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Folded Bivariate Distributions as Models for Magnitude Correlation

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Abstract:

- The concept of *magnitude correlation* requires the use of folded bivariate distributions. However, apart from the folded bivariate normal and folded bivariate t distributions (of these two only the former has received any real applications), nothing is known about folded bivariate distributions. Here, we introduce six new folded bivariate distributions. Applications involving stock indices of ten major economies show the value of the proposed distributions.

Keywords:

- *estimation; magnitude correlation of stock returns; value at risk.*

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- 62E15.

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1. INTRODUCTION

Let X and Y be two random variables taking values on the entire real line (for example, stock returns for two different commodities). Their *magnitude correlation* is defined as the correlation between $|X|$ and $|Y|$, i.e., the correlation between the absolute values of X and Y. Its derivation clearly requires the use of folded bivariate distributions.

The concept of magnitude correlation arises in many areas of the sciences, engineering and medicine. One prominent area is stock modeling. Let X denote the stock return for one commodity and Y the stock return for another commodity. It is of interest to know if large values of X in magnitude are associated with large values of Y in magnitude, if large values of X in magnitude are associated with small values of Y in magnitude, if small values of X in magnitude are associated with large values of Y in magnitude, or if small values of X in magnitude are associated with small values of Y in magnitude. The magnitude correlation of stock returns has been studied by many authors. Some recent examples are: Firth [22] observes that the "correlation between the magnitudes of the changes in dividends and changes in future earnings (over the next two years) is highly significant and positive"; While modeling volatility of Dhaka stock exchange, Islam *et al.* [32] observe that the "correlations between the magnitudes of returns on nearby days are positive and statistically significant"; While investigating the overnight effect on the Taiwan stock market, Tsai *et al.* [49] examine the "cross correlation between the magnitude of daytime (trading hour) returns, overnight (off-hour session) returns, and total (close-to-close) returns"; Tsai *et al.* [49] observe "a larger magnitude of overnight return implies a higher probability that the sign of the following daytime return is the opposite of the sign of overnight return"; Chabakauri [11] finds "a positive relationship between the amount of leverage in the economy and magnitudes of stock return correlations and volatilities"; Fukuda and Tanaka [23] show that "during the global financial crisis the magnitude correlation between TIBOR and LIBOR reversed depending on whether they were in yen or dollars"; Bhamra *et al.* [6] show that the magnitude of asset return correlation depends on "three structural parameters: the degree of market integration, the level of fundamental correlation and the rate of time preference"; While investigating equity market reactions to CreditWatch events, Gu *et al.* [25] find evidence of a "positive correlation between the magnitude of the cumulative abnormal returns prior to the listing day and the magnitude of the rating changes announced on the delisting day"; Hamalainen [28] says "Correlation between the magnitudes of asset returns is an overlooked concept in financial research. It affects portfolio variance explicitly when the directions of returns are predictable". See also Kutergin and Filimonov [39].

Magnitude correlation is one area requiring the use of folded bivariate and folded multivariate distributions. Other areas include: approximations to the mean and variance of the index of dissimilarity in contingency tables (Inman and Bradley [31], Mulekar *et al.* [41]); noise sensitivity of a new singularity index (Muralidhar *et al.* [42]); distribution and estimation of trading costs (Kourtis [38]); the joint distribution of indemnity payment and allocated loss adjustment expense for general liability claims (Guillou *et al.* [26]).

In all of these applications and others, only the folded bivariate/multivariate normal distribution appears to have been used. We are not aware of applications of any other folded bivariate/multivariate distribution. The normal distribution does not give good fits to many types of data including heavy tailed data (for example, stock returns). Hence, there is a need for folded bivariate/multivariate distributions for non-normal data.

Psarakis and Panaretos [45] were the first to introduce the folded bivariate normal distribution and study its statistical properties. They derived its marginal distributions and joint moment generating function. Chakraborty and Chatterjee [12] gave a multivariate form of the folded normal distribution and derived expressions for its mean vector, covariance matrix and joint moment generating function. They also discussed possible areas of applications of the folded multivariate normal distribution.

We are aware of only one folded bivariate distribution for non-normal data, the folded bivariate t distribution. We are aware of no folded multivariate distributions for non-normal data. The folded bivariate t distribution was also introduced by Psarakis and Panaretos [45]. They derived its marginal distributions and established its relationship to the folded bivariate normal distribution.

Neither of the papers (Psarakis and Panaretos [45] or Chakraborty and Chatterjee [12]) discussed real data applications or even simulation studies. The aim of this paper is to:

i) introduce six new folded bivariate distributions;

ii) illustrate real data applications of all of the folded bivariate distributions.

The six new folded bivariate distributions are based on the: bivariate skew normal distribution due to Azzalini and Dalla Valle [4]; bivariate skew t distribution due to Azzalini and Capitanio [3]; bivariate logistic distribution due to Gumbel [27]; bivariate Kotz type distribution due to Kotz [36]; bivariate Laplace distribution of the first kind due to Eltoft *et al.* [19]; bivariate Laplace distribution of the second kind due to Ernst [20]. We have chosen these distributions because they are some of the most tractable and applied bivariate distributions for non-normal data, see Balakrishnan and Lai [5] and references therein.

For each of the new distributions and for the two known ones, we give expressions for the joint probability density function. Expressions for the joint cumulative distribution function, joint moment generating function and the log-likelihood function can be obtained from the corresponding author. As a by product of the six new distributions, we also introduce two new univariate distributions: the folded univariate skew normal distribution and the folded univariate skew t distribution.

Our real data application involves forty five bivariate data sets on log returns of stocks. We show that:

i) the folded bivariate t and folded bivariate skew t distributions provide the best fits for the majority of the data sets;

ii) each of the folded distributions outperforms the corresponding truncated unfolded version for each of the forty five data sets.

The latter observation is a further advocate for the need for folded bivariate distributions.

The expressions in Section 2 involve standard normal cumulative distribution function defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$$

and the modified Bessel function of the second kind (Abramowitz and Stegun [1]) defined by

$$K_\nu(x) = \begin{cases} \dfrac{\pi \csc(\pi\nu)}{2}\left[I_{-\nu}(x) - I_\nu(x)\right], & \text{if } \nu \notin \mathbb{Z}, \\ \lim\limits_{\mu \to \nu} K_\mu(x), & \text{if } \nu \in \mathbb{Z}, \end{cases}$$

where $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)k!}\left(\frac{x}{2}\right)^{2k+\nu}.$$

2. NEW FOLDED BIVARIATE DISTRIBUTIONS

Let (X, Y) denote a random vector on $(-\infty, \infty) \times (-\infty, \infty)$ with joint probability density function $f_{X,Y}(x, y)$ and joint cumulative distribution function $F_{X,Y}(x, y)$. Set $(U, V) = (|X|, |Y|)$. Then the joint probability density function of (U, V) is

$$f_{U,V}(u, v) = f_{X,Y}(u, v) + f_{X,Y}(u, -v) + f_{X,Y}(-u, v) + f_{X,Y}(-u, -v). \tag{2.1}$$

The distribution given by $f_{U,V}(u, v)$ is said to be the folded version of the distribution given by $f_{X,Y}(x, y)$.

Let f_X, f_Y denote the marginal probability density functions of (X, Y). Let f_U, f_V denote the marginal probability density functions of (U, V). It follows from (2.1) that $f_U(u) = f_X(u) + f_X(-u)$ and $f_V(v) = f_Y(v) + f_Y(-v)$.

2.1. Folded bivariate normal distribution

The bivariate normal distribution due to the work of Laplace, Plana, Gauss and Bravais is given by the joint probability density function

$$f_{X,Y}(x, y) = \frac{1}{2\pi s_1 s_2 \sqrt{1-\rho^2}} \exp\left[-\frac{Q(x, y)}{2(1-\rho^2)}\right]$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$, where

$$Q(x, y) = \left(\frac{x-\mu_1}{s_1}\right)^2 + \left(\frac{y-\mu_2}{s_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{s_1}\right)\left(\frac{y-\mu_2}{s_2}\right) \tag{2.2}$$

for $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$ and $-1 < \rho < 1$. The corresponding folded version has the joint probability density function

$$\begin{aligned} f_{U,V}(u, v) = \frac{1}{2\pi s_1 s_2 \sqrt{1-\rho^2}} \Bigg\{ & \exp\left[-\frac{Q(u, v)}{2(1-\rho^2)}\right] + \exp\left[-\frac{Q(u, -v)}{2(1-\rho^2)}\right] \\ & + \exp\left[-\frac{Q(-u, v)}{2(1-\rho^2)}\right] + \exp\left[-\frac{Q(-u, -v)}{2(1-\rho^2)}\right] \Bigg\}. \end{aligned}$$

The marginals of the folded bivariate normal distribution are the folded univariate normal distributions due to Leone *et al.* [40].

2.2. Folded bivariate t distribution

The bivariate t distribution is given by the joint probability density function

$$f_{X,Y}(x,y) = \frac{\Gamma(1+\nu/2)}{\nu\pi\Gamma(\nu/2)s_1s_2\sqrt{1-\rho^2}}\left[1+\frac{Q(x,y)}{\nu(1-\rho^2)}\right]^{-1-\nu/2}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $\nu > 0$ and $-1 < \rho < 1$, where Q is given by (2.2). The corresponding folded version has the joint probability density function

$$\begin{aligned} f_{U,V}(u,v) &= \frac{\Gamma(1+\nu/2)}{\nu\pi\Gamma(\nu/2)s_1s_2\sqrt{1-\rho^2}}\left\{\left[1+\frac{Q(u,v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2}\right. \\ &\quad +\left[1+\frac{Q(u,-v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2}+\left[1+\frac{Q(-u,v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2} \\ &\quad \left.+\left[1+\frac{Q(-u,-v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2}\right\} \end{aligned} \tag{2.3}$$

for $u > 0$, $v > 0$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $\nu > 0$ and $-1 < \rho < 1$. The particular case for $\nu = 1$ is the folded bivariate Cauchy distribution, another new folded bivariate distribution. The marginals of the folded bivariate t distribution are the folded univariate t distributions due to Psarakis and Panaretos [44].

2.3. Folded bivariate skew normal distribution

The bivariate skew normal distribution due to Azzalini and Dalla Valle [4] has the joint probability density function specified by

$$f_{X,Y}(x,y) = \frac{1}{\pi s_1s_2\sqrt{1-\rho^2}}\exp\left[-\frac{Q(x,y)}{2(1-\rho^2)}\right]\Phi(\alpha_1x+\alpha_2y)$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $-\infty < \alpha_1 < \infty$, $-\infty < \alpha_2 < \infty$ and $-1 < \rho < 1$, where Q is given by (2.2). The corresponding folded version has the joint probability density function

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{\pi s_1s_2\sqrt{1-\rho^2}}\left\{\exp\left[-\frac{Q(u,v)}{2(1-\rho^2)}\right]\Phi(\alpha_1u+\alpha_2v)\right. \\ &\quad +\exp\left[-\frac{Q(u,-v)}{2(1-\rho^2)}\right]\Phi(\alpha_1u-\alpha_2v) \\ &\quad +\exp\left[-\frac{Q(-u,v)}{2(1-\rho^2)}\right]\Phi(-\alpha_1u+\alpha_2v) \\ &\quad \left.+\exp\left[-\frac{Q(-u,-v)}{2(1-\rho^2)}\right]\Phi(-\alpha_1u-\alpha_2v)\right\} \end{aligned}$$

for $u > 0$, $v > 0$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $-\infty < \alpha_1 < \infty$, $-\infty < \alpha_2 < \infty$ and $-1 < \rho < 1$. The marginals of the folded bivariate skew normal distribution are the folded univariate skew normal distributions, which appear to be new.

2.4. Folded bivariate skew t distribution

The bivariate skew t distribution due to Azzalini and Capitanio [3] has the joint probability density function specified by

$$f_{X,Y}(x,y) = \frac{2\Gamma(1+\nu/2)}{\nu\pi\Gamma(\nu/2)s_1s_2\sqrt{1-\rho^2}}\left[1+\frac{Q(x,y)}{\nu(1-\rho^2)}\right]^{-1-\nu/2} R(x,y)$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$, where Q is given by (2.2) and

$$R(x,y) = G\left(\left(\frac{\alpha_1(x-\mu_1)}{\sqrt{s_{11}}}+\frac{\alpha_2(y-\mu_2)}{\sqrt{s_{22}}}\right)\sqrt{\frac{\nu+2}{Q(x,y)+2}};\nu+2\right),$$

where $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $-\infty < \alpha_1 < \infty$, $-\infty < \alpha_2 < \infty$, $\nu > 0$, $-1 < \rho < 1$, and $G(\cdot;\nu)$ denotes the cumulative distribution function of a Student's t random variable with ν degrees of freedom. The corresponding folded version has the joint probability density function

$$\begin{aligned} f_{U,V}(u,v) \;=\; & \frac{2\Gamma(1+\nu/2)}{\nu\pi\Gamma(\nu/2)s_1s_2\sqrt{1-\rho^2}}\left\{\left[1+\frac{Q(u,v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2} R(u,v)\right. \\ & +\left[1+\frac{Q(u,-v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2} R(u,-v) \\ & +\left[1+\frac{Q(-u,v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2} R(-u,v) \\ & \left.+\left[1+\frac{Q(-u,-v)}{\nu(1-\rho^2)}\right]^{-1-\nu/2} R(-u,-v)\right\} \end{aligned}$$

for $u > 0$, $v > 0$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $-\infty < \alpha_1 < \infty$, $-\infty < \alpha_2 < \infty$, $\nu > 0$ and $-1 < \rho < 1$. The marginals of the folded bivariate skew t distribution are the folded univariate skew t distributions which appear to be new too. The G term does admit closed form expressions if ν is an integer, see Jamalizadeh *et al.* [33].

2.5. Folded bivariate logistic distribution

The bivariate logistic distribution due to Gumbel [27] has the joint probability density function specified by

$$f_{X,Y}(x,y) = \frac{2\exp[-(x-\mu_1)/s_1-(y-\mu_2)/s_2]}{s_1s_2\{1+\exp[-(x-\mu_1)/s_1]+\exp[-(y-\mu_2)/s_2]\}^3}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, and $s_2 > 0$. The corresponding folded version has the joint probability density function

$$\begin{aligned} f_{U,V}(u,v) &= \frac{2}{s_1 s_2}\left\{ \frac{\exp[-(u-\mu_1)/s_1 - (v-\mu_2)/s_2]}{\{1+\exp[-(u-\mu_1)/s_1] + \exp[-(v-\mu_2)/s_2]\}^3} \right. \\ &\quad + \frac{\exp[-(u-\mu_1)/s_1 + (v+\mu_2)/s_2]}{\{1+\exp[-(u-\mu_1)/s_1] + \exp[(v+\mu_2)/s_2]\}^3} \\ &\quad + \frac{\exp[(u+\mu_1)/s_1 - (v-\mu_2)/s_2]}{\{1+\exp[(u+\mu_1)/s_1] + \exp[-(v-\mu_2)/s_2]\}^3} \\ &\quad \left. + \frac{\exp[(u+\mu_1)/s_1 + (v+\mu_2)/s_2]}{\{1+\exp[(u+\mu_1)/s_1] + \exp[(v+\mu_2)/s_2]\}^3} \right\} \end{aligned}$$

for $u > 0$, $v > 0$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, and $s_2 > 0$. The marginals of the folded bivariate logistic distribution are the folded univariate logistic distributions due to Cooray *et al.* [13].

2.6. Folded bivariate Kotz type distribution

The bivariate Kotz type distribution is given by the joint probability density function

$$f_{X,Y}(x,y) = \frac{s r^{N/s} \left(1-\rho^2\right)^{1/2-N}}{s_1 s_2 \pi \Gamma(N/s)} [Q(x,y)]^{N-1} \exp\left\{ -\frac{r}{(1-\rho^2)^s} [Q(x,y)]^s \right\} \tag{2.4}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $N > 0$, $r > 0$, $s > 0$ and $-1 < \rho < 1$, where Q is given by (2.2). When $s = 1$, this is the original Kotz distribution introduced in Kotz [36]. When $N = 1$, $s = 1$ and $r = 1/2$, (2.4) reduces to the bivariate normal probability density function. When $N = 1$, $s = 1/2$ and $r = 1$, (2.4) reduces to the joint probability density function of the bivariate Laplace distribution of the second kind. The folded bivariate Kotz type distribution corresponding to (2.4) has the joint probability density function

$$\begin{aligned} f_{U,V}(u,v) &= \frac{s r^{N/s} \left(1-\rho^2\right)^{1/2-N}}{s_1 s_2 \pi \Gamma(N/s)} \left[[Q(u,v)]^{N-1} \exp\left\{ -\frac{r}{(1-\rho^2)^s} [Q(u,v)]^s \right\} \right. \\ &\quad + [Q(u,-v)]^{N-1} \exp\left\{ -\frac{r}{(1-\rho^2)^s} [Q(u,-v)]^s \right\} \\ &\quad + [Q(-u,v)]^{N-1} \exp\left\{ -\frac{r}{(1-\rho^2)^s} [Q(-u,v)]^s \right\} \\ &\quad \left. + [Q(-u,-v)]^{N-1} \exp\left\{ -\frac{r}{(1-\rho^2)^s} [Q(-u,-v)]^s \right\} \right] \end{aligned}$$

for $u > 0$, $v > 0$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $N > 0$, $r > 0$, $s > 0$ and $-1 < \rho < 1$. The marginals of the folded bivariate Kotz type distribution are the folded univariate exponential power distributions due to Nadarajah and Bakar [43].

2.7. Folded bivariate Laplace distribution of the first kind

The bivariate Laplace distribution of the first kind due to Eltoft *et al.* [19] has the joint probability density function specified by

$$f_{X,Y}(x,y) = \frac{1}{\pi r} K_0\left(\sqrt{\frac{2}{r}}\sqrt{Q(x,y)}\right)$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $r > 0$ and $-1 < \rho < 1$, where Q is given by (2.2). The corresponding folded version has the joint probability density function

$$\begin{aligned} f_{U,V}(u,v) \;=\; & \frac{1}{\pi r}\left[K_0\left(\sqrt{\frac{2}{r}}\sqrt{Q(u,v)}\right) + K_0\left(\sqrt{\frac{2}{r}}\sqrt{Q(u,-v)}\right)\right. \\ & \left. + K_0\left(\sqrt{\frac{2}{r}}\sqrt{Q(-u,v)}\right) + K_0\left(\sqrt{\frac{2}{r}}\sqrt{Q(-u,-v)}\right)\right] \end{aligned}$$

for $u > 0$, $v > 0$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $s_1 > 0$, $s_2 > 0$, $r > 0$ and $-1 < \rho < 1$. The marginals of the folded bivariate Laplace distribution of the first kind are the folded univariate Laplace distributions, special cases of the folded exponential power distribution due to Nadarajah and Bakar [43].

2.8. Folded bivariate Laplace distribution of the second kind

The folded bivariate Laplace distribution of the second kind is the particular case of the folded bivariate Kotz type distribution for $N = 1$, $s = 1/2$ and $r = 1$. So, the corresponding joint probability density function follows from the expression given in Section 2.6. The marginals of the folded bivariate Laplace distribution of the second kind are also the folded univariate Laplace distributions.

3. APPLICATION

In this section, we study the magnitude correlation for daily log returns of stock values from the 3rd of January 2000 to the 28th of February 2014 for the ten countries: the United States of America (S & P 500), Canada (S & P TSX), the United Kingdom (FTSE 100), Germany (DAX), China (SSE), Japan (Nikki), Brazil (BOVESPA), Argentina (MERVAL), South Africa (FTSE/JSE) and Nigeria (S & P). The data were obtained from the database Datastream.

The distributions in Section 2 assume that the data on each country are independent and identically distributed (i.e., randomness), have no serial correlation, and have no heteroskedasticity. We tested for randomness using Cox and Stuart [15]'s test, the rank test and the turning point test. We tested for no serial correlation using Durbin and Watson

[16, 17, 18]'s method and the method due to Godfrey [24] and Breusch [8]. We tested for no heteroskedasticity using Breusch and Pagan [9]'s test. The corresponding p-values not reported here showed no evidence are randomness, no serial correlation or no heteroskedasticity.

The magnitude correlation between returns for any two countries can be studied by:

i) fitting a folded bivariate distribution to positive log returns from the countries (that is, considering only those days where the log returns are positive for both countries);

ii) fitting a truncated unfolded bivariate distribution (truncation made to the positive quadrant) to positive log returns from the countries.

We have data for ten countries, so forty five pairs of bivariate data.

Section 2 describes eight folded bivariate distributions. Each of these has a corresponding truncated unfolded version. These sixteen distributions were fitted to positive log returns from: USA / CAD, USA / UK, USA / GER, USA / CHI, USA / JPN, USA / BRA, USA / ARG, USA / SA, USA / NG, CAD / UK, CAD / GER, CAD / CHI, CAD / JPN, CAD / BRA, CAD / ARG, CAD / SA, CAD / NG, UK / GER, UK / CHI, UK / JPN, UK / BRA, UK / ARG, UK / SA, UK / NG, GER / CHI, GER / JPN, GER / BRA, GER / ARG, GER / SA, GER / NG, CHI / JPN, CHI / BRA, CHI / ARG, CHI / SA, CHI / NG, JPN / BRA, JPN / ARG, JPN / SA, JPN / NG, BRA / ARG, BRA / SA, BRA / NG, ARG / SA, ARG / NG and SA / NG. The method of maximum likelihood was used. The maximization of the log-likelihood functions was performed using the routine `optim` in the R software package (R Development Core Team [46]).

The folded and the corresponding truncated unfolded distributions have the same number of parameters. So, criteria like the Akaike information criterion and the Bayesian information criterion reduce to comparing log-likelihood values. In other words, the one giving the larger log-likelihood value can be regarded as the better model. Boxplots of the differences between the log-likelihood values for the forty five pairs are shown in Figure 1. We see that the differences are huge. They range from: 575.7203 to 865.9098 when the truncated bivariate normal and folded bivariate normal distributions are compared; 436.8568 to 744.1456 when the truncated bivariate t and folded bivariate t distributions are compared; 380.7097 to 865.9098 when the truncated bivariate skew normal and folded bivariate skew normal distributions are compared; 180.1524 to 299.9072 when the truncated bivariate skew t and folded bivariate skew t distributions are compared; 1275.391 to 2159.847 when the truncated bivariate logistic and folded bivariate logistic distributions are compared; 1243.677 to 2298.231 when the truncated bivariate Laplace and folded bivariate Laplace distributions of the first kind are compared; 401.2748 to 644.6660 when the truncated bivariate Kotz type and folded bivariate Kotz type distributions are compared; 4229.656 to 7849.704 when the truncated bivariate Laplace and folded bivariate Laplace distributions of the second kind are compared. This is compelling evidence that the folded distributions are much better models.

The fit of the eight folded distributions were compared in terms of log-likelihood values as well as the Akaike information criterion due to Akaike [2], the Bayesian information criterion due to Schwarz [47], the consistent Akaike information criterion (CAIC) due to Bozdogan [7], the corrected Akaike information criterion (AICc) due to Hurvich and Tsai [30], the Hannan-Quinn criterion due to Hannan and Quinn [29] and p-values of the chisquared goodness of fit statistic.

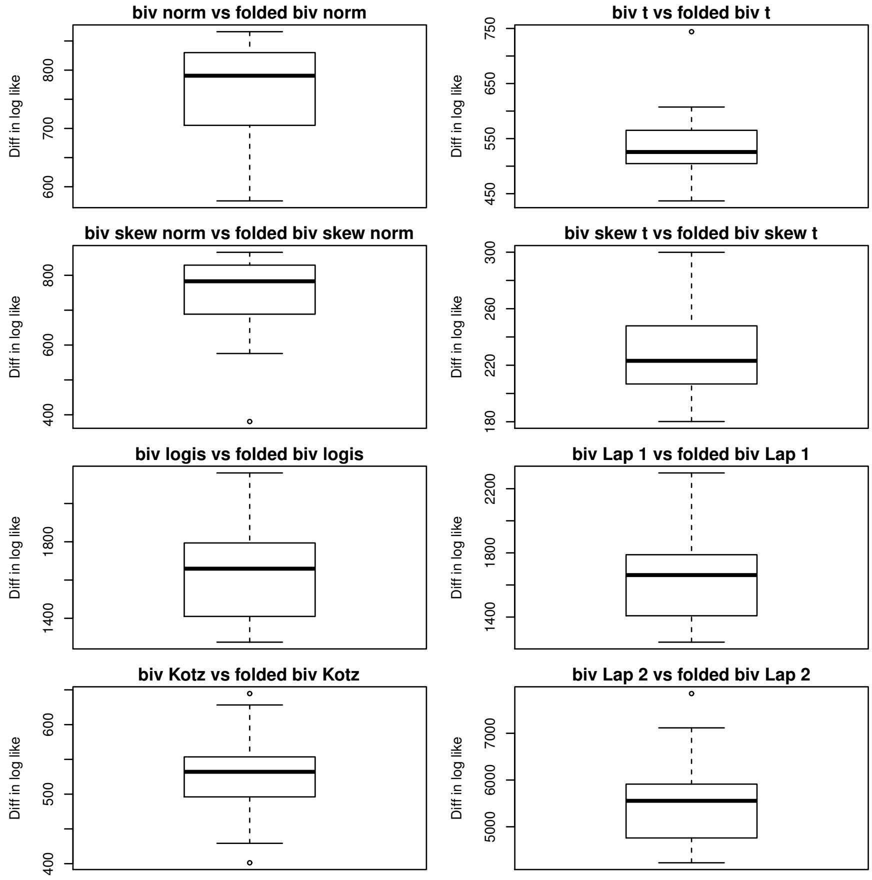

Figure 1: Boxplots of the differences between the log-likelihood values under the folded and truncated unfolded distributions for the forty five pairs.

The log-returns for each country exhibit heavy tails, presence of heavy tails was tested using the methods in Koning and Peng [35]. The p-values not reported here showed no evidence against heavy tails for each country. Hence, the log-returns for each pair of countries should be expected to exhibit heavy tails too. Of the eight distributions in Section 2, only the folded bivariate t and folded bivariate skew t distributions exhibit heavy tails.

For the following pairs (USA, CAD), (USA, ARG), (USA, SA), (USA, NG), (CAD, UK), (CAD, GER), (CAD, CHI), (CAD, JPN), (UK, GER), (UK, BRA), (UK, ARG), (GER, CHI), (GER, ARG), (CHI, JPN), (CHI, SA), (CHI, NG), (JPN, BRA), (JPN, ARG), (JPN, SA), (BRA, NG), (ARG, SA) and (SA, NG), the folded bivariate t distribution gave the best fit. The bivariate t distribution is heavy tailed but is also symmetric. The tests for bivariate symmetry (Snijders [48]) of the log-returns for these pairs showed no evidence against symmetry, see Table 3. This explains why the bivariate skew t distribution, a heavy tailed distribution accommodating for asymmetry, did not provide better fits for these pairs. None of the other distributions gave a significant p-value at the five percent significance level for each of these pairs.

For the following pairs (USA, UK), (USA, GER), (USA, CHI), (USA, JPN), (USA, BRA), (CAD, BRA), (CAD, ARG), (CAD, NG), (UK, SA), (GER, JPN), (GER, BRA), (GER, SA), (GER, NG), (CHI, ARG), (JPN, NG), (BRA, ARG), (BRA, SA) and (ARG, NG), the folded bivariate skew t distribution gave the best fit. The tests for bivariate symmetry of the log-returns for these pairs showed evidence against symmetry, see Table 3. This explains why the bivariate t distribution, a heavy tailed but symmetric distribution, did not provide better fits for these pairs. Again none of the other distributions gave a significant p-value at the five percent significance level for each of these pairs.

For the five remaining pairs of countries neither of the two heavy tailed distributions gave the best fit. For the pairs (CAD, SA), (UK, CHI), (UK, JPN) and (CHI, BRA), the folded bivariate Laplace distribution of the first kind gave the best fit. For the pair (UK, NG), the folded bivariate Kotz type distribution gave the best fit. The bivariate Laplace and Kotz type distributions are light tailed and are symmetric. We have not been to explain why these pairs were not best fitted by a heavy tailed distribution. However, the folded bivariate t distribution gave the second smallest values for AIC, BIC, CAIC, AICc, HQC and the second largest p-value for each of these pairs. Furthermore, the tests for bivariate symmetry for these pairs did not show evidence against symmetry, see Table 3.

Because of space concerns and to avoid repetitive discussion, we give details for only one of the forty five pairs, (USA, CAD). Table 1 gives the parameter estimates and standard errors for the fit of the eight folded distributions. Table 2 gives the log-likelihood values, AIC values, BIC values, CAIC values, AICc values, HQC values and p-values of the chisquared goodness of fit statistic for the fit of the eight folded distributions. We see that the folded bivariate t distribution gives the smallest values for AIC, BIC, CAIC, AICc and HQC and the largest p-value. The folded bivariate skew t distribution gives the second smallest values for AIC, BIC, CAIC, AICc and HQC. Contours of the joint probability density function of the best fitting distribution are shown in Figure 2. Also shown in this figure are the actual data values. The fit appears reasonable.

The magnitude correlations based on the best fits for the forty five pairs are given in Table 3. Also given in the table are p-values of the likelihood ratio test of the hypothesis that the absolute values of the components in each pair are independent. All of the correlations appear positive. This is expected since global economies are so inter dependent these days. One would not expect large stock values in magnitude for one country to be associated with small stock values in magnitude for another country or small stock values in magnitude for one country to be associated with large stock values in magnitude for another country. We also see that all of the correlations are significant except for (BRA, NG) and (ARG, NG). The strongest positive and significant correlations are for (UK, GER), (UK, SA) and (USA, CAD). The weakest positive and significant correlations are for (USA, NG), (CAD, NG), (UK, NG), (GER, NG), (CHI, NG), (JPN, NG) and (SA, NG).

We now give predictions based on bivariate value at risk curves. Under the folded t distribution, a bivariate value at risk curve with probability p is the solution of

$$\int_0^x \int_0^y f(u,v)\,dv\,du = p, \tag{3.1}$$

where $f(u,v)$ is given by (2.3) with $(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho,\nu)$ replaced by $(\widehat{\mu_1},\widehat{\mu_2},\widehat{\sigma_1},\widehat{\sigma_2},\widehat{\rho},\widehat{\nu})$, the maximum likelihood estimates of $(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho,\nu)$. So, (x,y) satisfying (3.1) can be interpreted as the positive log returns for (USA, CAD) occurring with probability p. The curves of (x,y) are plotted in Figure 3 for $p=0.9,0.95,0.99$.

Table 1: Fitted models, parameter estimates and standard errors for USA / CAD.

Model	Parameter estimates (ses)
folded biv norm	$\widehat{\mu_1}=2.598\times10^{-5}\left(1.102\times10^{-2}\right)$, $\widehat{\mu_2}=1.215\times10^{-4}\left(6.726\times10^{-3}\right)$, $\widehat{s_1}=1.374\times10^{-2}\left(2.724\times10^{-4}\right)$, $\widehat{s_2}=1.410\times10^{-2}\left(2.844\times10^{-4}\right)$, $\widehat{\rho}=7.102\times10^{-1}\left(1.630\times10^{-2}\right)$
folded biv t	$\widehat{\mu_1}=2.112\times10^{-3}\left(1.037\times10^{-3}\right)$, $\widehat{\mu_2}=5.417\times10^{-3}\left(6.712\times10^{-4}\right)$, $\widehat{s_1}=8.997\times10^{-3}\left(4.028\times10^{-4}\right)$, $\widehat{s_2}=8.852\times10^{-3}\left(5.069\times10^{-4}\right)$, $\widehat{\rho}=6.842\times10^{-1}\left(3.804\times10^{-2}\right)$, $\widehat{\nu}=3.589\left(3.350\times10^{-1}\right)$
folded biv skew norm	$\widehat{\mu_1}=2.599\times10^{-5}\left(1.437\times10^{-2}\right)$, $\widehat{\mu_2}=1.215\times10^{-4}\left(2.831\times10^{-3}\right)$, $\widehat{s_1}=1.374\times10^{-2}\left(2.724\times10^{-4}\right)$, $\widehat{s_2}=1.410\times10^{-2}\left(3.447\times10^{-4}\right)$, $\widehat{\rho}=7.102\times10^{-1}\left(1.989\times10^{-2}\right)$, $\widehat{\alpha_1}=1.789\times10^{-11}(3.257)$, $\widehat{\alpha_2}=-1.953\times10^{-12}\left(1.823\times10^{-1}\right)$
folded biv skew t	$\widehat{\mu_1}=2.112\times10^{-3}\left(1.354\times10^{-3}\right)$, $\widehat{\mu_2}=5.417\times10^{-3}\left(8.358\times10^{-4}\right)$, $\widehat{s_1}=8.997\times10^{-3}\left(3.266\times10^{-4}\right)$, $\widehat{s_2}=8.852\times10^{-3}\left(2.341\times10^{-4}\right)$, $\widehat{\rho}=6.842\times10^{-1}\left(3.144\times10^{-2}\right)$, $\widehat{\alpha_1}=0.000\left(6.279\times10^{-1}\right)$, $\widehat{\alpha_2}=0.000\left(6.301\times10^{-1}\right)$, $\widehat{\nu}=3.589\left(3.283\times10^{-1}\right)$
folded biv logis	$\widehat{\mu_1}=-3.405\times10^{-3}(2.427)$, $\widehat{\mu_2}=2.300\times10^{-3}(2.863)$, $\widehat{s_1}=60.4(4962.5)$, $\widehat{s_2}=72.3(6761.5)$
folded biv Lap 1	$\widehat{\mu_1}=13.4(8.4)$, $\widehat{\mu_2}=-13.0(2.3)$, $\widehat{s_1}=30.8(55.1)$, $\widehat{s_2}=34.7(77.3)$, $\widehat{\rho}=9.999\times10^{-1}\left(3.204\times10^{-2}\right)$, $\widehat{r}=3.870\times10^{-3}\left(3.202\times10^{-6}\right)$
folded biv Kotz	$\widehat{\mu_1}=4.678\times10^{-3}\left(4.393\times10^{-3}\right)$, $\widehat{\mu_2}=6.530\times10^{-3}\left(8.504\times10^{-3}\right)$, $\widehat{s_1}=9.928\times10^{-3}\left(2.001\times10^{-2}\right)$, $\widehat{s_2}=9.500\times10^{-3}\left(1.003\times10^{-3}\right)$, $\widehat{\rho}=5.823\times10^{-1}\left(5.332\times10^{-2}\right)$, $\widehat{r}=200.3(6.2)$, $\widehat{s}=5.821\times10^{-2}(11.2)$, $\widehat{N}=6.838(4.522)$
folded biv Lap 2	$\widehat{\mu_1}=1.726\times10^{-4}\left(1.912\times10^{-3}\right)$, $\widehat{\mu_2}=-2.919\times10^{-3}\left(2.455\times10^{-3}\right)$, $\widehat{s_1}=9.928\times10^{-3}\left(7.655\times10^{-2}\right)$, $\widehat{s_2}=9.500\times10^{-3}\left(5.366\times10^{-3}\right)$, $\widehat{\rho}=9.999\times10^{-1}\left(9.022\times10^{-2}\right)$

Table 2: Fitted models, log-likelihood values and selection criteria for USA / CAD.

Model	$-\ln L$	AIC	BIC	CAIC	AICc	HQC	p-value
folded biv norm	−10210.2	−20410.4	−20384.1	−20379.1	−20410.3	−20400.6	0.04
folded biv t	−10418.3	−20824.6	−20793.1	−20787.1	−20824.5	−20812.8	0.23
folded biv skew norm	−10210.2	−20406.4	−20369.6	−20362.6	−20406.3	−20392.7	0.05
folded biv skew t	−10418.3	−20820.6	−20778.6	−20770.6	−20820.5	−20804.9	0.23
folded biv logis	1710.3	3428.5	3449.5	3453.5	3428.5	3436.4	0.01
folded biv Lap 1	−6313.4	−12614.9	−12583.4	−12577.4	−12614.8	−12603.1	0.02
folded biv Kotz	−10406.9	−20797.8	−20755.8	−20747.8	−20797.7	−20782.1	0.04
folded biv Lap 2	−4518.0	−9026.0	−8999.7	−8994.7	−9025.9	−9016.2	0.03

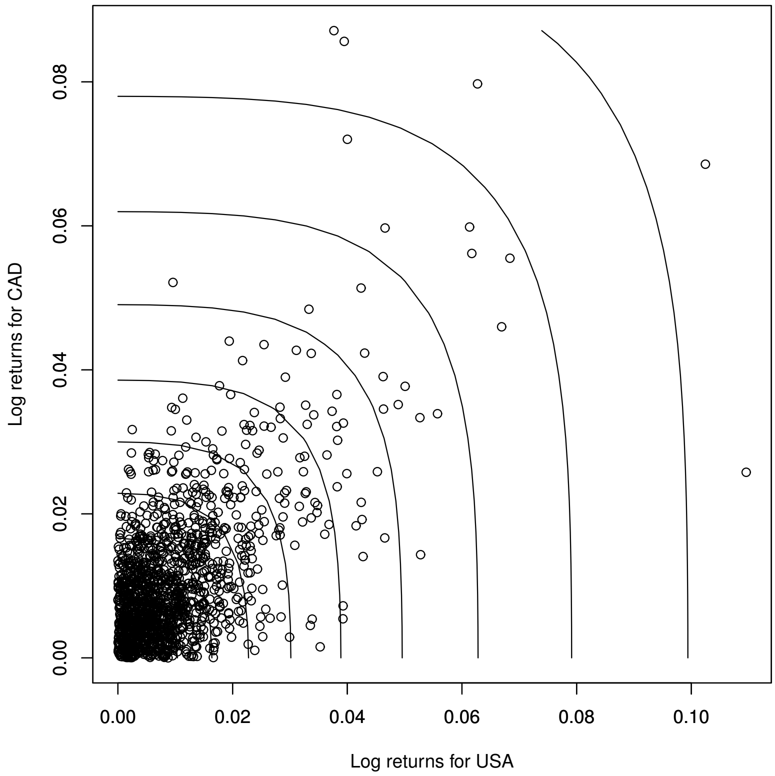

Figure 2: Contours of the joint probability density function of the fitted folded t distribution for (USA, CAD).

Finally, we check robustness of the fitted models by splitting the data into two halves. The first half was taken to be the data from 3rd January 2000 to 31 December 2007. The second half was taken to be the data from 1st January 2008 to 28th February 2014. We fitted the same models to each half. The results turned out to be the same as before. The folded bivariate t distribution gave the best fit for (USA, CAD), (USA, ARG), (USA, SA), (USA, NG), (CAD, UK), (CAD, GER), (CAD, CHI), (CAD, JPN), (UK, GER), (UK, BRA), (UK, ARG), (GER, CHI), (GER, ARG), (CHI, JPN), (CHI, SA), (CHI, NG), (JPN, BRA), (JPN, ARG), (JPN, SA), (BRA, NG), (ARG, SA) and (SA, NG) for each half. The folded bivariate skew t distribution gave the best fit for (USA, UK), (USA, GER), (USA, CHI), (USA, JPN), (USA, BRA), (CAD, BRA), (CAD, ARG), (CAD, NG), (UK, SA), (GER, JPN), (GER, BRA), (GER, SA), (GER, NG), (CHI, ARG), (JPN, NG), (BRA, ARG), (BRA, SA) and (ARG, NG) for each half. The folded bivariate Laplace distribution of the first kind gave the best fit for (CAD, SA), (UK, CHI), (UK, JPN) and (CHI, BRA) for each half. The folded bivariate Kotz type distribution gave the best fit for (UK, NG) for each half. The explanations for these best fits are the same as before.

Table 3: Estimated magnitude correlations, test for independence and test for bivariate symmetry.

Pair	Magnitude correlation	p-value for independence	p-value for symmetry
(USA, CAD)	0.596	0.000	0.112
(USA, UK)	0.499	0.000	0.006
(USA, GER)	0.533	0.000	0.048
(USA, CHI)	0.298	0.000	0.049
(USA, JPN)	0.153	0.000	0.001
(USA, BRA)	0.531	0.000	0.034
(USA, ARG)	0.317	0.000	0.086
(USA, SA)	0.366	0.000	0.057
(USA, NG)	0.049	0.003	0.074
(CAD, UK)	0.575	0.000	0.065
(CAD, GER)	0.509	0.000	0.095
(CAD, CHI)	0.342	0.000	0.077
(CAD, JPN)	0.189	0.000	0.084
(CAD, BRA)	0.507	0.000	0.021
(CAD, ARG)	0.312	0.000	0.013
(CAD, SA)	0.506	0.000	0.090
(CAD, NG)	0.057	0.001	0.004
(UK, GER)	0.743	0.000	0.068
(UK, CHI)	0.380	0.000	0.082
(UK, JPN)	0.210	0.000	0.062
(UK, BRA)	0.501	0.000	0.067
(UK, ARG)	0.268	0.000	0.081
(UK, SA)	0.626	0.000	0.042
(UK, NG)	0.051	0.002	0.066
(GER, CHI)	0.316	0.000	0.060
(GER, JPN)	0.175	0.000	0.006
(GER, BRA)	0.479	0.000	0.059
(GER, ARG)	0.259	0.000	0.055
(GER, SA)	0.529	0.000	0.008
(GER, NG)	0.043	0.009	0.002
(CHI, JPN)	0.370	0.000	0.075
(CHI, BRA)	0.306	0.000	0.076
(CHI, ARG)	0.152	0.000	0.034
(CHI, SA)	0.372	0.000	0.058
(CHI, NG)	0.064	0.000	0.091
(JPN, BRA)	0.145	0.000	0.094
(JPN, ARG)	0.097	0.000	0.056
(JPN, SA)	0.236	0.000	0.075
(JPN, NG)	0.075	0.000	0.009
(BRA, ARG)	0.351	0.000	0.028
(BRA, SA)	0.435	0.000	0.019
(BRA, NG)	0.022	0.191	0.077
(ARG, SA)	0.215	0.000	0.083
(ARG, NG)	0.020	0.222	0.012
(SA, NG)	0.041	0.012	0.067

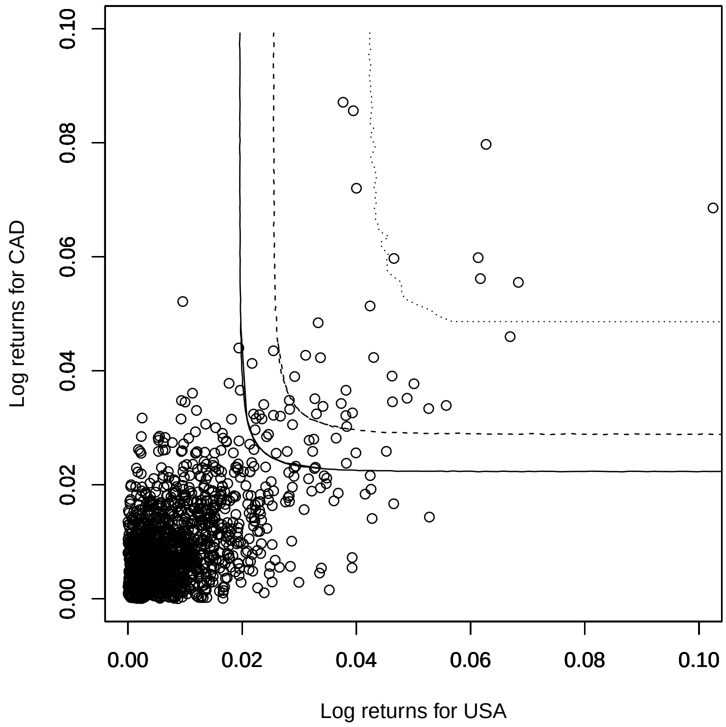

Figure 3: Value at risk curves of the fitted folded t distribution at $p = 0.9, 0.95, 0.99$ for (USA, CAD).

4. CONCLUSIONS

Motivated by the concept of magnitude correlation of stock returns, we have introduced the following folded bivariate distributions: the folded bivariate skew normal distribution; the folded bivariate skew t distribution; the folded bivariate logistic distribution; the folded bivariate Kotz type distribution; the folded bivariate Laplace distribution of the first kind; the folded bivariate Laplace distribution of the second kind. We have also introduced the following folded univariate distributions: the folded univariate skew normal distribution; the folded univariate skew t distribution.

We fitted eight folded bivariate distributions to forty five real data sets. The two heavy tailed distributions, the folded bivariate t and folded bivariate skew t distributions, gave the best fit for forty of the data sets. The remaining five data sets were best fitted by folded bivariate Laplace distribution of the first kind and the folded bivariate Kotz type distribution, two of the lighted tailed distributions. We have not been able to explain why these five data sets were best fitted by light tailed distributions when all of the data sets are heavy tailed.

We also compared the fits of the folded and truncated unfolded distributions using the same data sets. Remarkably each folded distribution outperformed the corresponding truncated unfolded distribution for each of the forty five data sets. This shows that magnitude correlations can be better modeled by folded bivariate distributions.

A future work is to extend the results of this paper for folded multivariate distributions, folded matrix variate distributions and folded complex variate distributions.

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