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Rényi Entropy of k-Records: Properties and Applications

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Abstract:

- In this paper, we discuss on Rényi entropy of k-records arising from any continuous distribution in detail. The relevance of constructing k-records from random sample in the context of information contained in a random variable has been described in the study. Some important properties of Rényi entropy of k-records have been derived as well. Two relevant applications of Rényi entropy of k-records are discussed in this work. Finally, a simple estimator is proposed for Rényi entropy of k-records and a numerical illustration has been carried out using a real life data set.

Keywords:

- *characterization; k-records; monotone property; Rényi entropy ordering; Rényi divergence; uniform distribution.*

AMS Subject Classification:

- 62B10, 94A15, 94A17.

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1. INTRODUCTION

Shannon [33] defined the entropy of a system which measures uncertainty contained in a random variable. The Shannon entropy measure of uncertainty is inversely related to the occurrence probability of the event. For a non-negative and absolutely continuous random variable X with probability density function (pdf) $f(x)$, the Shannon entropy is defined by

$$H(X) = -\int_0^\infty f(x)\ln f(x)dx.$$

Moreover, Rényi [30] introduced one parameter extension of Shannon entropy by defining an entropy of order α called Rényi entropy. The Rényi entropy of X with pdf $f(x)$ is defined by

$$H_\alpha(X) = \frac{1}{1-\alpha}\ln\int_{-\infty}^{\infty} f^\alpha(x)dx, \quad \alpha > 0,\ (\alpha \neq 1). \tag{1.1}$$

It can be easily shown that $\lim_{\alpha\to 1} H_\alpha(X) = H(X)$. Some important properties of Rényi entropy are as follows: $H_\alpha(X)$ can be negative, $H_\alpha(X)$ is invariant under a location transformation, $H_\alpha(X)$ is not invariant under a scale transformation and for any $\alpha_1 < \alpha_2$, we have $H_{\alpha_1}(X) \geq H_{\alpha_2}(X)$, the equality occurs if and only if X is uniformly distributed. The Rényi divergence of order α between two random variables X and Y with density functions $f(x)$ and $g(y)$, respectively, given by

$$D_\alpha(f,g) = \frac{1}{\alpha-1}\int_{-\infty}^{\infty}\left[\frac{f(x)}{g(x)}\right]^{\alpha-1} f(x)dx. \tag{1.2}$$

For details, see Golshani and Pasha [19] and Contreras-Reyes [8]. The intriguing properties and applications of Rényi entropy have been extensively studied in literature.

Morales *et al.* [27] studied properties of Rényi entropy with respect to testing of hypothesis in parametric models. The connection of Rényi information with log-likelihood of the random variable derived from the gradient of the spectrum of Rényi information is discussed in Song [34]. Csiszár [10] gave Rényi's entropy and divergence of order α operational characterizations in terms of block coding and hypothesis testing. In the field of statistical mechanics, the ergodic diffusion processes in terms of Rényi entropy has been discussed in De Gregorio and Iacus [12]. Further, Kirchanov [24] uses Rényi entropy to describe quantum dissipative systems. For more details about the application of Rényi entropy, one may refer Nadarajah and Zografos [28], Asadi *et al.* [5], Contreras-Reyes [8] and Contreras-Reyes and Cortés [9].

This paper is structured as follows: Section 2 gives a brief introduction about k-records. Section 3 expresses Rényi Entropy of k-records arising from any continuous distribution. In Section 4, we discuss some important properties of Rényi entropy of upper and lower k-records. Section 5 presents two applications of Rényi entropy of k-records. The overall findings are stated in Section 6.

2. BACKGROUND OF k-RECORDS

Chandler [7] defined records as successive extremes occuring in a sequence of independent and identically distributed (iid) random variables. Records are of great importance in several real life problems involving weather, economic studies, sports, etc. Prediction of next record value is an interesting problem in many real life situations. For example, the prediction of next record level of water that a dam can capture is helpful in holding or discharge of the water. Similarly, prediction of lowest share value in stock markets is essential to plan for the investment strategies. More applications of record values are available in Arnold *et al.* [4] and Ahsanullah [3].

In many events associated with athletics, temperature, wind velocity, etc., one is compelled to depend upon the available record data to deal with statistical inference problems of the parent distribution. But, statistical inferences based on records are difficult to make since the records occurs rarely in real life situations. We can observe that the expected waiting time for every record after the first observation is infinite. One may overcome this difficulty by the use of k-records introduced by Dziubdziela and Kopociński [13] which occur more frequently than the classical records. For example, consider first 10 observations from the data given in David and Nagaraja [11]: 0.464, 0.060, 1.486, 1.022, 1.394, 0.906, 1.179, −1.501, −0.690, 1.372. The records observed from the data are: 0.464 and 1.486. We can construct upper k-records from the data as given below:

Table 1: Sequences of k-records for $k = 2, 3, 4$.

2-Records	0.060, 0.464, 1.022, 1.394.
3-Records	0.060, 0.464, 1.022, 1.179, 1.372
4-Records	0.060, 0.464, 0.906, 1.022, 1.179

It is well known that if the number of observations on the random variable increases the statistical inferences becomes more reliable. In other words, the uncertainty contained in the random variable reduces.

Many works are going on to detect outliers in a data so as to delete them for devising more reasonable statistical methods to the problem of interest. The integer parameter k involved in k-records can be chosen in such a manner that the record data generated will exclude the specified number of outliers which are feared to be crept into the data. For example, if some initial scrutiny of the data reveals that there is a possibility of occurrence of only one outlier in terms of its largeness in the data, then it is enough to consider upper 2-records as the desirable record data that may be used for further analysis. Hence, it is beneficial to construct k-records from a sequence of random variables than constructing classical record values in such situations.

Suppose $\{X_i,\ i \geq 1\}$ is a sequence of iid random variables. If for a positive integer k, we collect those observations in the sequence which occupy the k-th largest position but exceeds in value for the first time the just previously recorded k-th largest value.

Then, the resulting sequence is known as the sequence of k-th upper records or simply k-records. We denote the times at which upper k-record values occur as $T_{n(k)}$ for $n = 1, 2, ...$ and are defined by $T_{1(k)} = k$ and for $n > 1$, $T_{n+1(k)} = \min\{j : j > T_{n(k)}, X[j : j + k - 1] > X[T_{n(k)} - k + 1 : T_{n(k)}]\}$, where $X[p : q]$ is the p-th order statistic in a random sample of size q. Then we define the sequence of upper k-record values denoted by $U_{n(k)}$ as $U_{n(k)} = X[T_{n(k)} - k + 1 : T_{n(k)}]$. If the parent distribution is absolutely continuous with survival function $\bar{F}_X(x)$ and pdf $f_X(x)$, then, the pdf of the n-th upper k-record value $U_{n(k)}$ is given by (see Arnold *et al.* [4])

$$f_{n(k)}(x) = \frac{k^n}{\Gamma(n)}\left[-\ln \bar{F}_X(x)\right]^{n-1}[\bar{F}_X(x)]^{k-1} f_X(x), \;\; n = 1, 2, ... \tag{2.1}$$

Similarly, we can define the lower k-records. For a positive integer k, if we denote the times at which lower k-records occur as $T^L_{n(k)}$ for $n = 1, 2, ...$ and are defined by $T^L_{1(k)} = k$ and for $n > 1$, $T^L_{n+1(k)} = \min\{j : j > T^L_{n(k)}, X[j : j + k - 1] < X[T^L_{n(k)} - k + 1 : T^L_{n(k)}]\}$. Then we define the sequence of lower k-records denoted by $L_{n(k)}$ as $L_{n(k)} = X[T^L_{n(k)} - k + 1 : T^L_{n(k)}]$. If the parent distribution is absolutely continuous with cumulative distribution function (cdf) $F_X(x)$ and pdf $f_X(x)$, then, the pdf of the n-th lower k-record value $L_{n(k)}$ is given by (see Ahsanullah [3])

$$g_{n(k)}(x) = \frac{k^n}{\Gamma(n)}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x), \quad n = 1, 2, ... \tag{2.2}$$

Several applications of k-records are available in the literature. In reliability, a k-out-of-n system breaks down at the time of the failure of $(n - k + 1)$-th component. The life time of a k-out-of-n system is the $(n - k + 1)$-th order statistic in a sample of size n. Consequently, the n-th upper k-record value can be regarded as the life length of a k-out-of-$T_{n(k)}$ system. In actuarial science, there arises situations where second or third largest set of values are of special interest when the insurance claim of some non-life insurance is considered. One may refer Kamps [23] for more details. Detailed description on the theoretical aspects as well as applications of k-records are available in Arnold *et al.* [4], Nevzorov [29] and Ahsanullah [3].

Many authors have discussed about the information measures of classical records and its generalized version (k-records) arising from probability distribution. Hofmann and Nagaraja [21] derived some general results on the Fisher information contained in the classical record values and Hofmann and Balakrishnan [20] derived some general results on the Fisher information contained in the k-record values generated from an iid sample of fixed size from a continuous distribution. Madadi and Tata [25] present results on the Shannon information contained in classical record values and Madadi and Tata [26] present results on the Shannon information contained in k-record values. They have established a relationship between the Shannon information content of a random sample of fixed size and the Shannon information in the data consisting of sequential maxima. Also, they have considered the information contained in the k-record data from an inverse sampling plan as well. Goel *et al.* [18] discussed the measure of entropy for past lifetime distributions based on k-records. Recently, Jose and Sathar [22] studied some important properties of residual extropy of k-record values as well. It is to be noted that, when $k = 1$, we can easily obtain classical record values from k-records. Hence, k-records can be also considered as a generalized version of classical records. Baratpour *et al.* [6] studied entropy properties of classical records. Abbasnejad and Arghami [2] have discussed about the information contained in classical record values in detail and have derived some important properties as well. But to the best of our knowledge, no attention has been paid to the study of Rényi information contained in k-records.

Through this paper, the Rényi entropy of k-records arising from any continuous distribution has been discussed in detail. We also explore some of its important properties and have presented two applications of Rényi entropy of k-records.

3. RÉNYI ENTROPY OF k-RECORDS

Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables with parent distribution $f(x)$. Then, analogous to (1.1), the Rényi entropy of n-th upper k-record value $\big(U_{n(k)}\big)$ is defined by

$$H_\alpha(U_{n(k)}) = \frac{1}{1-\alpha} \ln \int_x f^\alpha_{n(k)}(x)dx, \quad \alpha > 0,\ (\alpha \neq 1). \tag{3.1}$$

In the following example, we illustrate that Rényi entropy measure of uncertainty contained in the original random variable is more when compared to that of k-records arising from the observations on the original random variable.

Example 3.1. Assume X is a random variable following $U(2,4)$ with pdf given by

$$f_X(x) = \begin{cases} \frac{1}{2}, & 2 \leq x \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

We use the Rényi entropy to measure the uncertainty involved in the random variable X. Let $H_\alpha(X)$ denote the Rényi entropy of X. Then from (1.1), we get $H_\alpha(X) = \ln 2$. Also, the Rényi entropy of n-th upper k-record value arising from $U(2,4)$ is obtained from (3.1) as

$$H_\alpha\big(U_{n(k)}\big) = \frac{1}{1-\alpha} \ln\left[\frac{k^{\alpha n}}{\Gamma^\alpha(n)2^{\alpha-1}} \frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}\right].$$

It is to be noted that $H_\alpha(X)$ is independent of α. Moreover, $H_\alpha(X) - H_\alpha\big(U_{n(k)}\big) \geq 0$ for $\alpha > 0$. This means that the uncertainty of X is more than $U_{n(k)}$. Thus, the predictability of X is smaller than the predictability of $U_{n(k)}$. The graphical representation of Rényi entropy of X and the Rényi entropy of $U_{n(k)}$ for varying α is given in Figure 1.

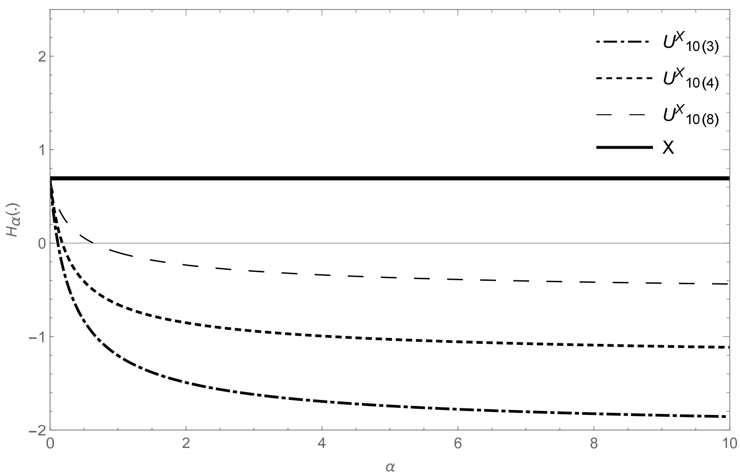


Figure 1: Rényi entropy of X and $U_{n(k)}$ for various values of α.

Fashandi and Ahmadi [15] have represented Rényi entropy of n-th upper k-record value in terms of Rényi entropy of n-th upper k-record value arising from $U(0,1)$. But they have not used that representation to study the properties of Rényi entropy of n-th upper k-record value arising from any continuous distribution. In this paper, we use the expression of Rényi entropy of n-th upper k-record value in terms of Rényi entropy of n-th upper k-record value arising from $U(0,1)$ to carry out investigation on properties and divergence of Rényi entropy of n-th upper k-record value. Let $\{X_i,\ i\geq 1\}$ be a sequence of iid random variables with a common distribution $U(0,1)$. Let $U^*_{n(k)}$ denote the n-th upper k-record value arising from the sequence $\{X_i,\ i\geq 1\}$. Using (2.1) in (3.1), we get

$$H_\alpha\left(U^*_{n(k)}\right)=\frac{1}{1-\alpha}\ln\int_{-\infty}^{\infty}\frac{k^{\alpha n}}{\Gamma^\alpha(n)}[\ln(1-x)]^{\alpha(n-1)}[1-x]^{\alpha(k-1)}dx.$$

Using the transformation $z=-\ln(1-x)$, we have

$$H_\alpha\left(U^*_{n(k)}\right)=\frac{1}{1-\alpha}\ln\int_{0}^{\infty}\frac{k^{\alpha n}}{\Gamma^\alpha(n)}z^{\alpha(n-1)}\mathrm{e}^{-z(\alpha(k-1)+1)}dz.$$

Then, the Rényi entropy of $U^*_{n(k)}$ is given by

$$H_\alpha\left(U^*_{n(k)}\right)=\frac{1}{1-\alpha}\ln\left[\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}\right]. \tag{3.2}$$

Then, for a sequence of iid random variables $\{X_i,\ i\geq 1\}$ with cdf $F(x)$ and pdf $f(x)$. If we denote $U_{n(k)}$ the n-th upper k-record value of the sequence $\{X_i\}$. Applying (2.1) in (3.1), we get

$$H_\alpha\left(U_{n(k)}\right)=\frac{1}{1-\alpha}\ln\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\int_{-\infty}^{\infty}[-\ln(1-F(x))]^{\alpha(n-1)}[1-F(x)]^{\alpha(k-1)}f^\alpha(x)dx.$$

Using the transformation $u=-\ln(1-F(x))$ and on integrating, we get

$$H_\alpha\left(U_{n(k)}\right)=\frac{1}{1-\alpha}\ln\left\{\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}E\left[f^{\alpha-1}\left(F^{-1}(1-\mathrm{e}^{-V})\right)\right]\right\},$$

where V follows gamma distribution with parameters $\alpha(n-1)+1$ and $\alpha(k-1)+1$ and we denote it by $V\sim\text{Gamma}(\alpha(n-1)+1,\alpha(k-1)+1)$. Then, from (3.2), the Rényi entropy of $U_{n(k)}$ is given by

$$H_\alpha\left(U_{n(k)}\right)=H_\alpha\left(U^*_{n(k)}\right)+\frac{1}{1-\alpha}\ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(1-\mathrm{e}^{-V})\right)\right]\right\}. \tag{3.3}$$

Similarly, the Rényi entropy of n-th lower k-record value arising from any continuous distribution can be expressed in terms of Rényi entropy of n-th lower k-record value arising from $U(0,1)$. Let $L_{n(k)}$ denote the n-th lower k-record value of the sequence $\{X_i\}$. Then, the Rényi entropy of $L_{n(k)}$ is given by

$$H_\alpha\left(L_{n(k)}\right)=H_\alpha\left(L^*_{n(k)}\right)+\frac{1}{1-\alpha}\ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(e^{-V})\right)\right]\right\}, \tag{3.4}$$

where $H_\alpha\left(L^*_{n(k)}\right)$ denote the Rényi entropy of n-th lower k-record value arising from $U(0,1)$ and $V\sim\text{Gamma}(\alpha(n-1)+1,\alpha(k-1)+1)$.

As an illustration, we obtain the Rényi entropy of k-records arising from exponential and Pareto distribution in the following examples.

Example 3.2. Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables having a common Pareto distribution with density function given by

$$f(x) = \frac{\beta}{\sigma}\left(\frac{x}{\sigma}\right)^{-\beta-1}, \quad x > \sigma.$$

Here,

$$F^{-1}(x) = \sigma[1-x]^{-\frac{1}{\beta}}.$$

Now, we have

$$E\big[f(F^{-1}(1-\mathrm{e}^{-V_n}))\big] = \frac{\beta^{\alpha n}}{\sigma^{\alpha-1}}\left[\frac{\alpha(k-1)+1}{\alpha(\beta k+1)-1}\right]^{\alpha(n-1)+1}.$$

Then, from (3.2) and (3.3), we get

$$H_\alpha\big(U_{n(k)}\big) = \frac{1}{1-\alpha}\ln\left[\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\frac{\beta^{\alpha n}\Gamma(\alpha(n-1)+1)}{\sigma^{\alpha-1}[\alpha(\beta k+1)-1]^{\alpha(n-1)+1}}\right].$$

The graphical representation of the Rényi entropy of $U^X_{n(k)}$ arising from Pareto distribution with shape parameter $\beta = 3$ and scale parameter $\sigma = 2$ is given in Figure 2, for varying α.

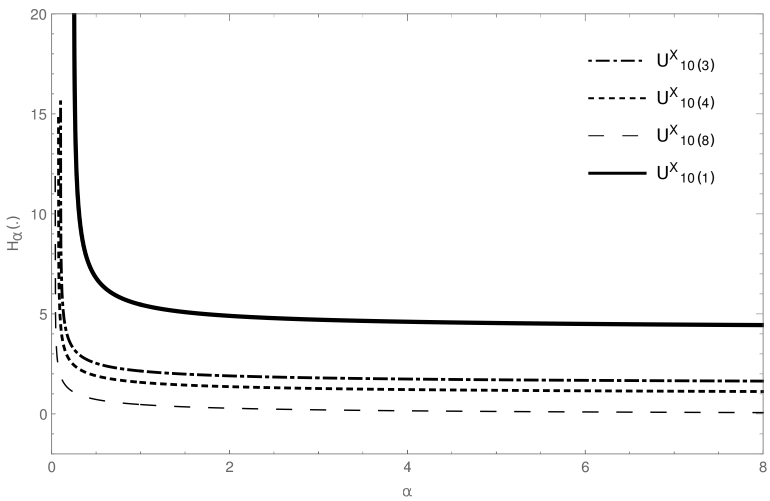


Figure 2: Rényi entropy of $U^X_{n(k)}$ for various values of α.

If we put $k = 1$, we can easily obtain the classical records from the sequence of k-records. From the figure, it can be observed that the Rényi entropy of classical upper record values (when $k = 1$) is greater than the Rényi entropy of upper k-records. This means that the uncertainty contained in classical records is more than that of k-records. Hence, one may observe certain situations where the predictability of classical records is less than the predictability of k-records when analyzed using Rényi entropy.

Example 3.3. Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables having a common exponential distribution with density function given by

$$f(x) = \theta \mathrm{e}^{-\theta x}, \quad x > 0,\ \theta > 0.$$

Here,

$$F^{-1}(x) = -\frac{1}{\theta}\ln(1-x).$$

Now, we have

$$E\big[f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-V}))\big] = \left[\frac{\alpha(k-1)+1}{\alpha k}\right]^{\alpha(n-1)+1}\theta^{\alpha-1}.$$

Then, from (3.2) and (3.3), we get

$$H_\alpha\big(U_{n(k)}\big) = \frac{1}{1-\alpha}\ln\left[\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\frac{\theta^{\alpha-1}\Gamma(\alpha(n-1)+1)}{(\alpha k)^{\alpha(n-1)+1}}\right].$$

4. PROPERTIES OF RÉNYI ENTROPY OF k-RECORDS

In this section, we discuss some important properties of Rényi entropy of upper and lower k-records arising from any continuous distribution. To determine the monotonicity of Rényi entropy of upper and lower k-records arising from any continuous distribution we make use of the following definitions of stochastic and likelihood ratio orders given in Shaked and Shanthikumar [32].

Definition 4.1. Let X and Y be two non-negative random variables with cdfs F and G and with pdfs f and g respectively, then X is said to be smaller than Y:

(1) in the likelihood ratio order, denoted by $X \leq_{\mathrm{lr}} Y$, if $\frac{f(x)}{g(x)}$ is decreasing in $x \geq 0$;

(2) in the usual stochastic order, denoted by $X \leq_{\mathrm{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \geq 0$, where $\bar{H}(\cdot)$ is the survival function.

It is well known that $X \leq_{\mathrm{lr}} Y \Longrightarrow X \leq_{\mathrm{st}} Y$ and $X \leq_{\mathrm{st}} Y$ if and only if $E[\phi(X)] \leq E[\phi(Y)]$ for all increasing functions ϕ.

Definition 4.2. The random variable X is said to be less than or equal to the random variable Y in Rényi entropy ordering, denoted by $X \leq_{\mathrm{RE}} Y$, if $H_\alpha(X) \leq H_\alpha(Y)$ for all $\alpha > 0$.

The following theorem reveals the monotone behaviour of Rényi entropy of upper k-record values based on n.

Theorem 4.1. *Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables with a common cdf $F(x)$ and pdf $f(x)$. Let $U_{n(k)}$ denote the n-th upper k-record value. If $f(x)$ is non-decreasing in x, then for $n > k$, $H_\alpha\big(U_{n(k)}\big)$ is non-increasing in n.*

Proof: The proof is straightforward as in Theorem 2.1 of Abbasnejad and Arghami [2]. □

In a similar way, we can state the monotone behaviour of Rényi entropy of lower k-records as given in the following theorem. The proof is not included since it easily follows as in Theorem 4.1.

Theorem 4.2. *Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables with a common cdf $F(x)$ and pdf $f(x)$. Let $L_{n(k)}$ denote the n-th lower k-record value. If $f(x)$ is non-increasing in x, then for $n > k$, $H_\alpha(L_{n(k)})$ is non-increasing in n.*

We will now discuss about the Rényi entropy ordering of n-th upper k-record value of two random variables. Abbasnejad and Arghami [2] have used Rényi entropy ordering of the random variables to establish their Rényi entropy ordering of classical record values. In the following theorem, we make use of Rényi entropy ordering of the random variables to establish their Rényi entropy ordering of n-th upper k-record value.

Theorem 4.3. *Let X and Y be two continuous random variables with cdfs $F(x)$ and $G(y)$ and pdfs $f(x)$ and $g(y)$ respectively. Suppose that $U^X_{n(k)}$ and $U^y_{n(k)}$ represents the n-th upper k record value arising from X and Y respectively. Assume that*

$$\Lambda_1 = \left\{ v > 0 \,\middle|\, \frac{g(G^{-1}(1-\mathrm{e}^{-v}))}{f(F^{-1}(1-\mathrm{e}^{-v}))} \leq 1 \right\},$$

$$\Lambda_2 = \left\{ v > 0 \,\middle|\, \frac{g(G^{-1}(1-\mathrm{e}^{-v}))}{f(F^{-1}(1-\mathrm{e}^{-v}))} > 1 \right\}$$

and $X \leq_{\mathrm{RE}} Y$. If $\inf \Lambda_1 \geq \sup \Lambda_2$, then $U^X_{n(k)} \leq_{\mathrm{RE}} U^Y_{n(k)}$, $\forall\ n \geq 1$ and $n > k$.

Proof: The proof is omitted since it is similar to that of Theorem 2.3 in Abbasnejad and Arghami [2]. □

In the following example, we apply Theorem 4.3 to obtain Rényi entropy ordering of two random variables following exponential distribution based on upper k-records.

Example 4.1. Let X and Y be two random variables having common exponential distribution with different scale parameters σ and λ respectively, where $\sigma > \lambda$. Then from (1.1), we get

$$H_\alpha(X) = \frac{1}{1-\alpha}\ln(\alpha) - \ln(\sigma).$$

It can be easily verified that $H_\alpha(X)$ is a decreasing function of σ. Thus, we have $H_\alpha(X) \leq H_\alpha(Y)$ and thereby $X \leq_{\mathrm{RE}} Y$. We have $f(F^{-1}(1-\mathrm{e}^{-x})) = \frac{1}{\sigma}\mathrm{e}^{-x}$ and $\inf \Lambda_1 = \sup \Lambda_2$. Hence, by Theorem 4.3 we get $U^X_{n(k)} \leq_{\mathrm{RE}} U^Y_{n(k)}$.

Similar to Theorem 4.3, we establish the Rényi entropy ordering of two random variables based on lower k-records.

Theorem 4.4. *Let X and Y be two continuous random variables with cdfs $F(x)$ and $G(y)$ and pdfs $f(x)$ and $g(y)$ respectively. Suppose*

$$\Lambda_1 = \left\{ v > 0 \,\middle|\, \frac{g(G^{-1}(\mathrm{e}^{-v}))}{f(F^{-1}(\mathrm{e}^{-v}))} \leq 1 \right\},$$

$$\Lambda_2 = \left\{ v > 0 \,\middle|\, \frac{g(G^{-1}(\mathrm{e}^{-v}))}{f(F^{-1}(\mathrm{e}^{-v}))} > 1 \right\}$$

and $X \leq_{\mathrm{RE}} Y$. If $\inf \Lambda_1 \geq \sup \Lambda_2$, then $L^X_{n(k)} \leq_{\mathrm{RE}} L^Y_{n(k)}$, $\forall\, n \geq 1$ and $n > k$.

The following lemma explains the effect of location-scale transformation on random variable in the case of Rényi entropy of k-records. The proof is simple and hence omitted.

Lemma 4.1. *Consider a non-negative random variable X with pdf f and cdf F. Let $Z = aX + b$ be a transformation on X, where $a > 0$ and $b \geq 0$ are constants. Then*

$$H_\alpha\left(U^Z_{n(k)}\right) = H_\alpha\left(U^X_{n(k)}\right) + \ln a, \tag{4.1}$$

where $U^Z_{n(k)}$ and $U^X_{n(k)}$ are the n-th k-record corresponding to Z and X respectively.

Thus, the Rényi entropy of k-records changes due to the change in scale, but it does not change due to the change in location. The next theorem will discuss on the Rényi entropy ordering of k-records under location-scale transformation.

Theorem 4.5. *Consider two absolutely continuous random variables X and Y. Assume that $U^Z_{n(k)}$ and $U^X_{n(k)}$ are the n-th upper k-record corresponding to X and Y respectively. Let $U^{Z_1}_{n(k)} = a_1 U^X_{n(k)} + b_1$ and $U^{Z_2}_{n(k)} = a_2 U^Y_{n(k)} + b_2$, where $a_1, a_2 > 0$ and $b_1, b_2 \geq 0$ are constants. If $U^X_{n(k)} \leq_{\mathrm{RE}} U^Y_{n(k)}$, then $U^{Z_1}_{n(k)} \leq_{\mathrm{RE}} U^{Z_2}_{n(k)}$ for $a_1 \leq a_2$.*

Proof: If $U^X_{n(k)} \leq_{\mathrm{RE}} U^Y_{n(k)}$, then

$$H_\alpha\left(U^X_{n(k)}\right) \leq H_\alpha\left(U^Y_{n(k)}\right).$$

Since $a_1 \leq a_2$, $\ln a_1 \leq \ln a_2$. Hence,

$$\ln a_1 + H_\alpha\left(U^X_{n(k)}\right) \leq \ln a_2 + H_\alpha\left(U^Y_{n(k)}\right).$$

Thus, from (4.1), we get $U^{Z_1}_{n(k)} \leq_{\mathrm{RE}} U^{Z_2}_{n(k)}$. Hence the theorem. □

We will now deduce the following corollary which removes the restriction on the scale constants.

Corollary 4.1. *Consider two absolutely continuous random variables X and Y. Assume that $U^Z_{n(k)}$ and $U^X_{n(k)}$ are the n-th upper k-record corresponding to X and Y respectively. Let $U^{Z_1}_{n(k)} = aU^X_{n(k)} + b$ and $U^{Z_2}_{n(k)} = aU^Y_{n(k)} + b$, where $a > 0$ and $b \geq 0$ are constants. If $U^X_{n(k)} \leq_{\mathrm{RE}} U^Y_{n(k)}$, then $U^{Z_1}_{n(k)} \leq_{\mathrm{RE}} U^{Z_2}_{n(k)}$.*

We will now discuss the effect of monotone transformation for Rényi entropy of k-records through the following theorem.

Theorem 4.6. *Assume a strictly convex function ψ having $\psi(0) = 0$ and $\psi(\infty) = \infty$. Consider, if $Y = \psi(X)$ then*

$$H_\alpha\left(U^Y_{n(k)}\right) = H_\alpha\left(U^*_{n(k)}\right) + \frac{1}{1-\alpha}\ln\left\{E\left[\frac{f(F^{-1}(1-\mathrm{e}^{-V_n}))}{\psi'(F^{-1}(1-\mathrm{e}^{-V_n}))}\right]^{\alpha-1}\right\}, \tag{4.2}$$

where $V_n \sim \mathrm{Gamma}(\alpha(n-1)+1, \alpha(k-1)+1)$. Here, $U^Y_{n(k)}$ are the n-th upper k-record value corresponding to Y.

Proof: Let $g_{n(k)}(y)$ and $\bar{G}_{n(k)}(y)$ be the pdf and survival function of n-th upper k-record value corresponding to Y. Then, from (2.1) we get

$$H_\alpha\left(U^Y_{n(k)}\right) = \frac{1}{1-\alpha}\ln\int_0^\infty \frac{k^{\alpha n}}{\Gamma^\alpha(n)}\left[-\ln\bar{G}(y)\right]^{\alpha(n-1)}[\bar{G}(y)]^{\alpha(k-1)}g^\alpha(y)dy.$$

Applying the transformation $Y = \psi(X)$, we have

$$H_\alpha\left(U^Y_{n(k)}\right) = \frac{1}{1-\alpha}\ln\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\int\limits_0^\infty\left[-\ln\bar{F}(x)\right]^{\alpha(n-1)}[\bar{F}(x)]^{\alpha(k-1)}\left(\frac{f(x)}{\psi'(x)}\right)^\alpha\psi'(x)dx.$$

Using the substitution $u = -\ln\bar{F}(x)$ in the integral, the theorem follows. □

The following theorem, establishes the Rényi entropy ordering of strictly increasing convex functions of two n-th upper k-records based on the Rényi entropy ordering of their respective k-records.

Theorem 4.7. *Suppose X and Y are non-negative random variables such that $U^X_{n(k)} \leq_{\mathrm{RE}} U^Y_{n(k)}$ and ψ be a strictly increasing convex function with $\psi(0) = 0$, $\psi(\infty) = \infty$, $\psi'(x)$ exists and is continuous with $\psi'(0) \geq 1$. Then $\psi\left(U^X_{n(k)}\right) \leq_{\mathrm{RE}} \psi\left(U^Y_{n(k)}\right)$, where $U^X_{n(k)}$ and $U^Y_{n(k)}$ denote the n-th upper k-record value corresponding to X and Y respectively.*

Proof: Since $U^X_{n(k)} \leq_{\mathrm{RE}} U^Y_{n(k)}$, we have $H_\alpha\left(U^X_{n(k)}\right) \leq H_\alpha\left(U^Y_{n(k)}\right)$. This implies

$$H_\alpha\left(U^{X*}_{n(k)}\right)E\left[f^{\alpha-1}\left(F^{-1}(1-\mathrm{e}^{-V_n})\right)\right] \leq H_\alpha\left(U^{Y*}_{n(k)}\right)E\left[g^{\alpha-1}\left(G^{-1}(1-\mathrm{e}^{-V_n})\right)\right], \tag{4.3}$$

where $V_n \sim \mathrm{Gamma}(\alpha(n-1)+1, \alpha(k-1)+1)$. Then, from (4.2), we have

$$\begin{aligned} H_\alpha\left(\psi\left(U^X_{n(k)}\right)\right) - H_\alpha\left(\psi\left(U^Y_{n(k)}\right)\right) &= \\ = H_\alpha\left(U^{X*}_{n(k)}\right) - H_\alpha\left(U^{Y*}_{n(k)}\right) &+ \frac{1}{1-\alpha}\ln\left\{\frac{E\left[\frac{f(F^{-1}(1-\mathrm{e}^{-V_n}))}{\psi'(F^{-1}(1-\mathrm{e}^{-V_n}))}\right]^{\alpha-1}}{E\left[\frac{g(G^{-1}(1-\mathrm{e}^{-V_n}))}{\psi'(G^{-1}(1-\mathrm{e}^{-V_n}))}\right]^{\alpha-1}}\right\}. \end{aligned}$$

Since $\psi'(0) \geq 1$ and from (4.3), we obtain $H_\alpha\left(\psi\left(U^X_{n(k)}\right)\right) - H_\alpha\left(\psi\left(U^Y_{n(k)}\right)\right) \leq 0$. Hence, $\psi\left(U^X_{n(k)}\right) \leq_{\mathrm{RE}} \psi\left(U^Y_{n(k)}\right)$. □

Therefore, we can observe that the Rényi entropy ordering of two random variables determine the Rényi entropy ordering of their respective k-records and the Rényi entropy ordering of the respective convex function of k-records are determined by the Rényi entropy ordering of their respective k-records. The following example discusses the same.

Example 4.2. Consider a convex function $\psi(x) = \beta x$, where $\beta \geq 1$. Hence ψ be a strictly increasing convex function with $\psi(0) = 0$, $\psi(\infty) = \infty$, $\psi'(x)$ exists and is continuous with $\psi'(0) \geq 1$. From Example 4.1, we have $U^X_{n(k)} \leq_{\text{RE}} U^Y_{n(k)}$. Thus, the assumptions of Theorem 4.7 are satisfied and therefore, we can directly obtain $\psi\Big(U^X_{n(k)}\Big) \leq_{\text{RE}} \psi\Big(U^Y_{n(k)}\Big)$ in which X and Y have common exponential distribution with different scale parameters σ and λ respectively, where $\sigma > \lambda$.

We will now study another property regarding the bound of Rényi entropy of k-records. Through the following theorem, we present a lower bound for the Rényi entropy of upper k-records arising from any continuous distribution.

Theorem 4.8. *Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables with a common distribution function $F(x)$ and density function $f(x)$. Let $H_\alpha\big(U_{n(k)}\big)$ denote the Rényi entropy of n-th upper k-record value arising from the sequence and $H_\alpha\Big(U^*_{n(k)}\Big)$ denote the Rényi entropy of n-th upper k-record value arising from $U(0,1)$. Suppose that $M = f(m)$ exists, where M is the mode of X, then for $\alpha > 0$*

$$H_\alpha\big(U_{n(k)}\big) \geq H_\alpha\Big(U^*_{n(k)}\Big) - \ln M. \tag{4.4}$$

Proof: Since M is the mode of X, we have

$$f(F^{-1}(y)) \leq M.$$

Using the transformation $y = 1 - \mathrm{e}^{-V}$, we get

$$\begin{aligned} f(F^{-1}(1-\mathrm{e}^{-U})) &\leq M, \\ f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-U})) &\leq M^{\alpha-1}. \end{aligned}$$

Taking expectations on both sides, we obtain

$$E\big[f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-U}))\big] \leq M^{\alpha-1}. \tag{4.5}$$

Similarly, for $0 < \alpha < 1$

$$E\big[f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-U}))\big] \geq M^{\alpha-1}. \tag{4.6}$$

From (4.5) and (4.6), for $\alpha > 0$, we have

$$\frac{1}{1-\alpha} \ln E\big[f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-U}))\big] \geq -\ln M. \tag{4.7}$$

Using (3.3) in (4.7), we get

$$\begin{aligned} H_\alpha\big(U_{n(k)}\big) - H_\alpha\Big(U^*_{n(k)}\Big) &\geq -\ln M \\ H_\alpha\big(U_{n(k)}\big) &\geq H_\alpha\Big(U^*_{n(k)}\Big) - \ln M. \end{aligned}$$

Hence the theorem. □

In the following example, we make use of Theorem 4.8 to obtain bound for Rényi entropy of upper k-record value arising from Gompertz distribution.

Example 4.3. The pdf of Gompertz distribution with shape parameter λ and scale parameter β is given by

$$f(x) = \beta\lambda \mathrm{e}^{\beta x + \lambda(1-\mathrm{e}^{\beta x})}, \quad x > 0, \ \beta, \ \lambda > 0.$$

We know that mode of this distribution is $\frac{1}{\beta}\ln\frac{1}{\lambda}$. Thus, from (4.4) we have

$$H_\alpha\big(U_{n(k)}\big) \geq \frac{1}{1-\alpha}\ln\left[\frac{k^{\alpha n}\beta}{\ln\lambda\Gamma^\alpha(n)}\frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}\right].$$

In the following theorem, similar to Theorem 4.8, we obtain lower bound for Rényi entropy of lower k-records arising from any continuous distribution.

Theorem 4.9. *Let $\{X_i, \ i \geq 1\}$ be a sequence of iid random variables with a common distribution function $F(x)$ and density function $f(x)$. Let $H_\alpha\big(L_{n(k)}\big)$ denote the Rényi entropy of n-th lower k-record value arising from the sequence and $H_\alpha\Big(L^*_{n(k)}\Big)$ denote the Rényi entropy of n-th lower k-record value arising from $U(0,1)$. Suppose that $M = f(m)$ exists, where M is the mode of X, then for $\alpha > 0$*

$$H_\alpha\big(L_{n(k)}\big) \geq H_\alpha\Big(L^*_{n(k)}\Big) - \ln M. \tag{4.8}$$

In the following example, we make use of Theorem 4.9 to obtain lower bound for Rényi entropy of lower k-records arising from Fréchet distribution.

Example 4.4. The density function of Frechet distribution with shape parameter a and scale parameter s is given by

$$f(x) = \frac{a}{s}\left(\frac{x}{s}\right)^{-1-a}\mathrm{e}^{-\left(\frac{x}{s}\right)^{-a}}, \quad x > 0; \ a, \ s > 0.$$

We know that mode of this distribution is $s\left(\frac{a}{1+a}\right)^{\frac{1}{a}}$. Thus, from (4.8), we get

$$H_\alpha\big(U_{n(k)}\big) \geq \frac{1}{1-\alpha}\ln\left\{\left[\frac{a}{a+1}\right]^a\frac{k^{\alpha n}}{s\,\Gamma^\alpha(n)}\frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}\right\}.$$

5. APPLICATIONS OF RÉNYI ENTROPY OF k-RECORDS

This section deals with the applications of Rényi entropy of k-records. One application of Rényi entropy of k-records is that it can be used to characterize a class of distributions of non-negative random variables. Another application of Rényi entropy of k-records is that it determines Rényi divergence between the distribution of k-record values and its parent distribution.

5.1. Characterization of exponential distribution

Ebrahimi [14] suggested that maximum entropy paradigm can be used to produce a model for the data generating distribution. In the maximum entropy procedure, a model that best approximates the unknown distribution is derived based on the partial knowledge about this distribution in terms of a set of information constraints. Then, the inference is based on the model that maximizes the entropy of the random variables subject to the information constraints. In this subsection, we derive exponential distribution as the distribution that maximizes the Rényi entropy of k-records under some information constraints.

Let ξ be a class of distributions $F(x)$ of non-negative random variables X with $F(0) = 0$ such that

(i) $r(x, \theta) = a(\theta)b(x)$,

(ii) $b(x) \geq \beta, \quad \beta > 0$,

where $b(x) = B'(x)$ is a non-negative function of x and $a(\theta)$ is a non-negative function of θ.

Abbasnejad and Arghami [2] derived exponential distribution as the distribution that maximizes the Rényi entropy of classical record values under some information constraints. In the following theorem we characterize ξ using the Rényi entropy of n-th upper k-record value.

Theorem 5.1. *Let $U_{n(k)}$ be the n-th upper k-record value of $F(x;\theta)$, where $F(x;\theta)$ is in class ξ. Then, the n-th upper k-record value of the distribution $F(x;\theta)$ has maximum Rényi entropy in ξ if and only if $F(x;\theta) = 1 - \mathrm{e}^{-a(\theta)\beta x}$.*

Proof: The proof follows similar steps to that of Theorem 4.1 in Abbasnejad and Arghami [2]. □

5.2. Rényi divergence of k-records

Several applications of entropy divergence measures in formulating test statistics for testing of hypotheses and goodness-of-fit tests are available in literature. Gil *et al.* [16] presented closed form expressions of Rényi divergence for nineteen commonly used univariate continuous distributions as well as those for multivariate Gaussian and Dirichlet distributions. Salicrú *et al.* [31] suggested test statistics using some families of divergence like ϕ-divergence. Vasicek [35] used the sample Shannon entropy estimate to test normality. Abbasnejad [1] obtained a test statistic for exponentiality based on Rényi divergence. Abbasnejad and Arghami [2] studied Rényi divergence between parent distribution and distribution of classical record value as well. Through the following theorem, we study Rényi divergence between parent distribution and distribution of n-th upper k-record value.

Theorem 5.2. *The Rényi divergence between distribution of n-th upper k-record value and its parent distribution is given by*

$$D_\alpha(f_{n(k)}, f) = -H_\alpha\Big(U^*_{n(k)}\Big),$$

where $f_{n(k)}$ is the pdf of $U_{n(k)}$ and $U^*_{n(k)}$ is the n-th upper k-record value arising from $U(0,1)$. Moreover, $D_\alpha(f_{n(k)}, f)$ is increasing in n.

Proof: Using (2.1) in (1.2) and by the transformation $u = -\ln \bar{F}(x)$, we get

$$\begin{aligned} D_\alpha(f_{n(k)}, f) &= \frac{1}{\alpha-1}\ln\int_0^\infty \frac{k^{\alpha n}}{\Gamma^\alpha(n)} u^{\alpha(n-1)} \mathrm{e}^{-u(\alpha(k-1)+1)}du, \\ &= -H_\alpha\Big(U^*_{n(k)}\Big). \end{aligned}$$

Hence, the Rényi divergence between the distribution of the n-th upper k-record value and the parent distribution is distribution free. Moreover, taking the derivative of $H_\alpha\Big(U^*_{n(k)}\Big)$ with respect to n, we get

$$\frac{dH_\alpha\Big(U^*_{n(k)}\Big)}{dn} = \frac{\alpha}{\alpha-1}(1-\ln k) - \frac{1}{\alpha-1}\xi(\alpha(n-1)+1) + \frac{\alpha}{\alpha-1}\xi(n),$$

where $\xi(u) = \dfrac{d\ln\Gamma(u)}{du}$. For every u, the function $\xi(u)$ is non-decreasing and therefore $H_\alpha\Big(U^*_{n(k)}\Big)$ is non-increasing in n. Thus the result follows. □

Thus, by increasing n, we expect that the divergence between the distribution of the n-th upper k-record value and the parent distribution increases.

5.3. Numerical illustration

In this subsection, we propose a simple estimator for the Rényi entropy of the n-th upper k-record value and discuss the merit of k-records over classical records and parent random variable in terms of uncertainty. To estimate the Rényi entropy based on n-th upper k-record value, kernel density has been applied to estimate the density function and empirical distribution has been used as an estimator for the distribution function. The estimator is proposed for Rényi entropy obtained by replacing the density of the parent random variable by the density of n-th upper k-record value and hence much complexities arises while deriving the properties of the proposed estimator directly. Therefore, the proposed simple estimator for Rényi entropy based on n-th upper k-record value can be analysed numerically by evaluating the average bias and MSE for different sample sizes which examines the bias and consistency characteristics of the proposed estimator. A numerical illustration has been presented with an intention to describe the benefit of applying Rényi entropy based on n-th k-record in comparison to that of the parent random variable. Using (2.1) in (3.1), the Rényi entropy of the n-th upper k-record can be expressed as

$$H_\alpha\big(U_{n(k)}\big) = \frac{1}{1-\alpha}\ln\int\limits_0^\infty \frac{k^{\alpha n}}{\Gamma^\alpha(n)}\big[-\ln\bar{F}(x)\big]^{\alpha(n-1)}\big[\bar{F}(x)\big]^{\alpha(k-1)} f^\alpha(x)dx. \tag{5.1}$$

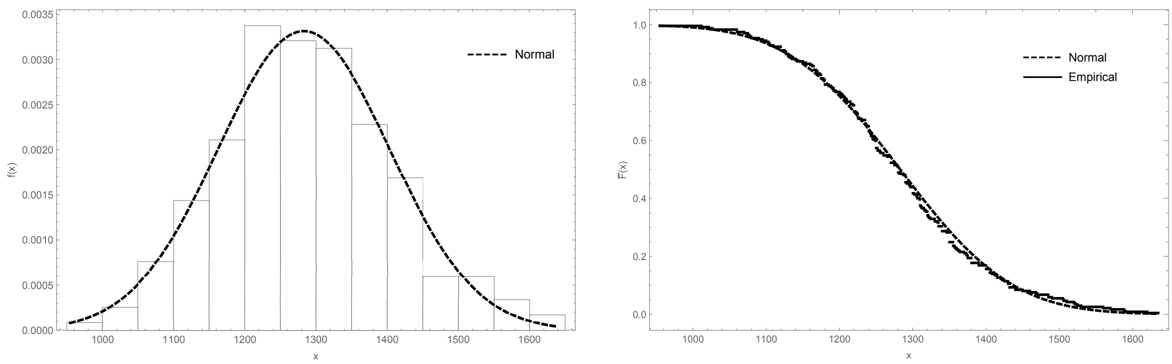
A simple estimator for the Rényi entropy of the n-th upper k-records value based on a random sample of size n is given by

$$\hat{H}_\alpha\big(U_{n(k)}\big) = \frac{1}{1-\alpha} \ln \int_0^\infty \frac{k^{\alpha n}}{\Gamma^\alpha(n)} \Big[-\ln \hat{\bar{F}}(x)\Big]^{\alpha(n-1)} \Big[\hat{\bar{F}}(x)\Big]^{\alpha(k-1)} \hat{f}^\alpha(x)dx, \tag{5.2}$$

where $\hat{f}(x) = \frac{1}{n b_n} \sum\limits_{j=1}^{n} K\big(\frac{x-X_j}{b_n}\big)$, denotes the kernel density estimator with the bandwidth b_n. Also $K(\cdot)$ is a kernel function satisfying the condition $\int_{-\infty}^{\infty} K(x)dx = 1$ and is usually a symmetric pdf. Also, $\hat{\bar{F}}(x) = \frac{1}{n} \sum\limits_{i=1}^{n} I(X_i \geq x)$ is the empirical survival function and $I(X_i \geq x)$ is the indicator function.

In the following illustration, we use a real life data set to compute Rényi entropy of the n-th upper k-record value and make a comparison with that of classical records and parent random variable.

Dataset 1: Let the random variable X represents the brain weight (in grams) of 237 adults discussed in Gladstone [17]. The brain weight of an adult is not so easy to obtain and hence for more reliable inferences on the random variable X, the distribution of X should possess less uncertainty. The study focus on the uncertainty contained in the distribution of the random variable X. Initial study on distribution of X suggests the normal distribution with location parameter $\mu = 1282.87$ and scale parameter $\sigma = 120.86$ is a good fit for the data set with Kolmogrove-Smirnov (K-S) statistic = 0.03914 and p-value = 0.9755. Since the normal distribution is a good fit for the proposed data, a Gaussian kernel can be chosen for estimation procedure using the given data set. The fit of normal distribution to data is depicted in Figure 3.



(a) Histogram and normal curve. **(b)** Normal and empirical CDF.

Figure 3: Modelling brain weight data using normal distribution.

To estimate Rényi entropy of the n-th upper k-record value the Gaussian kernel with $b_n = 120$ is applied in (5.2). The closeness of the estimators of Rényi entropy based on n-th upper k-record value and the parent random variable with the theoretical value of Rényi entropy which has been obtained by assuming normal distribution for the random variable with parameter values $\mu = 1282.87$ and $\sigma = 120.86$ (ML estimates) for different choices of α are presented in Table 2.

Table 2: Comparison of theoretical values and estimates of Rényi entropy based on X and $U_{n(k)}$ where $k = 1, 2, 5, 7, 9$ and 10.

α	$H_\alpha(X)$	$\hat{H}_\alpha(X)$	$\hat{H}_\alpha(U_{n(1)})$	$\hat{H}_\alpha(U_{n(2)})$	$\hat{H}_\alpha(U_{n(5)})$	$\hat{H}_\alpha(U_{n(7)})$	$\hat{H}_\alpha(U_{n(9)})$	$\hat{H}_\alpha(U_{n(10)})$
0.10	6.9885	8.9341	7.2943	6.4240	6.3835	6.2870	5.9981	5.8530
0.30	6.5692	9.5116	8.8650	8.7573	8.7103	8.6019	8.1354	7.5349
0.50	6.4024	11.1967	10.3061	10.2393	9.7591	9.6735	9.6341	9.5618
0.70	6.2846	20.6588	13.7849	13.7476	12.6784	12.6646	12.6296	12.5986
1.15	6.1556	17.3662	12.4395	12.3814	12.3616	12.2889	11.8381	11.7295
1.40	6.1147	10.8954	10.0227	9.2979	9.2531	9.1823	9.1673	9.1643
1.75	6.0823	3.7508	7.7072	7.5870	7.8013	7.8826	7.5907	7.6997
2.00	6.0558	3.1864	6.8005	6.7703	6.7178	6.7033	6.5973	6.5817
2.25	6.0336	2.2530	6.4741	6.1328	6.0768	6.0333	5.9477	5.8533
2.50	6.0147	1.6343	5.0568	4.9876	4.7800	4.6415	4.5031	4.4339
3.25	5.9839	0.8677	3.9306	3.8767	3.7152	3.6076	3.4999	3.4460
3.50	5.9598	0.4133	3.3674	3.3213	3.1828	3.0906	2.9983	2.9521

From Table 2, we can observe that the estimates of Rényi entropy based on n-th upper k-record value is closer to its theoretical value than the estimate of Rényi entropy based parent random variable. Also, when $k = 1$, k-records becomes classical records. In terms of uncertainty, we have compared three different estimates (based on parent random variable, classical records and k-records) for Rényi entropy which can be obtained from a random sample. Hence, from Table 2, one may conclude that there are situations where construction of k-records or classical records from random sample gives closer estimate than the estimate obtained based on random variable. Moreover, the k-records or classical records are ordered random variables which carry an additional information about their ranks when compared to the parent random variable.

Table 3: Average bias and MSE of the estimate of Rényi entropy of the n-th upper k-record value for different choices of α.

n	k	$\alpha = 0.25$		$\alpha = 0.75$		$\alpha = 1.50$		$\alpha = 3.0$	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	1	1.14072	1.09495	1.12052	1.06003	1.11085	1.03190	1.09435	1.03083
	3	1.07479	1.00485	1.07209	0.99914	1.06225	0.92091	1.05467	0.90367
	6	1.00195	0.89941	0.99823	0.85221	0.96188	0.81123	0.94057	0.80224
	8	0.92137	0.79354	0.91658	0.77677	0.91307	0.77391	0.89728	0.75497
	10	0.85722	0.74954	0.83132	0.73397	0.81957	0.71637	0.80547	0.68588
60	1	0.93818	0.84330	0.90040	0.84103	0.88101	0.82568	0.87493	0.82497
	3	0.85666	0.81577	0.83509	0.80718	0.82274	0.74541	0.79530	0.73895
	6	0.79185	0.73802	0.78635	0.73201	0.78544	0.71686	0.76749	0.70849
	8	0.76585	0.65507	0.75573	0.61861	0.72946	0.57439	0.70244	0.57347
	10	0.68611	0.53900	0.67284	0.50139	0.66842	0.49361	0.65933	0.44052
100	1	0.76797	0.69524	0.76709	0.68935	0.75349	0.68063	0.75052	0.68010
	3	0.74507	0.59512	0.73545	0.59152	0.72329	0.56915	0.71883	0.55927
	6	0.71429	0.53813	0.70983	0.52673	0.69858	0.51378	0.65525	0.48069
	8	0.64116	0.46515	0.63902	0.43912	0.62873	0.43155	0.61199	0.42260
	10	0.58717	0.41116	0.57277	0.40205	0.57109	0.38755	0.56469	0.37685

To study the effect of the estimator suggested for Rényi entropy of the n-th upper k-record value denoted as $H_\alpha\big(U_{n(k)}\big)$, we have obtained average bias and mean square error (MSE) of the estimator using bootstrapping procedure. The bias and MSE of the estimates are evaluated from value of Rényi entropy of the n-th upper k-record obtained using the parameter estimates $\mu = 1282.87$ and scale parameter $\sigma = 120.86$ in (5.1) which we have considered as the true value of $H_\alpha\big(U_{n(k)}\big)$. The average bias and MSE of $H_\alpha\big(U_{n(k)}\big)$ based on 100 bootstrap estimates from samples of sizes 20, 60 and 100 are presented in Table 3. It can be observed that the average bias and MSE of the estimator of Rényi entropy of the n-th upper k-record value diminishes as sample size becomes large.

6. CONCLUSION

The study explains the relevance of k-records in measuring uncertainty using Rényi entropy after comparing it with Rényi entropy of classical records as well as with Rényi entropy of original random variable. Fashandi and Ahmadi [15] have expressed Rényi entropy for k-records arising from any continuous distribution in terms of Rényi entropy of k-records arising from uniform distribution and we have used that representation to derive some important properties of Rényi entropy of k-records. The monotone behaviour of Rényi entropy of k-records have been derived. We have shown that the Rényi entropy ordering of random variables determines the Rényi entropy ordering of their respective k-record values. The Renyi entropy ordering of k-records determines the Renyi entropy ordering of their respective linear transformations of k-records as well as their convex function of k-records. A lower bound for the Rényi entropy of k-records have been obtained in this work. We have applied Rényi entropy of k-records to characterize exponential distribution by maximization of Rényi entropy based on certain information constraints. The study also establishes that the Rényi divergence between the distribution of k-records and its parent distribution is distribution free and the divergence increases with increase in n. A simple estimator for Rényi entropy of k-records has been proposed and compared estimates of Renyi entropy of k-records, classical records and parent random variable using a real life data set.

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