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CONSTRUCTION OF T_m-TYPE AND T_m-ASSISTED PBIB DESIGNS

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Abstract:

- Two-associate class triangular designs have been explored greatly but the T_m-type PBIB designs ($m \geq 3$) largely remains unexplored. The paper is written with an objective to construct a new series of T_m-type PBIB designs and to derive some more series of PBIB designs based on these PBIB designs, which we have called as T_m-assisted PBIB designs. For this, we begin by first constructing a series of T_m-type PBIB designs and then based on these designs, three series of T_m-assisted PBIB designs have been constructed. The association schemes of T_m-type and T_m-assisted PBIB designs have been discussed in their complete generalized form.

Keywords:

- *block designs; triangular association scheme; PBIB designs.*

AMS Subject Classification:

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1. INTRODUCTION

The first step before performing an experiment is to devise the way various treatments are allotted to different experimental units. Experimental designs assist us in performing this task keeping in mind various constraints like constraints on experimental material, constraints on the cost of the experimental setup, etc.

In the class of block designs, incomplete block designs are used whenever the constraints on the experimental materials do not allow us to use complete blocks or when the heterogeneity increases as a result of formation of complete blocks. Partially Balanced Incomplete Block (PBIB) designs, which fall under the category of incomplete block designs, are among the popular incomplete block designs which help in making treatment comparisons by utilizing lesser experimental material.

Since their introduction, the two-associate class PBIB designs have been studied to a greater extent by many authors. Notable among them are Bose and Nair ([1]), Bose and Shimamoto ([2]), Raghavarao ([12]), Clatworthy *et al.* ([5]), etc. Among the two associate class association scheme, triangular association scheme is interesting due to its specific arrangement of symbols in the form of a matrix. Ogaswara ([11]) generalized the triangular association scheme and introduced T_m – association scheme with m – associate classes. Later, John ([7]), Saha ([15]), Sinha ([22]), Sinha ([23]), Cheng *et al.* ([4]), Meitei ([8]), Sinha *et al.* ([24]), Singh ([19]), etc., studied the T_m-association scheme and constructed some T_m-type PBIB designs. Recently, Ruj and Roy (2007) have shown the applications of PBIB designs in key predistribution by using triangular association scheme.

Even today, the higher associate class T_m-type PBIB designs ($m \geq 3$) largely remains unexplored and because the higher associate class PBIB designs are needed as they provide us new and more efficient PBIB designs as discussed by Raghavarao ([12]), Sharma and Garg ([17]), etc., the present paper has been written to make some contributions towards the construction of higher-associate class T_m-type PBIB designs. For this, we will begin by discussing T_m-type association scheme and then construct a new series of T_m-type PBIB designs for which we have discussed a methodology to directly obtain the incidence matrix of the designs. After this, we will employ some easy and interesting techniques which utilize triangular type PBIB designs to construct some more four-associate class PBIB designs with different parametric combinations. We have called these designs as T_m-assisted PBIB designs. Their corresponding association schemes have also been discussed. We have also prepared a table containing PBIB designs constructed by using different methodologies discussed in this paper. This table compares the listed designs with designs having same parameters listed in Clatworthy *et al.* ([5]). The R programming codes for the computerized construction of all the series have been given in the Appendices.

2. T_m-TYPE ASSOCIATION SCHEME

T_m-type or m-dimensional triangular association scheme was first defined by Ogasawara ([11]) and we can refer to it for its original definition. We have also defined the T_m-type association scheme, for any arbitrary $m \geq 2$, in the following way:

Let us consider a vector of size b containing only binary elements 0's and 1's such that the vector contains r unit elements and (b – r) 0's. The number of associate classes, i.e., m is given by:

$$m = \min(r, (b - r)).$$

Let us say that this vector denotes our first treatment. The remaining (v – 1) treatments can be obtained by taking different combinations of the elements of this vector and in all we will get $v = \binom{b}{r}$ treatments. From this set of treatments, a treatment is the 0^{th}-associate of its own and two treatments are mutually i^{th}-associates if the corresponding vectors have (m – i) unit or null elements in common according to if $r < (b - r)$ or $(b - r) < r$ respectively. The parameters of the association scheme are:

$$v = \binom{b}{r},$$

$$n_i = \binom{m}{i}\binom{b-m}{i}; \quad i = 0, 1, 2, ..., m,$$

$$p^i_{jk} = \sum_{u=0}^{m-i} \binom{m-i}{u}\binom{i}{m-j-u}\binom{i}{m-k-u}\binom{b-m-i}{j+k+u-m}; \quad i, j, k = 0, 1, 2, ..., m.$$

3. CONSTRUCTION OF T_m-TYPE PBIB DESIGNS

Just like in association scheme, form a vector of size b. Now make all possible combinations of the elements of this vector and in all we will get $\binom{b}{r}$ different vectors. Now make a matrix N of order $\binom{b}{r} \times b$ matrix using these $\binom{b}{r}$ vectors as its rows such that any vector may form any row of this matrix. The resulting $\binom{b}{r} \times b$ matrix will be a matrix of 0's and 1's which will form the incidence matrix of a PBIB design following the T_m-type association scheme defined in Section 2 with the following parameters:

$$v = \binom{b}{r}, \quad b, \quad r, \quad k = \binom{b-1}{r-1}, \qquad \lambda_i = r - i; \quad i = 1, 2, ..., m.$$

Example 3.1. To illustrate the above construction methodology, let us consider a vector of size 6 having three unit (1) elements and three zero (0) elements. Following the above discussed methodology and taking all possible combinations of this vector, we will get 20 distinct vectors in all which will form the rows of the following incidence matrix N:

$$N = \begin{bmatrix} 1&0&0&1&0&1\\ 1&1&1&0&0&0\\ 1&1&0&0&0&1\\ 1&1&0&0&1&0\\ 1&1&0&1&0&0\\ 1&0&1&1&0&0\\ 1&0&1&0&0&1\\ 1&0&1&0&1&0\\ 1&0&0&0&1&1\\ 1&0&0&1&1&0\\ 0&1&1&1&0&0\\ 0&1&1&0&0&1\\ 0&1&1&0&1&0\\ 0&1&0&1&1&0\\ 0&1&0&0&1&1\\ 0&1&0&1&0&1\\ 0&0&1&1&1&0\\ 0&0&1&1&0&1\\ 0&0&1&0&1&1\\ 0&0&0&1&1&1 \end{bmatrix}.$$

The incidence matrix N corresponds to the following set of six blocks:

$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$, $(2, 3, 4, 5, 11, 12, 13, 14, 15, 16)$,
$(2, 6, 7, 8, 11, 12, 13, 17, 18, 19)$, $(1, 5, 6, 10, 11, 14, 16, 17, 18, 20)$,
$(4, 8, 9, 10, 13, 14, 15, 17, 19, 20)$, $(1, 3, 7, 9, 12, 15, 16, 18, 19, 20)$.

The above set of blocks will constitute a T_m-type PBIB design with the following parameters:

$v = 20$, $b = 6$, $r = 3$, $k = 10$,
$\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$,
$n_1 = 9$, $n_2 = 9$, $n_3 = 1$,

$$P_1 = \begin{bmatrix} 4&4&0\\ 4&4&1\\ 0&1&0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4&4&1\\ 4&4&0\\ 1&0&0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0&9&0\\ 9&0&0\\ 0&0&0 \end{bmatrix}.$$

4. CONSTRUCTION OF SOME T_m-ASSISTED PBIB DESIGNS

There is always a requirement for the construction of PBIB designs which are either new or more efficient than the already existing PBIB designs in the literature. For this purpose, the researchers are working towards the construction of higher associate class PBIB designs by devising new construction techniques and association schemes. In this section, we will use the T_m-type PBIB designs constructed above and apply some interesting techniques to construct some more PBIB designs. Already existing T_m-type PBIB designs can also be used. The association schemes of these derived PBIB designs have also been discussed.

4.1. Method – I

Using the incidence matrix of the T_m-type PBIB design constructed in Section 3 and its complementary matrix, we can obtain the incidence matrix of another PBIB design through matrix augmentation. Let us say that N be the incidence matrix of the T_m-type PBIB design as obtained in Section 3 and N^C be its complement. Then another matrix N_1 can be obtained through matrix augmentation as below:

$$N_1 = \left[\begin{array}{c|c} N & N^C \\ \hline N^C & N \end{array}\right].$$

Here, N is the incidence matrix of the T_m-type PBIB designs with the following parameters:

$$v = \binom{b}{r}, \quad b, \quad r, \quad k \quad \text{and} \quad \lambda_i; \ i = 1, 2, ..., m,$$

and n_i be the number of i^{th}-associates of any treatment. (A)
Therefore, N^C is the incidence matrix of the Tm-type PBIB designs with the parameters given below:

$$v^c = \binom{b^c}{r^c}, \quad b^c, \quad r^c, \quad k^c \quad \text{and} \quad \lambda_i^c; \ i = 1, 2, ..., m,$$

and n_i be the number of i^{th}-associates of any treatment. (B)
Then the matrix N_1 will form the incidence matrix of the PBIB design having (2m + 1) associate classes and with the following parameters:

$$v_1 = v + v^c, \qquad b_1 = b + b^c, \qquad r_1 = r + r^c, \qquad k_1 = k + k^c.$$

Let the difference between r and r^c be given by $d = |r - r^c|$.

(i) $d = 0$	(ii) $d = 1$	(iii) $d \geq 2$
$\lambda_i^1 = \lambda_i + \lambda_i^c$; $i = 1, 2, ..., m$ $\lambda_{m+1}^1 = r_1$ $\lambda_j^1 = \lambda_{j-(m+1)}^1$; $j = m+2, m+3, ..., 2m+1$	$\lambda_i^1 = r_1 - i$; $i = 1, 2, ..., 2m+1$	$\lambda_i^1 = \lambda_i + \lambda_i^c$; $i = 1, 2, ..., m$ $\lambda_{m+1}^1 = 0$ $\lambda_j^1 = \lambda_{j-(m+1)}^1 - (d-2)$; $j = m+2, m+3, ..., 2m+1$

On the basis of the value of d, we have the following three association schemes. Let $S_i(\alpha)$ denotes the set of i^{th}-associates of treatment α in T_m-association scheme (i = 1, 2, ..., m). Therefore,

$$p_{jk}^i = |S_j(\alpha) \cap S_k(\beta)|,$$

where α and β are mutually i^{th}-associates and $|S_i(\alpha)|$ represents the size of set $S_i(\alpha)$.

4.1.1. Association scheme for d = 0

We already know that the rows of the matrix N denote the corresponding treatment number. Now keeping in view, the parameters (A), (B) and incidence matrix N_1, we define the following association scheme:

- **i.** Two treatments are said to be i^{th}-associates if the inner product of the corresponding rows in the matrix N_1 is $\lambda_i^1 = \lambda_i + \lambda_i^c$ (i = 1, 2, ..., m).
- **ii.** Two treatments are said to be $(m+1)^{th}$-associates if the inner product of the corresponding rows in the matrix N_1 is r.
- **iii.** Two treatments are said to be j^{th}-associates if the inner product of the corresponding rows in the matrix N_1 is $\lambda_j^1 = \lambda_{(j-(m+1))}^1$; $j = m+2, m+3, ..., 2m+1$.

The above defined association scheme has the following parameters:

$$v_1 = \binom{b}{r} + \binom{b^c}{r^c}, \qquad n_i^1 = n_i; \quad i = 1, 2, ..., m,$$

$$n_{(m+1)}^1 = 1, \qquad n_{(m+j+1)}^1 = n_j; \quad j = 1, 2, ..., m,$$

$$p_{jk}^i = |S_j'(\alpha) \cap S_k'(\beta)|,$$

where α and β are mutually i^{th}-associates (i, j, k = 1, 2, ..., 2m+1), and
$S_i'(\alpha) = S_i(\alpha);\ i = 1, 2, ..., m,$
$S_{i+m}'(\alpha) = \{\alpha + v\},$
where v denotes the number of treatments in T_m-type association scheme,
$S_{j+m+1}'(\alpha) = S_j(\alpha + v);\ j = 1, 2, ..., m.$

4.1.2. Association scheme for d = 1

Like in association scheme 4.1.1, we keep in view the parameters (A), (B) and incidence matrix N_1 to define the following association scheme as:
Two treatments are said to be i^{th}-associates if the inner product of the corresponding rows in the matrix N is $\lambda_i^1 = r_1 - i; i = 1, 2, ..., 2m+1$.
Following are the parameters of the association scheme:

Suppose X is the set of values containing n_i's (i = 1, 2, ..., m) in the increasing order of their magnitudes. Therefore X contains m values with X(i) corresponding to the i^{th} value such that X(1) has the minimum and X(m) has the maximum value. Let $K_i(\alpha)$ denotes the sets $S_j(\alpha)$ arranged in ascending order of magnitudes for i, j = 1, 2, ..., m such that $K_1(\alpha)$ is minimum among $S_j(\alpha)$ and $K_m(\alpha)$ is maximum among $S_j(\alpha)$ (i, j = 1, 2, ..., m). Now the sets of i^{th}-associates in this association scheme are:

$S'_i(\alpha) = K_i(\alpha);\ \ i = 1, 2, ..., m,$
$S'_{i+m}(\alpha) = K_{(m-i+1)}(\alpha + v);\ \ i = 1, 2, ..., m,$
$S'_{2m+1}(\alpha) = \{\alpha + v\}.$
Keeping the cyclic order of treatments in mind.
Thus we have

$$p^i_{jk} = |S'_j(\alpha) \cap S'_k(\beta)|,$$

where α and β are mutually i^{th}-associates and i, j and k = 1, 2, ..., 2m+1,

$$v_1 = \binom{b}{r} + \binom{b^c}{r^c}, \qquad n^1_i = X(i);\ \ i = 1, 2, ..., m,$$

$$n^1_{(m+j)} = X(m - (j - 1));\ \ j = 1, 2, ..., m \qquad \text{and} \qquad n^1_{(2m+1)} = 1.$$

4.1.3. Association scheme for d ≥ 2

Similar to association schemes 4.1.1 and 4.1.2, we keep in mind the parameters (A), (B) and incidence matrix N_1 to define our third association scheme:

i. Two treatments are said to be i^{th}-associates if the inner product of the corresponding rows in the matrix N_1 is $\lambda^1_i = \lambda_i + \lambda^c_i$ $(i = 1, 2, ..., m)$.

ii. Two treatments are said to be $(m+1)^{th}$-associates if the inner product of the corresponding rows in the matrix N_1 is 0.

iii. Two treatments are said to be j^{th}-associates if the inner product of the corresponding rows in the matrix N_1 is $\lambda^1_j = \lambda^1_{(j-(m+1))} - (d - 2);\ j = m + 2, m + 3, ..., 2m + 1.$

This association scheme has the following parameters:

$$v_1 = \binom{b}{r} + \binom{b^c}{r^c}, \qquad n^1_i = n_i;\ \ i = 1, 2, ..., m,$$

$$n^1_{(m+1)} = 1 \qquad \text{and} \qquad n^1_{(m+j+1)} = n_{(m-(j-1))};\ \ j = 1, 2, ..., m,$$

$$p^i_{jk} = |S'_j(\alpha) \cap S'_k(\beta)|,$$

where α and β are mutually i^{th}-associates (i, j, k = 1, 2, ldots, 2m+1), and
$S'_i(\alpha) = S_i(\alpha);\ \ i = 1, 2, ..., m,$
$S'_{i+m}(\alpha) = \{\alpha + v\},$
where v denotes the number of treatments in T_m-type association scheme,
$S'_{j+m+1}(\alpha) = S_{(m-j+1)}(\alpha + v);\ \ j = 1, 2, ..., m.$

Example 4.1. Suppose that we have a T_m-type PBIB design, with $v = \binom{5}{2}$ treatments where m = 2, and its complementary PBIB design has $v^c = \binom{5}{3}$ treatments.

For $v = \binom{5}{2}$ treatments, let us consider the following vector of size 5 with two unit elements and three zeroes. From the methodology discussed in Section 3 by taking all possible combinations of this vector, we will get 10 distinct vectors in all which will form the rows of the following incidence matrix N:

$$N = \begin{bmatrix} 1&1&0&0&0\\ 1&0&0&0&1\\ 1&0&0&1&0\\ 1&0&1&0&0\\ 0&1&1&0&0\\ 0&1&0&0&1\\ 0&1&0&1&0\\ 0&0&1&1&0\\ 0&0&1&0&1\\ 0&0&0&1&1 \end{bmatrix}.$$

The complement of this matrix, i.e., N^C is given as:

$$N^C = \begin{bmatrix} 0&0&1&1&1\\ 0&1&1&1&0\\ 0&1&1&0&1\\ 0&1&0&1&1\\ 1&0&0&1&1\\ 1&0&1&1&0\\ 1&0&1&0&1\\ 1&1&0&0&1\\ 1&1&0&1&0\\ 1&1&1&0&0 \end{bmatrix}.$$

Therefore, according to the above discussed method, we can use the above T_m-type PBIB design and its complementary PBIB designs to construct another PBIB design with $v_1 = \binom{5}{2} + \binom{5}{3} = 20$ treatments having five-associate classes. Following is the incidence matrix of the five-associate class PBIB design constructed using the above discussed method which can be easily obtained using the incidence matrices of T_m-type PBIB design and its complementary PBIB design:

$$N_1 = \left[\begin{array}{c|c} N & N^C \\ \hline N^C & N \end{array}\right],$$

$$N_1 = \left[\begin{array}{ccccc|ccccc} 1&1&0&0&0&0&0&1&1&1\\ 1&0&0&0&1&0&1&1&1&0\\ 1&0&0&1&0&0&1&1&0&1\\ 1&0&1&0&0&0&1&0&1&1\\ 0&1&1&0&0&1&0&0&1&1\\ 0&1&0&0&1&1&0&1&1&0\\ 0&1&0&1&0&1&0&1&0&1\\ 0&0&1&1&0&1&1&0&0&1\\ 0&0&1&0&1&1&1&0&1&0\\ 0&0&0&1&1&1&1&1&0&0\\ \hline 0&0&1&1&1&1&1&0&0&0\\ 0&1&1&1&0&1&0&0&0&1\\ 0&1&1&0&1&1&0&0&1&0\\ 0&1&0&1&1&1&0&1&0&0\\ 1&0&0&1&1&0&1&1&0&0\\ 1&0&1&1&0&0&1&0&0&1\\ 1&0&1&0&1&0&1&0&1&0\\ 1&1&0&0&1&0&0&1&1&0\\ 1&1&0&1&0&0&0&1&0&1\\ 1&1&1&0&0&0&0&0&1&1 \end{array}\right].$$

The above incidence matrix corresponds to the following five-associate class PBIB design:

$$v_1 = 20, \quad b_1 = 10, \quad r_1 = 5, \quad k_1 = 10,$$
$$\lambda_1^1 = 4, \quad \lambda_2^1 = 3, \quad \lambda_3^1 = 2, \quad \lambda_4^1 = 1, \quad \lambda_5^1 = 0,$$
$$n_1^1 = 3, \quad n_2^1 = 6, \quad n_3^1 = 6, \quad n_4^1 = 3, \quad n_5^1 = 1,$$

$$P_1^1 = \begin{bmatrix} 0&2&0&0&0\\ 2&0&4&0&0\\ 0&4&0&2&0\\ 0&0&2&0&1\\ 0&0&0&1&0 \end{bmatrix}, \quad P_2^1 = \begin{bmatrix} 1&0&2&0&0\\ 0&3&0&2&0\\ 2&0&3&0&1\\ 0&2&0&1&0\\ 0&0&1&0&0 \end{bmatrix}, \quad P_3^1 = \begin{bmatrix} 0&2&0&1&0\\ 2&0&3&0&1\\ 0&3&0&2&0\\ 1&0&2&0&0\\ 0&1&0&0&0 \end{bmatrix},$$

$$P_4^1 = \begin{bmatrix} 0&0&2&0&1\\ 0&4&0&2&0\\ 2&0&4&0&0\\ 0&2&0&0&0\\ 1&0&0&0&0 \end{bmatrix}, \quad P_5^1 = \begin{bmatrix} 0&0&0&3&0\\ 0&0&6&0&0\\ 0&6&0&0&0\\ 3&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix}.$$

4.2. Method – II

4.2.1. Association scheme

Let us consider four quadrants and each of these contains exactly $n = \frac{n'(n'-1)}{2}$; $n' > 3$ treatments. These n treatments in each quadrant follow two-associate triangular association scheme with n treatments and can be arranged in a $n' \times n'$ matrix containing blank diagonal elements in such a way that all the n treatments are symmetric about the principal diagonal as shown in Figure 1. Thus, we have a total of $v = 4n$ treatments in all which follow the following four-associate class association scheme:

- **i.** Two treatments are said to be the first associates if they belong to the same row and same column of the matrix in the same quadrant.
- **ii.** The remaining treatments of the matrix in the same quadrant which are not the first associates of the treatment are said to be its second associates.
- **iii.** Two treatments are said to be the third associates if they are in the adjacent quadrants.
- **iv.** Treatments in the diagonal quadrants are the fourth associates.

Following are the parameters of the association scheme:

$$v = 4n, \qquad n = \frac{n'(n'-1)}{2}; n' > 3,$$
$$n_1 = 2(n'-2), \qquad n_2 = \frac{(n'-2)(n'-3)}{2},$$
$$n_3 = n'(n'-1) = 2n, \qquad n_4 = \frac{n'(n'-1)}{2} = n,$$

$$P_1 = \begin{bmatrix} (n'-2) & (n'-3) & 0 & 0 \\ (n'-3) & \frac{(n'-3)(n'-4)}{2} & 0 & 0 \\ 0 & 0 & n'(n'-1) & 0 \\ 0 & 0 & 0 & \frac{n'(n'-1)}{2} \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 4 & 2n'-8 & 0 & 0 \\ 2n'-8 & \frac{(n'-4)(n'-5)}{2} & 0 & 0 \\ 0 & 0 & n'(n'-1) & 0 \\ 0 & 0 & 0 & \frac{n'(n'-1)}{2} \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 2(n'-2) & 0 \\ 0 & 0 & \frac{(n'-2)(n'-3)}{2} & 0 \\ 2(n'-2) & \frac{(n'-2)(n'-3)}{2} & 0 & \frac{n'(n'-1)}{2} \\ 0 & 0 & \frac{n'(n'-1)}{2} & 0 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 2(n'-2) \\ 0 & 0 & 0 & \frac{(n'-2)(n'-3)}{2} \\ 0 & 0 & n'(n'-1) & 0 \\ 2(n'-2) & \frac{(n'-2)(n'-3)}{2} & 0 & 0 \end{bmatrix}.$$

Note: It has been observed that we can consider any number of, say t ($t \geq 3$), partitions instead of just four such that each partition has exactly $n = \frac{n'(n'-1)}{2}; n' > 3$ treatments following two-associate triangular association scheme arranged in $n' \times n'$ matrix in each partition. In such a situation, we have the following m-associate class association scheme:

i. Two treatments in the same row or same column of the matrix in the same partition are mutually first associates.

ii. Two treatments in the same partition but not in the same row or column of the matrix are second associates of each other.

iii. Treatments in the $(i-2)^{th}$ partition from the partition to which treatment, say α, belongs, are said to be the i^{th}-associates of treatment α ($i = 3, 4, ..., m$):

$$m = \begin{cases} \frac{t-1}{2} + 2; & \text{if t is odd,} \\ \frac{t}{2} + 2; & \text{if t is even.} \end{cases}$$

The parameters of the above association scheme are:

$$v = tn; \quad t \geq 3, \qquad n = \frac{n'(n'-1)}{2}; \quad n' > 3.$$

If t is odd

$$n_1 = 2(n'-2), \qquad n_2 = \frac{(n'-2)(n'-3)}{2},$$

$$n_3 = n_4 = n_5 = \cdots = n_m = 2n = n'(n'-1).$$

If t is even

$$n_1 = 2(n'-2), \qquad n_2 = \frac{(n'-2)(n'-3)}{2},$$

$$n_3 = n_4 = n_5 = \cdots = n_{(m-1)} = 2n = n'(n'-1), \qquad n_m = n = \frac{n'(n'-1)}{2}.$$

$*$	1	2	3	$\ldots$	$n'-1$	$*$	$n+1$	$n+2$	$n+3$	$\ldots$	$n+n'-1$
1	$*$	n'	$n'+1$	$\ldots$	$2n'-3$	$n+1$	$*$	$n+n'$	$n+n'+1$	$\ldots$	$n+2n'-3$
2	n'	$*$	$2n'-2$	$\ldots$	$3n'-6$	$n+2$	$n+n'$	$*$	$n+2n'-2$	$\ldots$	$n+3n'-6$
3	$n'+1$	$2n'-2$	$*$	$\vdots$	$\vdots$	$n+3$	$n+n'+1$	$n+2n'-2$	$*$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$*$	$n=\frac{n'(n'-1)}{2}$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$*$	$2n$
$n'-1$	$2n'-3$	$3n'-6$	$\ldots$	$n=\frac{n'(n'-1)}{2}$	$*$	$n+n'-1$	$n+2n'-3$	$n+3n'-6$	$\ldots$	$2n$	$*$
$*$	$3n+1$	$3n+2$	$3n+3$	$\ldots$	$3n+n'-1$	$*$	$2n+1$	$2n+2$	$2n+3$	$\ldots$	$2n+n'-1$
$3n+1$	$*$	$3n+n'$	$3n+n'+1$	$\ldots$	$3n+2n'-3$	$2n+1$	$*$	$2n+n'$	$2n+n'+1$	$\ldots$	$2n+2n'-3$
$3n+2$	$3n+n'$	$*$	$3n+2n'-2$	$\ldots$	$3n+3n'-6$	$2n+2$	$2n+n'$	$*$	$2n+2n'-2$	$\ldots$	$2n+3n'-6$
$3n+3$	$3n+n'+1$	$3n+2n'-2$	$*$	$\vdots$	$\vdots$	$2n+3$	$2n+n'+1$	$2n+2n'-2$	$*$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$*$	$4n$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$*$	$3n$
$3n+n'-1$	$3n+2n'-3$	$3n+3n'-6$	$\ldots$	$4n$	$*$	$2n+n'-1$	$2n+2n'-3$	$2n+3n'-6$	$\ldots$	$3n$	$*$

Figure 1: Arrangement of 4n treatments in four quadrants.

4.2.2. Series – I

Let us consider v = 4n treatments as discussed in association scheme defined in section 4.2.1 being arranged in four quadrants each containing exactly n treatments, where $n = \frac{n'(n'-1)}{2}$; $n' > 3$. We can form a set of b blocks from these v = 4n treatments such that i^{th} block has treatment i along with the treatments present in the quadrant diagonal to the quadrant containing treatment i. Thus in all we will obtain a set of b = 4n blocks and this set of b blocks constitute a four-associate class PBIB design following the association scheme 4.2.1 with the following parameters:

$$v = 4n = b, \qquad r = n + 1 = k, \quad n = \frac{n'(n'-1)}{2}; \quad n' > 3,$$

$$\lambda_1 = n = \lambda_2, \qquad \lambda_3 = 0, \qquad \lambda_4 = 2.$$

Example 4.2. Let us illustrate the above construction methodology by taking $n' = 4$. Thus, we have $n = \frac{4(4-1)}{2} = 6$ and we have a set of v = 24 treatments arranged as following:

*	1	2	3	*	7	8	9
1	*	4	5	7	*	10	11
2	4	*	6	8	10	*	12
3	5	6	*	9	11	12	*
*	19	20	21	*	13	14	15
19	*	22	23	13	*	16	17
20	22	*	24	14	16	*	18
21	23	24	*	15	17	18	*

Figure 2: Arrangement of 24 treatments.

From the above arrangement by taking the combinations of the i^{th} treatment with the treatments present in the diagonal quadrant, we will obtain the following set of blocks:

(1, 13, 14, 15, 16, 17, 18) (2, 13, 14, 15, 16, 17, 18) (3, 13, 14, 15, 16, 17, 18)
(4, 13, 14, 15, 16, 17, 18) (5, 13, 14, 15, 16, 17, 18) (6, 13, 14, 15, 16, 17, 18)
(7, 19, 20, 21, 22, 23, 24) (8, 19, 20, 21, 22, 23, 24) (9, 19, 20, 21, 22, 23, 24)
(10, 19, 20, 21, 22, 23, 24) (11, 19, 20, 21, 22, 23, 24) (12, 19, 20, 21, 22, 23, 24)
(13, 1, 2, 3, 4, 5, 6) (14, 1, 2, 3, 4, 5, 6) (15, 1, 2, 3, 4, 5, 6)
(16, 1, 2, 3, 4, 5, 6) (17, 1, 2, 3, 4, 5, 6) (18, 1, 2, 3, 4, 5, 6)
(19, 7, 8, 9, 10, 11, 12) (20, 7, 8, 9, 10, 11, 12) (21, 7, 8, 9, 10, 11, 12)
(22, 7, 8, 9, 10, 11, 12) (23, 7, 8, 9, 10, 11, 12) (24, 7, 8, 9, 10, 11, 12)

The above set of 24 blocks will form a four-associate class PBIB design with the following parameters:

$$v = 24 = b, \qquad r = 7 = k, \qquad \lambda_1 = 6, \qquad \lambda_2 = 6, \qquad \lambda_3 = 0, \qquad \lambda_4 = 2.$$

The parameters of the association scheme are:

$$n_1 = 4, \qquad n_2 = 1, \qquad n_3 = 12, \qquad n_4 = 6,$$

$$P_1 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & 0 & 6 \\ 0 & 0 & 6 & 0 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 12 & 0 \\ 4 & 1 & 0 & 0 \end{bmatrix}.$$

4.2.3. Series – II

As already stated earlier in the association scheme that n treatments, in each of the four quadrants, are arranged in $n' \times n'$ matrix with blank diagonal entries, we can form a set of blocks by taking a set of two rows of treatments from the adjacent quadrants and its treatments as the elements of block. In all, we will get a set of $b = 4n'^2$ blocks and this set of blocks will give us a four-associate class PBIB design following the association scheme 4.2.1 with the given below parameters:

$$v = 4n; \quad n = \frac{n'(n'-1)}{2}; \quad n' > 3,$$

$$b = 4n'^2, \qquad r = 4n', \qquad k = 2(n'-1).$$

$$\lambda_1 = 2n', \qquad \lambda_2 = 0, \qquad \lambda_3 = 4, \qquad \lambda_4 = 0.$$

Example 4.3. To illustrate the above construction methodology, let us take $n' = 4$. In this case, we have a set of $v = 24$ treatments. Now from Figure 2, by taking the combinations of rows with rows in adjacent quadrants we obtain the following set of $b = 64$ blocks:

$(1,2,3,7,8,9)$	$(1,2,3,7,10,11)$	$(1,2,3,8,10,12)$
$(1,2,3,9,11,12)$	$(1,2,3,19,20,21)$	$(1,2,3,19,22,23)$
$(1,2,3,20,22,24)$	$(1,2,3,21,23,24)$	$(1,4,5,7,8,9)$
$(1,4,5,7,10,11)$	$(1,4,5,8,10,12)$	$(1,4,5,9,11,12)$
$(1,4,5,19,20,21)$	$(1,4,5,19,22,23)$	$(1,4,5,20,22,24)$
$(1,4,5,21,23,24)$	$(2,4,6,7,8,9)$	$(2,4,6,7,10,11)$
$(2,4,6,8,10,12)$	$(2,4,6,9,11,12)$	$(2,4,6,19,20,21)$
$(2,4,6,19,22,23)$	$(2,4,6,20,22,24)$	$(2,4,6,21,23,24)$
$(3,5,6,7,8,9)$	$(3,5,6,7,1011)$	$(3,5,6,8,10,12)$
$(3,5,6,9,11,12)$	$(3,5,6,19,20,21)$	$(3,5,6,19,22,23)$
$(3,5,6,20,22,24)$	$(3,5,6,21,23,24)$	$(7,8,9,13,14,15)$
$(7,8,9,13,16,17)$	$(7,8,9,14,16,18)$	$(7,8,9,15,17,18)$
$(7,10,11,13,14,15)$	$(7,10,11,13,16,17)$	$(7,10,11,14,16,18)$
$(7,10,11,15,17,18)$	$(13,14,15,19,20,21)$	$(13,14,15,19,22,23)$
$(13,14,15,20,22,24)$	$(13,14,15,21,23,24)$	$(13,16,17,19,20,21)$
$(13,16,17,19,22,23)$	$(13,16,17,20,22,24)$	$(13,16,17,21,23,24)$
$(14,16,18,19,20,21)$	$(14,16,18,19,22,23)$	$(14,16,18,20,22,24)$
$(15,17,18,21,23,24)$	$(15,17,18,19,20,21)$	$(15,17,18,19,22,23)$
$(15,17,18,20,22,24)$	$(15,17,18,21,23,24)$	

The above set of 64 blocks constitutes a four-associate class PBIB design following association scheme 4.2.1 having the following parameters:

$$v = 24, \quad b = 64, \quad r = 16, \quad k = 6, \quad \lambda_1 = 8, \quad \lambda_2 = 0, \quad \lambda_3 = 4, \quad \lambda_4 = 0.$$

The parameters of the association scheme are same as those given in Example 4.2.

4.2.4. Series – III

From the arrangement of $v = 4n$ treatments in four quadrants as already discussed, if we form a set of blocks by taking two rows of treatments such that each of the row is placed in a quadrant diagonal to that of the other, and taking the treatment numbers as the elements of block, we will obtain a total of $b = 2n'^2$ blocks and this set of blocks will constitute a four-associate class PBIB design following the association scheme 4.2.1 with the following parameters:

$$v = 4n; \quad n = \frac{n'(n'-1)}{2}; \quad n' \geq 3,$$

$$b = 2n'^2, \qquad r = 2n', \qquad k = 2(n'-1),$$

$$\lambda_1 = n', \qquad \lambda_2 = 0, \qquad \lambda_3 = 0, \qquad \lambda_4 = 4.$$

Example 4.4. As an illustration, let us take of $n' = 4$. Now using Figure 2 and following the above discussed methodology, for the set of v = 24 treatments, we will get the following set of blocks:

$$
\begin{array}{lll}
(1,2,3,13,14,15) & (1,2,3,13,16,17) & (1,2,3,14,16,18) \\
(1,2,3,15,17,18) & (3,5,6,13,14,15) & (3,5,6,13,16,17) \\
(3,5,6,14,16,18) & (3,5,6,15,17,18) & (1,4,5,13,14,15) \\
(1,4,5,13,16,17) & (1,4,5,14,16,18) & (1,4,5,15,17,18) \\
(7,8,9,19,20,21) & (7,8,9,19,22,23) & (7,8,9,20,22,24) \\
(7,8,9,21,23,24) & (2,4,6,13,14,15) & (2,4,6,13,16,17) \\
(2,4,6,14,16,18) & (2,4,6,15,17,18) & (7,10,11,19,20,21) \\
(7,10,11,19,22,23) & (7,10,11,20,22,24) & (7,10,11,21,23,24) \\
(8,10,12,19,20,21) & (8,10,12,19,22,23) & (8,10,12,20,22,24) \\
(8,10,12,21,23,24) & (9,11,12,19,20,21) & (9,11,12,19,22,23) \\
(9,11,12,20,22,24) & (9,11,12,21,23,24) &
\end{array}
$$

The above set of 32 blocks constitutes a four-associate class PBIB design based on association scheme with the following parameters:

$$v = 24, \quad b = 32, \quad r = 8, \quad k = 6, \quad \lambda_1 = 4, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 4.$$

The parameters of the association scheme are same as given in Example 4.2.

5. APPLICATIONS

Not only PBIB designs have found their applications in agricultural experi-mentation, these designs are being largely explored in various other fields also, for example, in group testing, cryptography, medicine, clinical trials, reliability theory, etc. Relevant work in this direction is due to authors like Smith ([25]), Hinkelmann and Kempthorne ([6]), Narain and Arya ([9]), Braun ([3]), Singh and Hinkelmann ([20]), etc. The applications of PBIB designs in sample surveys have been discussed by Raghavarao and Singh ([13]), Singh *et al.* ([21]), See *et al.* ([16]), Sharma and Garg ([18]), etc.

Apart from this, the application of triangular designs having two associate classes (T_2-type designs) in the key predistribution has been discussed by Ruj and Roy ([14]) in their landmark paper thereby increasing the scope of T_m-type PBIB designs. Let us discuss the applications of PBIB designs using the illustration of testing of car tires as discussed by Naseer and Jawad ([10]). Suppose that we want to test 10 different kinds of car tires manufactured by different companies. In this case, we have a set of v = 10 treatments and the blocks are in the form of different cars. Since each car can have only four tires, our blocks in this case are capable of accommodating only four treatments and complete block designs cannot be used. Therefore, we need an incomplete block design to test the different types of tires. Among the incomplete block designs, balanced incomplete block (BIB) designs are the only designs which are both variance balanced and efficiency balanced but these designs sometimes require a large number of experimental units to test a particular set of treatments.

For example, in order to test v = 10 different types of tires using blocks of size k = 4, we need a minimum of b = 15 blocks, i.e., we require at least 60 experimental units. Now, if such a large number of experimental units are not available due to various constraints, BIB design cannot be used. In such a situation, we can opt for a PBIB design. For example, in order to test the above set of v = 10 treatments, we can use method discussed in section 3 to construct a PBIB design having parameters v = 10, b = 5, r = 2 and k = 4, i.e., we need only 20 experimental units which are $\frac{1}{3}^{\text{rd}}$ of the total experimental units required in above discussed BIB design. The layout of this PBIB design is as given below:

Position of tire	Cars				
	1	2	3	4	5
Front Left	A	A	D	C	B
Front Right	B	E	E	G	F
Back Left	C	F	H	H	I
Back Right	D	G	I	J	J

6. CONCLUSION

The higher-associate class PBIB designs are important because these provide designs with new parametric combinations of v, b, r and k. Moreover, higher-associate class PBIB designs, many times, come out to be more efficient than the corresponding lower-associate class PBIB designs having the same values of the parameters v, b, r and k. This give rise to the need for the study and construction of PBIB designs with associate classes $m \geq 3$. In this direction, the present paper has been written to make contributions towards higher associate-class PBIB designs and for this we have first constructed a new series of T_m-type PBIB designs and then based on this series, we have obtained some series of T_m-assisted PBIB designs by employing some easy and interesting construction techniques. Thus, we have enhanced the literature of PBIB designs. We have also discussed the corresponding association schemes in their fully generalized forms. We have also demonstrated how these designs can further be used to construct more PBIB designs with different parameters based on different types of association schemes. It shows how unique combinatorial properties of T_m-type PBIB designs can assist in deriving more series of PBIB designs which should inspire the present and future researchers to further explore these designs.

We have also provided a table which enlists some of the PBIB designs constructed in this paper and compares them with some of the PBIB designs listed in the table of PBIB designs provided by Clatworthy *et al.* ([5]) for the same values of parameters v, b, r and k.

Table 1: PBIB designs constructed in this paper.

Sr. No.	v	b	r	k	λ_i	O_{eff}	Obtained From
†1	6	4	2	3	$\lambda_1 = 1, \lambda_2 = 0$	0.6	Method 3
1 a	6	4	2	3	$\lambda_1 = 0, \lambda_2 = 1$	0.6	SR18
†2	10	5	2	4	$\lambda_1 = 1, \lambda_2 = 0$	0.5	Method 3
2 a	10	5	2	4	$\lambda_1 = 1, \lambda_2 = 0$	0.5	T28
†3	10	5	3	6	$\lambda_1 = 2, \lambda_2 = 1$	0.7778	Method 3
3 a	10	5	3	6	$\lambda_1 = 2, \lambda_2 = 1$	0.7778	T57
†4	15	6	2	5	$\lambda_1 = 1, \lambda_2 = 0$	0.4286	Method 3
4 a	15	6	2	5	$\lambda_1 = 1, \lambda_2 = 0$	0.4286	T48
*5	20	6	3	10	$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0$	0.6842	Method 3
5 a	20	6	3	10	$\lambda_1 = 3, \lambda_2 = 1$	0.6491	S106
§6	20	10	5	10	$\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2, \lambda_4 = 1, \lambda_5 = 0$	0.6758	Method 4.1
†7	21	7	2	6	$\lambda_1 = 1, \lambda_2 = 0$	0.375	Method 3
7 a	21	7	2	6	$\lambda_1 = 1, \lambda_2 = 0$	0.375	T65
‖8	21	7	5	15	$\lambda_1 = 4, \lambda_2 = 3$	0.9	Method 3
§9	24	24	7	7	$\lambda_1 = 6, \lambda_2 = 6, \lambda_3 = 0, \lambda_4 = 2$	0.3407	Method 4.2.2
‖10	24	64	16	6	$\lambda_1 = 8, \lambda_2 = 0, \lambda_3 = 4, \lambda_4 = 0$	0.3587	Method 4.2.3
11	24	32	8	6	$\lambda_1 = 4, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 4$	0.3261	Method 4.2.4
‖12	30	12	6	15	$\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = 4, \lambda_5 = 2$	0.6973	Method 4.1
‖13	35	7	3	15	$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0$	0.6078	Method 3
‖14	35	7	4	20	$\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$	0.7794	Method 3
‖15	40	12	6	20	$\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = 6, \lambda_5 = 4, \lambda_6 = 2, \lambda_7 = 0$	0.6923	Method 4.1
§16	40	50	10	8	$\lambda_1 = 5, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 4$	0.2795	Method 4.2.4
‖17	40	40	11	11	$\lambda_1 = 10, \lambda_2 = 10, \lambda_3 = 0, \lambda_4 = 2$	0.3136	Method 4.2.2
‖18	42	14	7	21	$\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 0, \lambda_4 = 4, \lambda_5 = 2$	0.7068	Method 4.1
‖19	56	8	3	21	$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0$	0.5454	Method 3
‖20	70	14	7	35	$\lambda_1 = 6, \lambda_2 = 5, \lambda_3 = 4, \lambda_4 = 3, \lambda_5 = 2, \lambda_6 = 1, \lambda_7 = 0$	0.7057	Method 4.1

* Design constructed in the chapter and has greater overall efficiency factor than the corresponding design listed in Clatworthy *et al.* ([5]) for same parametric combination v, b, r and k.

† Design constructed in the chapter and has exactly same overall efficiency factor than the corresponding design listed in Clatworthy *et al.* ([5]) for same parametric combination v, b, r and k.

§ Design constructed in the chapter and there is no listed design in Clatworthy *et al.* ([5]) for same parametric combination v, b, r and k.

‖ Design constructed in the chapter with r, k > 10.

A. R PROGRAMMING CODES

The R programming code for the computerized construction of T_m-type and T_m-assisted PBIB designs discussed are as follows:

A.1. Code 1

Code for developing the PBIB designs discussed in Section 3:

```
series1<-function(x){
b<-length(x)
r<-0
for(i in 1:b){
if (x[i]==1){
r<-r+1 } }
v<-factorial(b)/(factorial(r)*factorial(b-r))
k<-v*r/b
l<-c()
m<-min(r,b-r)
for(i in 1:m){
l[i]<-r-i }
p<-1
mat<-combn(b,r)
N<-matrix(nrow=v,ncol=b)
cat("The Blocks of the Tm type PBIB design are:\n")
for(ii in 1:v){
for(jj in 1:b){
if(mat[p,ii]==jj){
N[ii,jj]<-1
p<-p+1
if(p>r){
p<-p%%r } }
else
N[ii,jj]<-0 } }
for(j in 1:b){
cat("(")
for(kk in 1:v){
if (N[kk,j]==1)
cat(kk," ") }
cat(")")
cat("\n") }
cat("The parameters of the above design are:\n")
cat("v = ",v,"\t b = ", b, "\t r = ", r, "\t k = ", k, "\n")
for(i in 1:m)
cat("lambda ",i, " = ", l[i],"\t" ) }
cat("\n") }
```

A.2. Code 2

Code for developing the PBIB designs discussed in 4.1:

```
series2<-function(x){
b1<-length(x)
r1<-0
r2<-0
for(i in 1:b1){
if (x[i]==1){
r1<-r1+1 }
else
r2<-r2+1 }
m1<-min(r1,b1-r1)
l1<-c()
l2<-c()
for(i in 1:m1){
l1[i]<-r1-i
l2[i]<-r2-i }
v1<-factorial(b1)/(factorial(r1)*factorial(b1-r1))
k1<-v1*r1/b1
p<-1
mat<-combn(b1,r1)
N<-matrix(nrow=v1,ncol=b1)
for(ii in 1:v1){
for(jj in 1:b1){
if(mat[p,ii]==jj){
N[ii,jj]<-1
p<-p+1
if(p>r1){
p<-p%%r1 } }
else
N[ii,jj]<-0 } }
Nc<-matrix(nrow = v1, ncol = b1)
for(p in 1:v1){
for (q in 1:b1) {
if(N[p,q]==1){
Nc[p,q]<-0 }
else
Nc[p,q]<-1 } }
v<-2*v1
b<-2*b1
r<-r1+r2
k<-v1
N1<-cbind(N,Nc)
N2<-cbind(Nc,N)
Nf<-rbind(N1,N2)
```

```
cat("The Blocks of the Tm assisted PBIB design are:\n")
for(j in 1:b){
cat("(")
for(kk in 1:v){
if (Nf[kk,j]==1)
cat(kk," ") }
cat(")")
cat("\n") }
D<-abs(r1-r2)
m<-2*m1+1
l<-c()
if(D==0){
for (i in 1:m1) {
l[i]<-l1[i]+l2[i] }
l[m1+1]<-r
for(i in (m1+2):m){
l[i]<-l[i-(m1+1)]
} } else if(D==1){
for (i in 1:m) {
l[i]<-r-i
} }else{
for (i in 1:m1) {
l[i]<-l1[i]+l2[i] }
l[m1+1]<-0
for (i in (m1+2):m) {
l[i]<-l[i-(m1+1)]-(D-2) } }
cat("The parameters of the above design are:\n")
cat("v = ",v,"\t b = ", b, "\t r = ", r, "\t k = ", k, "\n")
for(i in 1:m){
cat("lambda ",i, " = ", l[i],"\t" ) } }
```

A.3. Code 3

To execute the following function first install and load the R package 'Matrix'. Following is the R programming Code for developing the PBIB designs discussed in Section 4.2.2:

```
series3<-function(n1){
n<-n1*(n1-1)/2
v<-4*n
a<-1:n
A1<-matrix(nrow=n1,ncol=n1)
diag(A1)<-0
A1[lower.tri(A1)]<-1:n
B1<-forceSymmetric(A1,"L")
A2<-matrix(nrow=n1,ncol=n1)
```

```
diag(A2)<-0
A2[lower.tri(A1)]<-(n+1):(2*n)
B2<-forceSymmetric(A2,"L")
A3<-matrix(nrow=n1,ncol=n1)
diag(A3)<-0
A3[lower.tri(A3)]<-(2*n+1):(3*n)
B3<-forceSymmetric(A3,"L")
A4<-matrix(nrow=n1,ncol=n1)
diag(A4)<-0
A4[lower.tri(A4)]<-(3*n+1):(4*n)
B4<-forceSymmetric(A4,"L")
C1<-cbind(B1,B2)
C2<-cbind(B4,B3)
D<-rbind(C1,C2)
b<-v
r<-n+1
k<-r
l<-c(n,n,0,2)
cat("The Blocks of the design are:\n")
p<-1
for (i in 1:4) {
for (j in ((i-1)*n+1):(i*n)) {
if(i==1){
cat("(",j,(2*n+1):(3*n),")")
cat("\n")
} else if(i==2){
cat("(",j,(3*n+1):(4*n),")")
cat("\n")
} else if(i==3){
cat("(",j,(1):(n),")")
cat("\n")
} else{
cat("(",j,(n+1):(2*n),")")
cat("\n") } } }
cat("The parameters of the design are:\n")
cat("v = ",v,"\t","b = ",b,"\t","r = ",r,"\t","k = ",k,"\n")
for(i in 1:4){
cat("lambda_",i," = ",l[i],"\t") } }
```

A.4. Code 4

To execute the following function first install and load the R package 'Matrix'. Following is the R programming code for developing the PBIB designs discussed in Section 4.2.3:

```
series4<-function(n1){
n<-n1*(n1-1)/2
v<-4*n
a<-1:n
A1<-matrix(nrow=n1,ncol=n1)
diag(A1)<-0
A1[lower.tri(A1)]<-1:n
B1<-forceSymmetric(A1,"L")
A2<-matrix(nrow=n1,ncol=n1)
diag(A2)<-0
A2[lower.tri(A1)]<-(n+1):(2*n)
B2<-forceSymmetric(A2,"L")
A3<-matrix(nrow=n1,ncol=n1)
diag(A3)<-0
A3[lower.tri(A3)]<-(2*n+1):(3*n)
B3<-forceSymmetric(A3,"L")
A4<-matrix(nrow=n1,ncol=n1)
diag(A4)<-0
A4[lower.tri(A4)]<-(3*n+1):(4*n)
B4<-forceSymmetric(A4,"L")
C1<-cbind(B1,B2)
C2<-cbind(B4,B3)
D<-rbind(C1,C2)
b<-4*n1*n1
r<-4*n1
k<-2*(n1-1)
l<-c(2*n1,0,4,0)
p<-c()
pp<-c()
cat("The blocks of the design are:\n")
for (i in 1:n1) {
ii<-1
for (j in 1:n1) {
if(D[i,j]!=0){
p[ii]<-D[i,j]
ii<-ii+1 } }
for (jj in (1):(n1)) {
iii<-1
for (ij in (n1+1):(2*n1)) {
if(D[jj,ij]!=0){
pp[iii]<-D[jj,ij]
iii<-iii+1 } }
```

```
cat(p,pp,”\n”) } }
for (i in (n1+1):(2*n1)) {
ii<-1
for (j in (1):(n1)) {
if(D[i,j]!=0){
p[ii]<-D[i,j]
ii<-ii+1 } }
for (jj in (n1+1):(2*n1)) {
iii<-1
for (ij in (n1+1):(2*n1)) {
if(D[jj,ij]!=0){
pp[iii]<-D[jj,ij]
iii<-iii+1 } }
cat(p,pp,”\n”) } }
for (i in 1:n1) {
ii<-1
for (j in 1:n1) {
if(D[j,i]!=0){
p[ii]<-D[j,i]
ii<-ii+1 } }
for (jj in (1):(n1)) {
iii<-1
for (ij in (n1+1):(2*n1)) {
if(D[ij,jj]!=0){
pp[iii]<-D[ij,jj]
iii<-iii+1 } }
cat(p,pp,”\n”) } }
for (i in (n1+1):(2*n1)) {
ii<-1
for (j in (1):(n1)) {
if(D[j,i]!=0){
p[ii]<-D[j,i]
ii<-ii+1 } }
for (jj in (n1+1):(2*n1)) {
iii<-1
for (ij in (n1+1):(2*n1)) {
if(D[ij,jj]!=0){
pp[iii]<-D[ij,jj]
iii<-iii+1 } }
cat(p,pp,”\n”) } }
cat(”The parameters of the above design are:\n”)
cat(”v = ”,v,”\t b = ”, b, ”\t r = ”, r, ”\t k = ”, k, ”\n”)
for(i in 1:4){
cat(”lambda_”,i, ” = ”, l[i],”\t” ) } }
```

A.5. Code 5

To execute the following function first install and load the R package 'Matrix'. Following is the R programming code for developing the PBIB designs discussed in Section 4.2.4:

```
series5<-function(n1){
n<-n1*(n1-1)/2
v<-4*n
a<-1:n
A1<-matrix(nrow=n1,ncol=n1)
diag(A1)<-0
A1[lower.tri(A1)]<-1:n
B1<-forceSymmetric(A1,"L")
A2<-matrix(nrow=n1,ncol=n1)
diag(A2)<-0
A2[lower.tri(A1)]<-(n+1):(2*n)
B2<-forceSymmetric(A2,"L")
A3<-matrix(nrow=n1,ncol=n1)
diag(A3)<-0
A3[lower.tri(A3)]<-(2*n+1):(3*n)
B3<-forceSymmetric(A3,"L")
A4<-matrix(nrow=n1,ncol=n1)
diag(A4)<-0
A4[lower.tri(A4)]<-(3*n+1):(4*n)
B4<-forceSymmetric(A4,"L")
C1<-cbind(B1,B2)
C2<-cbind(B4,B3)
D<-rbind(C1,C2)
b<-2*n1*n1
r<-2*n1
k<-2*(n1-1)
l<-c(n1,0,0,4)
p<-c()
pp<-c()
cat("The blocks of the design are:\n")
for (i in 1:n1) {
ii<-1
for (j in 1:n1) {
if(D[i,j]!=0){
p[ii]<-D[i,j]
ii<-ii+1 } }
for (jj in (n1+1):(2*n1)) {
iii<-1
for (ij in (n1+1):(2*n1)) {
if(D[jj,ij]!=0){
pp[iii]<-D[jj,ij]
iii<-iii+1 } }
```

```
cat(p,pp,"\n") } }
for (i in (n1+1):(2*n1)) {
ii<-1
for (j in (1):(n1)) {
if(D[j,i]!=0){
p[ii]<-D[j,i]
ii<-ii+1 } }
for (jj in (1):(n1)) {
iii<-1
for (ij in (n1+1):(2*n1)) {
if(D[ij,jj]!=0){
pp[iii]<-D[ij,jj]
iii<-iii+1 } }
cat(p,pp,"\n") } }
cat("The parameters of the above design are:\n")
cat("v = ",v,"\t b = ", b, "\t r = ", r, "\t k = ", k, "\n")
for(i in 1:4){
cat("lambda_",i, " = ", l[i],"\t" ) } }
```

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