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A Note on the Right Truncated Weibull Distribution and the Minimum of Power Function Distributions

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Abstract:

- In this note, the right truncated Weibull distribution is derived as the distribution of the minimum of a random number of independent and identically distributed random variables. Specifically, the independent random variables have a common power function distribution and the random number has a zero-truncated Poisson distribution.

Keywords:

- *right truncated Weibull distribution; power function distribution; truncated Poisson distribution; minimum of random variables.*

AMS Subject Classification:

- 60E05, 62E15, 62N05.

1. INTRODUCTION

The Weibull distribution is one of the most popular probability models, both from a theoretical and practical viewpoint, and it has been successfully used to model lifetime and failure data in a wide variety of areas. Prabhakar *et al.* [11] and Rinne [13] are two excellent monograph books that review the history, theory and applications of the Weibull distribution.

To be more precise, let X be a random variable having a two-parameter Weibull distribution, that is, its cumulative distribution function (cdf) is given by

$$F_X(x;\alpha,\beta) = 1 - \exp\bigl(-\alpha x^{\beta}\bigr)\,, \qquad x > 0\,, \tag{1.1}$$

where $\alpha > 0$ and $\beta > 0$ are the scale and shape parameters, respectively. Note that the domain of the Weibull model is the positive real line. However, there are many real situations in which the data take values in a bounded interval and then a truncated distribution may be preferred. In this note, the attention will be focussed on the Weibull distribution truncated to the interval $(0, c)$, $c > 0$, which is commonly referred to as the right –or upper– truncated Weibull (RTW) distribution. The cdf of a random variable Y having a RTW distribution on $(0, c)$ is easily deduced from (1.1), namely,

$$\begin{aligned} F_Y(y;\alpha,\beta,c) \;&=\; P\bigl(X \le y \,|\, X \le c\bigr) \;=\; \frac{F_X(y;\alpha,\beta) - F_X(0;\alpha,\beta)}{F_X(c;\alpha,\beta) - F_X(0;\alpha,\beta)} \\ &=\; \frac{1-\exp\bigl(-\alpha y^{\beta}\bigr)}{1-\exp\bigl(-\alpha c^{\beta}\bigr)}\,, \qquad 0 < y < c\,, \end{aligned} \tag{1.2}$$

where $\alpha > 0$ and $\beta > 0$. Statistical properties concerning the RTW model can be found in Martínez and Quintana [7], McEwen and Parresol [8], Rao [12], Wingo [16] and Zhang and Xie [18], among others.

On the other hand, let Z be a random variable having a power function (PF) distribution on the interval $(0, c)$, that is, its cdf is given by

$$F_Z(z;\beta,c) = \left(\frac{z}{c}\right)^{\beta}, \qquad 0 < z < c\,,$$

where $\beta > 0$ is a shape parameter. Recall that the PF distribution is obtained by inverting the Pareto distribution. Statistical properties of the PF distribution can be found in Forbes *et al.* [3, Chapter 36] and Johnson *et al.* [6, Chapter 20]. A detailed review of research concerning the PF law is given in Tahir *et al.* [15]. Practical applications in different areas can also be found in Ferreira and Andrade [2] (queuing theory), Meniconi and Barry [9] (electrical component reliability) and Wu *et al.* [17] (economics and finance), among others.

There exists a well-known relationship between the non-truncated Weibull distribution and the PF distribution. If a random variable Z follows a PF distribution on $(0, 1)$ with shape parameter $\alpha > 0$, then the random variable $(-\log Z)^{1/\beta}$ has a Weibull distribution with cdf (1.1). The aim of this note is to present a non-trivial connection between the distributions RTW and PF. In the next section, it is shown that the RTW model can be derived as the distribution of the minimum of a positive random number N of independent and identically distributed (iid) random variables having a common PF distribution. Specifically, the random number N follows a zero-truncated Poisson distribution.

Before going further, it is interesting to point out that families of distributions derived as the minimum of a positive random number N of iid random variables are common in statistical applications. For example, this stochastic representation arises in reliability analysis of series systems, in which the failure of the system is due to the presence of an unknown number of independent components of the same kind and it is assumed that the system fails if at least one component fails. Some of those families of distributions are listed in Nadarajah *et al.* [10] and some applications can be found in Silva *et al.* [14]. In addition, Bobotas and Koutras [1] have also studied the special case where N is a non-negative random number with $P(N=0) > 0$.

2. MAIN RESULT

Let N be a random variable having a zero-truncated Poisson distribution with parameter $\lambda > 0$. The probability mass function of N is given by

$$P(N=n) = \frac{\lambda^n \exp(-\lambda)}{\big(1-\exp(-\lambda)\big)\, n!}\,, \qquad n = 1, 2, \ldots \tag{2.1}$$

The following result provides a relationship between the RTW and the minimum of iid PF distributions. The zero-truncated Poisson distribution plays a crucial role.

Proposition 2.1. *For any $c > 0$, let $Z_1, \ldots, Z_N$ be iid random variables having a PF distribution on the interval $(0, c)$ with shape parameter $\beta > 0$. For any $\alpha > 0$, let N be a random variable having a zero-truncated Poisson distribution with parameter $\lambda = \alpha c^\beta$. Then, the random variable $T = \min\{Z_1, \ldots, Z_N\}$ has a RTW distribution on the interval $(0, c)$.*

Proof: For any $n = 1, 2, \ldots$, $c > 0$ and $\beta > 0$, the conditional cdf of the random variable $T|N=n$ is given by

$$F_{T|N=n}(t; \beta, c) \;=\; 1 - \prod_{i=1}^{n} \big(1 - F_{Z_i}(t; \beta, c)\big) \;=\; 1 - \left(1 - \left(\frac{t}{c}\right)^{\beta}\right)^{n}, \qquad 0 < t < c\,.$$

From the above equation together with (2.1), for any $\alpha > 0$ the marginal cdf of T is obtained as follows:

$$\begin{aligned} F_T(t; \alpha, \beta, c) \;&=\; \sum_{n=1}^{\infty} P\big(T \le t,\, N=n\big) \;=\; \sum_{n=1}^{\infty} F_{T|N=n}(t; \beta, c)\, P\big(N=n\big) \\ &= \sum_{n=1}^{\infty} \left[1 - \left(1 - \left(\frac{t}{c}\right)^{\beta}\right)^{n}\right] \frac{(\alpha c^\beta)^n \exp(-\alpha c^\beta)}{\big(1-\exp(-\alpha c^\beta)\big)\, n!} \\ &= \frac{1 - \exp\big(-\alpha t^\beta\big)}{1 - \exp\big(-\alpha c^\beta\big)}\,, \qquad 0 < t < c\,, \end{aligned}$$

which taking into account (1.2) implies the desired result. □

To conclude, it is interesting to note that by taking the minimum of a random number N of iid PF random variables on the unit interval $(0, 1)$, Jodrá [4] and Jodrá and Jiménez-Gamero [5] have introduced two new probability distributions depending on if N follows a shifted Poisson distribution or a zero-truncated geometric distribution, respectively. Surprisingly, the well-studied RTW distribution is obtained if the random number N has a zero-truncated Poisson distribution.

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