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On Analyzing Non-Monotone Failure Data

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Abstract:

- A new two-parameter distribution is defined for modeling non-monotone lifetime data. It is constructed based on the logistic-G family and the exponential distribution. Its hazard rate properties are different than those of the well-known distributions. Some of its statistical properties are presented. An extended regression based on the logarithm of the random variable of the introduced distribution is defined. The new regression can provide better fits than other special regressions for analyzing real data. The performance of the maximum likelihood estimates is investigated from a simulation study. Three lifetime data sets are used to prove empirically the usefulness of the new models.

Keywords:

- *bathtub failure rate; exponential distribution; logistic distribution; maximum likelihood estimation; regression model.*

AMS Subject Classification:

- 60E05, 62N05, 62F10.

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1. INTRODUCTION

Recently, many distributions have been defined for modeling lifetime data. The Weibull distribution has survival and hazard rate functions in closed-forms; see Murthy *et al.* [14]. Gupta and Kundu [8] introduced the exponentiated exponential (EE) distribution as an alternative to the gamma and Weibull distributions. It has many properties similar to those of the gamma and Weibull with closed-form survival and hazard rate functions; see Gupta and Kundu [9]. The hazard rate functions (hrfs) of the gamma, Weibull and EE distributions can not be upside-down bathtub and bathtub shapes but only monotonically increasing, monotonically decreasing or constant shapes.

Taking into account these points, we define a new two-parameter alternative to the above distributions to overcome the above-mentioned drawback. Further, it is common in practical situations to use an appropriate regression based on an asymmetric distribution for censored data and survival time data. Recently, various papers have been published on that subject such as those by Lanjoni *et al.* [10], Cordeiro *et al.* [5], among others. Another objective of this work is to propose a location-scale regression based on the logistic-exponential distribution named the log-logistic exponential regression. It is a new regression that can be applied to data sets with the presence of censored data.

The paper is outlined as follows. In Section 2, the new *logistic-G* (LG) family is introduced and some of its structural properties are studied. A special model of the LG family called the *logistic-exponential* (LE) distribution is presented in Section 3. Some of its mathematical properties are addressed in Section 4. The parameters of the LE distribution are estimated by maximum likelihood (ML) in Section 5. Further, a Monte Carlo simulation study is conducted to assess the performance of the ML method. An extended regression model is proposed and studied in Section 6. In Section 7, the usefulness of the new models is shown empirically by means of three real data sets. Finally, Section 8 offers some concluding remarks.

2. THE NEW LG FAMILY

Alzaatreh *et al.* [2] defined the *T-X family* of distributions as follows. Let $r(t)$ be the probability density function (pdf) of a random variable (rv) $T \in [a,b]$ for $-\infty \le a < b < \infty$ and let $W(\cdot)\colon [0,1] \to \mathbb{R}$ be an adequate link function. The cumulative distribution function (cdf) of the *T-X family* is

$$F(x;\boldsymbol{\xi}) = \int_a^{W\left[G(x;\boldsymbol{\xi})\right]} r(t)\,dt\,,$$

where $\boldsymbol{\xi}$ is the parameter vector of G.

Based on the above definition, if the function $W\left[G(x;\boldsymbol{\xi})\right]$ is monotonically non-increasing with $W(0) \to b$ and $W(1) \to a$, one can redefine the *T-X family* cdf as

$$F(x;\boldsymbol{\xi}) = 1 - \int_a^{W\left[G(x;\boldsymbol{\xi})\right]} r(t)\,dt\,. \tag{2.1}$$

Let T be a logistic rv with pdf $r(t) = \alpha\,\mathrm{e}^{-\alpha t}(1+\mathrm{e}^{-\alpha t})^{-2}$ and support in $\mathbb{R}$, where $\alpha > 0$. By setting $W\big[G(x;\boldsymbol{\xi})\big] = \log\big\{-\log\big[G(x;\boldsymbol{\xi})\big]\big\}$, a monotonically non-increasing function in $G(x;\boldsymbol{\xi})$, the cdf of the LG family follows from (2.1):

$$F(x;\alpha,\boldsymbol{\xi}) \,=\, 1 - \left[1 + \Big\{-\log\big[G(x;\boldsymbol{\xi})\big]\Big\}^{-\alpha}\right]^{-1}, \qquad x \in \mathbb{R}\,. \tag{2.2}$$

If $g(x;\boldsymbol{\xi}) = dG(x;\boldsymbol{\xi})/dx$, the associated pdf to (2.2) is

$$f(x;\alpha,\boldsymbol{\xi}) \,=\, \frac{\alpha\, g(x;\boldsymbol{\xi}) \Big\{-\log\big[G(x;\boldsymbol{\xi})\big]\Big\}^{-\alpha-1}}{G(x;\boldsymbol{\xi}) \left[1 + \Big\{-\log\big[G(x;\boldsymbol{\xi})\big]\Big\}^{-\alpha}\right]^{2}}\,. \tag{2.3}$$

The dependence on the baseline vector $\boldsymbol{\xi}$ and α is omitted and then $G(x) = G(x;\boldsymbol{\xi})$ and $f(x) = f(x;\alpha,\boldsymbol{\xi})$. Hereafter, a rv with pdf (2.3) is denoted by $X \sim \mathrm{LG}(\alpha,\boldsymbol{\xi})$.

The hrf of X has the form

$$h(x) \,=\, \frac{\alpha\, g(x) \Big\{-\log\big[G(x)\big]\Big\}^{-\alpha-1}}{G(x) \left[1 + \Big\{-\log\big[G(x)\big]\Big\}^{-\alpha}\right]}\,. \tag{2.4}$$

The quantile function (qf) of X follows by inverting $F(x) = u$ in (2.2):

$$Q(u) \,=\, Q_G\big(\mathrm{e}^{-v}\big)\,, \tag{2.5}$$

where $Q_G(v) = G^{-1}(v)$ is the parent qf and $v = \big[(1-u)/u\big]^{1/\alpha}$. Then, the solution of the nonlinear equation $X = Q(U)$ has density (2.3) if U has a uniform $U(0,1)$ distribution.

Equation (2.5) gives a simple interpretation for the LG family. If T has a logistic density $r(t)$ with shape parameter α, the LG family is obtained from the qf of the G distribution by $X = Q_G\big(\mathrm{e}^{-\mathrm{e}^{T}}\big)$.

Proposition 2.1. *Let $c = \inf\big\{x\colon G(x) > 0\big\}$. The asymptotics of Equations (2.2), (2.3) and (2.4) when $x \to c$ are:*

$$F(x) \,\sim\, \Big\{-\log\big[G(x)\big]\Big\}^{-\alpha},$$

$$f(x) \,\sim\, \frac{\alpha\, g(x)}{G(x)} \Big\{-\log\big[G(x)\big]\Big\}^{-\alpha-1},$$

$$h(x) \,\sim\, \frac{\alpha\, g(x)}{G(x)} \Big\{-\log\big[G(x)\big]\Big\}^{-\alpha-1}.$$

Proposition 2.2. *The asymptotics of Equations (2.2), (2.3) and (2.4) when $x \to \infty$ are given by*

$$1 - F(x) \sim \bar{G}(x)^{\alpha}\,, \qquad f(x) \sim \alpha\, g(x)\, \bar{G}(x)^{\alpha-1} \qquad \text{and} \qquad h(x) \sim \frac{\alpha\, g(x)}{\bar{G}(x)}\,.$$

Theorem 2.1. *The Shannon's entropy of the LG family takes the form*

$$\eta_X = E\left[\log\left\{g\left[G^{-1}\left(\mathrm{e}^{-\mathrm{e}^T}\right)\right]\right\}\right] - B\left(1-\frac{1}{\alpha}, 1+\frac{1}{\alpha}\right) - \log\alpha + 2\,, \tag{2.6}$$

where $B(\cdot,\cdot)$ is the beta function.

Proof: Alzaatreh *et al.* [2] obtained the Shannon entropy of the T-X family, where $W\left[G(x)\right] = -\log\left[1-G(x)\right]$. One can use their same technique to obtain this entropy for the LG family in (2.2) when $W\left[G(x)\right] = \log\left\{-\log\left[G(x)\right]\right\}$ as

$$\eta_X = \mathbb{E}\left[\log\left\{g\left[G^{-1}\left(\mathrm{e}^{-\mathrm{e}^T}\right)\right]\right\}\right] - \mathbb{E}\left(\mathrm{e}^T\right) + \mu_T + \eta_T\,, \tag{2.7}$$

where μ_T and η_T are the mean and Shannon entropy of the rv T, respectively. If T has the logistic distribution, (2.6) follows easily from (2.7). □

2.1. Linear representation

We can rewrite Equation (2.2) as

$$F(x) = \frac{\left\{-\log\left[G(x)\right]\right\}^{-\alpha}}{1+\left\{-\log\left[G(x)\right]\right\}^{-\alpha}}\,. \tag{2.8}$$

The power series $\left\{-\log\left[G(x)\right]\right\}^{-\alpha} = \sum_{k=0}^{\infty} p_k\left[1-G(x)\right]^k$ holds, where $p_0 = 1$, $p_1 = -\alpha/2$, $p_2 = (3\,\alpha^2 - 5\,\alpha)/24$, $p_3 = (-\alpha^3 + 5\,\alpha^2 - 6\,\alpha)/48$, etc. The radius of convergence of this series is infinite for $0 < G(x) < 1$ and then it converges for all real numbers x with great rapidity.

Then, we can express Equation (2.8) as a ratio of two convergent power series of $G(x)$:

$$F(x) = \frac{\sum_{k=0}^{\infty} p_k\left[1-G(x)\right]^k}{\sum_{k=0}^{\infty} q_k\left[1-G(x)\right]^k} = \sum_{k=0}^{\infty} b_k\left[1-G(x)\right]^k.$$

Here, $q_0 = 1 + p_0$, $b_0 = p_0/q_0$ and, for $k \geq 1$, $q_k = p_k$ and

$$b_k = \frac{1}{q_0}\left(p_k - \frac{1}{q_0}\sum_{r=1}^{k} q_r\, b_{k-r}\right).$$

Further, $F(x)$ can be rewritten as

$$F(x) = \sum_{k=0}^{\infty} b_k\left[1-G(x)\right]^k = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} (-1)^j\, b_k \binom{k}{j} G(x)^j$$

and then

$$F(x) = \sum_{j=0}^{\infty} d_j \, G(x)^j \,, \tag{2.9}$$

where $d_j = \sum_{k=j}^{\infty} (-1)^j \, b_k \binom{k}{j}$ and $G(x)^j$ denotes the exponentiated-G ("exp-G" for short) cdf with power parameter j.

Hence, the density of X has a linear representation in terms of exp-G densities, namely

$$f(x) = \sum_{j=0}^{\infty} d_{j+1} \, h_{j+1}(x) \,, \tag{2.10}$$

where $h_{j+1}(x) = (j+1) \, g(x) \, G(x)^j$ is the exp-G density with power parameter $j+1$. Some exp-G properties are addressed in more than 50 papers cited by Tahir and Nadarajah [19].

Clearly, some mathematical properties of the LG family can be derived from Equation (2.10) and those exp-G properties.

2.2. Moments

Let Y_{j+1} be a rv having density $h_{j+1}(x)$. The n-th moment of X follows from (2.10) as

$$\mathbb{E}(X^n) = \sum_{j=0}^{\infty} d_{j+1} \, \mathbb{E}(Y_{j+1}^n) = \sum_{j=0}^{\infty} (j+1) \, d_{j+1} \, \tau_{n,j} \,, \tag{2.11}$$

where $\tau_{n,j} = \int_{-\infty}^{\infty} x^n \, G(x)^j \, g(x) \, dx = \int_0^1 Q_G(u)^n \, u^j \, du$. Cordeiro and Nadarajah [4] determined the quantity $\tau_{n,j}$ for the normal, beta, gamma and Weibull distributions. Their developments can be used to other distributions.

The n-th incomplete moment of X, say $m_n(y) = \int_0^y x^n \, f(x) \, dx$, is given by

$$\begin{aligned} m_n(y) &= \sum_{j=0}^{\infty} d_{j+1} \int_0^y x^n \, h_{j+1}(x) \, dx \\ &= \sum_{j=0}^{\infty} (j+1) \, d_{j+1} \int_0^{G(y)} Q_G(u)^n \, u^j \, du \,. \end{aligned} \tag{2.12}$$

The main application of the first incomplete moment $m_1(y)$ refers to the deviations from the mean and median and the Bonferroni and Lorenz curves of X. A further important application is related to the mean residual life (MRL) of X, i.e. the function measuring the remaining life expectancy at age t, given by $\nu(t) = \big[1 - m_1(t)\big] / \big[1 - F(t)\big] - t$. This function is like the density and generating functions: for a distribution with a finite mean, it completely determines the distribution. The use of the MRL is a helpful tool in model building.

2.3. Generating function

The moment generating function (mgf) $M(t) = \mathbb{E}(e^{tX})$ of X can be determined from (2.10) as

$$M(t) \;=\; \sum_{j=0}^{\infty} d_{j+1}\, M_{j+1}(t) \;=\; \sum_{i=0}^{\infty} (j+1)\, d_{j+1}\, \rho(t,j), \tag{2.13}$$

where $M_{j+1}(t)$ is the mgf of Y_{j+1} and $\rho(t,j) = \int_0^1 \exp\big[t\, Q_G(u)\big]\, u^j\, du$.

Hence, $M(t)$ can be determined from the exp-G generating function. The characteristic function of X is simply $M(-\mathbf{i}\, t)$, where $\mathbf{i} = \sqrt{-1}$, and it always exists, even when the generating function does not.

3. THE LE DISTRIBUTION

Consider the baseline exponential with cdf $G(x) = 1 - e^{-\lambda x}$. The cdf of the LE distribution can be determined from (2.2) as

$$F(x) \;=\; F(x;\alpha,\lambda) \;=\; 1 - \left[1 + \Big\{-\log\big(1-e^{-\lambda x}\big)\Big\}^{-\alpha}\right]^{-1}. \tag{3.1}$$

Hereafter, let $X \sim \mathrm{LE}(\alpha,\lambda)$ have the cdf (3.1). The pdf of X is

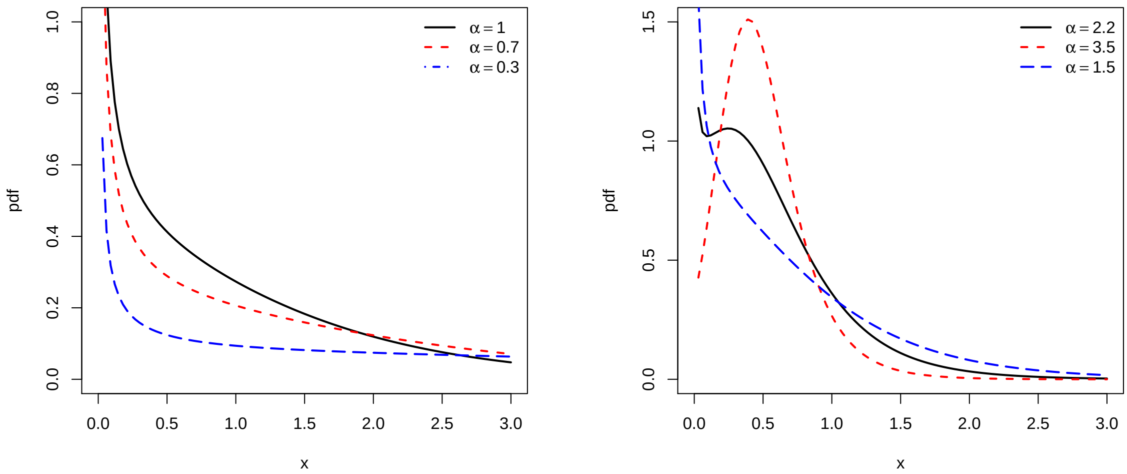
$$f(x) \;=\; \frac{\alpha\, \lambda \Big\{-\log\big(1-e^{-\lambda x}\big)\Big\}^{-\alpha-1}}{\big(e^{\lambda x}-1\big)\left[1 + \Big\{-\log\big(1-e^{-\lambda x}\big)\Big\}^{-\alpha}\right]^{2}}\,. \tag{3.2}$$

The hrf of X becomes

$$h(x) \;=\; \frac{\alpha\, \lambda \Big\{-\log\big(1-e^{-\lambda x}\big)\Big\}^{-\alpha-1}}{\big(e^{\lambda x}-1\big)\left[1 + \Big\{-\log\big(1-e^{-\lambda x}\big)\Big\}^{-\alpha}\right]}\,. \tag{3.3}$$

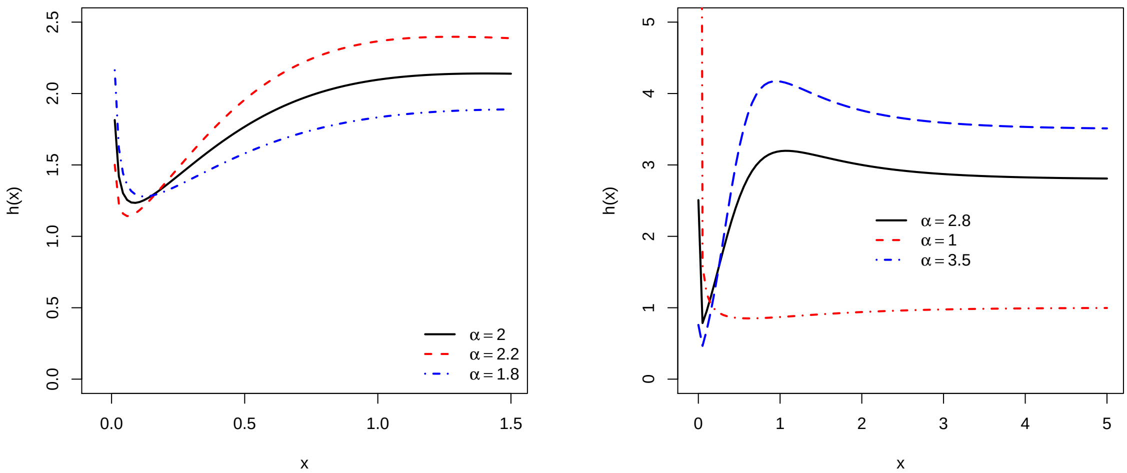
Equation (3.1) has two parameters α and λ such as the gamma, log-normal, Weibull and EE distributions. The LE model has closed-form survival and hazard functions like the Weibull and EE distributions.

Figures 1 and 2 display some plots of the density and hrf of X for selected values of α when $\lambda = 1$. Figure 1 shows that the LE density is a right-skewed distribution. The plots in Figure 2 indicate that the hrf of X can have decreasing failure rate (DFR), bathtub (BT) and decreasing-increasing-decreasing (DID) shapes. The limiting behavior of this hrf is $\lim_{x\to\infty} h(x) = \alpha$ and $\lim_{x\to 0} h(x) = \infty$, and it always approaches α when X goes to infinity.



(**a**) (**b**)

Figure 1: Plots of the LE density varying α with $\lambda = 1$.



(**a**) (**b**)

Figure 2: Plots of the LE hrf varying α for $\lambda = 1$.

4. PROPERTIES OF LE DISTRIBUTION

In this section, we obtain some properties of the LE distribution.

4.1. Asymptotics and shapes

Proposition 4.1. *The asymptotics of the cdf, pdf and hrf of X when $x \to 0$ are:*

$$\begin{aligned} F(x) &\sim 1-\Big\{1+\big[-\log(\lambda\,x)\big]^{-\alpha}\Big\}^{-1}, \\ f(x) &\sim \frac{\alpha}{x}\,\big[-\log(\lambda\,x)\big]^{-\alpha-1}\Big\{1+\big[-\log(\lambda\,x)\big]^{-\alpha}\Big\}^{-2}, \\ h(x) &\sim \frac{\alpha}{x}\,\big[-\log(\lambda\,x)\big]^{-\alpha-1}\Big\{1+\big[-\log(\lambda\,x)\big]^{-\alpha}\Big\}^{-1}. \end{aligned}$$

Proposition 4.2. *The asymptotics of the cdf, pdf and hrf of X when $x \to \infty$ are*

$$1-F(x) \sim \mathrm{e}^{-\alpha\lambda x}, \qquad f(x) \sim \alpha\lambda\,\mathrm{e}^{-\alpha\lambda x} \qquad \textit{and} \qquad h(x) \sim \alpha\lambda\,.$$

4.2. Transformation

If Y has the logistic distribution with parameter α, then $X=-\lambda^{-1}\log\big(1-\mathrm{e}^{-\mathrm{e}^{Y}}\big)$ follows the LE(α,λ) model.

4.3. Mode

Lemma 4.1. *The modes of the LE density are the solutions of $k(x)=0$, where*

$$k(x) \;=\; -\lambda-\frac{\lambda}{\mathrm{e}^{\lambda x}-1}\left[1-\frac{\alpha+1}{\Big\{-\log\big(1-\mathrm{e}^{-\lambda x}\big)\Big\}}+\frac{2\,\alpha\Big\{-\log\big(1-\mathrm{e}^{-\lambda x}\big)\Big\}^{-\alpha-1}}{1+\Big\{-\log\big(1-\mathrm{e}^{-\lambda x}\big)\Big\}^{-1}}\right].$$

4.4. Quantile function

The qf of X is $Q(u)=-\lambda^{-1}\log(1-\mathrm{e}^{-v})$, $u\in(0,1)$, where $v=\big[(1-u)/u\big]^{1/\alpha}$.

4.5. Shannon entropy

Theorem 4.1. *The Shannon entropy of X is*

$$\eta_X = \frac{\lambda}{\lambda-1} - B\left(1-\frac{1}{\alpha}, 1+\frac{1}{\alpha}\right) - \log\alpha + 2 . \tag{4.1}$$

Proof: For the LE distribution, the result holds:

$$\mathbb{E}\left[\log\left\{g\left[G^{-1}\left(\mathrm{e}^{-\mathrm{e}^{T}}\right)\right]\right\}\right] = \mathbb{E}(\mathrm{e}^{T}) = \frac{\lambda}{\lambda-1} .$$

Equation (4.1) follows by substituting the above result in (2.6). □

4.6. Moments and generating function

The LE density comes from (2.10) as

$$f(x) = \sum_{j=0}^{\infty} d_{j+1}\,(j+1)\,\lambda\,\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{j}.$$

The moments of X follow from the EE distribution and (2.11):

$$\mu'_n = \mathbb{E}(X^n) = n! \sum_{j,l=0}^{\infty} \frac{(-1)^l\,(j+1)\,d_{j+1}\,A(j,l)}{\lambda^{j+1}\,(l+1)^{n+1}} , \tag{4.2}$$

where $A(j,l) = j\,(j-1)\cdots(j-l)/l!$.

The skewness and kurtosis of X for some values of α by taking $\lambda=1$ are displayed in Figure 3. The distribution of X is right-skewed. For fixed λ, the skewness is a decreasing function of α, whereas the kurtosis decreases steadily towards asymptotic limits when α increases.

The n-th incomplete moment of X is obtained from (2.12):

$$m_n(y) = \lambda^{-n} \sum_{j=0}^{\infty} (j+1)\,d_{j+1}\,A^*_n(j+1) , \tag{4.3}$$

where

$$A^*_n(j+1) = \sum_{p=0}^{\infty} \frac{(-1)^p}{(p+1)^{r+1}} \binom{j}{p}\,\gamma\Big(n+1, (p+1)\,\lambda y\Big) , \qquad n = 1, 2, \ldots ,$$

and $\gamma(p,x) = \int_0^x w^{p-1}\,\mathrm{e}^{-w}\,dw$ (for $p>0$) is the incomplete gamma function.

The mgf of X follows from (2.13) as

$$M(t) = \Gamma\left(1-\frac{t}{\lambda}\right) \sum_{j=0}^{\infty} \frac{(j+1)!\,d_{j+1}}{\Gamma\left(j+2-\frac{t}{\lambda}\right)} . \tag{4.4}$$

Equations (4.2), (4.3) and (4.4) are the main results of this section.

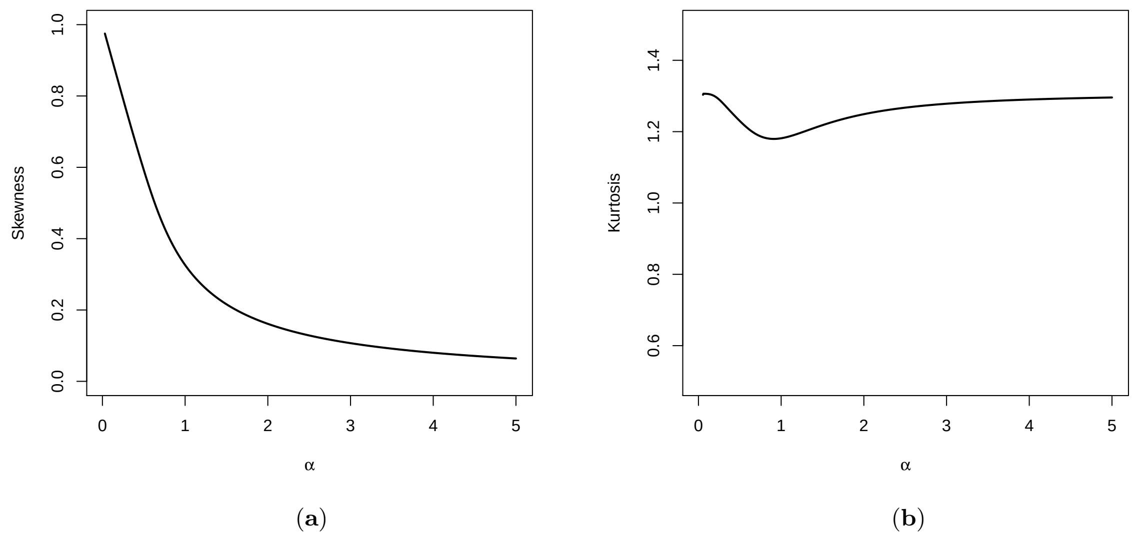


Figure 3: (a) Skewness and (b) Kurtosis plots of X for $\lambda=1$.

4.7. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, ..., X_n$ is a random sample from the LE distribution. Let $X_{i:n}$ denote the i-th order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) \;=\; K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1},$$

where $K = 1/B(i, n-i+1)$.

Gradshteyn and Ryzhik [7] provided a power series raised to a positive integer n:

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} b_{n,i}\, u^i\,, \tag{4.5}$$

where the coefficients $b_{n,i}$ (for $i=1,2,...$) satisfy the recurrence equation (with $b_{n,0} = a_0^n$)

$$b_{n,i} \;=\; (i\, a_0)^{-1} \sum_{m=1}^{i} \big[m\,(n+1) - i\big]\, a_m\, b_{n,i-m}\,.$$

The density function of $X_{i:n}$ can be reduced to

$$f_{i:n}(x) \;=\; \sum_{r,k=0}^{\infty} m_{r,k}\; \pi_{\mathrm{EE}}(x; \lambda, r+k+1)\,, \tag{4.6}$$

where $\pi_{\mathrm{EE}}(x; \lambda, r+k+1)$ (for $r,k \geq 0$) denotes the EE density function with parameters λ and $r+k+1$, and

$$m_{r,k} \;=\; \frac{n!\,(r+1)\,(i-1)!\, d_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, f_{j+i-1,k}}{(n-i-j)!\, j!}\,.$$

Here, d_r is defined in (2.9) and the quantities $f_{j+i-1,k}$ follow recursively from (for $k \geq 1$)

$$f_{j+i-1,k} = (k\, d_0)^{-1} \sum_{m=1}^{k} \big[m\,(j+i) - k\big]\, d_m\, f_{j+i-1,k-m}\,,$$

and $f_{j+i-1,0} = d_0^{j+i-1}$.

Equation (4.6) shows that the pdf of the LE order statistics is a double linear combination of EE densities. Therefore, several mathematical quantities of these order statistics can be derived from this result.

5. ESTIMATION

The maximum likelihood estimates (MLEs) enjoy desirable properties for constructing confidence intervals. We consider the estimation of the unknown parameters of the LE distribution by the maximum likelihood method. Further works could be addressed using different methods to estimate the LE parameters such as moments, least squares, weighted least squares, bootstrap, Jackknife, Cramér–von-Mises, Anderson–Darling, Bayesian, among others, and compare the estimators from these methods.

Let $x_1, ..., x_n$ be n observed values from the LE distribution given in Equation (3.2) with vector of parameters $\mathbf{\Theta} = (\alpha, \lambda)^{\top}$. The log-likelihood $\ell = \ell(\mathbf{\Theta})$ for $\mathbf{\Theta}$ is

$$\begin{aligned}\ell \;=\; & n\log(\alpha\lambda) - \sum_{i=1}^{n}\log\big(\mathrm{e}^{\lambda x_i}-1\big) - (\alpha+1)\sum_{i=1}^{n}\log\Big\{-\log\big(1-\mathrm{e}^{-\lambda x_i}\big)\Big\} \\ & - 2\sum_{i=1}^{n}\log\Big[1+\Big\{-\log\big(1-\mathrm{e}^{-\lambda x_i}\big)\Big\}^{-\alpha}\Big]\,. \end{aligned} \tag{5.1}$$

Equation (5.1) can be maximized either directly by using well-known platforms such as R (`optim` function), SAS (`PROC NLMIXED`) and Ox program (`MaxBFGS` subroutine).

5.1. Simulation results

We examine the accuracy of the MLEs of the parameters of the LE distribution using Monte Carlo simulations. The simulation analysis is carried out by generating 5,000 samples for some sample sizes and parameter combinations. Table 1 gives the average biases (Biases) of the MLEs, mean square errors (MSEs), coverage probabilities (CPs) and average widths (AWs) of 95% confidence intervals for α and λ. These results indicate that the MLEs are accurate. The biases, MSEs and AWs of X are small for large samples. Further, the CPs are quite close to the 95% nominal levels. So, we conclude that the MLEs can be used for estimating and constructing confidence intervals for the model parameters.

Table 1: Simulation results.

Parameter	n	$\alpha=0.3,\ \lambda=1$				$\alpha=0.8,\ \lambda=1$			
		Bias	MSE	CP	AW	Bias	MSE	CP	AW
α	25	−0.006	0.003	0.92	0.210	−0.091	0.029	0.92	0.599
	50	−0.004	0.002	0.93	0.149	−0.070	0.016	0.96	0.437
	75	−0.004	0.001	0.94	0.121	−0.073	0.013	0.96	0.354
	100	−0.002	0.001	0.95	0.106	−0.067	0.011	0.95	0.309
λ	25	0.013	0.018	0.93	0.510	−0.039	0.083	0.95	0.988
	50	0.011	0.009	0.95	0.360	−0.035	0.044	0.96	0.707
	75	0.006	0.006	0.96	0.292	−0.059	0.028	0.95	0.559
	100	0.005	0.004	0.95	0.254	−0.053	0.021	0.95	0.488
Parameter	n	$\alpha=1.5,\ \lambda=1$				$\alpha=3,\ \lambda=1$			
		Bias	MSE	CP	AW	Bias	MSE	CP	AW
α	25	0.175	0.149	0.96	1.362	0.156	0.400	0.94	2.260
	50	0.111	0.067	0.95	0.932	0.084	0.188	0.94	1.565
	75	0.097	0.046	0.96	0.758	0.048	0.104	0.95	1.267
	100	0.090	0.036	0.95	0.653	0.044	0.084	0.96	1.095
λ	25	0.002	0.068	0.93	0.968	0.020	0.025	0.92	0.572
	50	−0.007	0.034	0.95	0.677	0.011	0.012	0.93	0.401
	75	−0.013	0.021	0.96	0.551	0.007	0.007	0.96	0.326
	100	−0.027	0.016	0.96	0.465	0.002	0.005	0.95	0.280

6. THE LOG-LOGISTIC EXPONENTIAL REGRESSION WITH CENSORED DATA

If X follows the LE distribution (3.2), $Y=\log(X)$ will have the *log-logistic exponential* (LLE) distribution. The density function of Y (for $y\in\mathbb{R}$), parameterized in terms of $\lambda=\mathrm{e}^{-\mu}$, takes the form

$$f(y) = \frac{\alpha\,\exp\big[(y-\mu)-\exp(y-\mu)\big]\,\Big[-\log\Big\{1-\exp\big[-\exp(y-\mu)\big]\Big\}\Big]^{-\alpha-1}}{\Big\{1-\exp\big[-\exp(y-\mu)\big]\Big\}\,\bigg\{1+\Big[-\log\Big\{1-\exp\big[\exp(y-\mu)\big]\Big\}\Big]^{-\alpha}\bigg\}^{2}}\,, \tag{6.1}$$

where $\mu\in\mathbb{R}$ is a location parameter and α is a positive shape parameter.

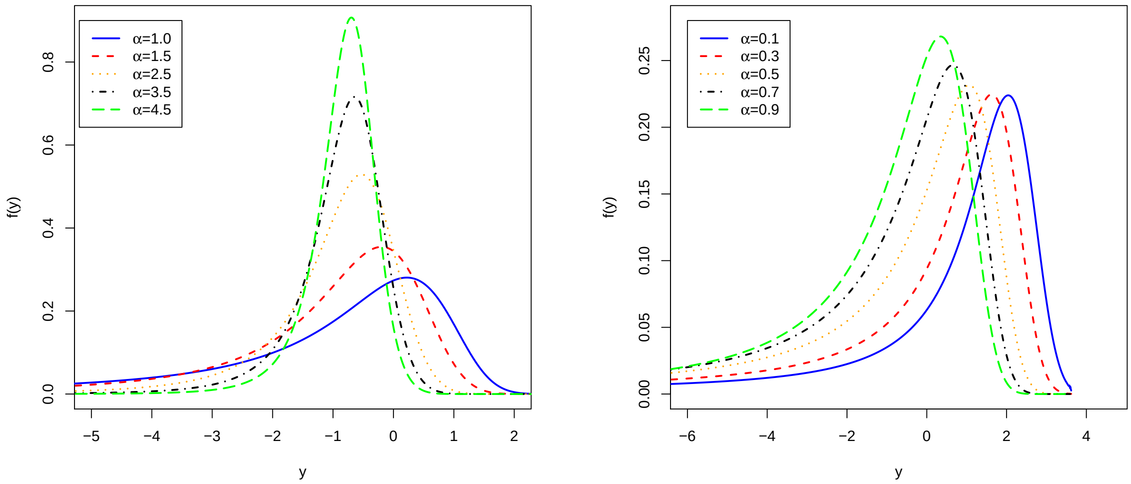
We refer to Equation (6.1) as the LLE distribution, say $Y\sim\mathrm{LLE}(\alpha,\mu)$. Thus,

$$\text{if}\quad X\sim\mathrm{LE}(\alpha,\lambda)\quad\text{then}\quad Y=\log(X)\sim\mathrm{LLE}(\alpha,\mu)\,.$$

Some shapes of the density function of Y are given in Figure 4.

The survival function of Y is

$$S(y) = \frac{1}{1+\Big[-\log\Big\{1-\exp\big[-\exp(y-\mu)\big]\Big\}\Big]^{-\alpha}}\,. \tag{6.2}$$



(**a**) (**b**)

Figure 4: The LLE density function. (a) For different values of $\alpha > 1$ with $\mu = 0$. (b) For different values of $\alpha < 1$ with $\mu = 0$.

The density function of $Z = (Y - \mu)$ is

$$\pi(z;\alpha) \;=\; \frac{\alpha\,\exp\big[z - \exp(z)\big]\,\Big[-\log\Big\{1 - \exp\big[-\exp(z)\big]\Big\}\Big]^{-\alpha-1}}{\Big\{1 - \exp\big[-\exp(z)\big]\Big\}\left\{1 + \Big[-\log\Big\{1 - \exp\big[-\exp(z)\big]\Big\}\Big]^{-\alpha}\right\}^{2}}\,, \qquad z \in \mathbb{R}\,. \tag{6.3}$$

Based on the LLE density, we propose the location-scale linear regression

$$y_i \;=\; \mathbf{v}_i^{\top}\boldsymbol{\beta} + z_i\,, \qquad i = 1, ..., n\,, \tag{6.4}$$

where the random error z_i has density function (6.3), $\mathbf{v}_i^{\top} = (v_{i1}, ..., v_{ip})$ is the vector of explanatory variables, $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)^{\top}$ and α are unknown parameters. The parameter $\mu_i = \mathbf{v}_i^{\top}\boldsymbol{\beta}$ is the location of y_i. The location parameter vector $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)^{\top}$ is represented by a linear model $\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\beta}$, where $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_n)^{\top}$ is a known model matrix. Equation (6.4) is referred to as the LLE regression for censored data and opens new possibilities for fitting several types of data. It is an extension of the log-exponential regression for censored data.

Consider a sample $(y_1, \mathbf{v}_1), ..., (y_n, \mathbf{v}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(X_i), \log(D_i)\}$ assuming that the observed lifetimes and censoring times are independent. Let F and D be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively.

The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = \big(\alpha, \boldsymbol{\beta}^{\top}\big)^{\top}$ from regression (6.4) is

$$\begin{aligned} l(\boldsymbol{\theta}) \;=\; & r\log(\alpha) + \sum_{i\in F} z_i - \sum_{i\in F}\exp(z_i) - (\alpha+1)\sum_{i\in F}\log\Big[-\log\Big\{1 - \exp\big[-\exp(z_i)\big]\Big\}\Big] \\ & - \sum_{i\in F}\log\Big\{1 - \exp\big[-\exp(z_i)\big]\Big\} - 2\sum_{i\in F}\log\left\{1 + \Big[-\log\Big\{1 - \exp\big[-\exp(z_i)\big]\Big\}\Big]^{-\alpha}\right\} \\ & - \sum_{i\in D}\log\left\{1 + \Big[-\log\Big\{1 - \exp\big[-\exp(z_i)\big]\Big\}\Big]^{-\alpha}\right\}, \end{aligned} \tag{6.5}$$

where $z_i = (y_i - \mathbf{v}_i^\top \boldsymbol{\beta})$, and r is the number of uncensored observations (failures). The MLE $\widehat{\boldsymbol{\theta}}$ of the vector of unknown parameters can be determined by maximizing the log-likelihood (6.5) using the subroutine `NLMixed` in SAS.

The `NLMixed` procedure of SAS has been exhaustively used to estimate the parameters for several distributions. Further, Molenberghs *et al.* [13] adopted this procedure to obtain the estimates in generalized linear models for repeated measures with normal and conjugate random effects, whereas Vangeneugden *et al.* [20] used it to calculate the estimates of extended random-effects models for repeated and overdispersed counts.

The estimated survival function for y_i ($\widehat{z}_i = y_i - \mathbf{v}_i^\top \widehat{\boldsymbol{\beta}}$) is

$$S(y_i;\widehat{\alpha},\widehat{\boldsymbol{\beta}}) = \frac{1}{1+\left[-\log\left\{1-\exp\left[-\exp\left(y_i-\mathbf{v}_i^\top\widehat{\boldsymbol{\beta}}\right)\right]\right\}\right]^{-\widehat{\alpha}}}\,. \tag{6.6}$$

We can adopt likelihood ratio (LR) statistics in the usual way for comparing some special models with the LLE regression.

7. EMPIRICAL ILLUSTRATIONS WITH LIFETIME DATA

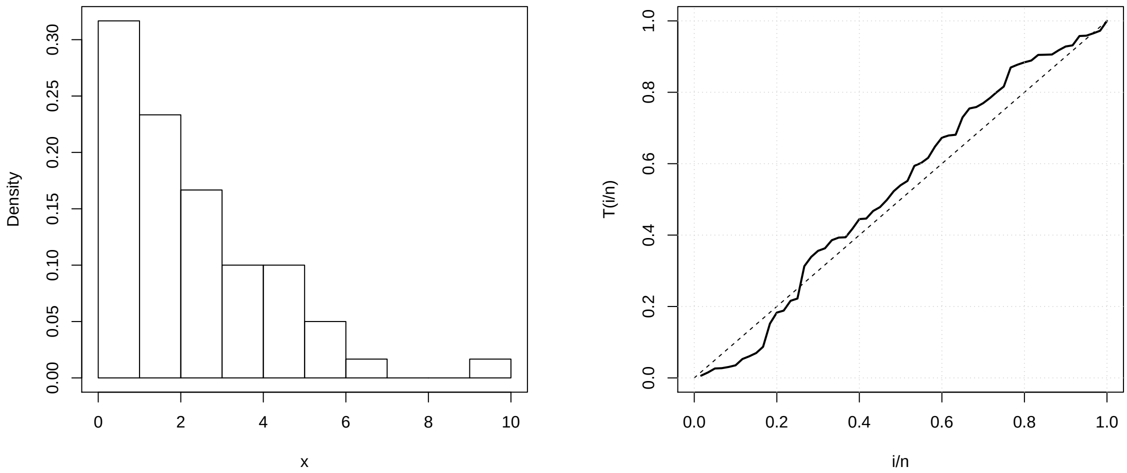
We now prove empirically that the LE distribution is a good alternative to the gamma, log-normal, Weibull, EE, Nadarajah–Haghighi (NH) introduced by Nadarajah and Haghighi [16], power Lindley (PL) defined by Ghitney *et al.* [6], exponentiated Lindley (EL) studied by Nadarajah *et al.* [15], Birnbaum–Saunders (BS) and inverse Gaussian (IG) distributions. For model comparison, we adopt the Anderson–Darling (A*), Cramér–von Mises (W*) and Kolmogorov–Smirnov (K-S) measures. The cdfs of the EE, NH, PL, EL, BS and pdf of the IG distributions (for $x > 0$) are, respectively,

$$\begin{aligned}
F_{\mathrm{EE}}(x;\alpha,\lambda) &= \left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha}, \qquad \alpha,\lambda>0\,,\\
F_{\mathrm{NH}}(x;\alpha,\lambda) &= 1-\mathrm{e}^{1-(1+\lambda x)^{\alpha}}, \qquad \alpha,\lambda>0\,,\\
F_{\mathrm{PL}}(x;\beta,\theta) &= 1-\left(\frac{1+\theta+\theta\,x^{\beta}}{1+\theta}\right)\mathrm{e}^{-\theta x^{\beta}}, \qquad \beta,\theta>0\,,\\
F_{\mathrm{EL}}(x;\alpha,\theta) &= \left[1-\left(\frac{1+\theta+\theta\,x}{1+\theta}\right)\mathrm{e}^{-\theta x}\right]^{\alpha}, \qquad \alpha,\theta>0\,,\\
F_{\mathrm{BS}}(x;\alpha,\beta) &= \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{1/2}-\left(\frac{\beta}{x}\right)^{1/2}\right\}\right], \qquad \alpha,\beta>0\,,\\
F_{\mathrm{IG}}(x;\mu,\lambda) &= \sqrt{\frac{\lambda}{2\pi\,x^{3}}}\;\exp\left[-\lambda(x-\mu)^{2}/(2\,x\,\mu^{2})\right], \qquad \mu,\lambda>0\,.
\end{aligned}$$

7.1. Application 1: Failure of electrical appliances in life test

The data set taken from Lawless [11] represents the 1000 cycles to failure for a group of 60 electrical appliances in a life test. These data were also analyzed by Chesneau *et al.* [3] and Mazucheli *et al.* [12]. Some descriptive statistics for these data are: $n = 60$, $\bar{x} = 2.19297$,

$s = 1.920062$, skewness $=1.2614$ and kurtosis $= 2.23207$. The histogram displayed in Figure 5(a) and the skewness indicates that the distribution is right-skewed. The TTT plot (Aarset [1]) is given in Figure 5(b). It is first convex and then concave, which suggests a bathtub failure rate. So, the LE distribution could in principle be appropriate for modeling the current data.



(a) (b)

Figure 5: (a) Histogram. (b) TTT plot for failure data.

Table 2: Estimated quantities and goodness-of-fit measures for failure data.

Distribution	Estimates		A*	W*	K-S	K-S p-value
LE(α, λ)	1.9798 (0.2555)	0.2625 (0.0357)	0.3258	0.0374	0.0547	0.9491
Gamma(α, θ)	0.9307 (0.1486)	2.3562 (0.4909)	0.7184	0.1042	0.0897	0.6860
Weibull(c, λ)	1.0008 (0.1066)	0.4555 (0.0814)	0.7154	0.1036	0.0777	0.8342
Log-normal(μ, σ)	0.1597 (0.1858)	1.4392 (0.1313)	2.5241	0.4291	0.1653	0.0666
NH(α, λ)	1.6133 (0.8016)	0.2274 (0.1575)	0.4574	0.0615	0.0914	0.6632
EE(α, λ)	0.9159 (0.1502)	0.4311 (0.0735)	0.7103	0.1028	0.0921	0.6543
PL(β, θ)	0.8883 (0.0891)	0.8042 (0.1031)	0.6467	0.0766	0.0766	0.8155
EL(α, θ)	0.7522 (0.1274)	0.6203 (0.0873)	0.4615	0.0644	0.0698	0.8522
IG(μ, λ)	2.1929 (0.7513)	0.3113 (0.1104)	4.6132	0.8576	0.30548	0.0000
BS(α, β)	1.9391 (0.1824)	0.6483 (0.1111)	2.4479	0.4343	0.3719	0.0000

Table 2 provides the MLEs of the parameters and the values of A*, W* and K-S statistics and associated p-value for each fitted model. We can conclude that the LE distribution provides the best fit and has the ability to fit right-skewed data with BT failure rate. We also provide QQ-plots for all fitted models in Figure 6. Clearly, the new model provides the closest fit to the data.

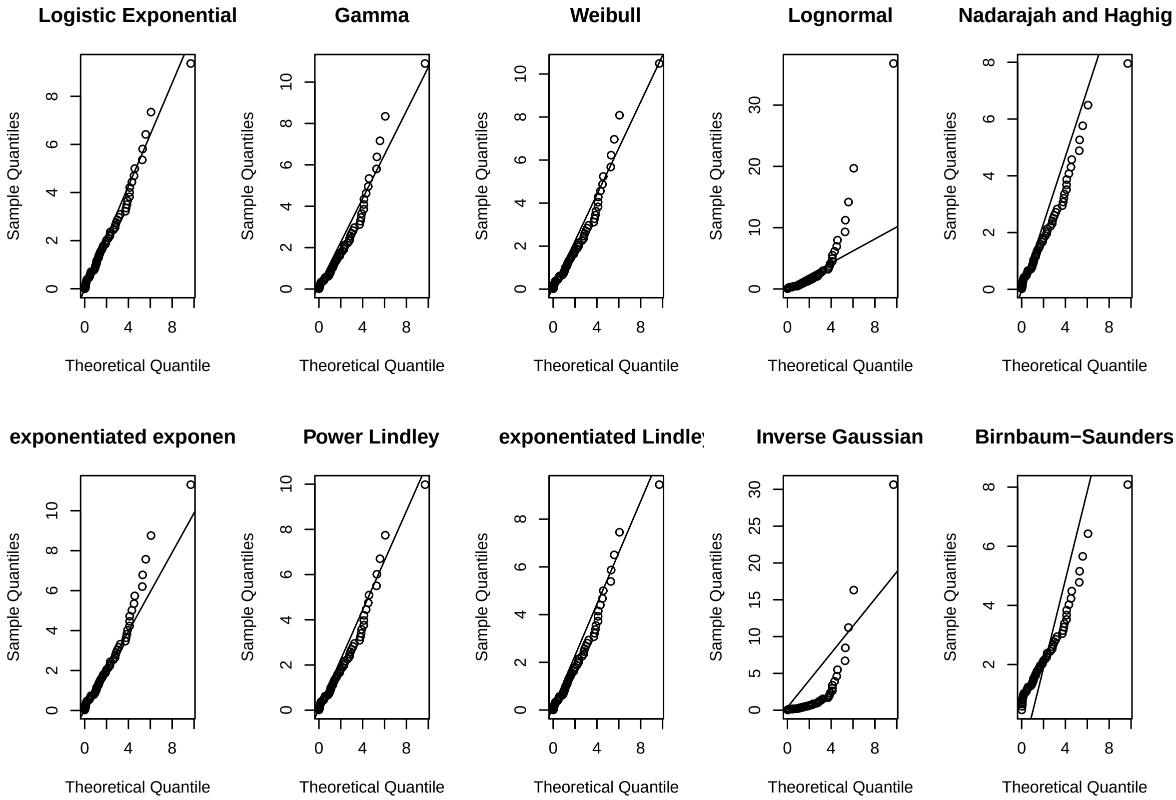
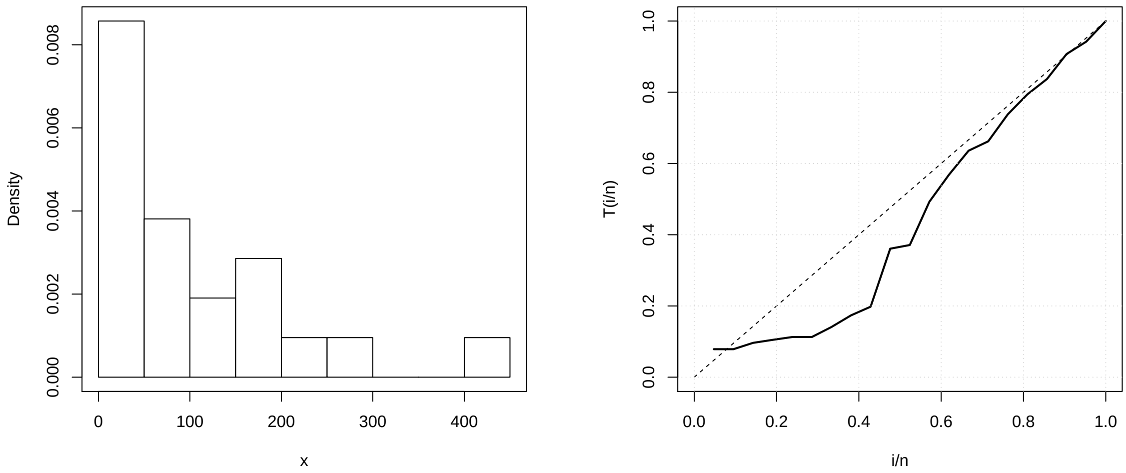


Figure 6: QQ-plots for failure data.

7.2. Application 2: Lung cancer patients data

This data is also taken from a study reported by Lawless [11]. These data represents 21 advanced lung cancer patients who were randomly assigned the chemotherapy treatments termed as standard. Survival times t, measured from the start of treatment for each patient. The main objective was to compare the effects of two chemotherapy treatments in prolonging survival time. The basic statistics for these data are: $n = 21$, $\bar{x} = 101.7619$, $s = 110.8147$, skewness $= 1.29047$ and kurtosis $= 1.00438$. The histogram displayed in Figure 7(a) and the skewness indicate that the distribution is right-skewed. The TTT plot of these data shown in Figure 7(b) indicates a decreasing failure rate.

The measures reported in Table 3 indicate that the LE model provides the most accurate fit to the data. Further, the QQ-plots for all fitted models in Figure 8 also suggest the same conclusion.



(**a**) (**b**)

Figure 7: (a) Histogram. (b) TTT plot for cancer data.

Table 3: Estimated quantities and goodness-of-fit measures for cancer data.

Distribution	Estimates		A^*	W^*	K-S	K-S p-value
LE(α, λ)	0.8417 (0.1641)	0.0108 (0.0028)	0.5871	0.0872	0.1574	0.8755
Gamma(α, θ)	1.2889 (0.2607)	57.7242 (10.7798)	0.6114	0.0912	0.1970	0.3887
Weibull(c, λ)	0.8757 (0.1462)	0.0185 (0.0142)	0.6120	0.0922	0.1616	0.4425
Log-normal(μ, σ)	3.9144 (0.2832)	1.2982 (0.2003)	0.7087	0.1130	0.1503	0.2299
NH(α, λ)	0.6437 (0.2855)	0.0217 (0.0192)	0.6364	0.0975	0.15307	0.2088
EE(α, λ)	0.8301 (0.2288)	0.0087 (0.0025)	0.6056	0.0905	0.1701	0.3776
PL(β, θ)	0.6293 (0.1253)	0.1195 (0.0023)	0.6343	0.0965	0.1595	0.5590
EL(α, θ)	0.4820 (0.1274)	0.0274 (0.0873)	0.6309	0.0928	0.3064	0.0386
IG(μ, λ)	101.0077 (38.6442)	32.1416 (9.9192)	0.6999	0.1152	0.4718	0.0150
BS(α, β)	1239.8960 (503.2201)	1880.1910 (642.4328)	1.0941	0.1844	0.5005	0.0000

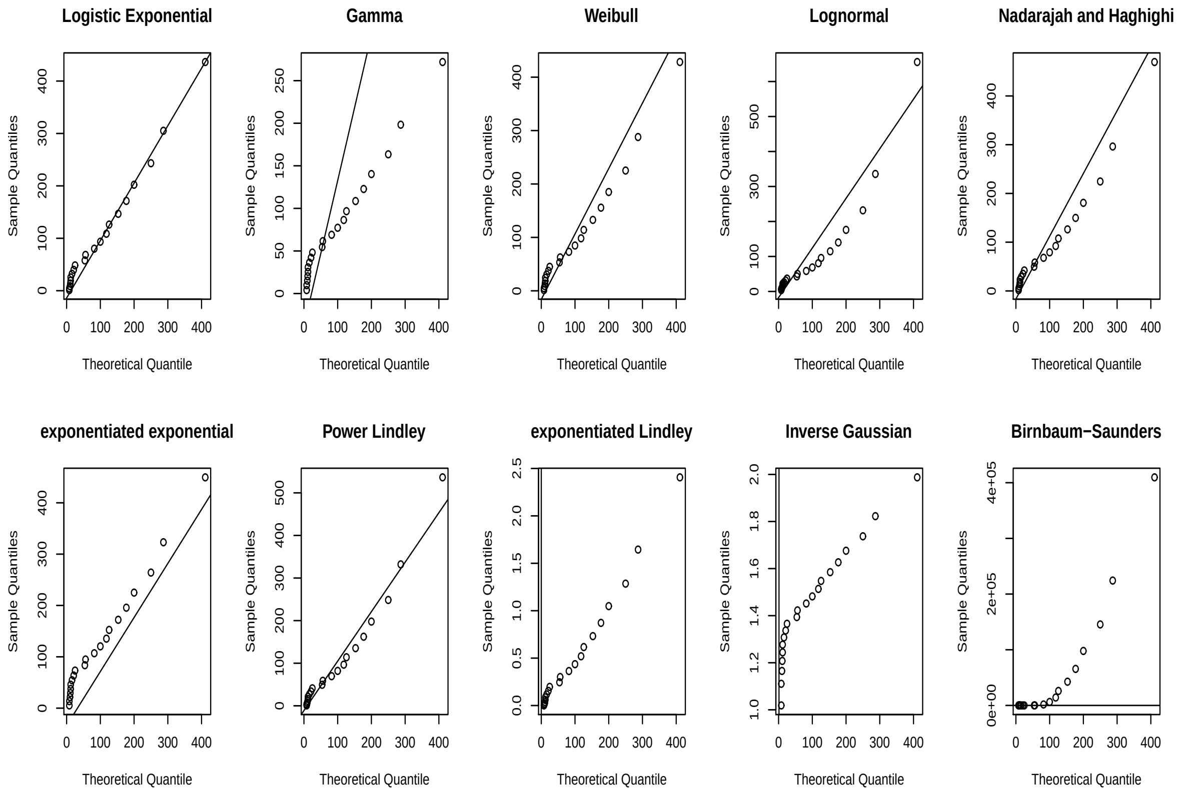


Figure 8: QQ-plots for cancer data.

7.3. Application 3: Entomology data

In this application, we take a data set from a study carried out at the Department of Entomology of the Luiz de Queiroz School of Agriculture, University of São Paulo. Such study aims to assess the longevity of the Mediterranean fruit fly (ceratitis capitata), which is considered a pest in agriculture. Instead of using an insecticide, Silva *et al.* [18] conducted a study using small portions of food containing substances extracted from a tree called *Azadirachta indica* which is best known internationally by the name "neem". The experiment was completely randomized with 11 treatments, consisting of different extracts of the neem tree at concentrations of 39, 225, and 888 ppm, where the response variable is the lifetime of the adult flies in days after exposure to the treatments. From the results of the experiment, these 11 treatments are allocated into two groups, namely:

Group 1: Control 1 (deionized water); Control 2 (acetone −5%); aqueous extract of seeds (AES) (39 ppm); AES (225 ppm); AES (888 ppm); methanol extract of leaves (MEL) (225 ppm); MEL (888 ppm); and dichloromethane extract of branches (DMB) (39 ppm) 425.

Group 2: MEL (39 ppm); DMB (225 ppm); and DMB (888 ppm).

Lanjoni *et al.* [10] analyzed these data by fitting the log-Burr XII geometric type I (LBXIIGI) and log-Burr XII geometric type II (LBXIIGII) models. Recently, these data were also analyzed by Cordeiro *et al.* [5] and Zubair *et al.* [21] using the generalized Weibull-logistic regression and log-power-Cauchy negative-binomial regressions, respectively. Following the same procedure from these surveys, we compare the proposed model with these regressions in this application.

The response variable in the experiment is the lifetime of the adult lies in days after exposure to the treatments. The total sample size is $n = 72$. So, the variables used in this study are:

- y_i : log-lifetime of ceratitis capitata adults in days;
- δ_i : censoring indicator;
- v_{i1} : sex of the larvae;
- v_{i2} : group $(0 = \text{group } 1,\ 1 = \text{group } 2)$, $i = 1, ..., 74$.

Lanjoni *et al.* [10] introduced two lifetime distributions by compounding the Burr XII and geometric distributions, and also defined two extended regressions based on the logarithms of these distributions. Let F and D be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. We adopt the classical log-Weibull (LW) regression as an example to illustrate that the LE regression can provides better fits. In this case, the total log-likelihood function for the parameters $\boldsymbol{\theta} = \left(\sigma, \boldsymbol{\beta}^\top\right)^\top$ is

$$l(\boldsymbol{\theta}) \;=\; r^* \log\left(\frac{1}{\sigma}\right) + \sum_{i\in F} z_i - \sum_{i\in F} \exp(z_i) - \sum_{i\in D} \exp(z_i)\,,$$

where $z_i = \left(y_i - \mathbf{v}_i^\top \boldsymbol{\beta}\right)/\sigma$.

Next, we present results by fitting the regression (for $i = 1, ..., 172$)

$$y_i \;=\; \beta_0 + \beta_1\, v_{i1} + \beta_2\, v_{21} + \sigma\, z_i\,,$$

where y_i can follow the LLE, LBXIIGII and LBXIIGI distributions. For some fitted regressions, Table 4 lists the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the following statistics: Akaike information criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC). The computations are performed using the `NLMixed` subroutine in SAS. These results indicate that the LLE regression model with censored data could be chosen as the best regression. So, this regression is really competitive to the log-Weibull regression.

The MLEs of the parameters and their standard errors are listed in Table 4. Note that the covariate (v_2) is significant at the 1% level, whereas the other covariate is not significant at the usual significance level.

Table 4: Estimated quantities, p-values in $[\cdot]$ and goodness-of-fit measures from some regressions fitted to entomology data.

Regression			α	β_0	β_1	β_2	AIC	CAIC	BIC
LLE			3.9444 (0.2774)	3.8567 (0.0607) [<0.0001]	0.0581 (0.0791) [0.4636]	−0.3474 (0.0882) [<0.0001]	334.5	334.7	347.1
LE				3.1724 (0.1198) [<0.0001]	0.1369 (0.1569) [0.3843]	−0.4430 (0.1766) [0.0130]	423.3	423.5	432.8
			σ	β_0	β_1	β_2			
LW			0.5151 (0.03256)	3.2435 (0.06309) [<0.0001]	0.1358 (0.08111) [0.0960]	−0.4158 (0.09124) [<0.0001]	344.3	344.6	356.9
	σ	k	p	β_0	β_1	β_2			
LBXIIGI	0.4877 (0.0596)	9.3993 (8.5578)	1E-8 (1E-9)	4.3085 (1.1316) [0.0002]	0.1104 (0.0962) [0.2525]	−0.4014 (0.0978) [<0.0001]	348.1	348.6	367.0
LBXIIGII	0.9107 (0.3379)	6.0541 (4.8403)	0.9798 (0.0247)	3.1649 (0.8847) [0.0005]	0.0354 (0.0803) [0.6605]	−0.3252 (0.0876) [0.0003]	335.7	336.2	354.6
LBXII	0.4877 (0.0597)	9.4002 (8.6141)	0	4.3085 (1.1349) [0.0002]	0.1104 (0.0963) [0.2528]	−0.4014 (0.0978) [<0.001]	346.1	346.4	361.8

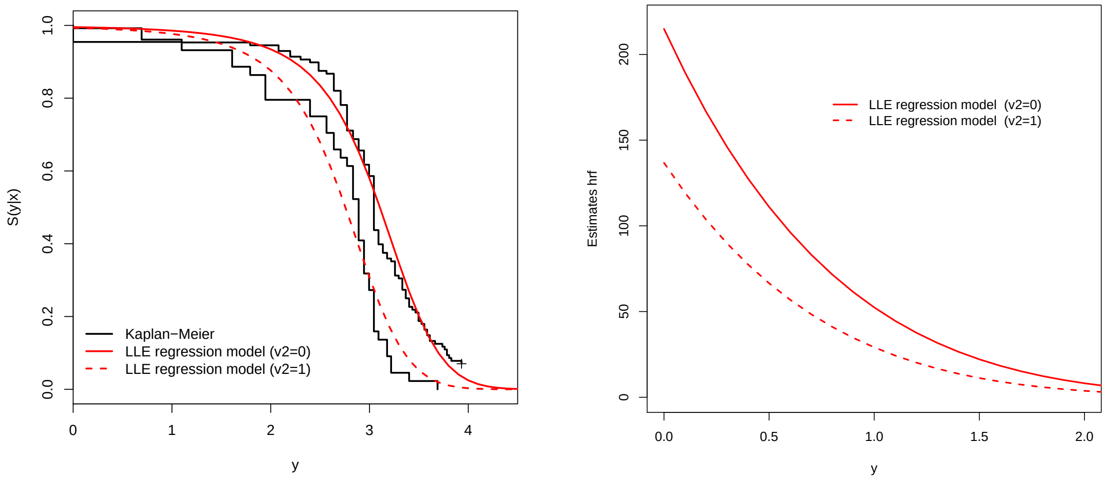
Finally, we turn to a simplified model retaining only v_2 as an explanatory variable

$$y_i = \beta_0 + \beta_2\, v_{i2} + \sigma\, z_i\,.$$

The MLEs for the LLE regression model fitted to the data are given in Table 5. In order to assess if the model is appropriate, Figure 9(a) displays the plots of the empirical survival function and the estimated survival function from the fitted LLE regression. The plots of its hrfs in Figure 9(b) reveal decreasing shapes. There is a significant difference between the levels of the covariable v_2. In fact, this regression provides a good fit to these data.

Table 5: MLEs of the parameters from the fitted LLE regression model to the entomology data.

Model	α	β_0	β_2
LLE	3.9489 (0.2777)	3.8845 (0.0475) [<0.001]	−0.3486 (0.0878) [0.0001]



(**a**) (**b**)

Figure 9: Entomology data: (a) Estimated LLE survival function and empirical survival. (b) Estimated hrf.

8. CONCLUDING REMARKS

The Weibull, gamma and exponentiated-exponential distributions have two parameters and they are used quite often in survival analysis. These distributions can have increasing or unimodal probability density functions, and monotone hazard functions. However, none of which can have non-monotone hazard rate function shape. In many practical situations, one might observe non-monotone hazard rate functions, and clearly in those cases, none of these distribution functions can be used. The proposed LE distribution can have decreasing or unimodal density function shapes. It is also interesting to note that the hazard rate function possesses three different shapes: decreasing failure rate, bathtub and decreasing-increasing-decreasing.

Moreover, the LE distribution has only two parameters which makes estimating the parameters not very difficult. It may be mentioned that not too many two-parameter distributions can have non-monotone hazard function shape. Therefore, the proposed distribution will be quite useful. Furthermore, its survival and hazard rate functions have closed-form representations. Accordingly, this model can readily be utilized to analyze censored data sets. We also propose a new regression model that can be useful to model real data sets. The importance of the new models is proved empirically by means of three real data sets.

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REFERENCES

[1] Aarset, M.V. (1987). How to identify bathtub hazard rate, *IEEE Transactions on Reliability*, **36**, 106–108.

[2] Alzaatreh, A.; Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions, *Metron*, **71**, 63–79.

[3] Chesneau, C.; Bakouch, H.S. and Hussain, T. (2019). A new class of probability distributions via cosine and sine functions with applications, *Communications in Statistics – Simulation and Computation*, **48**, 2287–2300.

[4] Cordeiro, G.M. and Nadarajah, S. (2011). Closed-form expressions for moments of a class of beta generalized distributions, *Brazilian Journal of Probability and Statistics*, **25**, 14–33.

[5] Cordeiro, G.M.; Ortega, E.M.M. and Ramires, T.H. (2015). A new generalized Weibull family of distributions: mathematical properties and applications, *Journal of Statistical Distributions and Applications*, **2**, 1–25.

[6] Ghitany, M.E.; Al-Mutairi, D.K.; Balakrishnan, N. and Al-Enezi, L.J. (2013). Power Lindley distribution and associated inference, *Computational Statistics & Data Analysis*, **64**, 20–33.

[7] Gradshteyn, I.S. and Ryzhik, I.M. (2000). *Tables of Integrals, Series and Products*, Academic Press, New York.

[8] Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions, *Australian and Newzealand Journal of Statistics*, **41**, 173–188.

[9] Gupta, R.D. and Kundu, D. (2007). Generalized exponential distribution: existing results and some recent developments, *Journal of Statistical Planning and Inference*, **137**, 3537–3547.

[10] Lanjoni, E.M.; Ortega, E.M.M. and Cordeiro, G.M. (2016). Extended Burr XII regression models: theory and applications, *Journal of Agricultural, Biological and Environmental Statistics*, **21**, 203–224.

[11] Lawless, J.F. (2003). *Statistical Models and Methods for Lifetime Data*, Wiley, New York.

[12] Mazucheli, J.; Ghitany, M.E. and Louzada, F. (2014). Estimation methods for a two-parameter bathtub-shaped lifetime distribution, *Australian Journal of Basic and Applied Sciences*, **10**, 189–198.

[13] Molenberghs, G.; Verbeke, G. and Demétrio, C.G.B. (2010). A family of generalized linear models for repeated measures with normal and conjugate random effects, *Statistical Science*, **25**, 325–347.

[14] Murthy, D.; Xie, M. and Jiang, R. (2004). *Weibull Models*, Wiley, New York.

[15] Nadarajah, S.; Bakouch, H. and Tahmasbi, R. (2011). A generalized Lindley distribution, *Sankhya*, **73**, 331–359.

[16] Nadarajah, S. and Haghighi, F. (2011). An extension of the exponential distribution, *Statistics*, **45**, 543–558.

[17] Silva, G.O.; Ortega, E.M.M.; Cancho, V.G. and Barreto, M.L. (2008). Log-Burr XII regression models with censored data, *Computational Statistics & Data Analysis*, **52**, 3820–3842.

[18] Silva, M.A.; Bezerra-Silva, G.C.D.; Vendramim, J.D. and Mastrangelo, T. (2013). Sublethal effect of neem extract on Mediterranean fruit fly adults, *Revista Brasileira de Fruticultura*, **35**, 93–101.

[19] Tahir, M.H. and Nadarajah, S. (2015). Parameter induction in continuous univariate distributions: well-established G families, *Anais da Academia Brasileira de Ciências*, **87**, 539–568.

[20] Vangeneugden, J.; De Maziere, P.A.; Van Hulle, M.M.; Jaeggli, T.; Van Gool, L. and Vogels, R. (2011). Distinct mechanisms for coding of visual actions in macaque temporal cortex, *Journal of Neuroscience*, **31**, 385–401.

[21] Zubair, M.; Tahir, M.H.; Cordeiro, G.M.; Alzaatreh, A. and Ortega, E.M.M. (2018). The power-Cauchy negative-binomial: properties and regression, *Journal of Statistical Distributions and Applications*, **5**, 1–17.