(u_8;\lambda,\phi) = (\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma < 1$ for $\gamma > 1$, then $r_1(u;\lambda,\phi) < 1$, for $u > 1$. Since $\log(u_8) > 0$, for $u_8 > 1$, from $r_1(u;\lambda,\phi)$ it is not possible to obtain the upper bound of the interval to which u_8 belongs. An added difficulty is that $r_1(u;\lambda,\phi) = \log(u) r_2(u;\lambda,\phi)$ can be negative, positive or even zero since $1-\phi < 1-\phi\mathrm{e}^{\lambda(1-u)} < 1$, which implies that $-1+\lambda u(1-\phi) < r_2(u;\lambda,\phi) < -1+\lambda u$, for $u > 1$, $\lambda > 1$ and $\phi > 0$. In addition, it is known that $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1-\phi\mathrm{e}^{\lambda(1-u)})\big(1+\log(u)\big) - \big(1 - \phi\lambda^2 u^2 \mathrm{e}^{\lambda(1-u)}\log(u)\big)\big]$ and $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1+(1-\phi)\lambda$. Therefore, this case will be separated into two sub-cases, depending on the relationship between the parameters λ and ϕ.

Sub-case 2.3.1. ($0 < \phi \leq 1 - \lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$)
If $0 < \phi \leq 1 - \lambda^{-1}$ with $\lambda > 1$, then $r_1'(u;\lambda,\phi) \geq 0$. It follows that $r_1(u;\lambda,\phi)$ is monotonically increasing and, consequently, $g(u;\lambda,\gamma,\phi) = \gamma - 1 - \gamma r_1(u;\lambda,\phi)$ is monotonically decreasing. However, in contrast to Sub-case 2.1.1, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$, for $\gamma > 1$. Thus, $g(u;\lambda,\gamma,\phi)$ is initially positive and, because it is monotonically decreasing with $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$, it has one root which will be denoted by $u_{8,1}$. Therefore, $f(u;\lambda,\gamma,\phi)$ is unimodal and the mode is equal to $\log(u_{8,1})^{1/\gamma}$, where $u_{8,1}$ is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $0 < \phi \leq 1 - \lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$.

Sub-case 2.3.2. ($\phi > 1 - \lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$)
If $\phi > 1-\lambda^{-1}$ with $\lambda > 1$, then $r_1'(u;\lambda,\phi)$ is initially negative and, given that $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$, it will eventually become positive as $u \to \infty$. From Sub-case 2.1.2, it follows that $r_1(u;\lambda,\phi)$ decreases until a given point and increases thereafter. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Sub-case 2.1.2, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$, for $\gamma > 1$. This means that, although $g(u;\lambda,\gamma,\phi)$ is unimodal, it is initially positive and, consequently, it has only one root which will be denoted by $u_{8,2}$. Thus, $f(t;\lambda,\gamma,\phi)$ is also unimodal. In this case, the mode is equal to $\log(u_{8,2})^{1/\gamma}$, where $u_{8,2}$ is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $\phi > 1 - \lambda^{-1}$, $\gamma > 1$ and $\lambda > 1$.

Case 2.4. ($\phi > 0$, $0 < \gamma < 1$ and $0 < \lambda \leq 1$)
From $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so this equation has at least one solution. Let u_9 be one of these solutions, that is, $r_1(u_9;\lambda,\phi) = (\gamma-1)/\gamma$. As in Case 2.1, given that $(\gamma-1)/\gamma < 0$ for $0 < \gamma < 1$ then $r_1(u_9;\lambda,\phi) < 0$, which implies that $\lambda u_9(1-\phi e^{\lambda(1-u_9)}) < 1$ since $\log(u_9) > 0$ for $u_9 > 1$. Then $\lambda(1-\phi e^{\lambda(1-u_9)}) < 1$ and so $e^{\lambda(1-u_9)} > \phi^{-1}(1-\lambda^{-1})$. However, in contrast to Case 2.1, from $r_1(u;\lambda,\phi)$ it is not possible to obtain the upper bound of the interval to which u_9 belongs for $0 < \lambda \leq 1$. Nonetheless, knowing that $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1+\log(u)\big) - \big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$ and since $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1+(1-\phi)\lambda < 0$, it is clear that $r_1'(u;\lambda,\phi)$ is always initially negative for $\phi > 0$ and $0 < \lambda \leq 1$. Furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$, it is seen that $r_1'(u;\lambda,\phi)$ has at least one root. Let u_{10} be one of these solutions, that is, $r_1'(u_{10};\lambda,\phi) = 0$, where $u_{10} > 1$. As in Sub-case 2.1.2, it can be seen that $r_1(u;\lambda,\phi)$ decreases until u_{10} and increases thereafter. So, u_{10} is the only solution of the non-linear equation $r_1'(u;\lambda,\phi) = 0$. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. Note that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 < 0$ for $0 < \gamma < 1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$. Consequently, as described in Sub-case 2.1.2, $f(u;\lambda,\gamma,\phi)$ can take two different shapes depending on whether the maximum of $g(u;\lambda,\gamma,\phi)$, that is, $g(u_{10};\lambda,\gamma,\phi)$, is negative or positive. If the maximum of $g(u;\lambda,\gamma,\phi)$ is less than or equal to 0, then $f(u;\lambda,\gamma,\phi)$ is monotonically decreasing. On the other hand, if the maximum of $g(u;\lambda,\gamma,\phi)$ is greater than 0, then $f(u;\lambda,\gamma,\phi)$ is DID because $g(u;\lambda,\gamma,\phi)$ has two zeros and it is unimodal. Thus, in this second situation, $g(u;\lambda,\gamma,\phi) = 0$ has two solutions, say $u_{9,1}$ and $u_{9,2}$, which can only be obtained using numerical methods.

Case 2.5. ($\phi > 0$, $\gamma = 1$ and $0 < \lambda \leq 1$)
From $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $\lambda u(1-\phi e^{\lambda(1-u)})$ takes values less or greater than 1 and so there exists at least one solution of this equation. Let u_{11} be one of these solutions, that is, $\lambda u_{11}(1-\phi e^{\lambda(1-u_{11})}) = 1$. Given that $u_{11} > 1$, then $\lambda(1-\phi e^{\lambda(1-u_{11})}) < 1$ and so $e^{\lambda(1-u_{11})} > \phi^{-1}(1-\lambda^{-1})$. In contrast to Case 2.2, it is not possible to obtain the upper bound of the interval to which u_{11} belongs for $0 < \lambda \leq 1$. Nonetheless, knowing that $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1-\phi e^{\lambda(1-u)})\big(1 + \log(u)\big) - \big(1-\phi\lambda^2u^2e^{\lambda(1-u)}\log(u)\big)\big]$ and $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1+(1-\phi)\lambda < 0$, it is clear that $r_1'(u;\lambda,\phi)$ is always initially negative for $\phi > 0$ and $0 < \lambda \leq 1$. Furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$ it is seen that $r_1'(u;\lambda,\phi)$ has at least one root. Let u_{12} be one of these solutions, that is, $r_1'(u_{12};\lambda,\phi) = 0$, where $u_{12} > 1$. As in Case 2.4, it can be seen that $r_1(u;\lambda,\phi)$ decreases until u_{12} and increases thereafter. So, u_{12} is the only solution of the non-linear equation $r_1'(u;\lambda,\phi) = 0$. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Case 2.4, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 = 0$ for $\gamma = 1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$. This means that, in this situation, $g(u;\lambda,\gamma,\phi) = 0$ has two solutions, say $u_{11,1}$ and $u_{11,2}$, which can only be obtained using numerical methods. Although $g(u;\lambda,\gamma,\phi)$ has two zeros, it is initially equal to 0 and, because it is unimodal, it crosses the horizontal axis only once, more precisely at $u_{11,2}$, with $u_{11,2} > u_{11,1}$. Therefore, $f(t;\lambda,\gamma,\phi)$ is unimodal and the mode is equal to $\log(u_{11,2})^{1/\gamma}$, where $u_{11,2}$ is the root of the non-linear equation $\lambda u(1-\phi e^{\lambda(1-u)}) = 1$, for $\gamma = 1$ and $0 < \lambda \leq 1$.

Case 2.6.($\phi > 0$, $\gamma > 1$ and $0 < \lambda \leq 1$)
From $g(u;\lambda,\gamma,\phi) = 0$ it is straightforward to see that $r_1(u;\lambda,\phi)$ takes values less or greater than $(\gamma-1)/\gamma$ and so there exists at least one solution of this equation. Let u_{13} be one of these solutions, that is, $r_1(u_{13};\lambda,\phi) = (\gamma-1)/\gamma$. Given that $(\gamma-1)/\gamma < 1$ for $\gamma > 1$, then $r_1(u;\lambda,\phi) < 1$, for $u > 1$. However, as in Case 2.3, from $r_1(u;\lambda,\phi)$ it is not possible to obtain the upper bound of the interval to which u_{13} belongs, since $\log(u_{13}) > 0$ for $u_{13} > 1$. An added difficulty is that $r_1(u;\lambda,\phi) = \log(u) r_2(u;\lambda,\phi)$, where $r_2(u;\lambda,\phi) = -1 + \lambda u(1 - \phi \mathrm{e}^{\lambda(1-u)})$, can be negative, positive or even zero since $1 - \phi < 1 - \phi \mathrm{e}^{\lambda(1-u)} < 1$, which implies that $-1 + \lambda u(1-\phi) < r_2(u;\lambda,\phi) < -1 + \lambda u$, for $u > 1$, $0 < \lambda \leq 1$ and $\phi > 0$. Nonetheless, knowing that $r_1'(u;\lambda,\phi) = u^{-1}\big[u\lambda(1 - \phi \mathrm{e}^{\lambda(1-u)})\big(1 + \log(u)\big) - \big(1 - \phi\lambda^2 u^2 \mathrm{e}^{\lambda(1-u)} \log(u)\big)\big]$ and $\lim_{u\to 1} r_1'(u;\lambda,\phi) = -1 + (1-\phi)\lambda < 0$, it is clear that $r_1'(u;\lambda,\phi)$ is always initially negative for $\phi > 0$ and $0 < \lambda \leq 1$. Furthermore, as $\lim_{u\to\infty} r_1'(u;\lambda,\phi) = \infty$ it is seen that $r_1'(u;\lambda,\phi)$ has at least one root. Let u_{14} be one of these solutions, that is, $r_1'(u_{14};\lambda,\phi) = 0$, where $u_{14} > 1$. As in Cases 2.4 and 2.5, it can be seen that $r_1(u;\lambda,\phi)$ decreases until u_{14} and increases thereafter. Therefore, u_{14} is the only solution of the non-linear equation $r_1'(u;\lambda,\phi) = 0$. Hence, $g(u;\lambda,\gamma,\phi)$ is unimodal. However, in contrast to Cases 2.4 and 2.5, it is seen that $\lim_{u\to 1} g(u;\lambda,\gamma,\phi) = \gamma - 1 > 0$ for $\gamma > 1$ and $\lim_{u\to\infty} g(u;\lambda,\gamma,\phi) = -\infty$. This means that, although $g(u;\lambda,\gamma,\phi)$ is unimodal, it is initially positive and, consequently, it has only one root, denoted by u_{13}. Thus, $f(t;\lambda,\gamma,\phi)$ is unimodal. In this case, the mode is equal to $\log(u_{13})^{1/\gamma}$, where u_{13} is the root of the non-linear equation $r_1(u;\lambda,\phi) = (\gamma-1)/\gamma$, for $\gamma > 1$ and $0 < \lambda \leq 1$. □

2. ELEMENTS OF THE OBSERVED INFORMATION MATRIX

The elements of the observed information, $\boldsymbol{I}(\lambda,\gamma,\phi)$, are given by

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \lambda^2} &= -\frac{m}{\lambda^2} - \phi^2 \sum_{i=1}^{n} (1-\delta_i) \frac{\left(\mathrm{e}^{t_i^\gamma} - 1\right)^2 \mathrm{e}^{2\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right) + \phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}}{\left(\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}} - 1\right)^2} \\
&\quad -\phi \sum_{i=1}^{n} \left(\delta_i \mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}} - 1\right) \frac{\left(\mathrm{e}^{t_i^\gamma} - 1\right)^2 \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}}{\mathrm{e}^{\phi \mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^\gamma}\right)}} - 1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial\lambda\partial\gamma} &= -\sum_{i=1}^{n}\delta_i t_i^{\gamma}\mathrm{e}^{t_i^{\gamma}}\log(t_i) - \lambda\phi\sum_{i=1}^{n}\delta_i\frac{t_i^{\gamma}\log(t_i)\mathrm{e}^{2t_i^{\gamma}+\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)+\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1}\\
&\quad+\lambda\phi^2\sum_{i=1}^{n}(1-\delta_i)\frac{t_i^{\gamma}\left(1-\mathrm{e}^{t_i^{\gamma}}\right)\log(t_i)\mathrm{e}^{t_i^{\gamma}+2\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)+\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{\left(\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)^2}\\
&\quad+\phi\sum_{i=1}^{n}\frac{\left(\lambda\mathrm{e}^{t_i^{\gamma}}+(\lambda+1)\left(\delta_i\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)\right)t_i^{\gamma}\log(t_i)\mathrm{e}^{t_i^{\gamma}+\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}{\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial\lambda\partial\phi} &= -\sum_{i=1}^{n}\left(\delta_i\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)\frac{\left(1-\mathrm{e}^{t_i^{\gamma}}\right)\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}{\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1}\\
&\quad-\phi\sum_{i=1}^{n}(1-\delta_i)\frac{\left(1-\mathrm{e}^{t_i^{\gamma}}\right)\mathrm{e}^{2\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)+\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{\left(\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial\gamma^2} &= -\frac{m}{\gamma^2}+\sum_{i=1}^{n}\delta_i t_i^{\gamma}\log(t_i)^2\left(1-\lambda\left(t_i^{\gamma}+1\right)\mathrm{e}^{t_i^{\gamma}}\right)\\
&\quad-\lambda^2\phi^2\sum_{i=1}^{n}(1-\delta_i)\frac{t_i^{2\gamma}\log(t_i)^2\ \mathrm{e}^{2t_i^{\gamma}+2\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)+\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{\left(\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)^2}\\
&\quad+\lambda\phi\sum_{i=1}^{n}\left(\delta_i\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)\frac{t_i^{\gamma}\log(t_i)^2\left(1+t_i^{\gamma}\left(1-\lambda\mathrm{e}^{t_i^{\gamma}}\right)\right)\mathrm{e}^{t_i^{\gamma}+\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}{\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial\gamma\partial\phi} &= \lambda\sum_{i=1}^{n}\left(\delta_i\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)\frac{t_i^{\gamma}\log(t_i)\mathrm{e}^{t_i^{\gamma}+\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}{\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1}\\
&\quad+\lambda\phi\sum_{i=1}^{n}(1-\delta_i)\frac{t_i^{\gamma}\log(t_i)\mathrm{e}^{t_i^{\gamma}+2\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)+\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{\left(\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)^2},
\end{aligned}
$$

$$
\frac{\partial^2 \ell}{\partial\phi^2} = -\frac{m}{\phi^2}+\frac{n\mathrm{e}^{\phi}}{\left(\mathrm{e}^{\phi}-1\right)^2}-\sum_{i=1}^{n}(1-\delta_i)\frac{\mathrm{e}^{2\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)+\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}}{\left(\mathrm{e}^{\phi\mathrm{e}^{\lambda\left(1-\mathrm{e}^{t_i^{\gamma}}\right)}}-1\right)^2},
$$

where $m=\sum_{i=1}^{n}\delta_i$ is the observed number of events.

3. SOME PROGRAMS DEVELOPED IN R SOFTWARE

This section provides the R programming codes to reproduce the results of the simulation study discussed in Section 3.4.

```
#=========================================================
# function to calculate the expected value of a variable
# with extended Chen-Poisson distribution
#=========================================================
Echenpois <- function(lambda, gamma, phi) {
  if ((!is.numeric(lambda)) || (!is.numeric(gamma))
      || (!is.numeric(phi)))
    stop("non-numeric argument")
  if ((min(lambda) <= 0) || (min(gamma) <= 0) ||
      (min(phi) == 0))
    stop("Invalid arguments")
  func <- function(y) {(phi*exp(-phi*y)*
  ((log(1-lambda^(-1)*log(y)))^(1/gamma)))/
  (1-exp(-phi))}
  integral<-integrate(Vectorize(func),
                      lower = 0, upper = 1)
  arr<-array(c(integral$value,integral$abs.error),
             dim=c(1,2))
  dimnames(arr)<-list("",c("estimate ",
                           " integral abs. error <"))
  return(arr)
}


#=========================================================
# function to generate pseudo-random data from an extended
# Chen-Poisson distribution, considering random censoring
#=========================================================
# lambda, gamma, phi: parameter values;
# n: sample size; p: percentage of censoring

rchenpoi <- function(lambda, gamma, phi, n, p) {
  temp <- matrix(0, nrow=n, ncol=1);
  t.event <- matrix(0, nrow=n, ncol=1);
  cens <- matrix(0, nrow=n, ncol=1)

  u<-runif(n,0,1) # for time-to-events
  t.event <- (log(1-(log(1-(log((exp(phi)-1)*u+1))/
                         phi))/lambda))^(1/gamma)

  # determine maux associated to percentage of censoring p
  if(p==0){temp<-t.event ;cens<-rep(1,n)}
  if(p!=0){maux <- Echenpois(lambda=lambda, gamma=gamma,
                             phi=phi)[1]/p
```

```
  # for random censoring
  cax<-runif(n,0,maux)
  for (i in 1:n) {
    if (t.event[i]<=cax[i]) {
      temp[i]<-t.event[i] ;cens[i]<-1}
    if (t.event[i]>cax[i])  {
      temp[i]<-cax[i]     ;cens[i]<-0}
  }}
  return(list(temp=temp, cens=cens))
}

#==========================================================
# log-likelihood function of the extended Chen-Poisson
# distribution
#==========================================================
# param: vector of parameter; cens: censoring vector;
# temp: times vector; n: sample size
# Note: In order to ensure that the estimate of phi is:
# positive, then consider exp(param[3])
# negative, then consider log(1/(1+exp(param[3])))

param = numeric(0)
fvero <- function(param, cens, temp, n) {
  vetsoma = 0

  p1 <- exp(param[1])  # lambda
  p2 <- exp(param[2])  # gamma
  p3 <- exp(param[3])  # phi

  vetsoma = lapply(1:n, function(z) {
    aux <- (-log(p3/(1-exp(-p3)))-cens[z]*(p1+log(p1*p2))-
    (p2-1)*cens[z]*log(temp[z])-cens[z]*(temp[z]^p2)+
    p1*cens[z]*exp(temp[z]^p2)-(1-cens[z])*
    log((1-exp(-p3*exp(p1*(1-exp(temp[z]^p2)))))/p3)+
    p3*cens[z]*exp(p1*(1-exp(temp[z]^p2)))); sum(aux)})
  llike <- sum(unlist(vetsoma))
  return(llike)
}

#==========================================================
# function to calculate the observed information matrix
#==========================================================
# param: vector of parameter; cens: censoring vector;
# temp: times vector; n: sample size

hess <- function(param, cens, temp, n) {
  aux11=0; aux12=0; aux13=0; aux22=0; aux23=0; aux33=0

  p1 <- param[1]  # lambda
  p2 <- param[2]  # gamma
  p3 <- param[3]  # phi

  # second derivative with respect to lambda
```

```
aux11 <- lapply(1:n, function(z) {
  hessiL = (((cens[z]*(-1 + exp(exp(p1 -
  exp((temp[z])^p2)*p1)*p3))^2)/(p1^2) +
  ((-1 + exp((temp[z])^p2))^2*p3^2)/
  exp(2*(-1 + exp((temp[z])^p2))*p1) +
  cens[z]*exp(p1 - 2*exp((temp[z])^p2)*p1 +
  exp(p1 - exp((temp[z])^p2)*p1)*p3)*
  (-1 + exp((temp[z])^p2))^2*p3*
  (-exp(exp((temp[z])^p2)*p1) +
  exp(exp((temp[z])^p2)*p1 +
  exp(p1 - exp((temp[z])^p2)*p1)*p3) - exp(p1)*p3) +
  exp(p1 - exp((temp[z])^p2)*p1)*
  (-1 + exp((temp[z])^p2))^2*
  (-1 + exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))*p3*
  (-1 + exp(p1 - exp((temp[z])^p2)*p1)*p3))/
  ((-1 + exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))^2));
sum(hessiL)})

a11 <- sum(unlist(aux11))

# second derivative of lambda with respect to gamma
aux12 <- lapply(1:n, function(z) {
  hessiLG = ((1/(-1+(exp(exp(p1-exp((temp[z])^p2)*p1)*
  p3)))^2)*(exp((temp[z])^p2-2*exp((temp[z])^p2)*p1)*
  (temp[z])^p2*(exp(p1)*p3*(exp((temp[z])^p2+
  exp((temp[z])^p2)*p1)*p1-exp((temp[z])^p2 +
  exp((temp[z])^p2)*p1+exp(p1- exp((temp[z])^p2)*p1)*
  p3)*p1-exp(exp((temp[z])^p2)*p1)*(1 + p1) +
  exp(exp((temp[z])^p2)*p1+exp(p1 - exp((temp[z])^p2)*
  p1)*p3)*(1 + p1) - exp(p1 + exp(p1 -
  exp((temp[z])^p2)*p1)*p3)*p1*p3 + exp((temp[z])^p2+
  p1 + exp(p1 - exp((temp[z])^p2)*p1)*p3)*p1*p3) +
  cens[z]*(exp(2*exp((temp[z])^p2)*p1) -
  2*exp(2*exp((temp[z])^p2)*p1 + exp(p1 -
  exp((temp[z])^p2)*p1)*p3)+exp(2*exp((temp[z])^p2)*p1+
  2*exp(p1-exp((temp[z])^p2)*p1)*p3)-exp((temp[z])^p2+
  p1+exp((temp[z])^p2)*p1+exp(p1-exp((temp[z])^p2)*p1)*
  p3)*p1*p3 + exp((temp[z])^p2+p1+exp((temp[z])^p2)*p1 +
  2*exp(p1 - exp((temp[z])^p2)*p1)*p3)*p1*p3 + exp(p1 +
  exp((temp[z])^p2)*p1+exp(p1-exp((temp[z])^p2)*p1)*p3)*
  (1 + p1)*p3 - exp(p1 + exp((temp[z])^p2)*p1 +
  2*exp(p1 - exp((temp[z])^p2)*p1)*p3)*(1 + p1)*p3 +
  exp(2*p1+exp(p1-exp((temp[z])^p2)*p1)*p3)*p1*p3^2 -
  exp((temp[z])^p2+2*p1+exp(p1-exp((temp[z])^p2)*p1)*
  p3)*p1*p3^2))*log(temp[z])));
sum(hessiLG)})

a12 <- sum(unlist(aux12))

# second derivative of lambda with respect to phi
aux13 <- lapply(1:n, function(z) {hessiLP = ((exp(p1 -
  2*exp((temp[z])^p2)*p1)*(1 - exp((temp[z])^p2))*
  (exp(exp((temp[z])^p2)*p1) - (1 + cens[z])*
```

```
    exp(exp((temp[z])^p2)*p1 + exp(p1 - exp((temp[z])^p2)*
    p1)*p3) + cens[z]*exp(exp((temp[z])^p2)*p1+2*exp(p1 -
    exp((temp[z])^p2)*p1)*p3) - (-1 + cens[z])*exp(p1 +
    exp(p1 - exp((temp[z])^p2)*p1)*p3)*p3))/((-1 +
    exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))^2));
  sum(hessiLP)})

  a13 <- sum(unlist(aux13))

  # second derivative with respect to gamma
  aux22 <- lapply(1:n, function(z) {
    hessiG = (cens[z]/(p2^2) + (1/((-1 + exp(exp(p1 -
    exp((temp[z])^p2)*p1)*p3))^2))*(((temp[z])^p2*
    (exp((temp[z])^p2+p1)*p1*p3*((-exp(exp((temp[z])^p2)*
    p1))*(1 + (temp[z])^p2) + exp(exp((temp[z])^p2)*p1 +
    exp(p1-exp((temp[z])^p2)*p1)*p3)*(1+(temp[z])^p2) +
    exp((temp[z])^p2 + exp((temp[z])^p2)*p1)*(temp[z])^p2
    *p1 - exp((temp[z])^p2+exp((temp[z])^p2)*p1+exp(p1 -
    exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*p1 +
    exp((temp[z])^p2+p1+exp(p1-exp((temp[z])^p2)*p1)*p3)*
    (temp[z])^p2*p1*p3)+cens[z]*(-exp(2*exp((temp[z])^p2)*
    p1) + 2*exp(2*exp((temp[z])^p2)*p1 + exp(p1 -
    exp((temp[z])^p2)*p1)*p3)-exp(2*exp((temp[z])^p2)*p1+
    2*exp(p1-exp((temp[z])^p2)*p1)*p3)+exp((temp[z])^p2+
    2*exp((temp[z])^p2)*p1)*(1+(temp[z])^p2)*p1-
    2*exp((temp[z])^p2 + 2*exp((temp[z])^p2)*p1 +
    exp(p1-exp((temp[z])^p2)*p1)*p3)*(1+(temp[z])^p2)*p1+
    exp((temp[z])^p2+2*exp((temp[z])^p2)*p1+2*exp(p1 -
    exp((temp[z])^p2)*p1)*p3)*(1 + (temp[z])^p2)*p1 +
    exp((temp[z])^p2+p1+exp((temp[z])^p2)*p1+exp(p1 -
    exp((temp[z])^p2)*p1)*p3)*(1 + (temp[z])^p2)*p1*p3 -
    exp((temp[z])^p2 + p1 + exp((temp[z])^p2)*p1 +
    2*exp(p1-exp((temp[z])^p2)*p1)*p3)*(1+(temp[z])^p2)*
    p1*p3 - exp(2*(temp[z])^p2+p1+exp((temp[z])^p2)*p1 +
    exp(p1 - exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*
    p1^2*p3+exp(2*(temp[z])^p2+p1+exp((temp[z])^p2)*p1 +
    2*exp(p1 - exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*
    p1^2*p3 - exp(2*(temp[z])^p2 + 2*p1 + exp(p1 -
    exp((temp[z])^p2)*p1)*p3)*(temp[z])^p2*p1^2*p3^2))*
    log((temp[z]))^2)/exp(2*exp((temp[z])^p2)*p1)));
  sum(hessiG)})

  a22 <- sum(unlist(aux22))

  # second derivative of gamma with respect to phi
  aux23 <- lapply(1:n, function(z) {
    hessiGP = ((exp((temp[z])^p2+p1-2*exp((temp[z])^p2)*
    p1)*(temp[z])^p2*p1*(-exp(exp((temp[z])^p2)*p1)+(1+
    cens[z])*exp(exp((temp[z])^p2)*p1 + exp(p1 -
    exp((temp[z])^p2)*p1)*p3)-cens[z]*
    exp(exp((temp[z])^p2)*p1+2*exp(p1-exp((temp[z])^p2)*
    p1)*p3)+(-1+cens[z])*exp(p1+exp(p1-exp((temp[z])^p2)*
    p1)*p3)*p3)*log(temp[z]))/((-1+exp(exp(p1-
```

```
    exp((temp[z])^p2)*p1)*p3))^2));
  sum(hessiGP)})

  a23 <- sum(unlist(aux23))

  # second derivative with respect to phi
  aux33 <- lapply(1:n, function(z) {
    hessiP = (-(exp(p3)/((-1+exp(p3))^2))-((-1+cens[z])*
    exp(2*p1-2*exp((temp[z])^p2)*p1+exp(p1-
    exp((temp[z])^p2)*p1)*p3))/
    ((-1 + exp(exp(p1 - exp((temp[z])^p2)*p1)*p3))^2) +
    cens[z]/(p3^2));
  sum(hessiP)})

  a33 <- sum(unlist(aux33))

  matrix(c(a11, a12, a13, a12, a22, a23, a13, a23, a33),
         nrow=3, byrow=T)
}

#==========================================================
# Set parameter values for the simulations scenarios
# n = 20, 50, 100, 500, 1000 ; p = 0, 0.1, 0.3
# lambda 0.2; gamma= 1.5; phi= 3    (hf is increasing)
# lambda 3;   gamma= 0.3; phi= 20   (hf is unimodal)
# lambda 1.3; gamma= 0.2; phi= -2   (hf is decreasing)
# lambda 0.6; gamma= 0.6; phi= -3.5 (hf is bathtub-shaped)
#==========================================================
# installing and loading library MASS to use ginv()
install.packages("MASS"); library(MASS)

# sample size
n <- c(20,50,100,500,1000)

# lambda, gamma and phi parameter values
lambda <- c(0.2,3)  # c(1.3,0.6)
gamma <-  c(1.5,0.3) # c(0.2,0.6)
phi <-  c(3,20)  #c(-2,-3.5)

# vector of initial values for parameters (see below)
# Note: If in the log-likelihood function was considered:
# exp(param[3]), then put log(phi)
# log(1/(1+exp(param[3]))), then put log(1-exp(phi))-phi
condinit.l = log(lambda)
condinit.g = log(gamma)
condinit.p = log(phi)

# percentage of censoring
p <- c(0,0.1,0.3)

# number of simulations
simul = 1000
```

```
#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
# Program for the simulation study
#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
# initializing table
table1 <- data.frame()

for (m in 1:length(p)){
for (a in 1:length(lambda)){
for (x in 1:length(n)){
set.seed(2143)
result=data.frame(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0);
names(result) <- c("lambda","Varlambda","LI","LS","gamma",
            "Vargamma","LI","LS","phi","Varphi","LI","LS")
s=1

options(warn=-1) #Note: warnings are disabled because we
    #have already dealt the problems in the simulations.

while (s <= simul) {
# generate the data
data = rchenpoi(lambda=lambda[a], gamma=gamma[a],
                phi=phi[a], n=n[x], p=p[m])

# fit model
# par=initial values for each lambda, gamma and phi
otim <- optim(par=c(condinit.l[a],condinit.g[a],
              condinit.p[a]), method="BFGS", fn=fvero,
              cens=data$cens, temp=data$temp, n=n[x],
              control=list(reltol=1e-5))

# compute the observed information matrix
# Note: If in the log-likelihood function was considered:
# exp(param[3]), then put exp(otim$par)
# log(1/(1+exp(param[3]))), then put c(exp(otim$par[1]),
# exp(otim$par[2]),log(1/(1+exp(otim$par[3]))))
Inf.Fisher <- hess(exp(otim$par), cens=data$cens,
                   temp=data$temp, n=n[x])

if (is.nan(sum(Inf.Fisher))) {  }
else {# compute the variance from the information matrix
  aux <- ginv(Inf.Fisher)
  vetvar <- diag(aux);
  if (is.nan(sqrt(vetvar[1]))||is.nan(sqrt(vetvar[2]))||
      is.nan(sqrt(vetvar[3]))) {  }
  else {
# compute the 95% CI of the parameters estimates
# Note: If in the log-likelihood function was considered:
# exp(param[3]), then here put
# matrix(c(exp(otim$par)- 1.96*sqrt(vetvar),
# exp(otim$par)+1.96*sqrt(vetvar)), ncol=2, byrow=F)
#
# log(1/(1+exp(param[3]))), then here put
# matrix(c(exp(otim$par[1])-1.96*sqrt(vetvar[1]),
```

```
# exp(otim$par[1])+1.96*sqrt(vetvar[1]),
# exp(otim$par[2])-1.96*sqrt(vetvar[2]),
# exp(otim$par[2])+1.96*sqrt(vetvar[2]),
# log(1/(1+exp(otim$par[3])))-1.96*sqrt(vetvar[3]),
# log(1/(1+exp(otim$par[3])))+1.96*sqrt(vetvar[3])),
# ncol=2, byrow=T)

     IC <- matrix(c(exp(otim$par)-1.96*sqrt(vetvar),
       exp(otim$par)+1.96*sqrt(vetvar)), ncol=2, byrow=F)

  # get the results for parameter lambda
  result[s,1] = exp(otim$par[1]); result[s,2] <- vetvar[1]
  result[s,3] <- IC[1,1]; result[s,4] <- IC[1,2]

  # get the results for parameter gamma
  result[s,5] = exp(otim$par[2]); result[s,6] <- vetvar[2]
  result[s,7] <- IC[2,1]; result[s,8] <- IC[2,2]

  # get the results for parameter phi
  # Note: If in the log-likelihood function was considered:
  # exp(param[3]), then here put exp(otim$par[3])
  # log(1/(1+exp(param[3]))), then here put
  # log(1/(1+exp(otim$par[3])))
  result[s,9] = exp(otim$par[3]); result[s,10] <- vetvar[3]
  result[s,11] <- IC[3,1]; result[s,12] <- IC[3,2]

  s=s+1}}}
  options(warn=0) # warnings turned on

  L1 <- length(which(result[,3] > lambda[a]))/simul
  U1 <- length(which(result[,4] < lambda[a]))/simul
  L2 <- length(which(result[,7] > gamma[a]))/simul
  U2 <- length(which(result[,8] < gamma[a]))/simul
  L3 <- length(which(result[,11] > phi[a]))/simul
  U3 <- length(which(result[,12] < phi[a]))/simul

  table1 <- rbind(table1,c(p[m]*100,lambda[a],gamma[a],
   phi[a],n[x],mean(result[,1]), mean(result[,5]),
   mean(result[,9]), mean(sqrt(result[,2])),
   mean(sqrt(result[,6])), mean(sqrt(result[,10])),
   sum(result[,1]-lambda[a])/simul,
   sum(result[,5]-gamma[a])/simul,
   sum(result[,9]-phi[a])/simul,
   sum((result[,1]-lambda[a])^2)/simul,
   sum((result[,5]-gamma[a])^2)/simul,
   sum((result[,9]-phi[a])^2)/simul,(1-(L1+U1))*100,
   (1-(L2+U2))*100, (1-(L3+U3))*100))

}}}

colnames(table1) <- c("% Cens","lambda","gamma","phi","n",
      "avg(l)","avg(g)","avg(p)","sd(l)","sd(g)","sd(p)",
      "bias(l)","bias(g)","bias(p)","mse(l)","mse(g)",
```

```
      "mse(p)","CP(l)","CP(g)","CP(p)")

#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
# show results for Table 1:
#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
table1
```