REVSTAT – Statistical Journal
Volume 20, Number 2, April 2022, 135–149

OPTIMAL REINSURANCE OF DEPENDENT RISKS

Authors: A. Bugalho de Moura
– ISEG – School of Economics and Management, Universidade de Lisboa;
REM – Research in Economics and Mathematics; CEMAPRE,
Lisbon, Portugal
amoura@iseg.ulisboa.pt

M.L. Centeno
– ISEG – School of Economics and Management, Universidade de Lisboa;
REM – Research in Economics and Mathematics; CEMAPRE,
Lisbon, Portugal
lcenteno@iseg.ulisboa.pt

Received: April 2019 Revised: December 2019 Accepted: January 2020

Abstract:

- We analyse the problem of finding the optimal combination of quota-share and stop loss treaties, maximizing the expected utility or the adjustment coefficient of the cedent, for each of two risks dependent through a copula structure. By risk we mean a line of business or a portfolio of policies. Results are obtained numerically, using the software *Mathematica*. Sensitivity of the optimal reinsurance strategy to several factors are investigated, including:
  i) the dependence level, by means of the Kendall's tau and the dependence parameter;
  ii) the type of dependence, using different copulas describing different tail behaviour;
  iii) the reinsurance calculation principles, where expected value, variance and standard deviation principles are considered.

  Results show that different dependence structures, yield significantly different optimal solutions. The optimal treaty is also very sensible to the reinsurance premium calculation principle. Namely, for variance related premiums the optimal solution is not the pure stop loss. In general, the maximum adjustment coefficient decreases when dependence increases.

Keywords:

- *reinsurance; dependent risks; copulas; premium calculation principles; expected utility; adjustment coefficient.*

AMS Subject Classification:

- 62P05.

1. INTRODUCTION

From the vast range of literature intended for the financial and insurance community, it is widely accepted that dependencies play a determinant role in risk assessment and management. Namely, reinsurance is a risk mitigating tool, constituting an important instrument in the management of risk of an insurance company where dependencies should be taken into account. When transferring risk, the cedent seeks a trade-off between profit and safety, which is dependent on the nature of the insured underlying risk and on the reinsurance premium calculation principle. This optimization problem has been largely studied in the literature, however only recently dependencies among risks have been considered. The goal is always to find the reinsurance strategy, which is usually defined by the forms of reinsurance to be considered and the specific retention levels, that minimizes a given measure of the underlying risk.

In [19] (for the aggregate claim model) and [13] (for the individual claim model) the authors obtain analytically the optimal reinsurance strategy maximizing the adjustment coefficient or the expected utility assuming independence. The premium calculation principle used is a convex functional, including the expected value, standard deviation and variance premium principles as special cases. In the case of "variance related" premium calculation principles, the optimal reinsurance contract is a specific, implicitly defined, non-linear function of the retained risk such that the tail of the underlying risk is shared by both the insurer and the reinsurer. If the expected value calculation principle is considered, the pure stop loss treaty is optimal. In fact, the pure stop loss, which appears as the optimal form of reinsurance in an innumerable amount of cases where the expected value premium principle is used, is not realistic in practice. It means all the risk in the tail is ceded to the reinsurer which will not accept it but at a very high premium loading, in which case the stop loss is probably not optimal anymore (as shown in [19, 13]). Other works considering convex premium principles include [21, 22] and [17], where convex risk measures (e.g. the variance or semi-variance of the retained risk) are used as optimality criteria. In all these works, independence is assumed. Indeed, while a large quantity of analytical studies can be found regarding optimal reinsurance, only a few number consider dependence. Notwithstanding, the interest in studying optimal reinsurance strategies under dependencies is increasing, driven by the need for real, robust and reliable quantitative risk models.

Article [12] is one of the first works including the effects of dependence when investigating analytical optimal forms or risk transfer. The optimal retention limit for the excess-loss (XL) reinsurance is studied considering two classes of insurance businesses, dependent through the number of claims by means of a bivariate Poisson, when the cedent intends to maximize the expected utility or the adjustment coefficient, using the expected value premium principle. Other authors have considered the optimal reinsurance problem under dependence between claim numbers, such as [28] and [5]. In [26] the impact of dependencies from year to year reinsurance payoffs are investigated using copulas and simulation, however optimal reinsurance is not directly addressed. In [6] positive dependencies in the individual risk are considered by means of the stochastic ordering. By considering a fixed reinsurance premium, calculated through the expected value principle, the authors demonstrate that in this case the optimal form of reinsurance is the XL treaty, when the optimality criterion is the maximization of the expectation of a convex function of the retained risk, including the expected utility for the exponential functional. In that paper, the authors refer to the non-proportional reinsurance as excess of loss (XL), assuming the risks are individual claims and then considering their sum. Accounting for dependence have protruded the use of numeral techniques, such as Dynamical Financial Analysis (DFA), Linear Programming, see e.g. [2], dynamic control problems, see e.g. [4], or simulation, see e.g. [26], which is often based on Monte Carlo simulation. Very recently, in [3], it has been advocated that when constraints on dependencies and economic and solvency factors are included in the optimal reinsurance problem,"the optimal contract can only be found numerically". Hence, they propose a numerical framework, based on the Second-Order Canonical Problem for numerical optimization. Other works regarding the application of numerical techniques to solve optimal reinsurance problems consider numerical methods for stochastic control theory (see for instance [29]). Most of these numerical works deal with real data.

In this work, we aim at studying the sensitiveness of the optimal reinsurance strategy, in presence of dependencies, to different factors such as premium calculation principles and dependence structures and levels. We account not only for the expected value principle, but also for the standard deviation and the variance principles. We consider two underlying risks and by risk we mean the aggregate claims of a line of business, a portfolio of policies or a policy. Dependence between the two risks is modelled through copulas, allowing to easily change the dependence structure and strength. We construct the optimal problem as finding the optimal combination of quota share (QS) and stop loss treaties, for each risk, that maximizes the expected utility or the adjustment coefficient of the total wealth of the first insurer. The analytical results in [6] for the expected value principle are not straightforwardly extendable to the variance related premiums, thus, we use numerical methods. To properly study the sensitivity of the optimal reinsurance strategy to several dependence structures and levels, and to a variety of reinsurance premium calculation principles, the problem setting is kept as simple as possible and no real data is used. The distributions of the underlying risks are assumed to be known and different distributions are considered. This controlled environment allows for a systematically analysis of the optimal reinsurance and its sensitivity to the several factors considered.

The layout of this paper is as follows. In Section 2 we set the optimization problem to be solved, introducing the copulas that will be used, the premium calculation principles and optimality criteria. In Section 3 we present the numerical results and their discussion. Finally, conclusions and future perspectives are drawn in Section 4.

2. SETTING THE OPTIMIZATION PROBLEM

We consider two risks, X_1 and X_2, with distribution functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, respectively. By risk we mean a line of business, a portfolio of policies or a policy. We assume that the two risks are dependent through a copula, denoted by C_α, $\alpha > 0$, such that the joint distribution is given by $C_\alpha(x_1, x_2) = C_\alpha(F_{X_1}(x_1), F_{X_2}(x_2))$, and the joint density function is given by $f_{X_1,X_2}(x_1, x_2)$. We use the notation $dC_\alpha(x_1, x_2) = f_{X_1,X_2}(x_1, x_2)\, dx_1 dx_2$. We assume that the insurer reinsures each risk by means of a QS contract topped by a stop loss treaty. Thus, the retained risks are $Y_i = Y_i(a_i, M_i) = \min(a_i X_i, M_i)$, $i = 1, 2$, where $0 \leqslant a_i \leqslant 1$, represents the QS retained level of risk i, $i = 1, 2$, and $M_i \geqslant 0$, denotes the stop loss retention limit, above which all the risk is ceded to the reinsurer, for risk i, $i = 1, 2$.

Therefore, the total wealth of the insurer after reinsurance is given by

$$\begin{aligned} W(a_1, M_1, a_2, M_2) &= W(a_1, M_1) + W(a_2, M_2) \\ &= (1 - e_1)\, P_1 - P_{R1} - Y_1 + (1 - e_2)\, P_2 - P_{R2} - Y_2\,, \end{aligned} \tag{2.1}$$

where $P_i > 0$, represents the premium received by the insurer for each risk i, $i = 1, 2$, and $e_i > 0$, $i = 1, 2$ are the corresponding insurer expenses; $P_{Ri} = P_{Ri}(a_i, M_i) > 0$ denotes the premium charged by the reinsurer for each risk i, $i = 1, 2$.

2.1. The dependence structure

When two risks are assumed not to be independent, an infinite range of possible dependencies between them can be at stake. The first question is, if they are dependent, what is the best model to explain the existing dependencies. Copulas constitute a convenient and elegant way of describing dependencies between two or more random variables. Also, using copulas, measures of non-linear dependence can be explored, such as the Kendall's rank correlation coefficient, which is a measure of concordance [14].

Our underlying risks[1], X_1 and X_2, are continuous random variables and the joint density function is given by $f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, c\big(F_{X_1}(x_1), F_{X_2}(x_2)\big)$, where $c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2)$, $(u_1, u_2) \in [0, 1]^2$ is the so-called copula density. In our case, the retained risk after the combination of QS and stop loss, $Y_i = \min(a_i X_i, M_i)$, $i = 1, 2$, $0 \leqslant a_i \leqslant 1$, $M_i \geqslant 0$, is non-decreasing function of X_i, hence the dependence structure is maintained for the retained risks (see [14]). That is, if the joint distribution of (X_1, X_2) is described by copula C, $F_{X_1,X_2}(x_1, x_2) = C\big(F_{X_1}(x_1), F_{X_2}(x_2)\big)$, then the joint distribution of (Y_1, Y_2) is also described by copula C, $F_{Y_1,Y_2}(y_1, y_2) = C\big(F_{Y_1}(y_1), F_{Y_2}(y_2)\big)$.

In this work, we consider Clayton's and Frank's copulas, which belong to the Archimedean family of copulas and have lower and no tail dependency, respectively. In these cases the Kendall's tau rank coefficient can be easily described by $\tau_\alpha = \frac{\alpha}{\alpha+2}$, for Clayton's copula, and $\tau_\alpha = 1 - 4\,\frac{1-D_1(\alpha)}{\alpha}$, with $D_1(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{t}{e^t-1}\, dt$, for Frank's copula (see [14]). We also consider the Pareto's copula which can be derived as the "natural" bivariate distribution of two Pareto distributions with the same shape parameter α and it is as heavy right tail copula. Indeed, the Pareto's copula is the survival Clayton's copula with dependence parameter $1/\alpha$. Thus, the Pareto's copula Kendall's tau is $\tau_\alpha = \frac{1}{1+2\alpha}$.

2.2. The reinsurance premium

We analyse optimal reinsurance strategies for the expected value calculation principle, where the loading is proportional to the expected value of the risk, and also for the variance and standard deviation calculation principles. The later belong to the so-called (see [19]) variance related premium principles, as the premium loading is an increasing function of the

[1] In this work the random variables of interest are the two risks considered. Hence, often the underlying random variables are designated by risks.

variance of the covered risk. Noticing that the amount of risk ceded to the reinsurer, per risk i, $i = 1, 2$, is $X_i - Y_i$, with $Y_i = \min(a_i X_i, M_i)$, we can compute the reinsurance premium on each total ceded risk.

Expected value principle: $P_{Ri} = E(X_i - Y_i) + \delta_i E(X_i - Y_i) = (1 + \delta_i) E(X_i - Y_i)$.

Variance principle: $P_{Ri} = E(X_i - Y_i) + \delta_i \operatorname{Var}(X_i - Y_i)$.

Standard deviation principle: $P_{Ri} = E(X_i - Y_i) + \delta_i \sqrt{\operatorname{Var}(X_i - Y_i)}$.

Here $\delta_i > 0$, $i = 1, 2$, is the loading coefficient. This is how the authors in [6], using the expected value principle, as well as in [19, 13], for variance related principles, compute the reinsurance premium. However, when a combination of QS and stop loss is taken into account, the QS and stop loss premiums can be considered separately. This is the procedure followed for instance in [8, 9], and it corresponds to many practical cases, where the stop-loss contract is independent of the QS treaty, coming on top of the QS. In fact, the QS premium is usually proportional to the ceded risk minus a commission. In this case, the QS premium is the proportion of the premium received by the insurer P_i correspondent to the ceded risk, $(1 - a_i) P_i$, subtracting the commission, $c_i > 0$: $P_{QSi} = (1 - a_i)(1 - c_i) P_i$. The stop loss premium will be computed on the ceded risk after QS: $Z_i = \max(a_i X_i - M_i, 0)$, $i = 1, 2$. Thereby, the total reinsurance premium turns out as follows.

Expected value principle: $P_{Ri} = P_{QSi} + (1 + \delta_i) E(Z_i)$.

Variance principle: $P_{Ri} = P_{QSi} + E(Z_i) + \delta_i \operatorname{Var}(Z_i)$.

Standard deviation principle: $P_{Ri} = P_{QSi} + E(Z_i) + \delta_i \sqrt{\operatorname{Var}(Z_i)}$.

Here, we will study and compare optimal reinsurance strategies in both cases where the premium is computed on the total ceded risk or separately for QS and stop loss.

2.3. The expected utility and the adjustment coefficient

Several authors have considered to use the expected utility of wealth as optimality criteria when ascertaining the optimal reinsurance strategy, e.g. [8, 9, 12, 19, 13, 23]. The adjustment coefficient can be regarded as a given coefficient of aversion of the exponential utility function. On the other hand, the adjustment coefficient is connected to the ultimate probability of ruin. From the well known Lundberg Inequality, the larger the adjustment coefficient is, the smaller the upper bound of the probability of ultimate ruin is. Thus, maximizing the adjustment coefficient R instead of minimizing the probability of ruin $\Phi(u)$ is reasonable. Because of this, many authors have considered maximizing the adjustment coefficient as optimality criteria for reinsurance, e.g. [7, 11, 12, 19, 13, 10, 28]. In [18] reinsurance strategies minimizing directly the probability of the insurer's ruin are studied. There, the authors consider that the reinsurance premium is an increasing function of the expected value of the transferred risk. They show that in this case the stop loss, or the truncated stop loss if there are reinsurance premium budget restrictions, is the optimal strategy.

In [27] the same problem, also considering the expected value premium principle, is analyzed in the presence of background risk. Other works can be found, where strategies minimizing directly the probability of ruin are obtained, such as in [20, 25, 1, 24]. However, in such works the framework is usually a dynamical setting, with a diffusion setup and a continuous time adaptation of the contract, which is not the case of the present paper.

Notice that the adjustment coefficient is independent from the initial capital, u, of the insurer. Thus, the optimal strategy that maximizes the adjustment coefficient is also independent of u. In [15] an upper bound for the probability of ruin, dependent on the initial capital, is provided. In [16] this inequality is further refined and used to approximate the probability of ruin in regime-switching Markovian models. This upper bound represents an improvement to the Lundberg bound, specially for the cases where the initial capital is small. Hence, it is expected that using such upper bound as optimality criteria will lead to different optimal retention levels, specially for small values of the initial capital. However, it requires the distribution of losses to be new worse than used (NWU) and represents a significantly more complex bound from the computational point of view, when compared to the Lundberg bound, as it includes the need to solve an extra minimization problem.

In this work we will consider maximizing the expected exponential utility and the adjustment coefficient. Interesting future works include the minimization of the improved upper bound for the probability of ruin provided in [15] as optimality criteria, and to compare it with the results here presented.

2.3.1. Maximizing the expected utility

The goal is to determine the optimal reinsurance contract for a risk-averse insurer which purpose is to maximize the expected utility of its wealth. We consider the exponential utility function, for risk averse investors, defined through $U(x) = \frac{1-e^{-\beta x}}{\beta}$, where $\beta = -U''(x)/U'(x) > 0$ is the coefficient of risk aversion. In this case, the expected utility of the wealth for a given (fixed) coefficient of aversion β is:

$$E\Big[U\Big(W\big(a_1, M_1, a_2, M_2\big)\Big)\Big] = \frac{1}{\beta}\Big(1 - E\Big[e^{-\beta W(a_1,M_1,a_2,M_2)}\Big]\Big). \tag{2.2}$$

Maximizing the expected utility (2.2) corresponds to find the reinsurance strategy, (a_1, M_1, a_2, M_2), that maximizes $E[U(W)]$ for a given (fixed) coefficient of risk aversion β. Recalling (2.1), this is equivalent to minimize the following functional:

$$\begin{aligned} E\Big[e^{-\beta W(a_1,M_1,a_2,M_2)}\Big] &:= G\big(\beta, a_1, M_1, a_2, M_2\big) = \\ &= e^{-\beta\big((1-e_1)P_1+(1-e_2)P_2\big)}\, e^{\beta\big(P_{R1}(a_1,M_1)+P_{R2}(a_2,M_2)\big)} \times \\ &\quad \times \int_0^{+\infty}\!\!\int_0^{+\infty} e^{\beta\big(Y_1(a_1,M_1)+Y_2(a_2,M_2)\big)}\, dC_\alpha(x_1,x_2) \end{aligned} \tag{2.3}$$

for a given (fixed) β.

2.3.2. Maximizing the adjustment coefficient

The adjustment coefficient, R, of the retained risk after reinsurance is defined as the unique positive root, if it exists, of $G(R, a_1, M_1, a_2, M_2) = 1$, where G is given by (2.3). The coefficient of adjustment is related to the coefficient of risk aversion of the exponential utility, as it corresponds to the value of the risk aversion coefficient for which the expected utility (2.2) is zero, see [19]. In [19] it is demonstrated that, under general regularity assumptions on the functional G verified in our case, a reinsurance policy maximizes the adjustment coefficient, $\widehat{R}$, if and only if:

- **i)** The expected utility, with coefficient of risk aversion $\widehat{R}$, is maximum for that policy, and
- **ii)** $G\big(\widehat{R}, a_1, M_1, a_2, M_2\big) = 1$.

Thus, as suggested in [19], the problem of maximizing the adjustment coefficient can be split in two sub problems:

1. For each $\beta > 0$, find the reinsurance strategy, (a_1, M_1, a_2, M_2) that minimizes G.
2. Solve $G\big(\beta, a_1, M_1, a_2, M_2\big) = 1$ with respect to the single variable β.

Whence, given the algorithm to find the optimal reinsurance maximizing the expected utility it is straightforward to obtain the reinsurance strategy maximizing the adjustment coefficient. However, maximizing the adjustment coefficient requires the solution of several expected utility maximization problems, until the desired root is found.

3. NUMERICAL RESULTS AND DISCUSSION

The numerical implementation was performed the *Mathematica*. All the double and single integrals involved in the evaluation of $G(\beta, a_1, M_1, a_2, M_2)$ are solved using *Mathematica* numerical integration, which applies global adaptive Gauss–Kronrod quadrature rules. The resolution of the minimization problems were carried out using numerical algorithms for non-linear constrained global optimization already implemented in *Mathematica*, namely the Nelder Mead and Differential Evolution algorithms. Strictly speaking, the Nelder Mead algorithm is not a global optimization method, but it tends to work quite well if the objective function does not have many local minima, which is the case here. The numerical procedure, namely the numerical optimization problem, is amenable for improvement as no particular features of the functional to minimize were taken into consideration and general global optimization was applied. The existence of *plateaux* regions in the functional to minimize, specially regarding the stop loss retention values, made the convergence to the optimal solution slower in some cases. Nevertheless, results were achieved and analysis of the sensitiveness to the several factors, such as premium calculation principles and dependence structures and levels, of the optimal reinsurance for two dependent risks were performed.

In the following, the premium received by the insurer is computed by means of the expected value principle with a loading coefficient of $\gamma_i = 0.2,\ i = 1, 2$. For the underlying risks, X_1 and X_2, we will consider different distributions, but in such way that the expected value is always 1. Hence, the premium loading charged by the insurer is $\gamma_i\, E(X_i) = \gamma_i$, $i = 1, 2$. We assume expenses are 5% of the premium, $e_i = 0.05$, $i = 1, 2$. Whenever the QS premium is computed on a proportional basis, separately from the stop-loss premium, the commission is $c_i = 0.03$, $i = 1, 2$. Indeed, the QS reinsurance commission should be lower than the insurer expenses $c_i < e_i$, meaning it is impossible to reinsure the whole risk through QS with a certain profit. This implies that the QS premium loading is $E(X_i)\,\big[(1-c_i)\,(1+\gamma_i)-1\big] = 0.164\, E(X_i)$. When maximizing the expected utility, we consider a coefficient of risk aversion $\beta = 0.1$.

In Table 1 are presented the premium loadings. With these values, the premium loading when all the risk is transferred by means of a pure stop loss contract, *i.e* when $a_i = 1$ and $M_i = 0$, is the same for all three premium principles. Indeed, in this case the moments involved in computing the reinsurance premiums, either for QS and stop loss together or separately, correspond to the moments of the underlying risk. However, if QS and stop loss are considered separately that is true only when $a_i = 1$ (and $M_i = 0$), whereas if the premium is computed for the QS and stop loss together that is true no matter the value of a_i (as long as $M_i = 0$).

Table 1: Loading coefficients for the three premium principles considered, where δ is the loading coefficient for the expected value principle and X is the underlying risk.

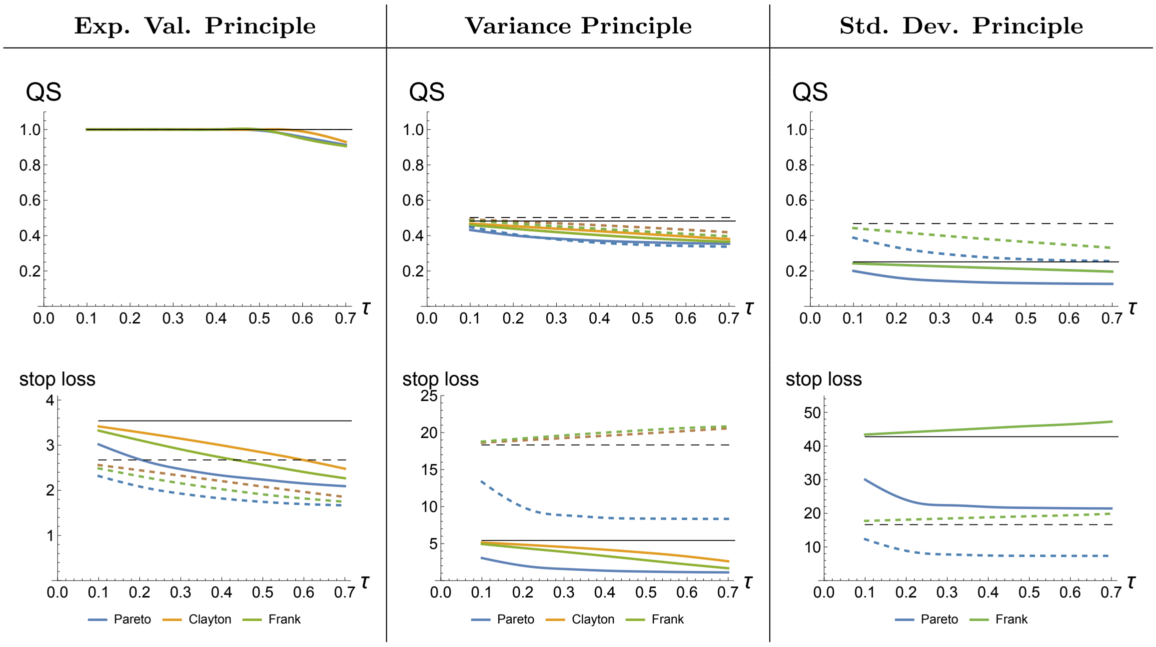
premium principle	loading coefficient
expected value	δ
variance	$\delta\, E(X)/\,\mathrm{Var}(X)$
standard deviation	$\delta\, E(X)/\sqrt{\mathrm{Var}(X)}$

We first consider two Pareto distributions with expected value 1 and shape parameter 3. The loading coefficients in this case are shown in Table 2. In this case, independently of the premium calculation principle, the optimal retention levels of QS and stop loss contracts are the same for both risks, as they are equal.

Table 2: Loading coefficients, for the three premium principles, considering two Pareto risks with expected value 1 and shape parameter 3.

premium principle	QS and stop loss separately	QS and stop loss together
expected value	0.3	0.2
variance	0.1	0.0666667
standard deviation	0.173205	0.11547

Results for the optimal reinsurance, as function of Kendall's tau coefficient, maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$ and the loading coefficients in Table 2 are presented in Figure 1.



horizontal lines: independent case;
dashed lines: premiums computed on the total ceded risk;
solid lines: premiums computed on the ceded risk after QS, with the QS premium computed on original terms;

Figure 1: Optimal reinsurance maximizing the expected utility with $\beta = 0.1$.

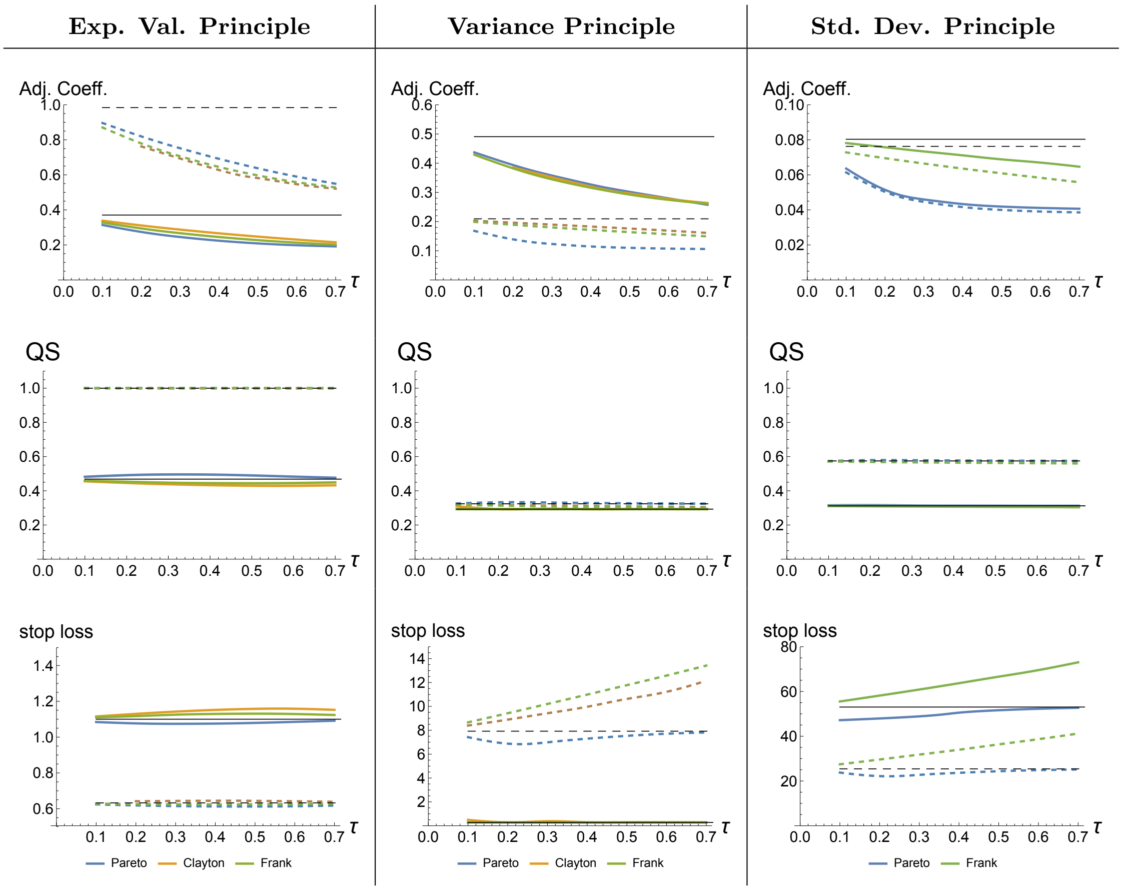
From the results, we can see that when the expected value principle is computed on the total ceded risk, the optimal reinsurance contract is always the pure stop loss, independently of the dependence structure and strength. This is expected, from the results in [6]. If the expected value principle is computed only on the ceded risk through stop-loss, after QS, the pure stop loss is no longer the optimal contract. In this case, for larger values of the Kendall's tau correlation, the optimal QS levels decrease below the independence optimal QS level. For larger values of the Kendall's tau, it compensates to cede part of the risk trough QS and to cede trough stop loss on top of that, independently of the dependence structure. This is related with the QS premium loading in this case, that for strong dependence compensates the stop-loss premium loading. This is not verified when independence is assumed, for this loading coefficients. Thus, the results suggest that affects the type of optimal contract even when the expected value principle is considered, if QS and stop-loss premiums are computed separately. We also observe that, no matter what the optimal contract is, the optimal stop loss limits for the expected value principle, computed together or separately for QS and stop loss, decrease as dependence strength increases.

Regarding the variance and standard deviation principles, we can see that the pure stop loss is never the optimal treaty, not even in the independent case. That is in accordance with the results in [19]. In these cases, the optimal QS levels always decrease as the dependence strength increases, independently of computing the reinsurance premium together, for the whole ceded risk, or separately for QS and stop-loss. However, the optimal stop-loss limit does not always decrease as dependence strength increases. For the variance principle, if the premium is computed together for QS and stop-loss, the optimal limit of stop loss decreases, as dependence increases, in the case of Pareto's copula, but it increases, as dependence increases (and the QS optimal level decreases) for Clayton's and Frank's copulas. For the standard deviation principle, this behaviour of Clayton's and Frank's copulas is observed not only for the premium computed on the whole ceded risk, but also when the premiums of QS and stop-loss are computed separately. These results show that dependence impacts the optimal levels of retention in non-intuitive ways, especially in the cases of variance related premium principles.

Next we consider the maximization of the adjustment coefficient as optimality criteria. As described in Chapter 2, to obtain the optimal reinsurance treaty maximizing the adjustment coefficient, it is enough to find the optimal solution for the expected utility problem with the coefficient of risk aversion $\beta > 0$ such that the expected utility value in (2.2) is equal to 0. In order to solve equation $G(R, a_1, M_1, a_2, M_2) = 1$, for (a_1, M_1, a_2, M_2) minimizing $G(R, a_1, M_1, a_2, M_2)$, a bisection method was applied. Amongst the root finding numerical methods, bisection is the simplest. Although its convergence is not very fast when compared with Newton-type methods, it has the advantage of not requiring the computation of derivatives of the functional. Also, convergence to a tolerance of 10^{-6} was reached within an average of 10 iterations, as the initial points were easily chosen close enough to the solution. Situations where convergence was more difficult regard instances where converge of the constraint global optimization algorithm to the minimum of functional G was slow. This was the case of Clayton's copula, when using the standard deviation principle computing QS and stop loss premiums separately. Results are depicted in Figure 2.

Again, as expected from the results in [6] and [19], the optimal treaty when the expected value principle is applied to QS and stop loss together is the pure stop loss, for all three copulas and all values of the dependence parameter. That is not the case for the variance related premium principles, even in the independent case. The optimal retention levels vary with the dependence parameter, although not so significantly as for the case where the risk aversion coefficient was fixed. Instead, the impact of dependence is very relevant in the value of the maximum adjustment coefficient of the optimal contract. It can be observed, for all copulas and premium principles considered, that the adjustment coefficient decreases as dependence strength increases. This means that the higher the dependence, the higher the upper bound of the ultimate probability of ruin.

The remarks made for the case of a fixed coefficient of aversion apply here, although now the differences in the standard deviation and variance principles, computing QS and stop loss premiums together, are more accentuated. The adjustment coefficient always decreases when dependence increases. It can be observed that the maximum adjustment coefficient using the standard deviation principle is always below those using the expected value and variance principles. If QS and stop loss premiums are computed together, then the maximum adjustment coefficient using the expected value principle is higher. If premiums are computed separately, then the maximum adjustment coefficient using the variance principle is higher.



horizontal lines: independent case;
dashed lines: premiums computed on the total ceded risk;
solid lines: premiums computed on the ceded risk after QS, with the QS premium computed on original terms;

Figure 2: Optimal reinsurance maximizing the adjustment coefficient, and corresponding optimal values of the adjustment coefficient.

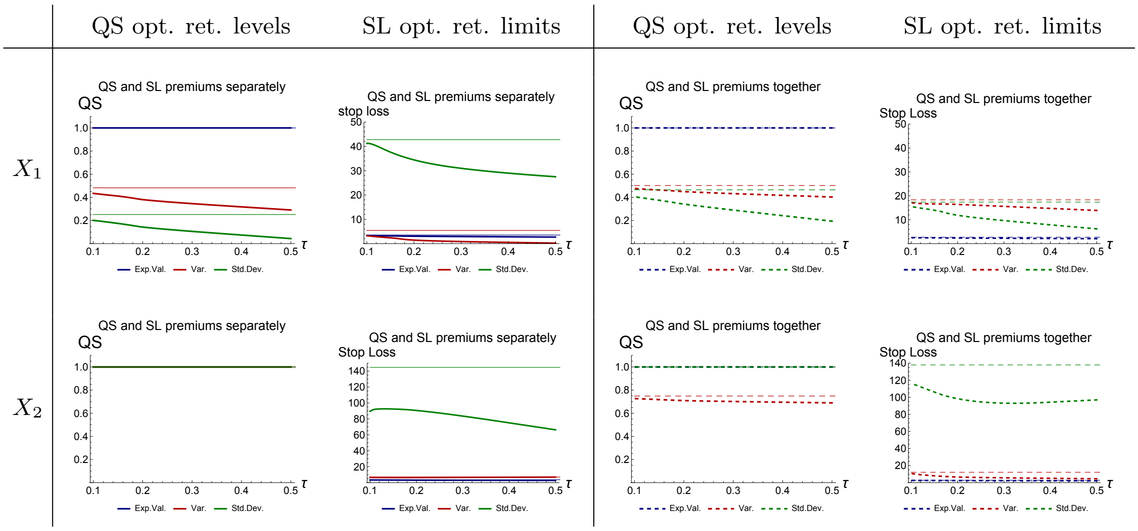
This is verified for all three copulas. When the standard deviation principle is considered, the maximum adjustment coefficient of the optimal contract is similar computing the premium together on the whole ceded risk, or separately for QS and stop-loss. It is worth noticing the differences in the optimal reinsurance for the different copulas. Differences are particularly significant between the Pareto's copula and Clayton's and Frank's copulas. This is because Pareto's copula has right tail dependence, while Clayton's and Frank's copulas do not.

Afterwards we have considered two risks with different tail heaviness: two Pareto distributions with expected value 1 and shape parameters 3 and 12, respectively. In this case, the variances are 3 and 1.2, respectively. For this case, we have considered dependence by means of the Pareto's copula, where dependence is stronger on the right tail, and we aim at maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$. Regarding the loading coefficients, we apply the same reasoning as before, which is described in Table 1 and leads to the loading coefficients presented in Table 3. Results are shown in Figure 3.

Table 3: Loading coefficients, for the three premium principles, considering two Pareto risks with expected value 1 and shape parameter 3 (X_1) and 12 (X_2).

premium principle	QS and stop loss separately		QS and stop loss together	
	X_1	X_2	X_1	X_2
expected value	0.3	0.3	0.2	0.2
variance	0.1	0.25	0.0666667	0.166667
standard deviation	0.173205	0.273861	0.11547	0.182574

Whenever the expected value principle is applied (blue lines of Figure 3), either on the total ceded risk or just on the stop loss contract, the pure stop loss treaty is optimal for both risks. The optimal stop loss retention limits are similar for both risks and decrease as dependence increases.



horizontal lines: independent case;
dashed lines: premiums computed on the total ceded risk;
solid lines: premiums computed on the ceded risk after QS, with the QS premium computed on original terms;

Figure 3: Optimal reinsurance maximizing the expected utility with $\beta = 0.1$ for two dependent risks with Pareto distributions with mean 1 and shape parameters 3 (X_1) and 12 (X_2).

Regarding the standard deviation principle (green lines in Figure 3), the pure stop loss contract is optimal for the second (lighter tailed) risk, when computing the reinsurance premium both on the total ceded risk or separately for QS and stop loss. The optimal stop loss retention values for this pure stop loss contract on the second risk are significantly high, compared to the first risk or with the other, expected value and variance, premium calculation principles, and decrease as dependence strength increases. For the first (heavier tailed) risk, the optimal reinsurance contract is not the pure stop loss anymore and the optimal stop loss retention limits are much lower than those of the second risk, though still higher compared to the expected value premium principle. The optimal QS levels are quite low and both QS and stop loss optimal retention limits decrease as dependence increases. With the standard deviation premium principle, much of the first risk is transferred, while much of the second risk is kept.

For what concerns the variance principle (red lines in Figure 3), the pure stop loss contract is optimal for the second risk only when the QS premium is computed on a proportional basis. Again, the optimal QS and stop loss retention values decrease with dependence. The optimal stop loss retention limits of the first risk are significantly different when QS and stop loss premiums are computed together or separately. This difference is less accentuated for the second risk, where the stop loss contract is optimal when computing QS and stop loss premiums separately.

In general, for all three premium principles and for both risks, the optimal QS and stop loss retained levels decrease as dependence increases. In most cases the pure stop loss contract is optimal for the second (lighter tailed) risk. Thus, in most cases only the tail of the second risk is transferred. On the contrary, for the first (heavier tailed) risk, the pure stop loss contract is optimal only for the expected value principle, meaning that for the standard deviation and variance principles it is optimal to transfer more of the first (heavier tailed) risk.

4. CONCLUSIONS

Clearly dependencies alter the optimal treaty, as compared with the independent case, and the impact of these dependences on the optimal treaty may be non-intuitive. Different dependence structures, yield significantly different optimal solutions. As expected, the optimal treaty is also highly sensitive to the premium calculation principle and relevant differences are encountered between premiums calculated on the total ceded risk or separately for QS and stop loss. In some cases, this behaviour is accentuated in the presence of dependencies. The results here presented can be useful in bringing insight on the impact of dependence on the optimal reinsurance strategy. Such insight can be helpful in the design of more general theoretical results on optimal reinsurance of dependent risks. It can also be beneficial when analysing *real world* case studies of applications.

ACKNOWLEDGMENTS

The authors were partially supported by the Project CEMAPRE–UID/MULTI/00491/2019 financed by FCT/MCTES through national funds.

REFERENCES

[1] Albrecher, H. and Haas, S. (2011). Ruin theory with excess of loss reinsurance and reinstatements, *Applied Mathematics and Computation*, **217**(8031):8043.

[2] Asimit, A.; Bignozzi, V.; Cheung, K.C.; Hu, J. and Kim, E.S. (2017). Robust and Pareto optimality of insurance contracts. *European Journal of Operational Research*.

[3] Asimit, A.V.; Gao, T.; Hu, J. and Kim, E.S. (2017). *Optimal risk transfer: a numerical optimisation approach*, Technical report, City University of London.

[4] Bai, L.; Cai, J. and Zhou, M. (2013). Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting, *Insurance: Mathematics and Economics*.

[5] Bi, J.; Liand, Z. and Xu, F. (2016). Optimal mean variance investment and reinsurance problems for the risk model with common shock dependence, *Insurance: Mathematics and Economics*.

[6] Cai, J. and Wei, W. (2012). Optimal reinsurance with positively dependent risks, *Insurance: Mathematics and Economics*, **50**, 57–63.

[7] Centeno, M.L. (1986). Measuring the effects of insurance by the adjustment coefficient, *Insurance: Mathematics and Economics*, **5**, 169–182.

[8] Centeno, M.L. (1986). Some mathematical aspects of combining proportional and non-proportional reinsurance, *Insurance and Risk Theory*, 247–266.

[9] Centeno, M.L. (1988). The expected utility applied to reinsurance, *Risk, Decision and Rationality*, 679–689.

[10] Centeno, M.L. (1997). Excess of loss reinsurance and the probability of ruin in finite horizon, *ASTIN Bulletin*, **27**(1), 59–70.

[11] Centeno, M.L. (2002). Measuring the effects of reinsurance by the adjustment coefficient in the Sparre Anderson model, *Insurance: Mathematics and Economics*, **30**, 37–49.

[12] Centeno, M.L. (2005). Dependent risks and excess of loss reinsurance, *Insurance: Mathematics and Economics*, **37**, 229–238.

[13] Centeno, M.L. and Guerra, M. (2010). The optimal reinsurance strategy – the individual case, *Insurance: Mathematics and Economics*, **46**, 450–460.

[14] Embrechts, P.; Lindskog, F. and McNeil, A. (2003). Modelling dependence with copulas and applications to risk management. In "Handbook of Heavy Tailed Distributions in Finance" (S. Rachev, Ed.), pp. 329–384, Elsevier, Amsterdam.

[15] Gajek, L. (2005). On the deficit distribution when ruin occurs – discrete time model, *Insurance: Mathematics and Economics*, **36**, 13–24.

[16] Gajek, L. and Rudz, M. (2018). Banach contraction principle and ruin probabilities in the regime-switching models, *Insurance: Mathematics and Economics*, **80**, 45–53.

[17] Gajek, L. and Zagrodny, D. (2000). Insurer's optimal reinsurance strategies, *Insurance: Mathematics and Economics*, **27**, 105–112.

[18] Gajek, L. and Zagrodny, D. (2004). Reinsurance arrangements maximizing insurer's survival probability, *The Journal of Risk and Insurance*, **71**(3), 421–435.

[19] Guerra, M. and Centeno, M.L. (2008). Optimal reinsurance policy: the adjustment coefficient and the expected utility criteria, *Insurance: Mathematics and Economics*, **42**, 529–539.

[20] Hipp, C. and Vogt, M. (2003). Optimal dynamic XL reinsurance, *Astin Bulletin*, **33**(2), 193–207.

[21] Kaluszka, M. (2004). An extension of Arrow's result on optimality of a stop-loss contract, *Insurance: Mathematics and Economics*, **35**, 527–536.

[22] Kaluszka, M. (2005). Optimal reinsurance under convex principles of premium calculations, *Insurance: Mathematics and Economics*, **36**, 375–398.

[23] Karageyik, B.B. and Dickson, D.C.M. (2015). Optimal reinsurance under multiple attribute decision making, *Annals of Actuarial Science*, **10**(1), 65–86.

[24] Liang, X. and Young, V.R. (2018). Minimizing the probability of ruin: optimal per-loss reinsurance, *Insurance: Mathematics and Economics*, **82**, 181–190.

[25] Schmidli, H. (2010). Optimal proportional reinsurance policies in a dynamic setting, *Scandinavian Actuarial Journal*, **2001**(1), 55–68.

[26] Wang, P. (2013). Risk modeling and multi-year, multi-line reinsurance using copulas, *Journal of Insurance Issues*, **36**(1), 58–81.

[27] Weng, C. and Zhuang, S.C. (2017). CDF formulation for solving an optimal reinsurance problem, *Scandinavian Actuarial Journal*, **2017**(5), 395–418.

[28] Zhang, L.; Hu, X. and Duan, B. (2015). Optimal reinsurance under adjustment coefficient measure in a discrete risk model based on Poisson MA(1) process, *Scandinavian Actuarial Journal*, **2015**(5), 455–467.

[29] Zhang, N.; Jin, Z.; Li, S. and Chen, P. (2016). Optimal reinsurance under dynamic VaR constraint, *Insurance: Mathematics and Economics*.