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A Multivariate Quantile Based on Kendall Ordering

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Abstract:

- We introduce the Kendall multivariate quantiles, which are a transformation of orthant quantiles by the Kendall function. Each quantile is then a set of vectors with some advantageous properties, compared to the standard orthant quantile:
  i) it induces a total order
  ii) the probability level of the quantile is consistent with the probability measure of the set of the dominated vectors,
  iii) the multivariate quantiles based on the distribution function or on the survival function have vectors in common which conciliate both upper- and lower-orthant approaches.

  Definition and properties of the Kendall multivariate quantiles are illustrated using Archimedean copulas.

Keywords:

- *multivariate quantile; copula; Archimedean copula; Kendall distribution; orthant quantile.*

AMS Subject Classification:

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1. INTRODUCTION

Given a random variable X defined on a probability space, the quantile of X at level α, Ψ_α, is such that $\alpha = \mathbb{P}[X \leq \Psi_\alpha]$. However, while univariate quantiles are well documented [3], the multivariate approach is not as straightforward. The multivariate analysis relies on a multivariate probability distribution. A useful tool for representing multivariate distributions is the copula which describes the dependence between random variables. A copula is a d-dimensional distribution function $[0,1]^d \to [0,1]$, where $d \in \mathbb{N}$ is the number of underlying random variables [31]. Let G be the multivariate probability distribution of a random vector $X = (X_1, ..., X_d)'$, i.e. the copula-based probability distribution:

$$G : (y_1, ..., y_d) \in \mathbb{R}^d \mapsto \mathbb{P}[X_1 \leq y_1, ..., X_k \leq y_d].$$

If $G_1, ..., G_d$ are the d univariate marginal distribution functions of X, then Sklar's theorem affirms the existence of a copula C such that $G(y_1, ..., y_d) = C(G_1(y_1), ..., G_d(y_d))$ [40]. Copulas, as a tool describing the dependence between random variables, have been applied in many fields, mostly in finance, but also in hydrology [20], astronomy [37, 36], or in telecommunication networks [33, 19].

The multivariate quantile of X can be defined as the set of vectors belonging to the boundary of the α-level set of G. In this set-valued approach, we distinguish the lower-orthant quantile, $\underline{\Psi}_\alpha(G)$, and the upper-orthant quantile, $\overline{\Psi}_\alpha(G)$. The lower-orthant quantile is defined by the set of vectors

$$\underline{\Psi}_\alpha(G) = \partial\{y \in \mathbb{R}^d | G(y) \geq \alpha\}, \tag{1.1}$$

where ∂ denotes the boundary of the mentioned set, whereas the upper-orthant quantile is

$$\overline{\Psi}_\alpha(G) = \partial\{y \in \mathbb{R}^d | \overline{G}(y) \leq 1 - \alpha\}, \tag{1.2}$$

where $\overline{G}$ is the survival function associated with G. If $(y_1, ..., y_d)' \in \underline{\Psi}_\alpha(G)$, then

$$\mathbb{P}[X_1 \leq y_1, ..., X_d \leq y_d] = \alpha.$$

In general, the lower-orthant quantile is more conservative than the upper-orthant quantile: this means that if a vector y belongs to both set-valued orthant quantiles $\overline{\Psi}_\alpha(G)$ and $\underline{\Psi}_{\alpha'}(G)$, then the probability level α' associated by the lower-orthant quantile to y is lower than the probability level α associated by the upper-orthant quantile to y.

The notion of multivariate quantile has been studied in various ways in the literature [38]. For instance we can mention the approach by Embrechts and Puccetti's [16] based on the orthant quantiles defined in equations (1.1) and (1.2) for applications in finance and insurance [11]. Outside the field of finance, the multivariate quantiles have been also studied [38], and applied in particular to meteorology, where extreme weather depends on a combination of parameters which cannot be aggregated, such as speed of wind, quantity of precipitation, temperature, and cloud cover [28] or in hydrology for frequency analysis [9]. These fields require advanced methodological and theoretical support with respect to multivariate analysis. Indeed, when dealing with multivariate data, no consensus arises about the definition of order statistics and quantiles. In particular, the question of quantiles of multivariate distributions has led to numerous interpretations often inspired by analogies with different ways of defining the quantiles of a univariate distribution. Among the various methods proposed, we can cite the spatial quantile [1, 39, 7, 14] or the geometric quantile [8, 6], with some applications in finance [24]. The inversion of a mapping is another kind of known multivariate quantile. In the unidimensional framework, a quantile is indeed defined as the generalized inverse of the cumulated distribution function. If one defines a mapping F from $\mathbb{R}^d$ to $\mathbb{R}$, then inversions can also define a quantile [27]. The exact definition of a multivariate quantile based on the inversion of a mapping is provided by equation (1.1), where the distribution function G is to be replaced by the mapping F. This method is linked to multivariate ordering based on a scalarization, which is the ordering of vectors by comparing scalars, such as isolated coordinates or a function of a linear combination of coordinates [34], or any mapping [41]. This is the method used for example in the orthant quantile, with F being in this case the joint distribution of the d coordinates.

Though extensions of orthant quantiles have been proposed, for instance by Cousin and Di Bernardino [12], who replaced both sets defined by the lower-orthant and the upper-orthant quantiles by their expected value, engendering a quantile defined by a simple vector instead of an infinite set of vectors. Also, replacing the set-valued orthant quantile by a vector-valued quantile has been made possible by selecting a particular direction [42]. The vector-valued multivariate quantile is then the intersection of the set-valued quantile with a line in $\mathbb{R}^d$, given the arbitrary choice of the direction of this line. These two singularizations of the orthant quantile show a need to be able to compare and order multivariate quantiles of different confidence levels. But a pitfall of the orthant approach is that it does not induce a total order, as defined below.

Definition 1.1. Given a random vector X defined in a probability space, we consider a set-valued multivariate quantile function $\alpha \in [0,1] \mapsto \Psi_\alpha(F)$ based on the inversion of a mapping $F : \mathbb{R}^d \to \mathbb{R}$ defined as the probability of a subset of $\mathbb{R}^d$. In other words, $\forall y \in \mathbb{R}^d$, $\exists\, \mathcal{S}_y \subset \mathbb{R}^d, F(y) = \mathbb{P}[X \in \mathcal{S}_y]$. We provide the following definitions:

- The vector $y \in \mathbb{R}^d$ is said to dominate the vector $z \in \mathbb{R}^d$ if $z \in \mathcal{S}_y$. We write it $z \preceq y$. If $y \in \Psi_\alpha(F)$, α is the probability of the set of all the vectors dominated by y.
- The order induced by this set-valued quantile is said to be total if $\preceq$ is a total order. In particular, $\forall y, z \in \mathbb{R}^d$, we have in this case $y \preceq z$ or $z \preceq y$. If this property does not hold, the order is said to be partial.

For the lower-orthant quantile, $\mathcal{S}_y$ is simply the lower-left orthant of y, that is the set of vectors for which each component is lower than the corresponding component of y. It could be interesting to extend the orthant approach, which induces a partial order, to a total order. We think that this total order is a desirable property for a multivariate set-valued quantile. Indeed, we consider that if y and z are vectors of $\Psi_\alpha(F)$ they should dominate the same set of vectors, this property leading to a total order. Furthermore, in this case, every vector $x \in \Psi_{\alpha'}(F)$, with $\alpha' < \alpha$, is dominated by y and z. This property does not hold for instance in the orthant approach. The direct consequence of this property is that α is solution of the equation

$$\alpha = \mathbb{P}[X \in \Psi_{\alpha'}(F) | \alpha' < \alpha]. \tag{1.3}$$

In other words, the probability measure of the set-valued quantiles $\Psi_{\alpha'}(F)$ for a probability level α' lower than α is exactly α, similarly to the univariate case. To our knowledge, existing set-valued multivariate quantiles, including orthant quantiles, do not fulfill this property.

In the family of multivariate quantiles based on the inversion of a mapping F, a proper choice of F may lead to a total order. We are interested in finding this proper F. In this quest, we are inspired by another setting of multivariate quantile known as centre-outward quantile surface. If one is given a statistical depth function, such as the likelihood depth [18] or the Mahalanobis depth [30], the centre-outward quantile surfaces are defined as concentric regions around the centre, which is the maximal-depth vector [29, 43]. More precisely, given a probability $p \in (0,1)$, the p-quantile of a distribution G is the set of vectors of depth α_p, which is defined such that the probability to have vectors of a higher depth than α_p is p: $p = \mathbb{P}(D(X,G) \geq \alpha_p)$, where X is a random vector of distribution G and $D(X,G)$ its depth.

The purpose of this paper is to propose an extension of the orthant quantile that also induces a total order and in particular for which equation (1.3) holds, and to study its properties. We propose to modify the centre-outward quantile surface to focus on tails instead of on the centre of the distribution. Instead of determining a spatial median first, we associate a metric for each vector. Vectors with a metric of the same value are gathered in an equivalence class. We can then order these classes with respect to this metric. The metric chosen is the multivariate distribution function and is thus consistent with the orthant quantile approach. In the quantile surface approach, if y belongs to the p-quantile, then the metric associated to any random vector X is lower with probability p than the one associated with y. For this reason, we will use the Kendall probability distribution

$$K : t \in [0,1] \mapsto \mathbb{P}[G(X) \leq t],$$

where G is the multivariate probability distribution function of X, applied to the the random vector X. The Kendall function indeed defines natural equivalence classes [32]. If vectors y and z are such that $G(y) = G(z)$, then the vector y is equivalent to z, and these vectors dominate every vector x such that $G(x) < G(y)$. Contrary to the orthant quantile, we affirm that y is a vector belonging to the set of the quantile of probability $K(G(y))$, instead of a probability $G(y)$, which is lower than $K(G(y))$ by construction. We base our new definition on the Kendall stochastic ordering [32] instead of the traditional product ordering. The first one is a total order, whereas the second one is only partial. An explanatory illustration is provided in Figure 1.

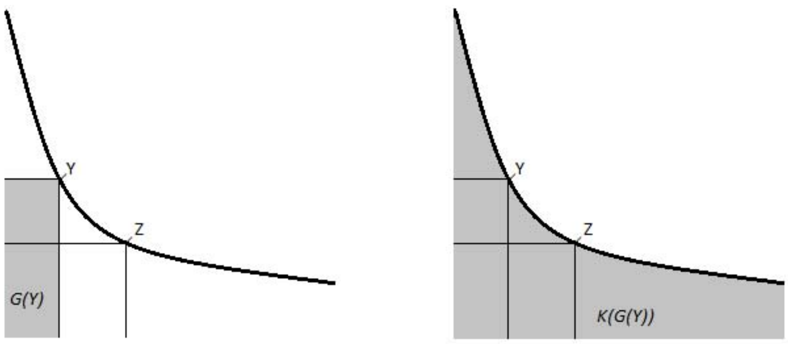

Figure 1: On the left, the thick line is a set of two-dimensional vectors having the same lower-left cumulated probability. In particular, $G(y) = G(z)$, which is the probability measure of the lower-left quadrant of y or z. However, some vectors of the lower-left quadrant of z are not in the lower-left quadrant of y and therefore cannot be compared to y in terms of dominance. On the right, the multivariate probability distribution only leads to the definition of equivalence classes. Therefore, every vector in the grey zone is dominated by every vector on the thick line. The vectors dominated by z are the same as those dominated by y. The probability associated with y and z is therefore the probability measure of the entire grey zone, that is $K(G(y))$, which is equal to $K(G(z))$ and which is greater than $G(y)$.

This blend of orthant quantile and quantile surface leads to a new definition of the multivariate quantile. We call it the lower-orthant or upper-orthant Kendall multivariate quantile, since it uses the Kendall distribution function. The lower-orthant Kendall multivariate quantile is $\underline{\Psi}_{\alpha}^{K}(G) = \partial\{y \in \mathbb{R}^d | K(G(y)) \geq \alpha\}$. The upper-orthant Kendall multivariate quantile is $\overline{\Psi}_{\alpha}^{\mathcal{K}}(G) = \partial\{y \in \mathbb{R}^d | \mathcal{K}(\overline{G}(y)) \leq 1-\alpha\}$, where $\mathcal{K} : t \in [0,1] \mapsto \mathbb{P}[\overline{G}(X) \leq t]$. According to Definition 1.1, the Kendall multivariate quantile induces a total order, contrary to the orthant quantile.

In this paper, we present some properties of this multivariate quantile. For instance, we will observe the extent to which the Kendall multivariate quantiles differ from the orthant quantiles. In particular, the Kendall multivariate quantile is less conservative than the lower-orthant quantile and more conservative than the upper-orthant quantile. Indeed, if a vector y belongs to $\underline{\Psi}_{\underline{\alpha}}(G)$, $\overline{\Psi}_{\overline{\alpha}}(G)$, $\underline{\Psi}_{\underline{\alpha}^K}^{K}(G)$, and $\overline{\Psi}_{\overline{\alpha}^{\mathcal{K}}}^{\mathcal{K}}(G)$, then the probability levels associated to this vector are ordered in the following way: $\underline{\alpha} \leq \underline{\alpha}^K \leq \overline{\alpha}$ as well as $\underline{\alpha} \leq \overline{\alpha}^{\mathcal{K}} \leq \overline{\alpha}$. Moreover, nothing indicates which of the lower-orthant quantile and of the upper-orthant quantile should be preferred. The Kendall quantile can then be seen as a way of diminishing the impact of such a choice, because there is also a smaller difference between both Kendall quantiles than between both orthant quantiles: $|\overline{\alpha}^{\mathcal{K}} - \underline{\alpha}^K| \leq |\overline{\alpha} - \underline{\alpha}|$.

In Section 2, we introduce the Kendall multivariate quantile and some of its properties. We provide theoretical results comparing the Kendall multivariate quantile with the orthant ones. In Section 3, we focus on the case of Archimedean copulas and present an application to simulated data. Section 4 concludes our findings.

2. KENDALL'S MULTIVARIATE QUANTILE

Two approaches relying on Kendall distributions are presented in the next subsection. Drawing a parallel with lower- and upper-orthant multivariate quantiles presented above, we formalise the notion of lower- and upper-orthant Kendall quantile. But before this, we state an assumption that will hold in the whole article.

Assumption 2.1. All the copulas considered have no singular components.

2.1. Definitions

1. *Lower-orthant Kendall quantile.*

As introduced in the section above, the Kendall distribution function is $K : t \in [0,1] \mapsto \mathbb{P}[G(X) \leq t]$, where G is the multivariate probability distribution of the random vector X, associated with a given copula. It is worth noting that the Kendall function does not depend on the full distribution of X but only on its dependence structure. The Kendall function has been used, for example, to estimate Archimedean copulas [21] or to create hierarchical Kendall copulas that deal with high-dimension problems [4]. Using this function, we define the lower-orthant Kendall quantile.

Definition 2.1. For a random vector X of dimension d, the lower-orthant Kendall quantile of probability $\alpha \in [0,1]$, denoted $\underline{\Psi}_\alpha^K$, is the boundary set of the set of vectors $y \in \mathbb{R}^d$ such that $K(G(y)) \geq \alpha$, where G is the multivariate distribution of X and K the corresponding Kendall function:

$$\underline{\Psi}_\alpha^K(G) = \partial\{y \in \mathbb{R}^d | K(G(y)) \geq \alpha\}.$$

2. *Upper-orthant Kendall quantile.*

Similar to the distinction between lower-orthant and upper-orthant multivariate quantiles, we can make a distinction between two kinds of Kendall quantiles, based either on the multivariate distribution function G or on the corresponding survival function $\overline{G}$. We thus introduce another Kendall function in Definition 2.2, $\mathcal{K} : t \in [0,1] \mapsto \mathbb{P}[\overline{G}(X) \leq t]$. We stress the fact that $\mathcal{K}$ is neither the survival Kendall function associated to G, nor K itself, but it is the standard Kendall function associated to $\overline{G}$.

Definition 2.2. For a random vector X of dimension d, the upper-orthant Kendall quantile of probability $\alpha \in [0,1]$, denoted $\overline{\Psi}_\alpha^{\mathcal{K}}$, is the boundary set of the set of vectors $y \in \mathbb{R}^d$ such that $\mathcal{K}(\overline{G}(y)) \leq 1-\alpha$, where $\overline{G}$ is the survival function associated to the multivariate distribution G of X, and $\mathcal{K}$ is the Kendall function corresponding to $\overline{G}$, that is $\mathcal{K} : t \in [0,1] \mapsto \mathbb{P}[\overline{G}(X) \leq t]$:

$$\overline{\Psi}_\alpha^{\mathcal{K}}(G) = \partial\{y \in \mathbb{R}^d | \mathcal{K}(\overline{G}(y)) \leq 1-\alpha\}.$$

These different definitions of multivariate quantiles are linked, as exposed in the following proposition. Indeed, contrary to lower-orthant and upper-orthant multivariate quantiles, both Kendall's multivariate quantiles have some vectors in common.

Proposition 2.1. *Let $\alpha \in [0,1]$, G be a non-atomic multivariate distribution function of dimension $d \in \mathbb{N}$, and having a density function whose support is $\mathbb{R}^d$, with K the Kendall function, supposed to be strictly monotonic, $\overline{G}$ the survival distribution, both associated with G, $\mathcal{K}$ the Kendall function of $\overline{G}$ as introduced in Definition 2.2, then:*

$$\overline{\Psi}_\alpha^{\mathcal{K}}(G) \cap \underline{\Psi}_\alpha^K(G) \neq \emptyset.$$

The proof of Proposition 2.1 is reported in the Appendix.

2.2. Properties

In this section, we focus on specific properties of the Kendall multivariate quantile. In particular, we specify the difference between the Kendall quantile and the orthant quantile.

As mentioned above, the probability associated with a vector by the Kendall function is higher (respectively lower) than in the lower-orthant (resp. upper-orthant) approach.

Using the Fréchet-Hoeffding bounds, it can be demonstrated that $t \leq K(t) \leq 1$ [21] and that $t \leq \mathcal{K}(t) \leq 1$ as well. For a vector y and a multivariate probability distribution G, the lower-orthant approach links y to the level of probability $G(y)$, whereas the Kendall approach associates it with a probability $K(G(y))$, which is, therefore, in the interval $[G(y), 1]$. In other words, the two approaches provides a different probability for a same vector. This probability is higher in the lower-orthant Kendall approach than in the lower-orthant one. We can compare both quantiles in the following manner:

Proposition 2.2. *Let K be strictly monotonic on a neighbourhood of a given probability $\alpha \in [0, 1]$. Then, the Kendall quantile and the lower-orthant quantile are linked by the following:*

$$\underline{\Psi}_{\alpha}(G) = \underline{\Psi}^{K}_{K(\alpha)}(G).$$

The proof of Proposition 2.2 is reported in the Appendix.

Similarly, we can show that the upper-orthant Kendall quantile (which has a non-empty intersection with the lower-orthant Kendall quantile as stated in Proposition 2.1) associates a vector with a lower probability than does the upper-orthant quantile. It is the meaning of the next proposition, since $\mathcal{K}$ is a growing function and since we have $1 - \mathcal{K}(1 - \alpha) \leq \alpha$.

Proposition 2.3. *Let $\mathcal{K}$ be strictly monotonic on a neighbourhood of a given probability $\alpha \in [0, 1]$. Then, the upper-orthant Kendall quantile and the upper-orthant quantile are linked by the following:*

$$\overline{\Psi}_{\alpha}(G) = \overline{\Psi}^{\mathcal{K}}_{1-\mathcal{K}(1-\alpha)}(G).$$

The proof is similar to the one of Proposition 2.2 and is thus omitted. We can also compare the level associated with the lower-orthant Kendall quantile to the level associated to the upper-orthant quantile, and the comparison can also be between the upper-orthant Kendall quantile and the lower-orthant quantile.

Proposition 2.4. *Let $\alpha, \alpha', \alpha'' \in [0, 1]$ and G be a probability distribution with no atoms.*

1. *If $\underline{\Psi}_{\alpha}(G) \cap \overline{\Psi}^{\mathcal{K}}_{\alpha''}(G) \neq \emptyset$, then $\alpha'' \geq \alpha$.*
2. *If $\overline{\Psi}_{\alpha'}(G) \cap \underline{\Psi}^{K}_{\alpha''}(G) \neq \emptyset$, then $\alpha'' \leq \alpha'$.*

The proof of Proposition 2.4 is reported in the Appendix.

The message conveyed by Propositions 2.2, 2.3, and 2.4 is that both the Kendal quantiles are a compromise between both orthant quantiles.

An interesting metric to compare the lower-orthant Kendall quantile and the lower-orthant quantile is given by the positive function $r : \alpha \in [0, 1] \mapsto K(\alpha) - \alpha$. This function r is the difference of probability associated with a same vector by the lower-orthant Kendall quantile and by the lower-orthant quantile, for a given level of probability. In other words, for a probability α, $\underline{\Psi}_{\alpha}(G)$ is a set of vectors corresponding to this probability α. For the same set of vectors, the lower-orthant Kendall quantile associates another level of probability, which is $K(\alpha)$ according to Proposition 2.2, and $r(\alpha)$ denotes this difference of probabilities.[1]

[1] For example, the Gumbel copula in Example 3.1 leads to $r(\alpha) = -\frac{\alpha \log(\alpha)}{\theta}$.

Generally, r can be linked to the Kendall rank correlation coefficient, known as Kendall's tau coefficient, as stated by the following proposition.

Proposition 2.5. *The average difference between the probabilities associated to the Kendall function and to the sole copula, for d-dimensional vectors and a continuous copula, is the following:*

$$\int_0^1 r(\alpha)d\alpha = (1-\tau)\left(\frac{1}{2}-\frac{1}{2^d}\right),$$

where τ is the Kendall rank correlation coefficient.

The proof of Proposition 2.5 is reported in the Appendix.

In the bivariate case, this average difference is $(1-\tau)/4$ which belongs to $[0,1/2]$, due to the fact that $\tau \in [-1,1]$. When d tends toward infinity, the average difference increases concomitantly with the dimension d, up to $(1-\tau)/2 \in [0,1]$. The case of the independent copula, for which $\tau = 0$, leads to an average r of $(1/2)-(1/2)^d$, whose value, $1/4$ for $d=2$, progressively increases with the dimension up to $1/2$. It confirms the analysis presented in Example 3.2. If we consider comonotonic coordinates, then $\tau = 1$ and the average r is equal to zero, whatever the dimension d. Graphically, it corresponds to a case where all the vectors dominated by a reference vector belong to the lower-left quadrant of this reference vector. The order implied by the orthant quantiles, which is partial in general, is total in this particular case, and there is no difference between the orthant and the Kendall quantiles. In the case of the opposite, if the coordinates are countermonotonic then $\tau = -1$ and the average r reaches its maximum, $1-(1/2)^{d-1}$, which goes from $1/2$, for $d=2$, to 1, when d goes toward infinity.

Additionally, we can quantify the difference between the probability associated to a vector by the upper-orthant method and by the upper-orthant Kendall method: $\overline{r}: \alpha \in [0,1] \mapsto \alpha - (1-\mathcal{K}(1-\alpha))$ which is a positive function. Proposition 2.6 states that the average twist $\overline{r}$ of the probability level between the upper-orthant quantile and the upper-orthant Kendall quantile is, in absolute value, exactly the same as the average twist r between the lower-orthant quantile and the standard Kendall quantile.

Proposition 2.6. *The average difference between the probabilities associated to the sole survival copula and to the Kendall function of the survival copula, for d-dimensional vectors and a continuous copula, is the following:*

$$\int_0^1 \overline{r}(\alpha)d\alpha = (1-\tau)\left(\frac{1}{2}-\frac{1}{2^d}\right),$$

where τ is the Kendall rank correlation coefficient.

The proof of Proposition 2.6 is reported in the Appendix.

In the framework of Proposition 2.1, where the upper-orthant and the lower-orthant Kendall quantiles have a non-empty intersection for a given probability level, the vectors belonging to both Kendall quantiles can thus be seen as a balanced compromise between lower- and upper-orthant quantiles. Indeed, in absolute value, they twist the probability associated with both in average over all the possible probability levels similarly, as stated in

Propositions 2.5 and 2.6. Nevertheless, for a particular level of probability, the lower-orthant Kendall quantile can be closer to one or to the other.

In addition to the average error, we can calculate an upper bound of the limit error $r(\alpha)$, when α tends to 0 or 1, and more widely of the probability distortion between the lower-orthant and the upper-orthant quantiles. We call probability distortion the difference of probability according to the lower-orthant and the upper-orthant approach for a vector x belonging to both sets: if $x \in \underline{\Psi}_\alpha(G) \cap \overline{\Psi}_{\alpha'}(G)$, the probability distortion is $\alpha' - \alpha$. It depends on α but also on the choice of x in $\underline{\Psi}_\alpha(G)$. To establish ideas, we will focus on a particular x corresponding to equal marginal probabilities: $x = (G_1^{-1}(u), ..., G_d^{-1}(u))$, where $u \in [0,1]$ is well chosen to have $x \in \underline{\Psi}_\alpha(G)$. So we have $u = \delta^{-1}(\alpha)$, where $\delta : v \in [0,1] \mapsto C(v, ..., v)$ is the diagonal section of the copula C associated to the joint distribution G. This choice is possible only if δ is invertible. As x is an element of the set $\underline{\Psi}_\alpha(G)$, the probability associated with x in the lower-orthant approach is α. By definition of the upper-orthant quantile, x is also an element of the set $\overline{\Psi}_{\alpha'}(G)$ with $\alpha' = 1 - \overline{G}(x) = 1 - \overline{G}\big(G_1^{-1}(\delta^{-1}(\alpha)), ..., G_d^{-1}(\delta^{-1}(\alpha))\big)$. If we note $\alpha \mapsto R(\alpha)$ the function of distortion of probability between the lower-orthant and the upper-orthant quantiles, then $R(\alpha) = \alpha' - \alpha$, which we can equivalently write:

$$R : \alpha \in [0,1] \mapsto 1 - \alpha - \overline{G}\big(G_1^{-1}(\delta^{-1}(\alpha)), ..., G_d^{-1}(\delta^{-1}(\alpha))\big). \tag{2.1}$$

In Proposition 2.7, we propose an upper bound for $R(\alpha)/\alpha$.

This distortion $R(\alpha)$ is directly linked to the notion of tail dependence. For a bivariate variable, the lower tail dependence λ_L is the following limit, if it exists: $\lim_{\alpha\to 0} \mathbb{P}(X_1 \leq G_1^{-1}(\alpha) | X_2 \leq G_2^{-1}(\alpha))$. Owing to Bayes' rule, this expression is symmetric in each component of the vector. Moreover, it only depends on the copula and not on the marginals. In higher dimension, one can define several lower tail dependence parameters corresponding to various choices of subsets $I_k \subset \{1, ..., d\}$ of size k: $\lambda_{L,I_k} = \lim_{\alpha\to 0} \mathbb{P}(X_i \leq G_i^{-1}(\alpha), \forall i \in I_k | X_j \leq G_j^{-1}(\alpha), \forall j \in \bar{I}_k)$ [13, 15]. Contrary to the case $d = 2$, this expression depends, in general, on the composition of I_k and not only on its cardinal. We will limit the study to a particular case of exchangeable copulas, for which $\lambda_{L,I_k} = \lambda_{L,I'_k}$ if $|I_k| = |I'_k|$. This assumption is in particular verified for Archimedean copulas [15], and we subsequently write $\lambda_{L,k}$ instead of λ_{L,I_k}. Symmetrically, one can define upper tail dependence parameters. For instance, for bivariate variables, it is $\lambda_U = \lim_{\alpha\to 1} \mathbb{P}(X_1 > G_1^{-1}(\alpha) | X_2 > G_2^{-1}(\alpha))$, if the limit exists.

Proposition 2.7. *Let R be defined as in equation (2.1) for an exchangeable copula C such that $\delta : v \in [0,1] \mapsto C(v, ..., v)$ is invertible. If all the lower and upper tail dependence parameters exist and are noted $\lambda_{L,k}$ and $\lambda_{U,k}$, for $k \in \{1, ..., d-1\}$, then the asymptotic difference $R(\alpha)$ between the probabilities associated to the lower- and upper-orthant quantiles is such that the following is applicable:*

$$\lim_{\alpha\to 0} \frac{R(\alpha)}{\alpha} \leq \frac{1}{\lambda_{L,d-1}} \sum_{k=1}^{d-1} \binom{d}{k} (1 - \lambda_{L,k}) \tag{2.2}$$

with equality only if the lower tail dependence parameters are all equal to 1, and

$$\lim_{\alpha\to 1} \frac{R(\alpha)}{1 - \alpha - R(\alpha)} \leq \frac{1}{\lambda_{U,d-1}} \sum_{k=1}^{d-1} \binom{d}{k} (1 - \lambda_{U,k}), \tag{2.3}$$

with equality only if the upper tail dependence parameters are all equal to 1.

The proof of Proposition 2.7 is reported in the Appendix.

Proposition 2.7 gives an upper bound for the difference of probability associated to the lower-orthant and the upper-orthant approaches. Naturally, the level of probability associated to the corresponding Kendall quantile is between lower-orthant and upper-orthant measures. In particular, $r(\alpha) \leq R(\alpha)$. This provides an upper bound for $r(\alpha)$. When $d = 2$, inequalities in Proposition 2.7 are simplified and upper bounds in equations (2.2) and (2.3) are $2(\lambda_L^{-1} - 1)$ and $2(\lambda_U^{-1} - 1)$ respectively. In special cases, if the lower tail dependence is strong, λ_L is close to 1 and the upper bound in equation (2.2) is close to 0: the lower-orthant, upper-orthant and Kendall quantile are very close in the lower tail. On the contrary, when the lower tail dependence is weak, λ_L is close to 0 and the upper bound in equation (2.2) tends to infinity: the lower-orthant, upper-orthant and Kendall quantiles are very disparate in the lower tail.

In Proposition 3.1, we use the result of Proposition 2.7 in the particular framework of Archimedean copulas with regularly varying generators.

3. KENDALL'S MULTIVARIATE QUANTILE FOR AN ARCHIMEDEAN COPULA

In this section, we assume that the multivariate distribution of the random vector X of dimension d is provided by an Archimedean copula C of generator ϕ:

$$C : (u_1, ..., u_d) \in [0,1]^d \mapsto \phi^{-1}\left(\sum_{j=1}^{d} \phi(u_j)\right).$$

It is a wide class of copulas which includes the following copulas: independent, Gumbel, Clayton, Frank, Joe, and Ali-Mikhail-Haq, among others. Moreover, this framework leads to simple expressions for the Kendall function, so that it is an interesting illustration of our theory.[2]

3.1. Theoretical results

In this Archimedean framework, we make some assumptions regarding ϕ.

Assumption 3.1. The generator ϕ is such that:

- $\phi : (0,1] \to [0,\infty)$,
- $\phi(1) = 0$,
- ϕ is strict, that is $\lim_{t\to 0^+} \phi(t) = \infty$,
- $(-1)^i (\phi^{-1})^{(i)}(x) > 0$ for all $1 \leq i \leq d$ and all $x \geq 0$,[3]
- $\lim_{t\to 0^+} \phi(t)^i (\phi^{-1})^{(i)}(\phi(t)) = 0$ for all $1 \leq i \leq d-1$.

[2] It is known that Archimedean copulas can be difficult to use in high dimensions for the purpose of estimation. Nevertheless, the vine approach permits bypassing this problem. Vine copulas are indeed based on nested bivariate copulas instead of a sole high-dimension copula [10, 25, 26]. Statistical selection techniques may help to truncate the vine so as to reduce the dimension of the problem in a relevant way [5]. For non-Archimedean copulas, semi-parametric methods may be used to estimate the Kendall function [35].

[3] In particular, ϕ is strictly decreasing and convex.

In Assumption 3.1, the fact that the generator is strict is intended to avoid that the zero curve of the copula may have a non-zero probability. The other assumptions are required by equation (3.1), which derives the Kendall distribution function in the Archimedean case with the help of the generator ϕ:

$$K : t \in (0,1] \mapsto t + \sum_{i=1}^{d-1} \frac{(-\phi(t))^i}{i!} (\phi^{-1})^{(i)}(\phi(t)), \tag{3.1}$$

where $f^{(i)}$ denotes the i-th derivative of f [2, 22]. We now apply this formula in two examples.

Example 3.1. The Gumbel copula is an Archimedean copula of parameter $\theta \geq 1$, generated by the function

$$\phi : t \mapsto (-\log(t))^\theta.$$

When $\theta = 1$, the Gumbel copula is equal to the independent copula. The inverse generator is $\phi^{-1}(x) = \exp(-x^{1/\theta})$. According to equation (3.1), if we consider the bivariate case, the Kendall function is as follows:

$$K : t \mapsto t - \frac{t \log(t)}{\theta}.$$

In Figure 2, we demonstrate how the Kendall function behaves when θ changes: the greater θ, the closer the Kendall function and the identity. In particular, when θ tends to infinity, K converges toward the identity, so that the lower-orthant Kendall quantile and the lower-orthant quantile are equal in this limit case.

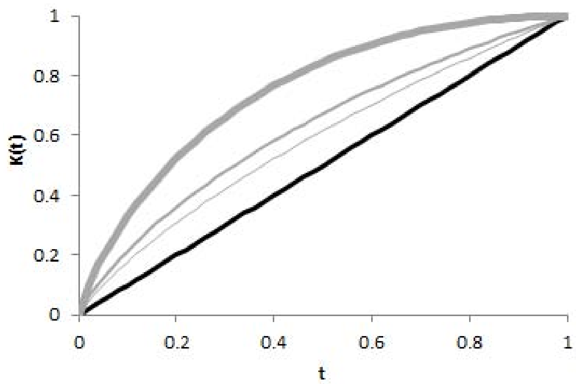

Figure 2: Kendall function (in grey) of the Gumbel copula for $\theta = 1$ (thick line), $\theta = 2$ (medium line), and $\theta = 3$ (thin line). The greater the difference between the Kendall function and the identity (in black, corresponding to $\theta \to \infty$), the greater the difference between the lower-orthant Kendall quantile and the lower-orthant quantile.

Example 3.2. The independent copula leads to easy formulas in higher dimensions. It is a particular case of the Gumbel copula with $\theta = 1$. According to equation (3.1), for a dimension $d \geq 2$, we get the following formula for K:

$$K : t \mapsto t \left(1 + \sum_{i=1}^{d-1} \frac{(-\log(t))^i}{i!} \right).$$

When d goes to infinity, $\sum_{i=1}^{d-1} \frac{(-\log(t))^i}{i!}$ tends toward $-1 + e^{-\log(t)} = -1 + 1/t$, for every $t \neq 0$.

Therefore, the limit behaviour of K, for $d \to \infty$, is a discontinuous function equal to 0 for $t = 0$ and equal to 1 everywhere else. It leads to the maximal difference possible between the lower-orthant Kendall quantile and the lower-orthant quantile. In Figure 3, we observe the Kendall function for various values of d.

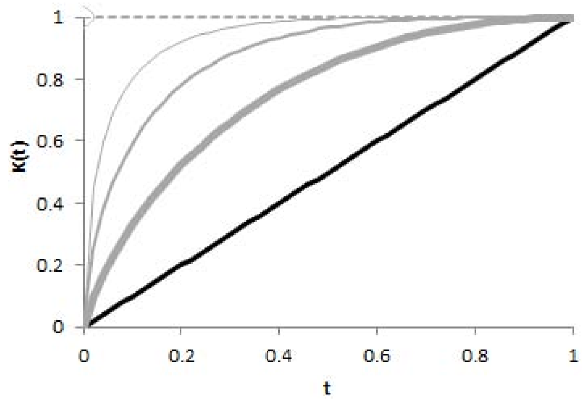

Figure 3: Kendall function (in grey) of the independent copula for $d = 2$ (thick line), $d = 3$ (medium line), $d = 4$ (thin line), and the limit case $d \to \infty$ (dotted line). The greater the difference between the Kendall function and the identity (in black), the greater the difference between the lower-orthant Kendall quantile and the lower-orthant quantile.

In Proposition 2.7, we saw that the probability distortion between the orthant and Kendall approaches was linked to the tail dependence. This is true, regardless of what the copula is. In the case of Archimedean copulas, we have an additional result allowing to link the probability distortion to the regular variations of the inverse generator. We first recall the definition of a regularly varying function:

Definition 3.1. A function f is regularly varying at 0, with index ρ, if

$$\forall s > 0, \lim_{x \to 0^+} \frac{f(sx)}{f(x)} = s^\rho.$$

Then, we note $f \in \mathcal{RV}_\rho(0)$.

Proposition 3.1. *If, for a given bivariate Archimedean copula, the inverse generator* $\phi^{-1} \in \mathcal{RV}_{-\rho}(0)$, *with* $\rho > 0$, *then the asymptotic difference* $R(\alpha)$ *between the probabilities associated to the lower- and upper-orthant quantiles, as defined in equation* (2.1), *is such that the following holds:*

$$\lim_{\alpha \to 0} \frac{R(\alpha)}{\alpha} \leq \sum_{k=1}^{d-1} \binom{d}{k} \left(d^{1/\rho} - (d-k)^{1/\rho} \right). \tag{3.2}$$

Proposition 3.1 is a direct consequence of Proposition 2.7 with Theorem 2.1 of lower tail coefficients in [15]. In equation (3.2), for bivariate variables, the upper bound is $2(2^{1/\rho}-1)$. The faster ϕ^{-1} varies, the greater $|\rho|$ and the closer to zero is the upper bound of equation (3.2). On the contrary, slowly varying inverse generators are associated with a big probability distortion between lower-orthant, upper-orthant and Kendall approaches.

3.2. Simulation experiments

In this section, we apply the methodology presented above and evaluate the Kendall quantile using various Archimedean copulas. More precisely, we first illustrate the probability transformation implied by the Kendall distribution. Then, we present and compare orthant and Kendall quantiles with the help of simulations.

The different existing versions of Archimedean copulas are intended to depict various types of tail dependence [17]. Figure 4 shows how the Kendall distribution evolves with the type of copula we are using. It is interesting to note that the shape of the Kendall distribution obtained with a particular copula is consistent with the nature of the tail dependence of the copula. In other words, if the copula captures an upper-tail dependence behaviour, that is if extreme positive events have a tendency to occur simultaneously while others are independent, the Kendall distribution inflexion point is located in the left tail of the distribution. It is for instance the case of the Gumbel or, more sharply, of the Joe copula. If the copula captures a lower-tail dependence behaviour, as it is the case for the Clayton copula, the Kendall distribution inflexion point is located in the right tail of the distribution. The Frank copula is more body-centred, i.e. events present in the body are more dependent than those present in the tails. In this case, the twist in the Kendall function is similar for lower and upper tails.

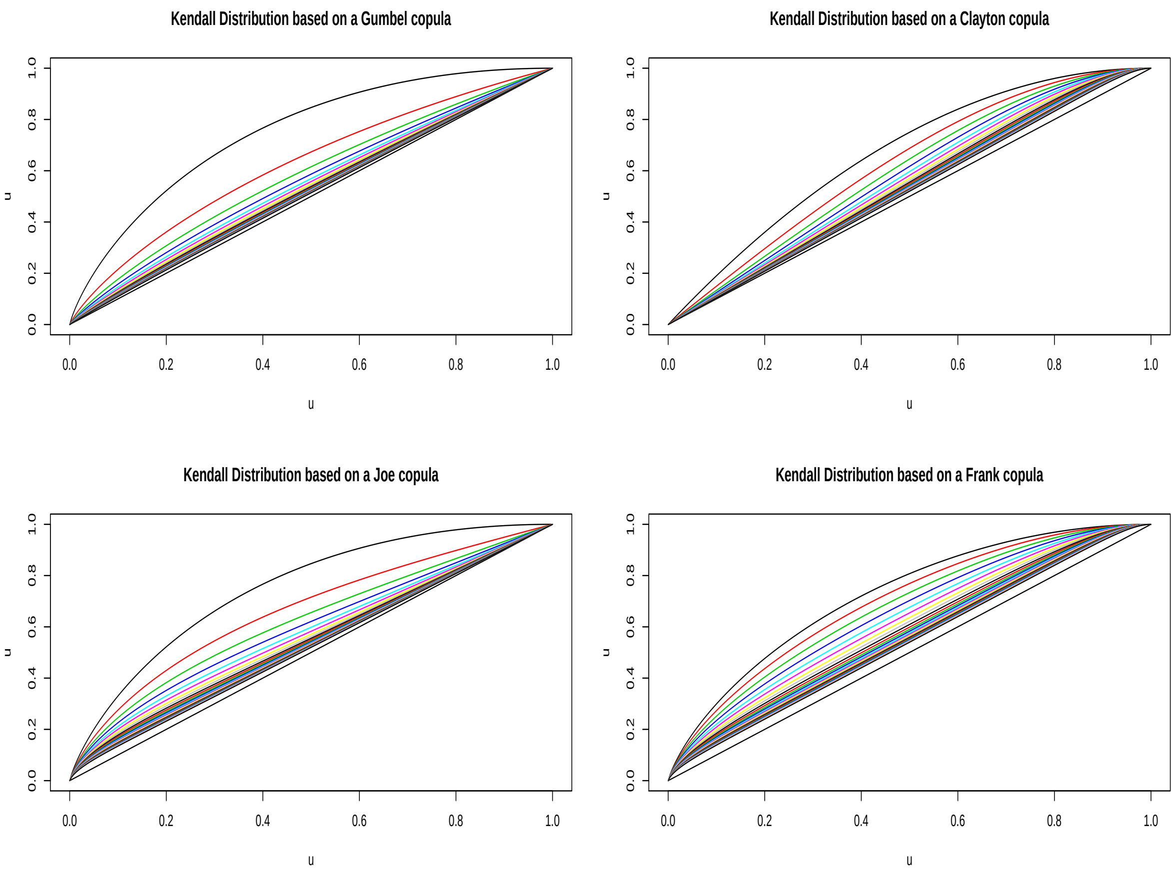

Figure 4: Kendall functions of four Archimedean copulas: Clayton, Gumbel, Frank, and Joe. The parameters of each copula are varying from 1 to 25. The bisector of the unit square corresponds to the orthant approach.

We now compare the quantiles obtained from the orthant and the Kendall approaches. Recall that multivariate quantiles will be represented by sets of vectors. To initiate our experimentation, we build a Clayton copula function, with parameter equal to 3. We used two lognormal marginal distributions, with the following sets of parameters: ($\mu = 5$, $\sigma = 2$) and ($\mu = 8$, $\sigma = 1.2$). Lower-orthant quantiles are obtained by calculating all the combinations of all pairs of margins providing the same bivariate probability. As analysed above, the Kendall distribution transforms the natural probabilities, taking into account the shape of the copula. This transformation allows us to calculate, in a similar fashion, the lower-orthant quantile and the Kendall quantile, transforming the lower-orthant percentile into the Kendall one.

Figure 5 shows lower-orthant, upper-orthant, and lower-orthant Kendall quantiles using the Clayton copula. In this figure, the dotted line, which is a set of vectors, represents the lower-orthant Kendall quantile, the continuous line located above[4] the dotted line represents the lower-orthant quantile, and the continuous line located below represents the upper-orthant quantile. In this figure, the quantile is given at the same percentile, but we see that the lower-orthant Kendall quantile is not equivalent to the lower-orthant quantile, as the Kendall function twists the probabilities. As a result, the lower-orthant quantile curve which is identical to the lower-orthant Kendall quantile curve of probability level α is below the lower-orthant quantile obtained at the same percentile α. The opposite is observed for upper-orthant quantiles. Indeed, Figure 5 shows the orthant quantiles obtained for α equal to 71%, as well as their Kendall equivalent, i.e. for $K(\alpha)$ also equal to 71%. To obtain this specific value for $K(\alpha)$, α has to be equal to 56%. In other results not displayed in figures, the lower-orthant Kendall quantile of probability level 86% or 98% is equal to the lower-orthant quantile of probability level 80% or 89%. This illustrates the probability distortion induced by the choice of copula and the Kendall function.

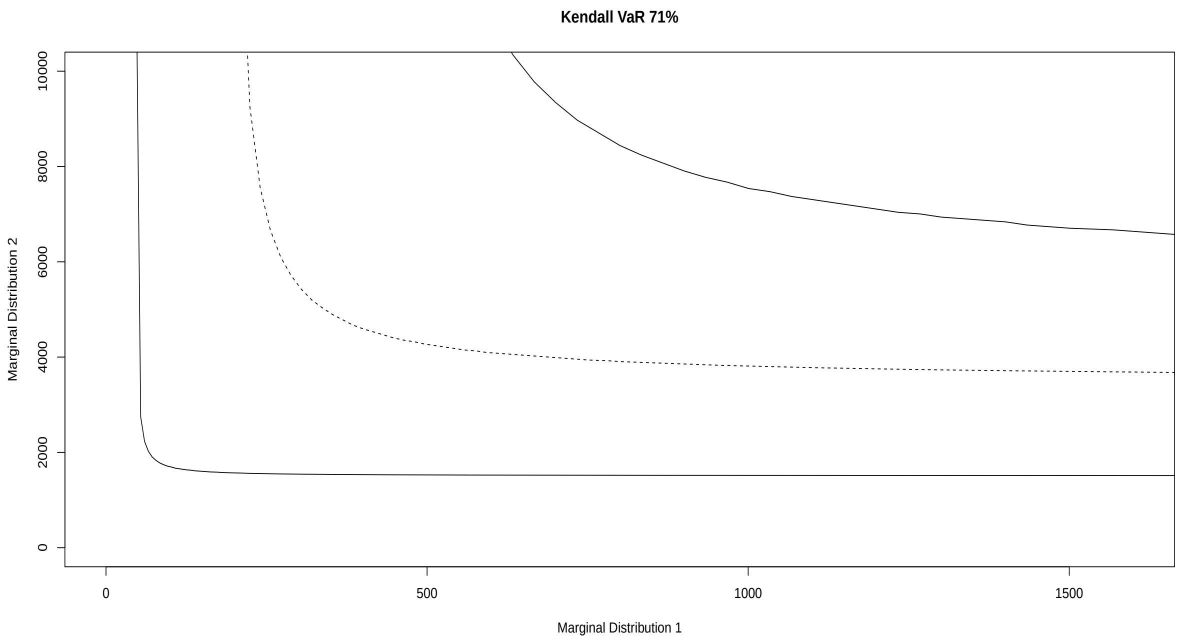

Figure 5: Lower-orthant Kendall quantile obtained using a Clayton copula with two different lognormal marginal distributions. The probability level is 71% for the Kendall quantile as well as for its lower-orthant (above) and upper-orthant (below) counterparts.

[4] The comparison between these sets of vectors must be understood in the sense of the lexicographical order for each pair of vectors.

4. CONCLUSION

In this paper, we focus on theoretical results on the concept of Kendall multivariate quantile, which allows for the calculation of a multivariate quantile using the probability transformation implied by the Kendall distribution. For instance, the Kendall distribution captures the intrinsic characteristics of the dependence architecture represented by the selected copula (non-linearity, upper or lower tail dependence etc.) and transfers it in one dimension. Therefore, the Kendall distribution allows the operation of a percentile transformation. We provide a simple relationship between the Kendall quantiles and the orthant quantiles, which allows to define the Kendall quantiles as a compromise between the bounds represented by both orthant quantiles. We also quantify the differences between the Kendall quantiles and the orthant quantiles, and link these asymptotic differences to tail dependence parameters.

For the orthant quantiles as well as for the Kendall quantiles, we observed that the non-linearity of the copulas implies that the sums of each set representing a given percentile are not constant. This phenomenon will have an important impact if any of these methodologies are used within financial institutions (for instance, banks or insurance companies), as if these approaches are used to evaluate the diversified capital pertaining to the various risks faced by them, the accurate value of the capital as well as the allocation of this capital will be problematic. Indeed, multiple sets of values will be representative of the same level of risks going from one end to the other.

In terms of applications, this result provides a variety of possible interpretations which will be the purpose of a companion paper, though as mentioned in introduction, it is an important topic considering the implications in terms of financial and climatic risks measurement.

A. APPENDIX – Proofs

Proof of Proposition 2.1: Let $L = \{y \in \mathbb{R}^d \mid K(G(y)) \leq \alpha\}$ and $U = \{y \in \mathbb{R}^d \mid \mathcal{K}(\overline{G}(y)) \leq 1-\alpha\}$. Let g be the probability measure associated with G. Then, by inversibility of the strictly monotonic K, $g(L) = \mathbb{P}[K(G(X)) \leq \alpha] = \mathbb{P}[G(X) \leq K^{-1}(\alpha)] = K(K^{-1}(\alpha)) = \alpha$ and $g(U) = 1-\alpha$. If $\overline{\Psi}_\alpha^{\mathcal{K}}(G) \cap \underline{\Psi}_\alpha^K(G) = \emptyset$, there are two possibilities: L and U are overlapping or they are not, but in both cases they have no boundary in common.

1. Either $L \cap U$ is an infinite and closed set. Since $\overline{\Psi}_\alpha^{\mathcal{K}}(G) \cap \underline{\Psi}_\alpha^K(G) = \emptyset$, every vector is at least in one of the two sets L and U. The probability measure of $L \cap U$ is then strictly positive, thanks to the assumptions regarding G, but it is contradictory to the fact that the measure of $\mathbb{R}^d$, equal to 1, is then $g(L) + g(U) - g(L \cap U) = \alpha + (1-\alpha) - g(L \cap U) = 1 - g(L \cap U)$.
2. Or the set S of vectors, defined by $S = \mathbb{R}^d \setminus (L \cup U)$, is an infinite and closed set. Then, similar to the previous case, $1 = g(L) + g(U) + g(S) = 1 + g(S)$, which is contradictory to the fact that its probability measure is expected to be strictly greater than 0. □

Proof of Proposition 2.2: Let $A = \{y \in \mathbb{R}^d \mid G(y) \geq \alpha\}$ and $B = \{y \in \mathbb{R}^d \mid K(G(y)) \geq K(\alpha)\}$.

- K is an increasing function, since it is a probability distribution function. Therefore, it follows immediately that $A \subset B$.
- The reciprocal inclusion does not hold in general. However, with the assumption of strict monotonicity of K in a neighbourhood V_α of α, the restriction of K to V_α is invertible. Let $y \in B$ and $\alpha' \in V_\alpha$ such that $\alpha' < \alpha$ (it does not exist if $\alpha = 0$ but this case is trivial). Therefore, $K(\alpha') < K(\alpha)$. Let's assume $y \notin A$. Then $G(y) < \alpha$. Two cases arise here. First, if $G(y) \leq \alpha'$, then $K(G(y)) \leq K(\alpha') < K(\alpha)$, which is contradictory to the assumption $y \in B$. Second, if $G(y) \in (\alpha', \alpha)$, then $G(y)$ is in V_α, so that $K(G(y)) < K(\alpha)$; the contradiction also holds. Therefore, the assumption $y \notin A$ was absurd, and we can conclude that $B \subset A$.
- Finally, $A = B$.

As a consequence, when considering the definition of both quantiles, we get the following:

$$\begin{aligned} \underline{\Psi}_\alpha(G) &= \partial\{y \in \mathbb{R}^d | G(y) \geq \alpha\} \\ &= \partial\{y \in \mathbb{R}^d | K(G(y)) \geq K(\alpha)\} \\ &= \underline{\Psi}_{K(\alpha)}^K(G). \end{aligned}$$
□

Proof of Proposition 2.4: We prove the second assertion, the proof for the first one being similar. Whatever y and z in $\underline{\Psi}_{\alpha''}^K(G)$, z cannot be in the interior of the upper orthant of y. Indeed, in such a case, $G(z) > G(y)$ or, if $G(z) = G(y)$, y would not be on the border of $A = \{x \in \mathbb{R}^d \mid K(G(x)) \geq \alpha''\}$, since all the lower orthant of z, in the interior of which is y, belongs to A.

Let $y \in \overline{\Psi}_{\alpha'}(G) \cap \underline{\Psi}_{\alpha''}^K(G)$. The probability measure of the upper orthant U of y is $1-\alpha'$. Since no vector of A is in the interior of U, the probability measure of A, which is equal to α'', is lower than the measure of the complement set of U, since G has no atoms. Therefore, $\alpha'' \leq 1 - (1-\alpha') = \alpha'$. □

Proof of Proposition 2.5: Kendall's tau and the Kendall function are linked by the following relation, for a continuous copula [22]:

$$\tau = \frac{2^d - 1 - 2^d \int_0^1 K(\alpha)d\alpha}{2^{d-1} - 1}.$$

Therefore:

$$\begin{aligned}\int_0^1 r(\alpha)d\alpha &= \int_0^1 (K(\alpha) - \alpha)d\alpha \\ &= \frac{2^d - 1 - (2^{d-1} - 1)\tau}{2^d} - \frac{1}{2} \\ &= (1-\tau)\left(\frac{1}{2} - \frac{1}{2^d}\right).\end{aligned}$$

□

Proof of Proposition 2.6: By a change of variable, we have the following:

$$\begin{aligned}\int_0^1 \overline{r}(\alpha)d\alpha &= \int_0^1 (\mathcal{K}(\alpha) - \alpha)d\alpha \\ &= \int_0^1 \mathcal{K}(\alpha)d\alpha - \frac{1}{2}.\end{aligned}$$

Moreover, we note that $\mathcal{K}$ is, according to Definition 2.2, the Kendall function corresponding to the survival distribution function. It can thus be written in terms of the Kendall's tau of the survival copula, $\overline{\tau}$:

$$\int_0^1 \mathcal{K}(\alpha)d\alpha = \frac{2^d - 1 - (2^{d-1} - 1)\overline{\tau}}{2^d}.$$

Besides, we know that the Kendall's tau of the survival copula is equal to the Kendall's tau of the copula itself [23], so that $\overline{\tau} = \tau$. This immediately leads to the result stated in the proposition. □

Proof of Proposition 2.7: First, we look for $u \in [0,1]$, such that $\alpha \underset{\alpha \to 0}{\sim} G(G_1^{-1}(u), ..., G_d^{-1}(u))$. By using the corresponding copula, this is equivalent to $\alpha \underset{\alpha \to 0}{\sim} C(u, .., u)$. By Bayes' rule, we thus should have

$$\alpha \underset{\alpha \to 0}{\sim} u\lambda_{L,d-1}.$$

Therefore, we define u as $\alpha/\lambda_{L,d-1}$.

Then, we define a vector $(x_1, ..., x_d) = (G_1^{-1}(u), ..., G_d^{-1}(u))$. Since $G(G_1^{-1}(u), ..., G_d^{-1}(u)) = \alpha$, this vector belongs to $\underline{\Psi}_\alpha(G)$. It also belongs to $\overline{\Psi}_{\alpha+R(\alpha)}(G)$, by the definition of $R(\alpha)$. Incidentally, the probability measure of the complement of the upper right quadrant of this vector $(x_1, ..., x_d)$, that is to say $\alpha + R(\alpha)$, is such that the following is applicable:

$$\begin{aligned}\alpha + R(\alpha) &= 1 - \mathbb{P}(X_1 > x_1, ..., X_d > x_d) \\ &= 1 - \mathbb{P}(G_1(X_1) > u, ..., G_d(X_d) > u) \\ &= 1 - \mathbb{P}(U_1 > u, ..., U_d > u),\end{aligned}$$

with $U_1, ..., U_d$ uniform variables linked by the same copula C as $X_1, ..., X_d$. Then

$$\begin{aligned}\alpha + R(\alpha) = & \ \mathbb{P}(U_1 \le u, \ ..., \ U_d \le u) \\ & + \sum_{i=1}^{d} \mathbb{P}(U_1 \le u, \ ..., \ U_i > u, \ ..., \ U_d \le u) \\ & + \sum_{i=1}^{d} \sum_{j=1, j \ne i}^{d} \mathbb{P}(U_1 \le u, \ ..., \ U_i > u, \ ..., \ U_j > u, \ ..., \ U_d \le u) \\ & + \cdots\end{aligned}$$

whose asymptote, as $\alpha \to 0$, is $\sum_{k=0}^{d-1} \binom{d}{k} P_{d-k}$, where $P_k = \mathbb{P}(U_1 \le u, \ ..., \ U_k \le u, \ U_{k+1} > u,$ $..., \ U_d > u)$, owing to the assumption that the lower tail dependence parameter is constant for a given size of I_k, whatever the composition of the subset I_k.

Last, we observe that $P_d = C(u, ..., u) \underset{\alpha \to 0}{\sim} \alpha$ and that, for $k \ge 1$,

$$\begin{aligned}P_k & = \mathbb{P}(U_{k+1} > u, \ ..., \ U_d > u | \ U_1 \le u, \ ..., \ U_k \le u) \ \mathbb{P}(U_1 \le u, \ ..., \ U_k \le u) \\ & = \Big(1 - \mathbb{P}(U_{k+1} \le u, \ ..., \ U_d \le u | \ U_1 \le u, \ ..., \ U_k \le u)\Big) \ \mathbb{P}(U_1 \le u, \ ..., \ U_k \le u),\end{aligned}$$

according to Bayes' rule. The asymptote of P_k, as $\alpha \to 0$, is thus $(1 - \lambda_{L,d-k}) \, \mathbb{P}(U_1 \le u, ...,$ $U_k \le u) \le (1 - \lambda_{L,d-k})u$ according to the upper Fréchet-Hoeffding bound, with equality only if $\lambda_{L,d-k} = 1$. The equality also holds if, when focusing on the Fréchet-Hoeffding inequality, the variables are comonotonic, which also implies that lower tail dependence parameters are equal to 1. This leads to equation (2.2).

Concerning equation (2.3), we observe that the upper tail dependence parameters of a random vector are equal to the lower tail dependence of the opposite of the vector. We can thus directly apply equation (2.2) to $(-X_1, ..., -X_d)$, for a probability level $\overline{\alpha}$, when it tends to 0, a difference function $\overline{R}$ and lower tail dependence parameters $\overline{\lambda}_{L,k} = \lambda_{U,k}$. For a vector $(-x_1, ..., -x_d)$ belonging to the lower-orthant quantile of level $\overline{\alpha}$ of the distribution of $(-X_1, ..., -X_d)$, if the probability measure of its upper orthant is α, then $\overline{\alpha} = 1 - (\alpha + R(\alpha))$, noting that this upper orthant is the lower orthant of $(x_1, ..., x_d)$. Moreover, $\overline{R}(\overline{\alpha})$ is 1 minus the probability measure of both the lower and upper orthants, therefore $\overline{R}(\overline{\alpha}) = 1 - \alpha - \overline{\alpha} =$ $R(\alpha)$. □

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