REVSTAT – Statistical Journal
Volume 20, Number 1, January 2022, 85–100

THE SOLUTION TO A DIFFERENTIAL–DIFFERENCE EQUATION ARISING IN OPTIMAL STOPPING OF A JUMP–DIFFUSION PROCESS

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Received: September 2018 Revised: July 2019 Accepted: November 2019

Abstract:

- In this paper we present a solution to a second order differential–difference equation that occurs in different contexts, specially in control engineering and finance. This equation leads to an ordinary differential equation, whose homogeneous part is a Cauchy–Euler equation. We derive a particular solution to this equation, presenting explicitly all the coefficients.
The differential–difference equation is motivated by investment decisions addressed in the context of real options. It appears when the underlying stochastic process follows a jump–diffusion process, where the diffusion is a geometric Brownian motion and the jumps are driven by a Poisson process. The solution that we present — which takes into account the geometry of the problem — can be written backwards, and therefore its analysis is easier to follow.

Keywords:

- *differential–difference equation; differential equation, jump–diffusion process.*

AMS Subject Classification:

- 37H10, 60G40.

1. MOTIVATION AND RELATED WORK

In this paper we present a solution to a second order differential–difference equation. This equation may appear when one solves an optimal stopping problem in which the state process follows a jump–diffusion process, where the diffusion is a geometric Brownian motion and the jumps are driven by a Poisson process. The main difficulty of working with this type of equation is due to the jump process, which makes the equation not local in one point — see, for instance, Murto [25]. This characteristic is not universal, i.e., there are optimal stopping problems involving jump–diffusions processes for which the differential–difference equation does not exhibit this behavior. Thus, on these cases to find a closed form solution to the differential–difference equation can be easier. However, as we will see later, this is not the case when the jumps may lead directly to the stopping region, across the boundary.

The seminal works in financial options — such as the classical work of Black and Scholes [5], where for the first time a pricing formula was derived — and in real options — as the seminal book of Dixit and Pindyck [12] — assume that the sample path of the involved state process is continuous, with probability one.

In recent times investors often need to take decisions facing uncertainty and there is higher likelihood of financial crashes, which are the climax of the so-called log-periodic power law signatures associated with speculative bubbles (see Johansen and Sornette [18]). One example of this occurred in February 2015, when due to a cyber-attack, a high-frequency trading company started uncontrollably buying oil futures, causing a downward jump in the oil prices[1]. Here, a crash is a significant drop in the total value of the market, creating a situation wherein the majority of investors are trying to flee the market at the same time and consequently incurring massive losses. Indeed, in the presence of a crash investors likely take the decision to sell their assets. As the crash means that there is a significant drop, we borrow the probabilistic terminology and we call it a *jump* (in the above example, a downward jump).

The sudden changes in the state variable can also be found when one decides about investments in projects, often addressed in the context of real options. In this context, usually the temporal term is long, and therefore unexpected events may occur, leading to a disruption of the market. One example of a disrupt event is the introduction or the abolition of public subsidies. There are many economical sectors where subsidies play an important role, such as agriculture.

Due to the interest of the equation that we solve in this paper in the framework of real options, we mainly focus in problems and questions arising in such context. The following are examples of decisions regarding investments where investors face the likelihood of sudden events.

It is well established that agricultural pricing policies (taxes, subsidies) have a substantial influence on farmer production decisions[2]. For example, USA has been supporting farming since early times. But after several decades, these incentive policies have proved to

[1] https://www.businessinsider.com/investigation-into-hft-firm-for-using-an-algo-gone-wild-that-caused-oil-trading-mayhem-in-just-5-seconds-2010-8

[2] http://www.pbl.nl/en/publications/the-impact-of-taxes-and-subsidies-on-crop-yields

be unsuccessful [3]. In 2005 Bush administration decided to change the farm incentive policy, cutting in agricultural subsidies[4]. Evidently, this decision led to changes in private investment farming projects.

Another area where subsidies play an important role is the renewable energy (RE) sector. In an effort to reach the ambitious targets of the EU Strategic Energy Technology Plan (SET-Plan), EU member states have implemented support mechanisms of various forms (e.g., price mechanisms, like carbon tax or permit trading schemes) intended to incentive and accelerate adoption of RE technologies. These climate change policies have introduced a new factor that has to be included into the investment decision and have become a major source of uncertainty in energy strategy. The problem is that policies designed to stimulate the investment in green energies have frequently and unexpectedly been changed for a number of reasons. For instance, change of governments, collapse of the international cooperation for reducing GHG emissions, arrival of new information about climate sensitivity, and fiscal pressure. In the last decade we have seen many studies on the impact of wrong investment decisions. We refer, for instance, to Boomsma and Linnerud [6], Boomsma *et al.* [7], and Hagspiel *et al.* [14].

These examples show that when taking decisions regarding investments in new projects, the investor needs to take into account these sudden changes. The area of real options soon realized the importance of such events, and therefore the interest of real options literature in problems involving jump–diffusion processes is not new. We refer to Kou [21] for a survey on jump–diffusion models for finance engineering. In the area of real options, there has been an increasing interest about jump–diffusion processes in the context of technology adoption (see, for instance, Hagspiel *et al.* [16]).

Furthermore, Kwon [22] and Hagspiel *et al.* [15] consider a combination of a continuous process with a jump-process, but they do not consider a sequence of innovations arriving over time. Instead, they assume a one-single innovation opportunity, with other involved options (like the option to exit the market). Kwon [22] work is generalized in Hagspiel *et al.* [15], by considering capacity optimization, and by Matomäki [23], considering different stochastic processes representing the profit uncertainty.

In another context, Couto *et al.* [11] and Nunes and Pimentel [26] consider the investment problem in a high-speed railway service, assuming that both the demand and the investment cost are modeled by jump–diffusion processes. Although these papers start by assuming two sources of uncertainty, they end up with the study of a one-dimensional problem. This happens because they assume that the value of the firm is homogeneous, and therefore it is possible to consider a change of variables that will turn the two-dimensional problem in a one-dimensional one. Murto [25] also consider two stochastic processes, in order to model technological and revenue uncertainties, motivated by wind power investment. He assumes that the investment cost depends on the technological progress, driven by a pure Poisson process, whereas the price of the output is a geometric Brownian motion. As the value of the project is homogeneous, the same type of approach as in Nunes and Pimentel [26] is proposed.

In all the above examples, it is of the most crucial importance to assess the impact of the jumps in the decision, and, in particular, in case the jumps anticipate the optimal decision.

[3] https://grist.org/article/farm_bill2/
[4] https://www.agpolicy.org/weekpdf/258.pdf

Moreover, the impact of such jumps has to be reflected in the *value* of the project, which is a quantitative measure of the value that the firm has as a result of its option to invest. Under an optimal strategy in terms of the investment timing, such value is, before the investment, solution of a differential equation (that we will present in section 2). Mathematically, the possibility of occurrence of jumps leads to this value being solution of particular types of differential equations.

Our contribution to the state of the art is two-fold: on one side, we provide an analytical solution to a non-homogeneous differential equation. As it turns out, some optimal stopping problems found in real options lead to a differential–difference equations that are exactly as the form of such differential equation, for a subset of the state space. Therefore, one may use this analytical solution to provide a characterization of the value of a firm, which is given by a piecewise function.

The paper is organized as follows: in Section 2 we motivate the differential–difference equation that we address in this paper, presenting also the basic assumptions. In Section 3 we show how we can find a general solution for such equation, using a backwards procedure. This procedure presents the solution as a piecewise function. For each branch, the function is the solution of a non-homogeneous differential equation. Therefore, in Section 4 we provide the particular solution to it. Finally, in Section 5 we conclude.

2. DIFFERENTIAL–DIFFERENCE EQUATION

In order to motivate the meaningfulness of the differential–difference equation solved in this paper, we consider that we want to derive the value of a firm that has the option to undertake an investment. As we briefly explain in this section, these type of problems leads to a variational inequality known as the Hamilton–Jacobi–Bellman (HJB, for short) equation, where one of the members is a differential–difference equation. To solve such equation, we also need to be able to find the solution of a differential equation of the following type:

$$x^2 y''(x) + axy'(x) + by(x) = Ax^{\alpha}(\ln x)^n, \tag{2.1}$$

with $x > 0$, $a, b \in \mathbb{R}$, $\alpha, A \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}_0$.

We note that the corresponding homogeneous equation to (2.1) is an Euler–Cauchy equation and its solution is known. The difficulty lays in the particular solution, consequence of the non-homogeneous term, $Ax^{\alpha}(\ln x)^n$.

The result that we provide in this paper is *per se* interesting, as it provides a contribution to the area of ordinary differential equations (ODE). Besides this contribution, being able to compute the solution of such equation is also relevant for the applications. Next we motivate the mathematical problem by an investment problem, using the terminology and notation of real options.

Real options is a theory on how to make decisions under uncertainty about future returns. These decisions share the following two characteristics: they are irreversible and can be postponed.

One of the most relevant problems in real options regards the characterization of the optimal time to undertake some investment decision. This leads to an *optimal stopping problem*, which is formally defined as follows: given a stochastic process $\mathbf{X} = \{X(t), t > 0\}$, find $V(x)$ and $\tau^\star \in \mathcal{T}$ such that

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^x \left[e^{-r\tau} g(X(\tau)) \chi_{\{\tau < +\infty\}} \right], \quad x \in \mathbb{R}^+, \tag{2.2}$$

with $\mathcal{T}$ being the set of all stopping times adapted to the filtration generated by the process $\mathbf{X}$, $r > 0$ states for the discount factor and $\chi_{\{A\}}$ represents the indicator function on set A. The function g is usually called *running function*, which accounts for the return of the investment.

The class of stochastic processes that lead to the type of equations that we study in this paper — equation (2.1) — is an one-dimensional jump–diffusion, which is the strong solution of the following stochastic differential equation:

$$\frac{dX(t)}{X(t^-)} = \mu dt + \sigma dW(t) + \kappa dN(t),$$

with initial value $X(0) = x > 0$, where $\{W(t), t > 0\}$ is a standard one-dimensional Brownian motion, and $\{N(t), t > 0\}$ is a centered time-homogeneous Poisson process, with intensity $\lambda > 0$. Moreover, μ is the drift of the process $\mathbf{X}$, $\sigma > 0$ is its volatility and κ is the multiplicative factor, in case a jump occurs. The notation $X(t^-)$ means that whenever there is a jump, the value of the process before the jump is considered. Motivated by the references mentioned in Section 1, we assume that the jumps are multiplicative and with constant magnitude.

One way to solve the optimal stopping problem defined in (2.2) is to solve the variational inequality HJB (we do not provide further details, referring instead to Peskir and Shiryaev [28]). In this case, the corresponding HJB equation is the following:

$$\min \{rV(x) - \mathcal{L}V(x), V(x) - g(x)\} = 0, \tag{2.3}$$

where $\mathcal{L}$ is the infinitesimal generator of the process $\mathbf{X}$. As $\mathbf{X}$ is a jump–diffusion process, it follows that

$$\mathcal{L}v(x) = \frac{\sigma^2}{2} x^2 v''(x) + (\mu - \lambda\kappa) x v'(x) + \lambda \left(v(x(1+\kappa)) - v(x) \right), \tag{2.4}$$

for $v \in C^1$ and $x \in \mathbb{R}^+$ (see Øksendal and Sulem [27] for more details).

In general, the use of these inequalities leads to a differential equation, which in some cases may be solved analytically. Besides the possible difficulty to find the analytical solution to the differential equation, one faces also the problem to find the boundary conditions, as the set of values where the differential equation holds is also unknown. For this reason the problem presented in (2.2) when solved by the use of variational inequalities is known in the literature as a *free boundary problem.*

Considering an investment problem, the differential equation holds in the region where it is not optimal to stop (in our case to invest). For that reason, this region is usually called *continuation region*, and in opposition its complementary is called *stopping region*. In some cases, one can provide a guess for the shape of the continuation set. For example, if g is a non-decreasing function, the firm takes the decision to invest for large values of x, whereas for small values of x the firm postpones its investment decision. Thus, the stopping region is of the form $\mathcal{S} = [x^*, +\infty)$ and the continuation region is $\mathcal{C} = (0, x^*)$, where x^* is the exercise threshold.

When $\mathbf{X}$ is a jump–diffusion process with positive jumps, the stopping region can be reached in two different ways:

(**i**) Either due to a continuous change, caused by the diffusion. In this case the state process *hits* the boundary threshold x^*.

(**ii**) Or due to the occurrence of a jump. In this case the state process *crosses* the boundary threshold.

In the literature, the majority of the authors address the case that either there is just the jump process (for which it is possible to solve the corresponding difference equation, as there is no differential part) — this is the case of Huisman [17] — or the process is a jump–diffusion but the jumps always lead to the continuation region — which is the case of Nunes and Pimentel [26].

Our work is related with Merton [24], who considers a model to price American call options. He assumes multiplicative independent and identically distributed jumps and $g(x) = \max(x - K, 0)$ (the payoff of an American call option). For this case, he provides in Equation (16) a semi-analytical result, as it involves a series with infinite number of terms that depend, each one, on the cumulative distribution of a normal random variable. More recently, Murto [25] considers a problem with a similar setting as ours. However, in view of the impossibility to derive an analytical solution, he provides solutions only for some particular cases (namely, if the volatility parameter of the diffusion is zero, or when the jump process is in fact deterministic, with an exponential decay).

In the current paper, we assume a non-decreasing g function. Then it follows that on the one hand, in the stopping region V is equal to g, i.e. $V(x) = g(x)$ for $x \geq x^\star$. On the other hand, in the continuation region the value function V must be the solution of the left-hand side of the HJB Equation (2.3), which combined with Equation (2.4), leads to the following equation:

$$x^2V''(x) + a\, xV'(x) + b\, V(x) - c\, V(x(1+k)) = 0, \tag{2.5}$$

where $a = \frac{2(\mu-\lambda k)}{\sigma^2}$, $b = -\frac{2(r+\lambda)}{\sigma^2}$ and $c = -\frac{2\lambda}{\sigma^2}$. This is called in the literature *mixed partial differential–difference equation*, and it is known to be difficult to solve (see Merton [24]).

3. BACKWARDS ANALYSIS

In this section we provide a backwards procedure that can be used to solve the Equation (2.5). This procedure is motivated by the geometry of the stopping/continuation regions previously presented, when g is non-decreasing.

Firstly, we note that the homogeneous part of Equation (2.5) has an analytical solution, hereby denoted by V_h, which is given by

$$V_h(x) = \delta_1 x^{\beta_1} + \delta_2 x^{\beta_2}, \tag{3.1}$$

where β_1 and β_2 are the roots of the characteristic polynomial

$$Q(\beta) = \beta\,(\beta - 1) + a\beta + b. \tag{3.2}$$

In our case, given that $b < 0$, there are two distinct real roots:

$$\beta_1 = \frac{1}{2}\left[1 - a + \sqrt{(1-a)^2 - 4b}\right] > 0; \tag{3.3}$$

$$\beta_2 = \frac{1}{2}\left[1 - a - \sqrt{(1-a)^2 - 4b}\right] < 0. \tag{3.4}$$

As presented before, $V(x) = g(x)$ for $x \in [x^\star, +\infty)$. Therefore, one needs to solve the problem for $0 < x < x^\star$. For that, we start by considering $x \in \left[\frac{x^\star}{1+\kappa}, x^\star\right)$, meaning that $x(1+\kappa) \geq x^\star$. So, the interval $\left[\frac{x^\star}{1+\kappa}, x^\star\right)$ is the set of values of x where stopping will surely happen if a jump occurs. Thus $V(x(1+\kappa)) = g(x(1+\kappa))$. In this case Equation (2.5) can be re-written as

$$x^2 V''(x) + a\, xV'(x) + b\, V(x) = c\, g(x(1+\kappa)).$$

and therefore its solution, hereby denoted by V_1, is given by

$$V(x) := V_1(x) = V_h(x) + V_p^1(x) = \delta_1 x^{\beta_1} + \delta_2 x^{\beta_2} + f_g^1(x), \tag{3.5}$$

Note that the superscript in V_p^1 and f_g^1 represents how many jumps we are away from the stopping region[5] (see Figure 1 for an illustration). Moreover, the bottom index in f_g^1 emphasizes that this function depends explicitly on g.

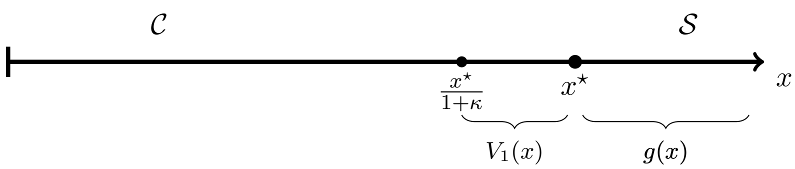

Figure 1: Representation of V in the last interval before stopping.

Next we derive the value of V when we are two jumps away from the stopping region. Following the same notation, we denote this function by V_2, defined for $x \in \left[\frac{x^\star}{(1+\kappa)^2}, \frac{x^\star}{1+\kappa}\right)$. In this case $x(1+\kappa) \in \left[\frac{x^\star}{1+\kappa}, x^\star\right)$, so $V(x(1+\kappa)) = V_1(x(1+\kappa))$. This means that (2.5) can be re-written as follows:

$$x^2 V''(x) + a\, xV'(x) + b\, V(x) = c\, V_1(x(1+\kappa)).$$

The homogeneous part of the previous equation is the same as before, and thus the solution is provided in (3.1). We just need to take into account the particular solution, which we denote by V_p^2. This particular solution depends on V_p^1 (and thus depends on g) but also depends on V_h (then also depends on the roots of Q, β_1 and β_2), as V_1 is given by (3.5). Therefore, both the homogeneous and the particular solution for this case *share* the powers β_1 and β_2. Using Theorem 3.5 of Sabuwala and De Leon [29], we end up with the following particular solution:

$$V_p^2(x) = \eta_1^2 \ln x\, x^{\beta_1} + \eta_2^2 \ln x\, x^{\beta_2} + f_g^2(x).$$

[5] We use this type of notation for all particular solutions.

We write f_g^2 to denote the part of the solution that depends strictly on g (following the same reasoning as for f_g^1), whereas η_1^2 and η_2^2 depend on the parameters from the homogeneous solution. So, for $x \in \left[\frac{x^\star}{(1+\kappa)^2}, \frac{x^\star}{1+\kappa}\right)$ (see Figure 2 for an illustration), we have

$$V(x) := V_2(x) = \delta_1 x^{\beta_1} + \delta_2 x^{\beta_2} + \eta_1^2 \ln x \, x^{\beta_1} + \eta_2^2 \ln x \, x^{\beta_2} + f_g^2(x). \tag{3.6}$$

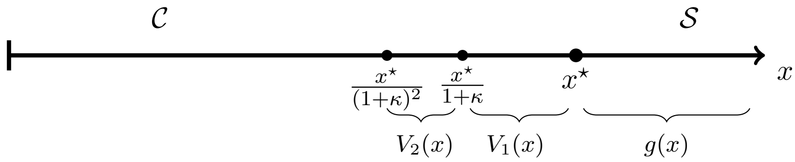

Figure 2: Representation of V in the last two intervals before stopping.

Proceeding one step back, we determine the value of V when we are three jumps away from the stopping region, which we call V_3. When $x \in \left[\frac{x^\star}{(1+\kappa)^3}, \frac{x^\star}{(1+\kappa)^2}\right)$, then $x(1+\kappa) \in \left[\frac{x^\star}{(1+\kappa)^2}, \frac{x^\star}{1+\kappa}\right)$ and $V(x(1+\kappa)) = V_2(x(1+\kappa))$. Then, Equation (2.5) is re-written as

$$x^2 V''(x) + a \, xV'(x) + b \, V(x) = c \, V_2(x(1+\kappa)). \tag{3.7}$$

As before, the homogeneous equation is the same and therefore V_h is part of the solution of this equation. Once more, the problem is reduced to the derivation of a particular solution, which is not trivial, as the function V_2 involves polynomials of power β_1 and β_2 multiplied by a logarithm (see Equation (3.6)). After some calculations, one may find that the particular solution of (3.7) is of the following form:

$$V_p^3(x) = \eta_1^3 \ln x \, x^{\beta_1} + \eta_2^3 \ln x \, x^{\beta_2} + \eta_3^3 \, (\ln x)^2 \, x^{\beta_1} + \eta_4^3 \, (\ln x)^2 \, x^{\beta_2} + f_g^3(x).$$

Also here f_g^3 stands for the part of the solution that depends strictly on g whereas $\eta_1^3, \eta_2^3, \eta_3^3$ and η_4^3 depend on the parameters from the homogeneous solution. As previously, for $x \in \left[\frac{x^\star}{(1+\kappa)^3}, \frac{x^\star}{(1+\kappa)^2}\right)$, we have

$$\begin{aligned} V(x) := V_3(x) \; = \; & \delta_1 x^{\beta_1} + \delta_2 x^{\beta_2} + \eta_1^3 \ln x \, x^{\beta_1} + \eta_2^3 \ln x \, x^{\beta_2} \\ & + \eta_3^3 \, (\ln x)^2 \, x^{\beta_1} + \eta_4^3 \, (\ln x)^2 \, x^{\beta_2} + f_g^3(x). \end{aligned}$$

A similar reasoning applies for other intervals of x. When we are i (with $i \in \mathbb{N}$) jumps away from the stopping region, we have $\frac{x^\star}{(1+\kappa)^i} \leq x < \frac{x^\star}{(1+\kappa)^{i-1}}$ and V is represented by V_i, which may be obtained using a similar procedure as the one used for V_1, V_2 and V_3. Indeed, V is a piecewise function, given by

$$V(x) = \begin{cases} V_i(x) & \text{if} \quad \frac{x^\star}{(1+\kappa)^i} \leq x < \frac{x^\star}{(1+\kappa)^{i-1}} \\ g(x) & \text{if} \quad x \geq x^\star \end{cases},$$

where

$$V_i(x) = \delta_1 x^{\beta_1} + \delta_2 x^{\beta_2} + V_p^i(x)$$

with

$$V_p^1(x) = f_g^1(x) \quad \text{and}$$

$$(3.8) \qquad V_p^i(x) = \sum_{j=1}^{i-1} \left[\eta_{2j-1}^i \, x^{\beta_1} + \eta_{2j}^i \, x^{\beta_2}\right] (\ln x)^j + f_g^i(x), \text{ for } i \in \mathbb{N} \setminus \{1\}\,.$$

Clearly, one needs to find functions that are solutions of certain differential equations, that depend intrinsically on the function g, considered in the definition of the problem.

For example, for

$$(3.9) \qquad g(x) = \rho x^\theta - I$$

we obtain the following particular solutions V_p^i, for $i = 1, 2, 3$:

$$V_p^1(x) = \xi_1^1 x^\theta + \xi_2^1, \quad \text{with} \quad \xi_1^1 = \frac{c\rho(1+\kappa)^\theta}{Q(\theta)},\ \xi_2^1 = -\frac{cI}{b}.$$

$$\begin{aligned} V_p^2(x) =& \eta_1^2 \ln x \, x^{\beta_1} + \eta_2^2 \ln x \, x^{\beta_2} + \xi_1^2 x^\theta + \xi_2^2, \quad \text{with} \\ & \eta_1^2 = \delta_1 \frac{c(1+\kappa)^{\beta_1}}{Q'(\beta_1)},\ \eta_2^2 = \delta_2 \frac{c(1+\kappa)^{\beta_2}}{Q'(\beta_2)}, \\ & \xi_1^2 = \rho \left[\frac{c(1+\kappa)^\theta}{Q(\theta)}\right]^2,\ \xi_2^2 = -\left(\frac{c}{b}\right)^2 I. \end{aligned}$$

$$\begin{aligned} V_p^3(x) =& \eta_1^3 \ln x \, x^{\beta_1} + \eta_2^3 \ln x \, x^{\beta_2} + \eta_3^3 (\ln x)^2 \, x^{\beta_1} + \eta_4^3 (\ln x)^2 \, x^{\beta_2} + \xi_1^3 x^\theta + \xi_2^3, \quad \text{with} \\ & \eta_1^3 = \delta_1 \frac{c(1+\kappa)^{\beta_1}}{Q'(\beta_1)} \left[1 + \frac{c(1+\kappa)^{\beta_1}}{Q'(\beta_1)} \left(\ln(1+\kappa) - \frac{1}{Q'(\beta_1)}\right)\right], \\ & \eta_2^3 = \delta_2 \frac{c(1+\kappa)^{\beta_2}}{Q'(\beta_2)} \left[1 + \frac{c(1+\kappa)^{\beta_2}}{Q'(\beta_2)} \left(\ln(1+\kappa) - \frac{1}{Q'(\beta_2)}\right)\right], \\ & \eta_3^3 = \frac{\delta_1}{2} \left[\frac{c(1+\kappa)^{\beta_1}}{Q'(\beta_1)}\right]^2,\ \eta_4^3 = \frac{\delta_2}{2} \left[\frac{c(1+\kappa)^{\beta_2}}{Q'(\beta_2)}\right]^2, \\ & \xi_1^3 = \rho \left[\frac{c(1+\kappa)^\theta}{Q(\theta)}\right]^3,\ \xi_2^3 = -\left(\frac{c}{b}\right)^3 I. \end{aligned}$$

For simplicity, in the above calculations we assume that θ is not a root of the characteristic polynomial Q. This example is motivated by the relevance of this analysis in real options context. In fact, functions such that the one presented in (3.9) are frequently used in this context and describe the profit of a firm. This function is called in the literature an iso-elastic demand function (see, for instance, Nunes and Pimentel [26]).

This example also shows that a more systematic way to find the solution to the nonhomogeneous differential Equation (2.1) is quite valuable. We address this issue in the next section.

4. MAIN RESULTS

We want to find a particular solution to the Equation (2.1). The type of solution is understandable from the special case solved at the end of the previous section. However, a systematic way to obtain all the coefficients is not so easy to develop.

We start deriving a recursive expression for the particular solution of (2.1). Later, using this result, we will be able to present explicit expressions for the involved coefficients.

Theorem 4.1 (recursive)**.** *Consider the second order ODE presented in* (2.1)*, with the corresponding characteristic polynomial Q given by* (3.2)*. Then the following cases occur:*

- *If α is not a root of Q, the particular solution of* (2.1) *is*

$$y_p(x) = x^\alpha \sum_{i=0}^{n} c_i \, (\ln x)^i \,,$$

where $c_n = \frac{A}{Q(\alpha)}$, $c_{n-1} = -nA\frac{Q'(\alpha)}{Q(\alpha)^2}$ *and* $c_i = -\frac{i+1}{Q(\alpha)}\left[Q'(\alpha)c_{i+1} + (i+2)c_{i+2}\right]$ *for* $i = 0, 1, 2, ..., n-2$.

- *If α is a simple root of Q, the particular solution of* (2.1) *is*

$$y_p(x) = x^\alpha \sum_{i=0}^{n} c_i \, (\ln x)^{i+1} \,,$$

where $c_n = \frac{A}{(n+1)Q'(\alpha)}$ *and* $c_i = -\frac{i+2}{Q'(\alpha)}c_{i+1}$*, for* $i = 0, 1, 2, ..., n-1$.

- *If α is a root of Q with multiplicity two, the particular solution of* (2.1) *is*

$$y_p(x) = x^\alpha c_n \, (\ln x)^{n+2} \,,$$

where $c_n = \frac{A}{(n+1)(n+2)}$.

Proof: We start by proposing that the particular solution of Equation (2.1) is of the form $y_p(x) = x^\alpha P(x)$. Calculating first and second derivatives, we obtain

$$\begin{aligned} y_p'(x) &= x^{\alpha-1}\left[xP'(x) + \alpha P(x)\right] \\ y_p''(x) &= x^{\alpha-2}\left[x^2P''(x) + 2\alpha x P'(x) + \alpha(\alpha-1)P(x)\right], \end{aligned}$$

from where

$$x^2y_p''(x) + axy_p'(x) + by_p(x) \;=\; x^\alpha\left[x^2P''(x) + \big(Q'(\alpha)+1\big)\,xP'(x) + Q(\alpha)P(x)\right].$$

Thus $P(x)$ is such that

$$x^2P''(x) + \big(Q'(\alpha)+1\big)\,xP'(x) + Q(\alpha)P(x) = A(\ln x)^n. \tag{4.1}$$

Taking into account whether $Q(\alpha)$ is null or not, we end up with different cases, described hereafter:

1. If α is not a root of Q, then $P(x) = \sum_{i=0}^{n} c_i (\ln x)^i$, as we prove next. For that, we compute the first and second derivatives:

$$\begin{aligned} P'(x) &= \frac{1}{x}\sum_{i=1}^{n} ic_i (\ln x)^{i-1} \\ P''(x) &= \frac{1}{x^2}\left[\sum_{i=2}^{n} i(i-1)c_i(\ln x)^{i-2} - \sum_{i=1}^{n} ic_i(\ln x)^{i-1}\right]. \end{aligned}$$

Thus, $x^2P''(x) + (Q'(\alpha)+1)\, xP'(x) + Q(\alpha)P(x)$ is given by

$$\begin{aligned} &\sum_{i=0}^{n-2}\left[(i+2)(i+1)c_{i+2} + Q'(\alpha)(i+1)c_{i+1} + Q(\alpha)c_i\right](\ln x)^i \\ &+ \left[Q'(\alpha)nc_n + Q(\alpha)c_{n-1}\right](\ln x)^{n-1} + Q(\alpha)c_n(\ln x)^n. \end{aligned}$$

Therefore, (4.1) holds if $Q(\alpha)c_n = A$, $Q'(\alpha)nc_n + Q(\alpha)c_{n-1} = 0$ and $(i+2)(i+1)c_{i+2} + Q'(\alpha)(i+1)c_{i+1} + Q(\alpha)c_i = 0$, for $i = 0, 1, ..., n-2$, which leads to the result.

2. If α is a root of Q with multiplicity one, then $P(x) = \sum_{i=0}^{n} c_i (\ln x)^{i+1}$. In fact, calculating first and second derivatives, we obtain

$$\begin{aligned} P'(x) &= \frac{1}{x}\sum_{i=0}^{n}(i+1)c_i (\ln x)^i, \\ P''(x) &= \frac{1}{x^2}\left[\sum_{i=1}^{n}(i+1)ic_i(\ln x)^{i-1} - \sum_{i=0}^{n}(i+1)c_i(\ln x)^i\right]. \end{aligned}$$

Given that $Q(\alpha) = 0$, then $x^2P''(x) + (Q'(\alpha)+1)\, tP'(x) + Q(\alpha)P(x)$ is given by $\sum_{i=0}^{n-1}\left[(i+2)c_{i+1} + Q'(\alpha)c_i\right](\ln x)^i + Q'(\alpha)(n+1)c_n(\ln x)^n$.
Assuming that α has multiplicity one we have $Q'(\alpha) \neq 0$. Thus, in order to have (4.1), we need to set that $Q'(\alpha)(n+1)c_n = A$ and $(i+2)c_{i+1} + Q'(\alpha)c_i = 0$, for $i = 0, 1, ..., n-1$, and the result follows.

3. If α is a root of Q with multiplicity two, then $P(x) = c_n (\ln x)^{n+2}$ as

$$\begin{aligned} P'(x) &= \frac{1}{x}c_n(n+2)(\ln x)^{n+1}, \\ P''(x) &= \frac{1}{x^2}c_n(n+2)\left[(n+1)(\ln x)^n - (\ln x)^{n+1}\right]. \end{aligned}$$

Since $Q(\alpha) = 0$ and $Q'(\alpha) = 0$, then $x^2P''(x) + (Q'(\alpha)+1)\, tP'(x) + Q(\alpha)P(x)$ is given by $c_n(n+2)(n+1)(\ln x)^n$. Finally, in order to have (4.1) we conclude that $c_n = \frac{A}{(n+1)(n+2)}$. □

This theorem is useful in two ways: first it provides a way to compute (recursively) the particular solution of the differential equation (2.1). Second, it provides the tool to derive explicit expressions for the involved coefficients. In the following theorem we present such result.

Theorem 4.2 (non-recursive)**.** *Consider the second order ODE presented in (2.1), with the corresponding characteristic polynomial Q given by (3.2).*

- *If α is not a root of Q, the particular solution of (2.1) is given by $y_p(x) = x^\alpha \sum_{i=0}^{n} c_i (\ln x)^i$, with*

$$c_i = (-1)^{n-i} \frac{n!}{i!} \frac{A}{Q(\alpha)^{n-i+1}} \sum_{\substack{j=0 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}} (-1)^j \binom{n-i-j}{j} Q'(\alpha)^{n-i-2j}\, Q(\alpha)^j, \tag{4.2}$$

for $i = 0, 1, 2, ..., n$, where $\binom{k}{r} = \frac{k!}{r!(k-r)!}$, with $k \geq r \geq 0$.

- *If α is a simple root of Q, the particular solution of (2.1) is $y_p(x) = x^\alpha \sum_{i=0}^{n} c_i (\ln x)^{i+1}$, with*

$$c_i = (-1)^{n-i} \frac{n!}{(i+1)!} \frac{A}{Q'(\alpha)^{n-i+1}}, \text{ for } i = 0, 1, 2, ..., n. \tag{4.3}$$

- *If α is a root of Q with multiplicity two, the particular solution of (2.1) is $y_p(x) = x^\alpha c_n (\ln x)^{n+2}$, with $c_n = \frac{A}{(n+1)(n+2)}$.*

Proof: The last case coincides with the one presented in Theorem 4.1. For the other two cases, we use backwards mathematical induction to prove it, taking advantage of the recursive solutions presented in Theorem 4.1.

1. If α is not a root of Q, we already know that, the particular solution is of the form $y_p(x) = x^\alpha \sum_{i=0}^{n} c_i (\ln x)^i$, where $c_n = \frac{A}{Q(\alpha)}$, $c_{n-1} = -nA\frac{Q'(\alpha)}{Q(\alpha)^2}$ and $c_i = -\frac{i+1}{Q(\alpha)} [Q'(\alpha)c_{i+1} + (i+2)c_{i+2}]$ for $i = 0, 1, 2, ..., n-2$. We want to prove that, for $i = 0, 1, 2, ..., n$, the coefficients c_i can be written in the general form presented in (4.2).
   Using backwards mathematical induction we have two base cases to be verified, c_n and c_{n-1}, which we know from Theorem 4.1 that are $\frac{A}{Q(\alpha)}$ and $-nA\frac{Q'(\alpha)}{Q(\alpha)^2}$, respectively. Taking into account (4.2), we have

$$\begin{aligned} c_n &= (-1)^0 \frac{n!}{n!} \frac{A}{Q(\alpha)} (-1)^0 \binom{0}{0} Q'(\alpha)^0\, Q(\alpha)^0 = \frac{A}{Q(\alpha)}, \\ c_{n-1} &= (-1) \frac{n!}{(n-1)!} \frac{A}{Q(\alpha)^2} (-1)^0 \binom{1}{0} Q'(\alpha)^1\, Q(\alpha)^0 = -nA\frac{Q'(\alpha)}{Q(\alpha)^2}, \end{aligned}$$

   which means that the base cases are verified. For the inductive step, we assume that, for $i = 0, 1, 2, ..., n-2$, c_{i+1} and c_{i+2} are given by (4.2), and we want to prove that c_i is also given by (4.2).
   From Theorem 4.1, we know that $c_i = -\frac{i+1}{Q(\alpha)} [Q'(\alpha)c_{i+1} + (i+2)c_{i+2}]$ for $i = 0, 1, 2, ..., n-2$. Plugging the expressions of c_{i+1} and c_{i+2}, which are defined by (4.2), in the expression of c_i we obtain

$$\begin{aligned} -\frac{i+1}{Q(\alpha)} \Bigg[& Q'(\alpha)(-1)^{n-i-1} \frac{n!}{(i+1)!} \frac{A}{Q(\alpha)^{n-i}} \sum_{\substack{j=0 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-\frac{1}{2}} (-1)^j \binom{n-i-j-1}{j} Q'(\alpha)^{n-i-2j-1} Q(\alpha)^j \\ & + (i+2)(-1)^{n-i-2} \frac{n!}{(i+2)!} \frac{A}{Q(\alpha)^{n-i-1}} \sum_{\substack{j=0 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-1} (-1)^j \binom{n-i-j-2}{j} Q'(\alpha)^{n-i-2j-2} Q(\alpha)^j \Bigg]. \end{aligned}$$

Rearranging the terms and changing the variable in the second sum, we get

$$(-1)^{n-i}\frac{n!}{i!}\frac{A}{Q(\alpha)^{n-i+1}}\left[\sum_{\substack{j=0\\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-\frac{1}{2}}(-1)^j\binom{n-i-j-1}{j}Q'(\alpha)^{n-i-2j}\,Q(\alpha)^j\right.$$
$$\left.+\sum_{\substack{j=1\\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}}(-1)^j\binom{n-i-j-1}{j-1}Q'(\alpha)^{n-i-2j}\,Q(\alpha)^j\right].$$

Joining the two sums and taking into account some permutation's properties, we end up with the following expression:

$$(-1)^{n-i}\frac{n!}{i!}\frac{A}{Q(\alpha)^{n-i+1}}\left[\sum_{\substack{j=1\\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-\frac{1}{2}}(-1)^j\binom{n-i-j}{j}Q'(\alpha)^{n-i-2j}\,Q(\alpha)^j\right.$$
$$\left.+\,Q'(\alpha)^{n-i}+(-1)^{\frac{n-i}{2}}\,Q(\alpha)^{\frac{n-i}{2}}\chi_{\{n-i\text{ is even}\}}\right].$$

Finally, we conclude that

$$c_i\;=\;(-1)^{n-i}\frac{n!}{i!}\frac{A}{Q(\alpha)^{n-i+1}}\sum_{\substack{j=0\\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}}(-1)^j\binom{n-i-j}{j}Q'(\alpha)^{n-i-2j}\,Q(\alpha)^j,$$

which coincides with the expression given by (4.2). Thus the proof for the first case is finished.

2. If α is a root of Q with multiplicity one, as we proved before, the particular solution is of the form $y_p(x)=x^{\alpha}\sum_{i=0}^{n}c_i\,(\ln x)^{i+1}$, where $c_n=\frac{A}{(n+1)Q'(\alpha)}$ and $c_i=-\frac{i+2}{Q'(\alpha)}c_{i+1}$, for $i=0,1,2,...,n-1$. We want to prove that we can write the coefficients c_i in the general way presented in (4.3).
As before, we use backwards mathematical induction. Starting with the base case and taking into account (4.3), we have

$$c_n=(-1)^0\,\frac{n!}{(n+1)!}\,\frac{A}{Q'(\alpha)}=\frac{A}{(n+1)Q'(\alpha)},$$

which coincides with the expression given by Theorem 4.1. Thus, the base case is verified. To prove the induction step, for $i=0,1,2,...,n-1$, we assume that c_{i+1} is given by (4.3) and we want to prove that c_i is also given by (4.3).
From Theorem 4.1, we know that $c_i=-\frac{i+2}{Q'(\alpha)}c_{i+1}$, for $i=0,1,2,...,n-1$. Plugging in c_i the expression of c_{i+1}, which is given by (4.3), we obtain

$$c_i=-\frac{i+2}{Q'(\alpha)}\,(-1)^{n-i-1}\,\frac{n!}{(i+2)!}\,\frac{A}{Q'(\alpha)^{n-i}}=(-1)^{n-i}\,\frac{n!}{(i+1)!}\,\frac{A}{Q'(\alpha)^{n-i+1}},$$

and therefore the induction step is proved. With this we conclude the proof. □

A special case of the previous theorem is when $n = 0$. In this case the Equation (2.1) is simply

$$x^2 y''(x) + axy'(x) + by(x) = Ax^\alpha .$$

Using the results proved before, the corresponding particular solution is given by

$$y_p(x) = \varphi \, x^\alpha \, (\ln x)^r \, ,$$

where $\varphi = \frac{A}{Q^{(r)}(\alpha)}$[6], with r[7] being the multiplicity of α as a root of Q.

In the following corollary, we use the results presented in Theorem 4.2 for the case that the non-homogeneous part of the differential equation is a sum of power and log functions (as it is the case, for example, of (3.8)).

Corollary 4.1. *Consider the following second order differential equation:*

$$x^2 y''(x) + axy'(x) + by(x) = \sum_{k=1}^{m} A_k x^{\alpha_k} \, (\ln x)^{n_k} \, , \tag{4.4}$$

with $x > 0$, $a, b \in \mathbb{R}$, $\alpha_k, A_k \in \mathbb{R} \setminus \{0\}$ *and* $n_k \in \mathbb{N}_0$, *for* $k = 1, 2, ..., m$, *with* $m \in \mathbb{N}$. *Then the particular solution of* (4.4) *is of the form* $y_p(x) = \sum_{k=1}^m y_{p_k}(x)$[8], *where* $y_{p_k}(x)$ *is the solution of the equation*

$$x^2 y_k''(x) + axy_k'(x) + by_k(x) = A_k x^{\alpha_k} \, (\ln x)^{n_k} \, ,$$

which is presented in Theorem 4.2.

Proof: The result follows from the superposition principle. □

5. CONCLUSIONS

In this paper we provide a solution to a differential–difference equation that can be found, for instance, when one studies an investment problem with the underlying following a jump–diffusion process. This problem is particularly important from the point of view of the application, as nowadays the prices and demand are often subject to external shocks that cause a disruptive behavior on the state variables. Analytical solutions or quasi-analytical solutions are scarce or even non-existent. Our results contribute to the state of the art in this area.

As our results show, the solution to the differential–difference equation is a piecewise function, where each branch depends on the next one. Therefore, to find the expression for each branch a non-homogeneous ODE needs to be solved. In this paper we also provide the expression for each coefficient involved in the particular solution of this family of ODEs.

[6] $Q^{(r)}(\alpha)$ is the derivative of order r of Q w.r.t. α. In particular, if $r = 0$ we consider that $Q^{(r)}(\alpha)$ is exactly $Q(\alpha)$.

[7] r can take the values $0, 1$ or 2. We consider $r = 0$ when α is not a root of Q.

[8] Note that y_p has at least m parcels and at most $m + \sum_{k=1}^m n_k$ parcels. When $\alpha_1 = \alpha_2 = ... = \alpha_m$ are roots of Q all with multiplicity two, y_p has m parcels. Oppositely, when none of the α_k (with $k = 1, 2, ..., m$) has multiplicity two, y_p has $m + \sum_{k=1}^m n_k$ parcels.

As future work, we want to apply these results to solve the original optimal stopping problem. We highlight that this is a challenging question, as in order to find the optimal value function, we need to use enough conditions to define all the unknown parameters of the solution. Indeed, the expressions that we provide in this paper define classes of solutions, and only considering the boundary and initial conditions we are able to derive the solution.

ACKNOWLEDGMENTS

Most of Rita Pimentel's research was supported by her Ph.D. research grant from FCT with reference number SFRH/BD/97259/2013. The final part of the work was carried out during the tenure of an ERCIM 'Alain Bensoussan' Fellowship Programme.

Most of Cláudia Nunes's research was supported by the FCT research grants FARO_PTDC/EGE-ECO/30535/2017 and UID/MULTI/04621/2019. She also acknowledge Invest ExL Project No.: 268093/E20, funded by the Norwegian research agency.

The authors also thank to Carlos Oliveira for the discussions and all invaluable suggestions.

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