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# A New Bivariate Birnbaum–Saunders Type Distribution Based on the Skew Generalized Normal Model

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Abstract:

- It is well known that it is possible to represent a Birnbaum–Saunders variable as a relatively simple (and invertible) function of a standard normal random variable. Marginal transformations of this kind are applied in this paper to a bivariate distribution with generalized skew-normal conditionals (and normal marginals), to obtain a new bivariate Birnbaum–Saunders distribution. Parameter estimation for this model is implemented using an EM algorithm. A simulation study sheds light on the performance of the estimation strategy. Data from a cancer risk study is used to illustrate use of the model. For this data set, the new model exhibits better performance than does a competing skew-normal based model already discussed in the literature. Possible multivariate extensions of the new model are outlined.

Keywords:

- *Birnbaum–Saunders distribution; bivariate distribution; conditional specifications; EM algorithm.*

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## 1. INTRODUCTION

While there are many univariate models available for analysis of survival data, the same cannot be said for cases involving bivariate or, even more challenging, multivariate cases. In the univariate case there has been a flurry of recent activity focused on the Birnbaum–Saunders (BS) distribution (see, Birnbaum and Saunders [14]). A particularly attractive feature of the BS model is its representation as a monotone transformation of a standard normal variable.

Analogous distributions, which can be called distributions of the BS type, can be constructed by assuming that the normal random variable which is transformed to obtain a BS distributed variable, is replaced by a random variable with a different distribution. Recent papers dealing with the BS distribution and its close relatives include those of Balakrishnan and Kundu [13], Athayde [8], Bourguignon *et al.* [15], Arrué *et al.* [7], Carrasco *et al.* [16], Dasilva *et al.* [17] and Martínez-Flórez *et al.* [30]. See also the book by Leiva [27].

In particular we may consider replacing the normal component that is transformed to yield a BS variable by some skewed normal random variable. The traditional skew-normal (SN) distribution was introduced by Azzalini [9]. The skew-generalized-normal (SGN) distribution, introduced by Arellano-Valle *et al.* [2] (see also Arnold *et al.* [4]), includes an additional parameter. The SGN model can be viewed as a shape parameter mixture of SN distributions, where the shape parameter is endowed with a standard normal distribution. The model contains the SN model as particular case. The parameter space for the SGN distribution is $\{(\lambda, \theta) : -\infty < \lambda < \infty, \theta \geq 0\}$. As discussed in Arellano-Valle *et al.* [2], this model has identifiability problems which can be circumvented by restricting the parameter space, resulting in a distribution known as the skew-curved-normal (SCN) distribution (see, Gómez *et al.* [23]) . It is this SCN distribution that we propose to use instead of a standard normal distribution in order to arrive at a flexible extension of the BS model, which we will call a skew-curved-normal-BS (SCNBS) model.

While there has been much discussion of univariate variations on the BS theme, there is much less available for analyzing higher dimensional survival data. The present paper will make a contribution towards filling this gap. Our interest, then, is in the development of flexible bivariate and multivariate BS distributions. The bivariate case will be discussed in detail. Our goal is to seek models which have BS marginals and, in addition, will exhibit well behaved conditional structure. As will be seen, approaches involving conditional specification of joint distributions will prove to be fruitful. A convenient reference for discussion of conditionally specified models is Arnold *et al.* [6].

The paper will be organized as follows. Section 2 reviews the construction of the univariate BS distribution and its variants, and introduces the new bivariate SCNBS (BSCNBS) model and its main properties. Section 3 presents the inference for the BSCNBS model based on a classical approach and includes discussion of residuals for this model from both a marginal and a bivariate point of view. Section 4 includes a simulation study to assess the performance of the estimators obtained with the EM algorithm in finite samples and includes a real data application to illustrate the performance of the BSCNBS model. Multivariate extensions are not difficult to envision and are described briefly in Section 5. In Section 6 we present the main conclusions of the paper, together with discussion of related topics.

## 2. THE MODEL

In this Section, we introduce the BSCNBS model, where the conditional distributions are SCNBS and the marginal distribution are BS. Some properties of the model are discussed, as is a procedure to draw values from the model.

### 2.1. A background of related univariate distributions

A random variable $T$ is said to have a $\mathrm{BS}(\alpha,\beta)$ distribution if it can be represented in the form

$$T = \beta\left[\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right]^2, \tag{2.1}$$

where $Z \sim \mathrm{N}(0,1)$, i.e., the standard normal distribution. The density function of such a random variable is given by

$$f_T(t) = \phi\left(\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right]\right)\frac{t^{-3/2}(t+\beta)}{2\alpha\sqrt{\beta}}, \qquad t > 0, \tag{2.2}$$

where $\phi$ denotes the density function for the standard normal distribution. As mentioned in the introduction, we may replace $Z$ by some skewed normal random variable.

The density function of a random variable with SGN distribution is given by

$$f_U(u) = 2\phi(u)\Phi\left(\frac{\lambda u}{\sqrt{1+\theta u^2}}\right), \qquad u \in \mathbb{R}, \tag{2.3}$$

where $\Phi$ denotes the cumulative distribution function for the standard normal model. Note that if $\theta = 0$, this simplifies to the form of the traditional SN model. The parameter space for the SGN distribution is $\{(\lambda,\theta) : -\infty < \lambda < \infty, \theta \geq 0\}$, which is reduced to the SCN model for $\theta = \lambda^2$.
In addition, the SCNBS model is obtained considering $Z$ with SCN distribution in the transformation in equation (2.1). The associated density function for the SCNBS distribution is

$$f_T(t) = 2\phi(a)\Phi\left(\frac{\lambda a}{\sqrt{1+\lambda^2 a^2}}\right)A, \qquad t > 0,$$

where $a = \alpha^{-1}\left[(t/\beta)^{1/2} - (\beta/t)^{1/2}\right]$ and $A = t^{-3/2}(t+\beta)/(2\alpha\sqrt{\beta})$. We use the notation $\mathrm{SGN}(\lambda,\theta)$ to refer to a random variable with this density function. Those distributions are very relevant to the construction of the our proposal.

### 2.2. A Bivariate SCNBS distribution

Before describing the proposed bivariate distribution, which has Birnbaum Saunders marginals and SCNBS conditional distributions, we review two bivariate BS distributions that have been discussed in the literature. As will be seen, both the existing bivariate BS models and the one proposed in this paper are constructed by means of marginal transformations applied to bivariate densities with normal marginals and normal or skew-normal conditionals.

The first bivariate BS (BVBS) model was proposed by Kundu *et al.* [26]. They began by assuming that $(Z_1, Z_2)$ has a classical bivariate normal distribution with standard normal marginals and correlation $\rho$. They then defined, as in (2.1)

$$T_i = \beta_i \left[ \frac{\alpha_i Z_i}{2} + \sqrt{\left(\frac{\alpha_i Z_i}{2}\right)^2 + 1} \right]^2, \quad i = 1, 2. \tag{2.4}$$

It is evident that the bivariate random variable $(Z_1, Z_2)$ so defined has BS marginal distributions and BS conditional distributions.

In order to enhance the flexibility of the model (2.4), Lemonte *et al.* [28] proposed a more general model using a parallel construction which utilizes a distribution with SN conditionals introduced by Arnold *et al.* [3]. This bivariate density is of the form

$$f(z_1, z_2) = 2\phi(z_1)\phi(z_2)\Phi(\lambda z_1 z_2), \qquad (z_1, z_2) \in \mathbb{R}^2. \tag{2.5}$$

Lemonte *et al.* [28] then apply the marginal transformation (2.4) to a bivariate random variable $(Z_1, Z_2)$ with joint density of the form (2.5). Since the Arnold *et al.* [3] density (2.5) is readily verified to have standard normal marginals and SN conditionals, it follows that the Lemonte *et al.* [28] distribution will have BS marginals and skew-normal-Birnbaum–Saunders (SNBS) conditionals. The bivariate SNBS (BSNBS) density studied by Lemonte *et al.* [28] is

$$f_{T_1,T_2}(t_1, t_2) = 2\phi(a_1)\phi(a_2)\Phi(\lambda a_1 a_2) A_1 A_2, \qquad (t_1, t_2) \in \mathbb{R}_+^2,$$

where $a_j = a_j(\alpha_j, \beta_j) = \alpha_j^{-1}\left[\sqrt{t_j/\beta_j} - \sqrt{\beta_j/t_j}\right]$ and

$$A_j = A_j(\alpha_j, \beta_j) = t_j^{-3/2}(t_j + \beta_j)/(2\alpha_j\sqrt{\beta_j}), \quad j = 1, 2.$$

In addition to the density (2.5), Arnold *et al.* [4] proposed a more general two parameter model of the form

$$f(z_1, z_2) = 2\phi(z_1)\phi(z_2)\Phi\left(\frac{\lambda z_1 z_2}{\sqrt{1 + \theta z_1^2 z_2^2}}\right), \qquad (z_1, z_2) \in \mathbb{R}_+^2, \tag{2.6}$$

where $\lambda \in (-\infty, \infty)$ and $\theta \in [0, \infty)$. In this paper, we will consider the case $\theta = \lambda^2$. This distribution then has standard normal marginals and has generalized skew-normal conditional distributions. Specifically we have, if $(Z_1, Z_2)$ has density (2.6) with $\theta = \lambda^2$ then

$$Z_1 | Z_2 = z_2 \sim \mathrm{SCN}(\lambda z_2) \tag{2.7}$$

and

$$Z_2 | Z_1 = z_1 \sim \mathrm{SCN}(\lambda z_1). \tag{2.8}$$

It is to this joint distribution that we apply the marginal transformations (2.4) to obtain a flexible bivariate distribution with BS marginals that will be the focus of the remainder of this paper. The resulting joint density is of the form

$$f_{T_1,T_2}(t_1, t_2) = 2\phi(a_1)\phi(a_2)\Phi\left(\frac{\lambda a_1 a_2}{\sqrt{1 + \lambda^2 a_1^2 a_2^2}}\right) A_1 A_2, \qquad (t_1, t_2) \in \mathbb{R}_+^2. \tag{2.9}$$

If a random variable $(T_1, T_2)$ has its density of the form (2.9) then we write $(T_1, T_2) \sim \mathrm{BSCNBS}(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)$.

For the BSCNBS$(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)$ distribution, we have the following properties:

**(1)** $T_i \sim$BS$(\alpha_i, \beta_i), \quad i = 1, 2$.

**(2)** $T_1 \,|\, T_2 = t_2 \sim$SCNBS$(\alpha_1, \beta_1, \lambda a_2)$ and $T_2 \,|\, T_1 = t_1 \sim$SCNBS$(\alpha_2, \beta_2, \lambda a_1)$.

**(3)** $(c_1T_1, c_2T_2) \sim$BSCNBS$(\alpha_1, \alpha_2, c_1\beta_1, c_2\beta_2, \lambda), c_i > 0,\ i = 1, 2$.

**(4)** $(c_1T_1, c_2T_2^{-1}) \sim$BSCNBS$(\alpha_1, \alpha_2, c_1\beta_1, c_2\beta_2^{-1}, -\lambda), c_i > 0,\ i = 1, 2$.

**(5)** $(c_1T_1^{-1}, c_2T_2) \sim$BSCNBS$(\alpha_1, \alpha_2, c_1\beta_1^{-1}, c_2\beta_2, -\lambda), c_i > 0,\ i = 1, 2$.

**(6)** $(c_1T_1^{-1}, c_2T_2^{-1}) \sim$BSCNBS$(\alpha_1, \alpha_2, c_1\beta_1^{-1}, c_2\beta_2^{-1}, \lambda), c_i > 0,\ i = 1, 2$.

**(7)** And, going back, if $Z_i = \alpha_i^{-1}\left[\sqrt{T_i/\beta_i} - \sqrt{\beta_i/T_i}\right],\ i = 1, 2,$ then $Z_1 \,|\, Z_2 = z_2 \sim$ SCN$(\lambda z_2)$ and $Z_2 \,|\, Z_1 = z_1 \sim$SCN$(\lambda z_1)$. Using Proposition 10 in Arellano-Valle *et al.* [2], the first conditional distribution is equivalent to $Z_1 \,|\, Z_2 = z_2, U = u \sim$SN$(u)$ and $U \sim$N$(\lambda z_2, \lambda^2 z_2^2)$. Similarly, $Z_2 \,|\, Z_1 = z_1, U = u \sim$SN$(u)$ and $U \sim$N$(\lambda z_1, \lambda^2 z_1^2)$.

Parts (1) and (2) are obtained directly from the definition of the distribution and the results given in (3) to (6). are obtained using appropriate transformations in the density given in (2.9). The representation given in (7) of the conditional distributions are useful for simulation purposes, as illustrated in the following Sub-Section. Figure 1 shows the contour levels and Figure 2 shows the density for BSCNBS model for some combinations of the parameters. Note that the contours exhibit different and a greater variety of shapes than the BSNBS model.

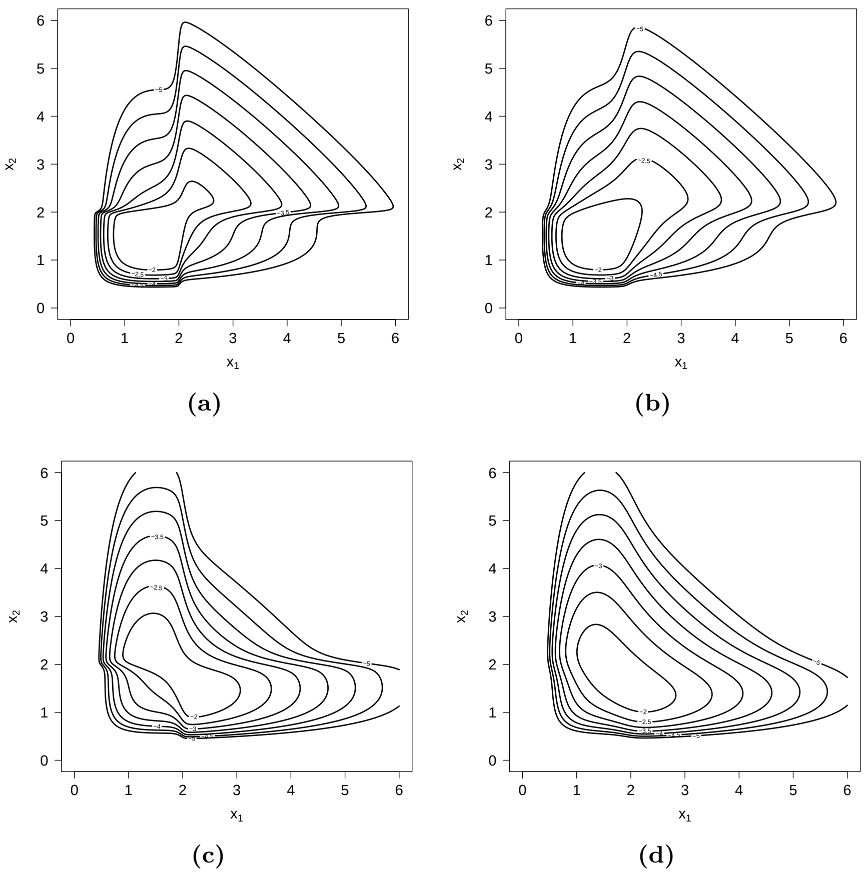



**Figure 1**: Contours levels for BSCNBS$(\alpha_1 = 0.5, \alpha_2 = 0.5, \beta_1 = 2, \beta_2 = 2, \lambda)$ distribution considering: (a) $\lambda = 7$; (b) $\lambda = 2.5$; (c) $\lambda = -3.5$ and; (d) $\lambda = -1$.

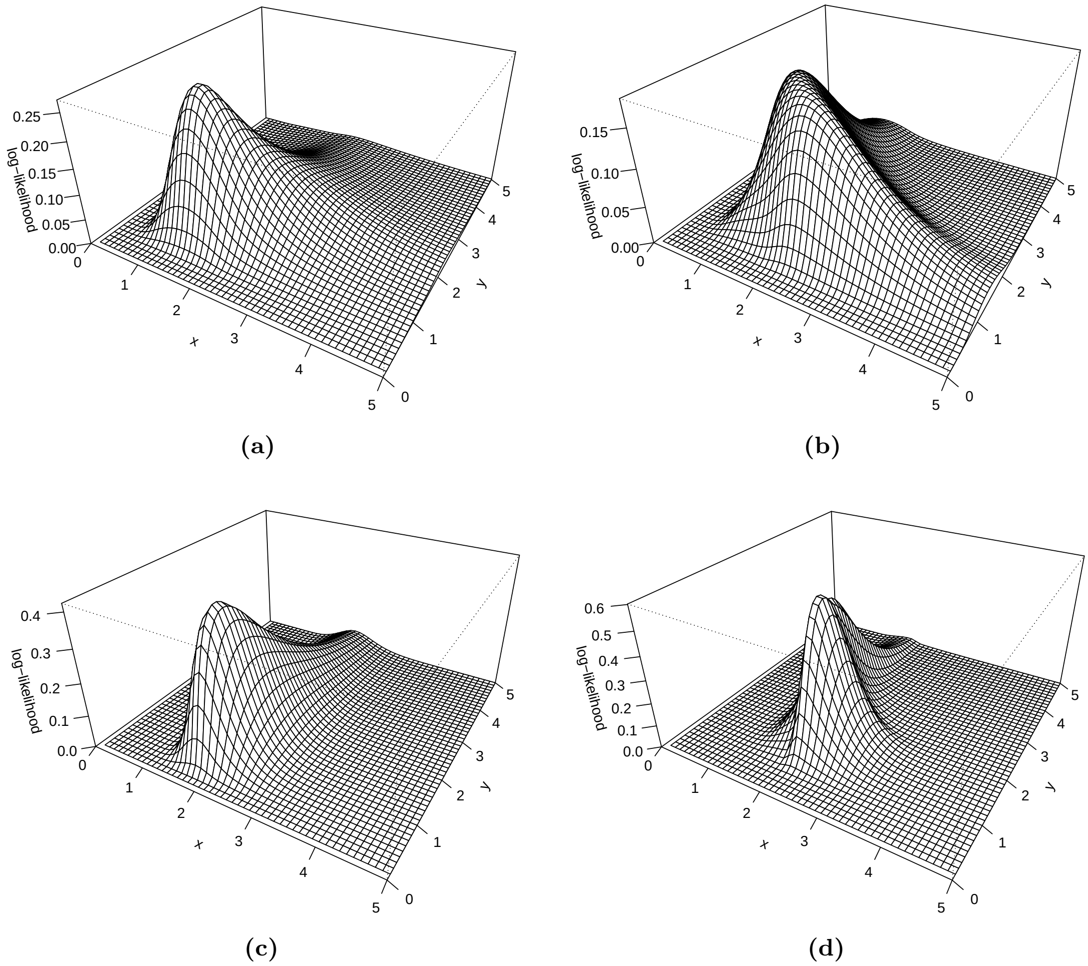



**Figure 2**: Density for BSCNBS($\alpha_1, \alpha_2, \beta_1 = 2, \beta_2 = 2, \lambda$) distribution considering: (a) $\alpha_1 = \alpha_2 = 0.5$ and $\lambda = 1$; (b) $\alpha_1 = \alpha_2 = 0.5$ and $\lambda = -1$; (c) $\alpha_1 = 0.2$, $\alpha_2 = 0.7$ and $\lambda = 0.5$; (d) $\alpha_1 = 0.2, \alpha_2 = 0.4$ and $\lambda = -2$.

The non-singularity of the Fisher information matrix (FIM) for $\lambda = 0$ is verified in the Appendix. This point is very important because $\lambda = 0$ represents the case where the model is reduced to two independent BS variates. Therefore, the non-singularity of the FIM allows to apply usual hypothesis test such as maximum likelihood ratio, score and Wald tests to decide between BSCNBS and independent BS variates.

## 2.3. Drawn values from BSCNBS distribution

The fact that this model has conditional and marginal distributions in closed form allows one to draw values from the distribution BSCNBS in a relatively simple way using the following Algorithm 1.

**Algorithm 1** Simulate a value from the BSCNBS$(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)$ distribution.

1: Draw $Z_1 \sim$N$(0, 1)$.
2: Draw $Z_2 \,|\, Z_1 = z_1 \sim$ SCN$(\lambda z_1)$.
  2.1: Draw $U \sim$ N$(\lambda z_1, \lambda^2 z_1^2)$.
  2.2: Draw $Z_2 \,|\, U = u \sim$ SN$(u)$.
    2.2.1: Draw $V_1, V_2 \sim$ N$(0, 1)$ (independent).
    2.2.2: Let $Z_2 = \left(\frac{u}{\sqrt{1+u^2}}\right)|V_1| + \left(\frac{1}{\sqrt{1+u^2}}\right)V_2$.
3: Make the usual BS-type transformation

$$T_j = \beta_j \left[ \frac{\alpha_j Z_j}{2} + \sqrt{\left(\frac{\alpha_j Z_j}{2}\right)^2 + 1} \right]^2, \quad j = 1, 2.$$

**Remark 2.1.** This algorithm requires only obvious minor modification to produce a drawn value from the Lemonte *et al.* [28] distribution.

## 3. ESTIMATION

In this Section we consider the parameter estimation problem based on a classical approach. An EM algorithm is developed for this problem. Initial values for such procedures and two kind of residuals also are presented.

### 3.1. Estimation based on the EM algorithm

For the BSCNBS model, the log-likelihood function for $\boldsymbol{\psi} = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)$ in a random sample $\boldsymbol{t} = \boldsymbol{t}_1, \boldsymbol{t}_2, ..., \boldsymbol{t}_n$ (where $\boldsymbol{t}_i = (t_{i1}, t_{i2})$) is, up to a constant, given by

$$\begin{aligned} \ell(\boldsymbol{\psi}) = & -\frac{n}{2}\sum_{j=1}^{2}\left[\frac{1}{\alpha_j^2}\left(\frac{\bar{s}_j}{\beta_j} + \bar{r}_j\beta_j - 2\right) + 2\log(\alpha_j) + \log(\beta_j)\right] + \sum_{j=1}^{2}\sum_{i=1}^{n}\log(\beta_j + t_{ij}) \\ & + \sum_{i=1}^{n}\log\Phi\left(\frac{\lambda a_{i1}a_{i2}}{\sqrt{1+(\lambda a_{i1}a_{i2})^2}}\right), \end{aligned} \tag{3.1}$$

where

$$\bar{s}_j = \frac{1}{n}\sum_{i=1}^{n} t_{ji} \qquad \text{and} \qquad \bar{r}_j = \left(\frac{1}{n}\sum_{i=1}^{n} t_{ji}^{-1}\right)^{-1}, \qquad j = 1, 2.$$

and $a_{ij} = \alpha_j^{-1}\left[(t_{ij}/\beta_j)^{1/2} - (\beta_j/t_{ij})^{1/2}\right]$. The maximum likelihood (ML) estimation requires the maximization of eq. (3.1) in relation to $\boldsymbol{\psi}$. However, such a procedure can be difficult to implement because it involves maximization over a parameter space of dimension 5. For this reason, we discuss the development of an EM-type algorithm (Dempster *et al.* [18]) for this problem. This algorithm has been applied satisfactorily in BS models and their extension by Balakrishnan *et al.* ([10],[11],[12]), Pradhan and Kundu [34], Reyes *et al.* ([36], [37]), Romeiro *et al.* [38], among others.

A hierarchical representation of the BSCNBS model is given by

$$
\begin{aligned}
T_{1i} \,|\, T_{2i} = t_{2i}, U_i = u_i &\sim \mathrm{SCNBS}(\alpha_1, \beta_1, u_i) \\
U_i \,|\, T_{2i} = t_{2i} &\sim \mathrm{N}(\lambda a_{2i}, \lambda^2 a_{2i}^2) \\
T_{2i} &\sim \mathrm{BS}(\alpha_2, \beta_2), \quad i = 1, ..., n.
\end{aligned} \tag{3.2}
$$

Let $\boldsymbol{t}$ and $\boldsymbol{u} = (u_1, ..., u_n)$ the observed values and the unobserved latent values, respectively. The complete data set then is $\mathbf{t}_c = (\mathbf{t}^T, \mathbf{u}^T)^T$. Using (3.2), the log-likelihood of the complete data set is given by

$$
\begin{aligned}
\ell_c(\boldsymbol{\psi} \,|\, \boldsymbol{t}_c) \;\propto\; & \sum_{j=1}^{2}\sum_{i=1}^{n}\left[-\frac{1}{\alpha_j^2}\left(\frac{t_{ji}}{\beta_j} + \frac{\beta_j}{t_{ji}} - 2\right) + \log(t_{ji} + \beta_j) - \log(\alpha_j) - \frac{1}{2}\log(\beta_j)\right] \\
& - \, n\log\lambda + n\log\alpha_2 - \frac{1}{2}\sum_{i=1}^{n}\left[\log\left(\frac{t_{1i}}{\beta_1} + \frac{\beta_1}{t_{1i}} - 2\right) + \log\Phi(u_i a_{1i})\right. \\
& \left. - \, \frac{1}{2\lambda^2 a_{2i}^2}(u_i^2 + \lambda^2 a_{2i}^2 - 2\lambda u_i a_{2i})\right].
\end{aligned}
$$

Let $\widehat{u}_i^k = E(U_i^k \,|\, t_i, \boldsymbol{\psi} = \widehat{\boldsymbol{\psi}})$, $k = 1, 2$. Note that

$$
\begin{aligned}
f(u_i \,|\, t_{1i}, t_{2i}, \boldsymbol{\psi}) &\propto f(t_{1i} \,|\, t_{2i}, u_i, \boldsymbol{\psi}) f(u_i \,|\, \boldsymbol{\psi}), \\
&\propto \phi\left(\frac{u_i - \lambda a_{2i}}{\lambda a_{2i}}\right)\Phi(u_i a_{1i}), \qquad u_i \in \mathbb{R}.
\end{aligned}
$$

Defining $C_{ki} = \int_{-\infty}^{\infty} u_i^k \phi\big((\lambda a_{2i})^{-1}(u_i - \lambda a_{2i})\big)\Phi(u_i a_{1i})\mathrm{d}u_i$, we have $\widehat{u}_i^r = C_{ri}/C_{0i}$, $r = 1, 2$. Note that the existence of $C_{ri}$ is guaranteed since

$$
\int_{-\infty}^{\infty} u_i^r \phi\left(\frac{u_i - \lambda a_{2i}}{\lambda a_{2i}}\right)\Phi(u_i a_{1i})\mathrm{d}u_i \le \lambda a_{2i}\int_{-\infty}^{\infty} u_i^r \frac{1}{\lambda a_{2i}}\phi\left(\frac{u_i - \lambda a_{2i}}{\lambda a_{2i}}\right)\mathrm{d}u_i < \infty.
$$

In this manner, the estimation process for this model, using the EM algorithm, may be described as follows in Algorithm 2.

The process is repeated iteratively until convergence is attained. For instance, we considered $\epsilon = 10^{-4}$.

**Remark 3.1.**

**i)** The application of the ECM algorithm requires only uni-dimensional procedures, instead of the original problem which required a maximization of dimension 5.

**ii)** The integrations involved in the E-step can be easily computed in the R software (R Core Team, [35]) with the `integrate` function.

**iii)** The CM steps of the algorithm explicitly update $\lambda$, $\alpha_1$ and $\alpha_2$ and require the solution of a non-linear equation for $\beta_1$ and $\beta_2$. Such equations can be solved using the `uniroot` function in the R software.

**Algorithm 2** Provide the ML estimates based on the EM algorithm for the BSCNBS distribution.

Set initial values $\boldsymbol{\psi}^{(0)} = (\alpha_1^{(0)}, \alpha_2^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}, \lambda^{(0)})$
$k \leftarrow 0$
dif $\leftarrow 1$
**while** dif $> \epsilon$ **do**
  $i \leftarrow 1$
  **while** $i \leq n$ **do**
    (E-step) Compute the expected values for $U_i$ and $U_i^2$
    $\widehat{u}_i^{(k+1)} = \frac{C_{1i}^{(k)}}{C_{0i}^{(k)}}$ and $\widehat{u}_i^2 = \frac{C_{2i}^{(k)}}{C_{0i}^{(k)}}$.
    $i \leftarrow i+1$
  **end while**
  (CM-step I) Update $\lambda$

$$\widehat{\lambda}^{(k+1)} = \frac{\sum_{i=1}^{n} u_i^{(k+1)}}{\sum_{i=1}^{n} a_{2i}^{(k)}}.$$

  $j \leftarrow 1$
  **while** $j \leq 2$ **do**
    (CM-step II) Update $\alpha_j$
    $\widehat{\alpha}_j^{2(k+1)} = \frac{S_j}{\widehat{\beta}_j^{(k)}} + \frac{\widehat{\beta}_j^{(k)}}{R_j} - 2$
    (CM-step III) Update $\beta_j$ as the solution of the equation
    $\widehat{\beta}_j^{2(k+1)} - \widehat{\beta}_j^{(k+1)}\left[K_j(\widehat{\beta}_j^{(k+1)}) + 2R_j\right] + R_j\left[K_j(\widehat{\beta}_j^{(k+1)}) + S_j\right] = 0$
    where $K_j(x) = \left\{\frac{1}{n}\sum_{i=1}^{n}(x + t_{ji})\right\}^{-1}$.
    $j \leftarrow j+1$
  **end while**
  $\boldsymbol{\psi}^{(k+1)} = (\alpha_1^{(k+1)}, \alpha_2^{(k+1)}, \beta_1^{(k+1)}, \beta_2^{(k+1)}, \lambda^{(k+1)})$.
  dif$= ||\boldsymbol{\psi}^{(k+1)} - \boldsymbol{\psi}^{(k)}||$, where $||x||$ denotes the Euclidean norm of the vector $x$.
  $k \leftarrow k+1$
**end while**

## 3.2. Initial values of the algorithm

Since $T_j \sim$BS$(\alpha_j, \beta_j)$, we can use modified moment estimators of the BS distribution to estimate $\alpha_j$ and $\beta_j$, $j = 1, 2$ (see Ng *et al.* [33]). Thus, the initial values for those parameters are

$$\widehat{\alpha}_j^{(0)} = \sqrt{2\left(\sqrt{\frac{\bar{s}_j}{\bar{r}_j}} - 1\right)} \quad \text{and} \quad \widehat{\beta}_j^{(0)} = \sqrt{\bar{s}_j \bar{r}_j}, \qquad j = 1, 2. \tag{3.3}$$

With those values, we can construct a profile version of (3.1) for $\lambda$ and choose the value of $\lambda$ that maximizes that function.

### 3.3. Residuals for the BSCNBS model

In order to check the goodness of fit of the BSCNBS model, we evaluate the marginal quantile residuals (MQR; Dunn and Smyth, [19]) and the bivariate quantile residuals (BQR; Kalliovirta, [24]). Such theoretical residuals are given by

$$r_{ij}^{\text{MQR}} = a_{ij} \quad \text{and} \quad r_i^{\text{BQR}} = \Phi^{-1}(\nu_i(1-\log\nu_i)),$$

respectively, for $i = 1, ..., n$, $j = 1, 2$, where

$$\nu_i = \Phi(a_{i1}) \int_{-\infty}^{a_{i2}} 2\phi(u)\Phi\left(\frac{\lambda a_{1i} u}{\sqrt{1+\lambda^2 a_{1i}^2 u^2}}\right) du, \quad i = 1, ..., n,$$

where $a_{ij} = \alpha_j^{-1}\left[(t_{ij}/\beta_j)^{1/2} - (\beta_j/t_{ij})^{1/2}\right]$, $i = 1, ..., n$, $j = 1, 2$. The observed MQR and BQR (say $\widehat{r}_{i1}^{\text{MQR}}$, $\widehat{r}_{i2}^{\text{MQR}}$ and $\widehat{r}_i^{\text{BQR}}$) are the theoretical MQR and BQR, respectively, evaluated as functions of the estimated parameters.
If the BS model is correctly specified for the $j$-th variable, then $\widehat{r}_{1j}^{\text{MQR}}, ..., \widehat{r}_{nj}^{\text{MQR}}$ has a $\mathrm{N}(0,1)$ distribution.
In a similar way, if the BSCNBS model is correctly specified for the two variables (jointly), then $r_1^{\text{BQR}}, ..., r_n^{\text{BQR}}$ has a $\mathrm{N}(0,1)$ distribution. Such hypothesis can be tested considering, for instance, the Kolomogorov–Smirnov (KS; Kolmogorov, [25]) test.

## 4. NUMERICAL RESULTS

In this Section we present details computational aspects used for this work. We also present a simulation study to assess the performance of the ML estimators obtained by the ECM algorithm discussed previously and a real data illustration in order to show the performance of the BSCNBS model. For the sake of comparison, we also consider the BSNBS model of Lemonte *et al.* [28] and the BVBS model of Kundu *et al.* [26].

### 4.1. Computational aspects

All the programs used in this work were run in R Core Team [35] in a computer with processor Intel(R) Core(TM) i7-7700HQ CPU 2.8GHz with 16 GB of RAM memory. The used packages for the development were the `VGAM` package [40] which includes some functions related to the BS model, the `mvtnorm` package [22] which includes some functions related to the multivariate normal model, the `goftest` package [20] including some functions related to goodness-of-fit tests and the `DAAG` package [29] which include the data used in the application presented in subsection 4.3. Codes for the application are included as supplementary material.

## 4.2. Simulation study

In this Section we report on a small simulation study with the objective of verifying that the EM-based estimation procedure is capable of recovering, approximately, the parameter values used to simulate data sets from the model (2.9). To simulate the data sets, we use the procedure described in Section 2.1. Then we use as the initial values, those discussed in Section 3.1 together with the EM algorithm outlined in Section 3. In particular we consider the parameter values $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ in all cases while $\lambda$ ranges over the set $\{-5, -2, -1, -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75, 1, 2, 5\}$. In addition, we consider three sample sizes: $n = 100$, $n = 250$ and $n = 500$. In each case we make 1,000 replications and calculate the mean absolute bias (AB), the mean of the standard errors ($\mathrm{SE}_1$), the standard deviation of the estimated parameters ($\mathrm{SE}_2$), and the coverage proportion (CP) of the nominal 95% intervals for the parameters. The results are presented in Tables 1 and 2. In these Tables we see that the biases of the estimates of $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$ are negligible in all cases considered. However, the bias of the estimates of $\lambda$ can be considerable in cases in which the true value of $\lambda$ is far from 0. Although, as expected, the biases decrease as sample size increases.

**Table 1**: Simulation study for the BSCNBS model.

| Case | Parameter | $n=100$ | | | | $n=250$ | | | | $n=500$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | AB | $\mathrm{SE}_1$ | $\mathrm{SE}_2$ | CP | AB | $\mathrm{SE}_1$ | $\mathrm{SE}_2$ | CP | AB | $\mathrm{SE}_1$ | $\mathrm{SE}_2$ | CP |
| $\lambda = -5.00$ | $\alpha_1$ | −0.007 | 0.070 | 0.073 | 0.940 | −0.004 | 0.045 | 0.044 | 0.952 | −0.001 | 0.032 | 0.031 | 0.956 |
| | $\alpha_2$ | −0.005 | 0.071 | 0.072 | 0.936 | −0.004 | 0.045 | 0.047 | 0.935 | −0.001 | 0.032 | 0.033 | 0.941 |
| | $\beta_1$ | 0.001 | 0.067 | 0.072 | 0.908 | −0.001 | 0.042 | 0.043 | 0.943 | 0.000 | 0.030 | 0.031 | 0.939 |
| | $\beta_2$ | 0.005 | 0.069 | 0.072 | 0.921 | 0.002 | 0.042 | 0.044 | 0.924 | 0.001 | 0.029 | 0.030 | 0.931 |
| | $\lambda$ | −1.758 | 5.891 | 5.391 | 0.863 | −1.075 | 3.030 | 3.371 | 0.935 | −0.637 | 1.847 | 2.510 | 0.943 |
| $\lambda = -2.00$ | $\alpha_1$ | −0.006 | 0.070 | 0.069 | 0.939 | −0.004 | 0.045 | 0.044 | 0.959 | −0.001 | 0.032 | 0.032 | 0.939 |
| | $\alpha_2$ | −0.006 | 0.070 | 0.070 | 0.939 | −0.003 | 0.045 | 0.044 | 0.947 | −0.002 | 0.032 | 0.032 | 0.949 |
| | $\beta_1$ | −0.001 | 0.079 | 0.079 | 0.943 | 0.002 | 0.050 | 0.051 | 0.945 | 0.000 | 0.035 | 0.035 | 0.944 |
| | $\beta_2$ | 0.004 | 0.079 | 0.082 | 0.925 | 0.003 | 0.050 | 0.052 | 0.936 | 0.001 | 0.035 | 0.036 | 0.938 |
| | $\lambda$ | −0.725 | 2.052 | 2.679 | 0.904 | −0.263 | 0.875 | 1.071 | 0.931 | −0.128 | 0.543 | 0.572 | 0.956 |
| $\lambda = -1.00$ | $\alpha_1$ | −0.009 | 0.070 | 0.070 | 0.930 | −0.005 | 0.045 | 0.044 | 0.942 | −0.003 | 0.032 | 0.032 | 0.947 |
| | $\alpha_2$ | −0.010 | 0.070 | 0.070 | 0.941 | −0.001 | 0.045 | 0.045 | 0.943 | −0.001 | 0.032 | 0.032 | 0.940 |
| | $\beta_1$ | 0.008 | 0.084 | 0.087 | 0.933 | 0.003 | 0.053 | 0.053 | 0.952 | 0.000 | 0.038 | 0.037 | 0.947 |
| | $\beta_2$ | 0.004 | 0.084 | 0.087 | 0.937 | 0.000 | 0.053 | 0.055 | 0.937 | 0.001 | 0.038 | 0.037 | 0.954 |
| | $\lambda$ | −0.324 | 0.842 | 1.237 | 0.906 | −0.074 | 0.377 | 0.409 | 0.927 | −0.036 | 0.251 | 0.260 | 0.941 |
| $\lambda = -0.75$ | $\alpha_1$ | −0.011 | 0.070 | 0.069 | 0.935 | −0.004 | 0.045 | 0.046 | 0.949 | −0.001 | 0.032 | 0.029 | 0.957 |
| | $\alpha_2$ | −0.009 | 0.070 | 0.074 | 0.918 | −0.003 | 0.045 | 0.044 | 0.944 | −0.002 | 0.032 | 0.032 | 0.939 |
| | $\beta_1$ | 0.008 | 0.086 | 0.089 | 0.934 | 0.001 | 0.054 | 0.053 | 0.958 | −0.002 | 0.038 | 0.039 | 0.941 |
| | $\beta_2$ | −0.003 | 0.085 | 0.088 | 0.937 | 0.001 | 0.054 | 0.057 | 0.938 | 0.001 | 0.038 | 0.038 | 0.954 |
| | $\lambda$ | −0.175 | 0.559 | 0.773 | 0.913 | −0.046 | 0.283 | 0.304 | 0.941 | −0.025 | 0.191 | 0.198 | 0.942 |
| $\lambda = -0.50$ | $\alpha_1$ | −0.008 | 0.070 | 0.074 | 0.930 | −0.003 | 0.045 | 0.046 | 0.948 | −0.003 | 0.032 | 0.031 | 0.952 |
| | $\alpha_2$ | −0.002 | 0.071 | 0.070 | 0.947 | −0.003 | 0.045 | 0.045 | 0.949 | −0.002 | 0.032 | 0.031 | 0.952 |
| | $\beta_1$ | 0.006 | 0.087 | 0.087 | 0.947 | 0.001 | 0.055 | 0.056 | 0.946 | 0.002 | 0.039 | 0.040 | 0.944 |
| | $\beta_2$ | 0.004 | 0.087 | 0.086 | 0.947 | −0.002 | 0.055 | 0.054 | 0.945 | −0.001 | 0.039 | 0.039 | 0.952 |
| | $\lambda$ | −0.150 | 0.410 | 0.565 | 0.931 | −0.039 | 0.203 | 0.220 | 0.947 | −0.020 | 0.137 | 0.143 | 0.945 |
| $\lambda = -0.25$ | $\alpha_1$ | −0.010 | 0.070 | 0.070 | 0.942 | −0.003 | 0.045 | 0.044 | 0.935 | −0.001 | 0.032 | 0.031 | 0.953 |
| | $\alpha_2$ | −0.007 | 0.070 | 0.070 | 0.951 | −0.002 | 0.045 | 0.045 | 0.945 | −0.002 | 0.032 | 0.032 | 0.950 |
| | $\beta_1$ | 0.002 | 0.087 | 0.089 | 0.947 | 0.003 | 0.056 | 0.055 | 0.955 | 0.001 | 0.039 | 0.041 | 0.939 |
| | $\beta_2$ | 0.007 | 0.088 | 0.087 | 0.946 | −0.001 | 0.055 | 0.056 | 0.942 | 0.003 | 0.039 | 0.039 | 0.944 |
| | $\lambda$ | −0.054 | 0.229 | 0.261 | 0.945 | −0.023 | 0.130 | 0.135 | 0.955 | −0.006 | 0.087 | 0.089 | 0.950 |
| $\lambda = 0.00$ | $\alpha_1$ | −0.009 | 0.070 | 0.069 | 0.949 | −0.004 | 0.045 | 0.044 | 0.951 | 0.000 | 0.032 | 0.031 | 0.960 |
| | $\alpha_2$ | −0.010 | 0.070 | 0.070 | 0.938 | −0.001 | 0.045 | 0.044 | 0.947 | −0.002 | 0.032 | 0.031 | 0.952 |
| | $\beta_1$ | 0.001 | 0.087 | 0.089 | 0.943 | 0.000 | 0.056 | 0.054 | 0.951 | −0.001 | 0.039 | 0.041 | 0.933 |
| | $\beta_2$ | 0.004 | 0.088 | 0.088 | 0.939 | 0.005 | 0.056 | 0.056 | 0.941 | 0.001 | 0.039 | 0.039 | 0.938 |
| | $\lambda$ | 0.006 | 0.156 | 0.176 | 0.996 | 0.001 | 0.086 | 0.086 | 0.990 | 0.002 | 0.059 | 0.057 | 0.978 |

Note that the values of $SE_1$ y $SE_2$ are very similar for $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$ in all cases considered, which suggests that the standard errors of the estimates are themselves well estimated. However, for estimates of $\lambda$ in most cases we have $SE_2 > SE_1$, suggesting that the standard errors of the $\lambda$ estimates are underestimated, especially, once again, when the true value of $\lambda$ is far from 0. We note that the coverage percentages of the interval estimates are close to the nominal values for all parameters in all cases, except for the intervals for $\lambda$ when the true value of $\lambda$ satisfies $|\lambda| \geq 1$ in the case in which $n = 100$.

**Table 2**: Simulation study for the BSCNBS model (continuation).

| Case | Parameter | $n=100$ | | | | $n=250$ | | | | $n=500$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | AB | $SE_1$ | $SE_2$ | CP | AB | $SE_1$ | $SE_2$ | CP | AB | $SE_1$ | $SE_2$ | CP |
| $\lambda = 0.25$ | $\alpha_1$ | −0.012 | 0.070 | 0.073 | 0.926 | −0.003 | 0.045 | 0.045 | 0.944 | −0.003 | 0.032 | 0.030 | 0.957 |
| | $\alpha_2$ | −0.010 | 0.070 | 0.068 | 0.949 | −0.003 | 0.045 | 0.046 | 0.934 | −0.003 | 0.032 | 0.031 | 0.947 |
| | $\beta_1$ | 0.007 | 0.087 | 0.087 | 0.947 | 0.002 | 0.056 | 0.054 | 0.955 | −0.001 | 0.039 | 0.040 | 0.944 |
| | $\beta_2$ | 0.003 | 0.087 | 0.091 | 0.940 | 0.002 | 0.056 | 0.055 | 0.958 | 0.000 | 0.039 | 0.039 | 0.955 |
| | $\lambda$ | 0.078 | 0.247 | 0.356 | 0.944 | 0.013 | 0.126 | 0.126 | 0.952 | 0.011 | 0.088 | 0.091 | 0.950 |
| $\lambda = 0.50$ | $\alpha_1$ | −0.006 | 0.070 | 0.072 | 0.928 | −0.005 | 0.045 | 0.044 | 0.954 | −0.001 | 0.032 | 0.032 | 0.954 |
| | $\alpha_2$ | −0.007 | 0.070 | 0.070 | 0.933 | −0.002 | 0.045 | 0.045 | 0.939 | −0.001 | 0.032 | 0.032 | 0.949 |
| | $\beta_1$ | 0.009 | 0.087 | 0.085 | 0.961 | 0.003 | 0.055 | 0.054 | 0.953 | 0.001 | 0.039 | 0.041 | 0.940 |
| | $\beta_2$ | 0.006 | 0.087 | 0.090 | 0.942 | 0.003 | 0.055 | 0.056 | 0.949 | 0.001 | 0.039 | 0.039 | 0.947 |
| | $\lambda$ | 0.103 | 0.374 | 0.469 | 0.943 | 0.030 | 0.199 | 0.209 | 0.949 | 0.016 | 0.137 | 0.140 | 0.947 |
| $\lambda = 0.75$ | $\alpha_1$ | −0.008 | 0.070 | 0.070 | 0.940 | −0.002 | 0.045 | 0.044 | 0.946 | −0.002 | 0.032 | 0.031 | 0.961 |
| | $\alpha_2$ | −0.010 | 0.070 | 0.071 | 0.937 | 0.000 | 0.045 | 0.046 | 0.945 | −0.002 | 0.032 | 0.031 | 0.938 |
| | $\beta_1$ | 0.003 | 0.085 | 0.089 | 0.941 | 0.004 | 0.054 | 0.055 | 0.940 | 0.003 | 0.038 | 0.039 | 0.946 |
| | $\beta_2$ | 0.004 | 0.085 | 0.088 | 0.949 | 0.001 | 0.054 | 0.056 | 0.941 | 0.001 | 0.038 | 0.039 | 0.947 |
| | $\lambda$ | 0.171 | 0.568 | 0.775 | 0.916 | 0.041 | 0.284 | 0.306 | 0.939 | 0.026 | 0.192 | 0.196 | 0.947 |
| $\lambda = 1.00$ | $\alpha_1$ | −0.008 | 0.070 | 0.071 | 0.932 | −0.005 | 0.045 | 0.044 | 0.946 | −0.001 | 0.032 | 0.033 | 0.928 |
| | $\alpha_2$ | −0.004 | 0.071 | 0.074 | 0.935 | −0.003 | 0.045 | 0.046 | 0.937 | −0.002 | 0.032 | 0.032 | 0.942 |
| | $\beta_1$ | 0.002 | 0.084 | 0.088 | 0.931 | 0.001 | 0.053 | 0.056 | 0.936 | 0.002 | 0.038 | 0.038 | 0.952 |
| | $\beta_2$ | 0.005 | 0.084 | 0.085 | 0.950 | 0.002 | 0.053 | 0.054 | 0.956 | 0.002 | 0.038 | 0.038 | 0.940 |
| | $\lambda$ | 0.346 | 0.847 | 1.526 | 0.908 | 0.080 | 0.379 | 0.462 | 0.933 | 0.032 | 0.250 | 0.265 | 0.945 |
| $\lambda = 2.00$ | $\alpha_1$ | −0.009 | 0.070 | 0.070 | 0.949 | −0.005 | 0.045 | 0.045 | 0.948 | −0.001 | 0.032 | 0.032 | 0.941 |
| | $\alpha_2$ | −0.006 | 0.070 | 0.071 | 0.937 | −0.002 | 0.045 | 0.045 | 0.943 | −0.002 | 0.032 | 0.032 | 0.940 |
| | $\beta_1$ | 0.004 | 0.078 | 0.082 | 0.933 | 0.003 | 0.050 | 0.054 | 0.930 | 0.000 | 0.035 | 0.034 | 0.951 |
| | $\beta_2$ | 0.003 | 0.079 | 0.084 | 0.923 | 0.000 | 0.050 | 0.051 | 0.939 | 0.000 | 0.035 | 0.037 | 0.940 |
| | $\lambda$ | 0.875 | 2.158 | 3.217 | 0.886 | 0.229 | 0.841 | 1.038 | 0.939 | 0.085 | 0.531 | 0.602 | 0.947 |
| $\lambda = 5.00$ | $\alpha_1$ | −0.006 | 0.071 | 0.072 | 0.935 | −0.005 | 0.045 | 0.046 | 0.940 | −0.001 | 0.032 | 0.032 | 0.944 |
| | $\alpha_2$ | −0.006 | 0.071 | 0.073 | 0.928 | −0.002 | 0.045 | 0.045 | 0.952 | −0.002 | 0.032 | 0.032 | 0.940 |
| | $\beta_1$ | 0.005 | 0.067 | 0.073 | 0.909 | −0.001 | 0.042 | 0.044 | 0.940 | 0.000 | 0.029 | 0.030 | 0.937 |
| | $\beta_2$ | 0.005 | 0.068 | 0.074 | 0.915 | 0.000 | 0.042 | 0.045 | 0.920 | 0.000 | 0.029 | 0.032 | 0.929 |
| | $\lambda$ | 1.668 | 5.676 | 5.215 | 0.873 | 1.184 | 3.227 | 3.772 | 0.919 | 0.675 | 1.839 | 2.071 | 0.949 |

## 4.3. Real data set: Ais data set

The *ais* data set (see DAAG package, Maindonald and Braun, [29]) includes information about 13 characteristics measured in 202 Australian athletes. We considered two of those variables: the red blood cell count (`rcc`) and the lean body mass (in kg, `lbm`). We model such variables jointly with the BSCNBS distribution. From the data we obtain $s_1 = 4.7186, s_2 = 64.8737, r_1 = 4.6753$ and $r_2 = 62.2709$, providing the following initial values for the estimation algorithm: $\widehat{\alpha}_1^{(0)} = 0.0961, \widehat{\alpha}_2^{(0)} = 0.2034, \widehat{\beta}_1^{(0)} = 4.6969$, $\widehat{\beta}_2^{(0)} = 63.5590$ and $\lambda^{(0)} = 4.53$. Table 3 shows the estimates for the three considered models. We also use two tests to verify the improved performance of the BSCNBS model compared to the BSNBS and BVBS models.

**Table 3**: ML estimates for BSCNBS and BSNBS models in betaplasma data set (standard errors in parenthesis).

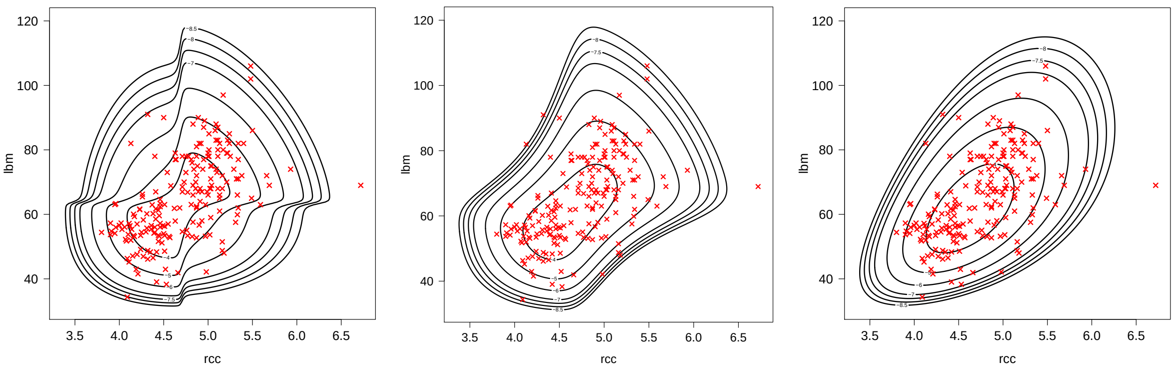
| parameter | BSCNBS | BSNBS | BVBS |
|---|---|---|---|
| $\alpha_1$ | 0.0962 (0.0048) | 0.0961 (0.0048) | 0.0962 (0.0048) |
| $\alpha_2$ | 0.2034 (0.0101) | 0.2035 (0.0101) | 0.2034 (0.0101) |
| $\beta_1$ | 4.6969 (0.0279) | 4.6946 (0.0277) | 4.6967 (0.0317) |
| $\beta_2$ | 63.5588 (0.6720) | 63.2352 (0.7882) | 63.5568 (0.9049) |
| $\lambda$ | 4.5003 (1.8784) | 1.2132 (0.1843) | — |
| $\rho$ | — | — | 0.5573 (0.0485) |
| log-likelihood | −881.3825 | −885.485 | −890.9184 |
| AIC | 1772.77 | 1780.97 | 1791.84 |
| BIC | 1789.31 | 1797.51 | 1808.38 |

The Kolmogorov–Smirnov (KS) statistics are used to verify marginally the BS fit of the `rcc` and `lbm` variates. In addition, it is very important to also consider the bivariate fit of the data to the model. For this, we use an empirical goodness-of-fit test for multivariate distributions proposed in McAssey [31]. We denote $A_T$ as the statistic for such test. Note that both the Akaike information criterion (AIC) (Akaike [1]) and Bayesian information criterion (BIC) (Schwarz [39]) are lower for the BSCNBS model. Additionally, Table 4 shows that both, marginal and bivariate tests provides greater p-values for the BSCNBS model. Thus both, marginal and bivariate tests, suggest better performance for the BSCNBS model.

**Table 4**: Goodness-of-fit to betaplasma data set (p-values in parenthesis).

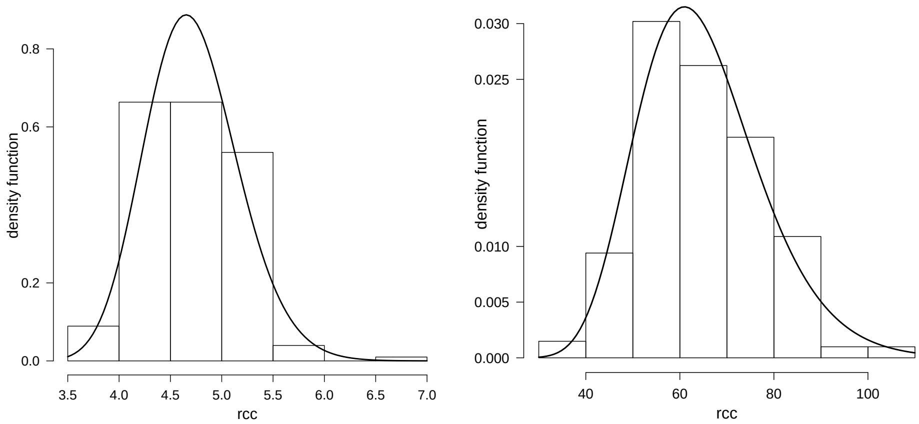
| Test | BSCNBS | | BSNBS | | BVBS | |
|---|---|---|---|---|---|---|
| | rcc | lbm | rcc | lbm | rcc | lbm |
| KS (marginal) | 0.078 (0.172) | 0.060 (0.456) | 0.079 (0.152) | 0.066 (0.334) | 0.078 (0.170) | 0.060 (0.455) |
| $A_T$ (bivariate) | 5.723 (0.150) | | 6.574 (0.021) | | 6.158 (0.063) | |

Figure 3 also shows the scatterplot for this data set superimposed on contours of the three fitted models and Figure 4 shows the histogram and estimated density function based on the marginal BS for `rcc` and `lbm` variables. A visual inspection indicates a somewhat better fit of the BSCNSBS relative to the BSNBS model and that the BS distribution is appropriate marginally for this data set. Figure 5 presents the MQR for both variables, the BQR and the respective p-values for the KS test to check the normality hypothesis. Note that, under the usual significance levels, the hypothesis for both, marginal and bivariate residuals, is not rejected, reinforcing the idea that the BSCNBS is appropriate for this data set.



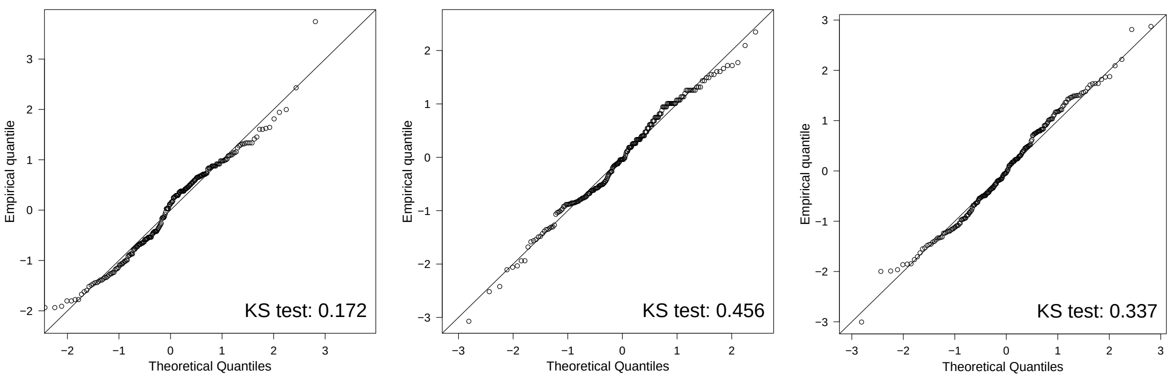

**(a)** **(b)** **(c)**

**Figure 3**: Scatterplot of rcc versus lbm for ais data set:
(a) BSCNBS; (b) BSNBS and; (c) BVBS models.




**(a)** **(b)**

**Figure 4**: Histogram and density function for: (a) `rcc`; and (b) `lbm`, and their estimated density function based on the marginal BS distributions.




**(a)** **(b)** **(c)**

**Figure 5**: QQ-plot for the MQR for: (a) the variable `rcc`; (b) the `lbm` variable and; (c) the BQR, based on the fitted BSCNBS model in the ais data set. Also is presented the p-value for the KS test to check if the residuals have standard normal distribution.

## 5. MULTIVARIATE EXTENSIONS

To obtain a $k$-dimensional extension of the model discussed in this paper, it is only necessary to identify a specific $k$-dimensional skewed distribution with normal marginals and skew-normal conditionals to which appropriate marginal transformations are to be applied. For example, one might consider the following joint density.

$$f_1(x_1, x_2, ..., x_k; \lambda) = 2\left[\prod_{i=1}^{k} \phi(x_i)\right] \Phi\left(\frac{\lambda \prod_{i=1}^{k} x_i}{\sqrt{1+\lambda^2 \prod_{i=1}^{k} x_i^2}}\right), \tag{5.1}$$

which, when marginally transformed, yields a natural extension of the bivariate model discussed in the present paper. Instead of (5.1) we might consider

$$f_2(x_1, x_2, ..., x_k; \lambda) = 2\left[\prod_{i=1}^{k} \phi(x_i)\right] \Phi\left(\lambda \prod_{i=1}^{k} x_i\right), \tag{5.2}$$

which yields the k-dimensional version of the Lemonte *et al.* [28] model.

Both of these models suffer from the fact that only a single dependence parameter, $\lambda$, is present. Based on our experience with multivariate normal models and their close relatives, we might prefer to have perhaps $k(k-1)/2$ dependence parameters, if not more, to ensure sufficient flexibility of the model. An extreme example is one which involves use of a $k$-dimensional joint density which has skew generalized normal conditionals with $2^{k-1}$ or $3^{k-1}$ parameters, which is to be transformed to have BS marginals. In practice, some intermediate configuration of dependence parameters might be expected to be appropriate in a particular data setting.

## 6. CONCLUSIONS, LIMITATIONS AND FUTURE RESEARCH

The model (2.9) that has been investigated in this paper is, of course, only one of the many bivariate models with BS marginals. In complete generality, one could consider two BS quantile functions and apply them to any copula (i.e., any distribution with standard uniform marginals). Perusal of Nelsen [32] will reveal that essentially there are a limited number of copulas with analytic forms that are readily available for such constructions. Moreover many of the well known copula families have only a single dependence parameter, as is the case with the families of distributions discussed in the present paper. It does thus seem reasonable to consider some of these copula based bivariate BS models as competitors to the models of this paper.

Another approach that might be considered for data sets with BS characteristics, is to take advantage of the fact that the family of univariate BS distributions is an exponential family. Following Arnold and Strauss [6] we might wish to consider the exponential family of bivariate densities with BS conditionals (rather than marginals) as competitors of the models in this paper. Such models have been investigated in Arnold *et al.* [5].

Yet a third general class of models might be considered. For it, assume that $X \sim$BS$(\alpha, \beta)$ and that, for each $x > 0$, $Y|X = x \sim$BS$(A(x; \underline{\theta}), B(x; \underline{\theta}))$, for certain positive functions $A(x; \underline{\theta})$ and $B(x; \underline{\theta})$. Filus and Filus [21] have investigated models of this genre, in cases in which the roles of the BS distributions are played by normal or exponential distributions.

Multivariate extensions of all the concepts alluded to in this Discussion are readily envisioned.

## A. APPENDIX

The Fisher information matrix for the BSCNBS model is given by

$$I(\boldsymbol{\psi}) = \begin{pmatrix} I_{\alpha_1\alpha_1} & I_{\alpha_1\alpha_2} & I_{\alpha_1\beta_1} & I_{\alpha_1\beta_2} & I_{\alpha_1\lambda} \\ I_{\alpha_2\alpha_1} & I_{\alpha_2\alpha_2} & I_{\alpha_2\beta_1} & I_{\alpha_2\beta_2} & I_{\alpha_2\lambda} \\ I_{\beta_1\alpha_1} & I_{\beta_1\alpha_2} & I_{\beta_1\beta_1} & I_{\beta_1\beta_2} & I_{\beta_1\lambda} \\ I_{\beta_2\alpha_1} & I_{\beta_2\alpha_2} & I_{\beta_2\beta_1} & I_{\beta_2\beta_2} & I_{\beta_2\lambda} \\ I_{\lambda\alpha_1} & I_{\lambda\alpha_2} & I_{\lambda\beta_1} & I_{\lambda\beta_2} & I_{\lambda\lambda} \end{pmatrix},$$

with

$$I_{\alpha_1\alpha_1} = \frac{2}{\alpha_1^2} + \frac{\lambda^3}{\alpha_1^2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{\lambda^2}{\alpha_1^2}\mathbb{E}\left(\frac{\omega^2 a_1^2 a_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) + \frac{3\lambda^3}{\alpha_1^2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) - \frac{2\lambda}{\alpha_1^2}\mathbb{E}\left(\frac{\omega a_1 a_2}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\alpha_1\alpha_2} = \frac{\lambda^3}{\alpha_1\alpha_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{\lambda^2}{\alpha_1\alpha_2}\mathbb{E}\left(\frac{\omega^2 a_1^2 a_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) - \frac{\lambda}{\alpha_1\alpha_2}\mathbb{E}\left(\frac{\omega a_1 a_2}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) + \frac{2\lambda^3}{\alpha_1\alpha_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right)$$

$$I_{\alpha_1\beta_1} = \frac{2\lambda^3}{\alpha_1^2\beta_1}\mathbb{E}\left(\frac{\omega a_1^2 a_2^3 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{6\lambda^3}{\alpha_1^2\beta_1}\mathbb{E}\left(\frac{\omega a_1^2 a_2^3 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) - \frac{\lambda}{2\alpha_1^2\beta_1}\mathbb{E}\left(\frac{\omega a_2 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\alpha_1\beta_2} = \frac{\lambda^3}{2\alpha_1\alpha_2\beta_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^2 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) - \frac{\lambda}{2\alpha_1\alpha_2\beta_2}\mathbb{E}\left(\frac{\omega}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) + \frac{\lambda^3}{\alpha_1\alpha_2\beta_1}\mathbb{E}\left(\frac{\omega a_1^2 a_2^2 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right)$$

$$I_{\alpha_1\lambda} = \frac{\lambda^2}{\alpha_1}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) - \frac{\lambda}{\alpha_1}\mathbb{E}\left(\frac{\omega^2 a_1^2 a_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) + \frac{1}{\alpha_1}\mathbb{E}\left(\frac{\omega a_1 a_2}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\alpha_2\alpha_2} = \frac{2}{\alpha_2^2} + \frac{\lambda^3}{\alpha_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{\lambda^2}{\alpha_2}\mathbb{E}\left(\frac{\omega^2 a_1^2 a_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) + \frac{3\lambda^3}{\alpha_2^2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) - \frac{2\lambda}{\alpha_2^2}\mathbb{E}\left(\frac{\omega a_1 a_2}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\alpha_2\beta_1} = \frac{\lambda^3}{2\alpha_1\alpha_2\beta_1}\mathbb{E}\left(\frac{\omega a_1^2 a_2^3 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) - \frac{\lambda}{2\alpha_1\alpha_2\beta_1}\mathbb{E}\left(\frac{\omega a_2 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) + \frac{\lambda^3}{\alpha_1\alpha_2\beta_1}\mathbb{E}\left(\frac{\omega a_1^2 a_2^2 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right)$$

$$I_{\alpha_2\beta_2} = \frac{\lambda^3}{2\alpha_2^2\beta_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^2 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{3\lambda^3}{2\alpha_2^2\beta_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^2 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) - \frac{\lambda}{2\alpha_2^2\beta_2}\mathbb{E}\left(\frac{\omega a_1 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\alpha_2\lambda} = \frac{\lambda^2}{\alpha_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) - \frac{\lambda}{\alpha_2}\mathbb{E}\left(\frac{\omega^2 a_1^2 a_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) + \frac{1}{\alpha_2}\mathbb{E}\left(\frac{\omega a_1 a_2}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\beta_1\beta_1} = \frac{1}{\alpha_1^2\beta_1^2} + \mathbb{E}[(T_1+\beta_1)^{-2}] + \frac{\lambda^3}{4\alpha_1^2\beta_1^2}\mathbb{E}\left(\frac{\omega a_1 a_2^3 d_1^2}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{\lambda^2}{4\alpha_1^2\beta_1^2}\mathbb{E}\left(\frac{\omega^2 a_2^2 d_1^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) + \frac{3\lambda^3}{4\alpha_1^2\beta_1^2}\mathbb{E}\left(\frac{\omega a_1 a_2^3 d_1^2}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) - \frac{\lambda}{4\alpha_1^2\beta_1^2}\mathbb{E}\left(\frac{\omega a_2 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right) - \frac{\lambda}{2\alpha_1^2\beta^{5/2}}\mathbb{E}\left(\frac{\omega a_2 T_1^{1/2}}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\beta_1\beta_2} = \frac{\lambda^3}{4\alpha_1\alpha_2\beta_1\beta_2}\mathbb{E}\left(\frac{\omega a_1^2 a_2^2 d_1 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) - \frac{\lambda}{4\alpha_1\alpha_2\beta_1\beta_2}\mathbb{E}\left(\frac{\omega d_1 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) + \frac{\lambda^3}{2\alpha_1\alpha_2\beta_1\beta_2}\mathbb{E}\left(\frac{\omega a_1^2 a_2^2 d_1 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right)$$

$$I_{\beta_1\lambda} = \frac{\lambda^2}{2\alpha_1\beta_1}\mathbb{E}\left(\frac{\omega a_1^2 a_2^3 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{1}{2\alpha_1\beta_1}\mathbb{E}\left(\frac{\omega a_2 d_1}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\beta_2\beta_2} = \frac{1}{\alpha_2^2\beta_2^2} + \mathbb{E}[(T_2+\beta_2)^{-2}] + \frac{\lambda^3}{4\alpha_2\beta_2^2}\mathbb{E}\left(\frac{\omega a_1^3 a_2 d_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{\lambda^2}{4\alpha_2\beta_2^2}\mathbb{E}\left(\frac{\omega^2 a_1^2 d_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) + \frac{3\lambda^3}{4\alpha_2\beta_2^2}\mathbb{E}\left(\frac{\omega a_1^3 a_2 d_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{5/2}}\right) - \frac{\lambda}{4\alpha_2^2\beta_2^2}\mathbb{E}\left(\frac{\omega a_1 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right) - \frac{\lambda}{2\alpha_2^2\beta_2^{5/2}}\mathbb{E}\left(\frac{\omega a_1 T_2^{1/2}}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\beta_2\lambda} = \frac{\lambda^2}{2\alpha_2\beta_2}\mathbb{E}\left(\frac{\omega a_1^3 a_2^2 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{7/2}}\right) + \frac{1}{2\alpha_2\beta_2}\mathbb{E}\left(\frac{\omega a_1 d_2}{(1+\lambda^2 a_1^2 a_2^2)^{3/2}}\right)$$

$$I_{\lambda\lambda} = \mathbb{E}\left(\frac{\omega^2 a_1^2 a_2^2}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right) + 3\lambda\mathbb{E}\left(\frac{\omega a_1^3 a_2^3}{(1+\lambda^2 a_1^2 a_2^2)^{3}}\right),$$

where $d_j = (T_j/\beta_j)^{1/2} + (\beta_j/T_j)^{1/2}$, $j = 1, 2$, $\omega = \phi(b)/\Phi(b)$ and $b = \lambda a_1 a_2/\sqrt{1+\lambda a_1^2 a_2^2}$. Note that for $\boldsymbol{\psi}_0 = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda = 0)$, this matrix is reduced to

$$I(\boldsymbol{\psi}_0) = \text{diag}\left(\frac{2}{\alpha_1^2}, \frac{2}{\alpha_2^2}, \frac{1}{\alpha_1^2\beta_1^2} + \mathbb{E}[(T_1+\beta_1)^{-2}], \frac{1}{\alpha_2^2\beta_2^2} + \mathbb{E}[(T_2+\beta_2)^{-2}], \frac{1}{2\sqrt{2\pi}}\right).$$

Then, the determinant of $I(\boldsymbol{\psi}_0)$ is

$$|I(\boldsymbol{\psi}_0)| = \frac{2}{\sqrt{2\pi}\alpha_1^2\alpha_2^2}\prod_{j=1}^{2}\left\{(\alpha_j\beta_j)^{-2} + \mathbb{E}\left[(T_j+\beta_j)^{-2}\right]\right\} > 0.$$

Therefore, the Fisher information matrix is not singular at $\lambda = 0$.

## ABBREVIATIONS

The following abbreviations are used in this manuscript:

| | |
|---|---|
| BS | Birnbaum–Saunders |
| SN | Skew-normal |
| SGN | Skew-generalized-normal |
| SCN | Skew-curved-normal |
| SCNBS | Skew-curved-normal-Birnbaum–Saunders |
| BSCNBS | Bivariate skew-curved-normal-Birnbaum–Saunders |
| BVBS | Bivariate Birnbaum–Saunders |
| SNBS | Skew-normal-Birnbaum–Saunders |
| BSNBS | Bivariate skew-normal-Birnbaum–Saunders |
| FIM | Fisher information matrix |
| ML | Maximum likelihood |
| MQR | Marginal quantile residuals |
| BQR | Bivariate quantile residuals |
| AB | Absolute bias |
| $SE_1$ | Mean of the standard errors |
| $SE_2$ | Standard deviation of the estimated parameters |
| CP | Coverage proportion |
| KS | Kolmogorov–Smirnov |
| AIC | Akaike information criterion |
| BIC | Bayesian information criterion |

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