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GENERALIZED MATRIX t DISTRIBUTION BASED ON NEW MATRIX GAMMA DISTRIBUTION

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Abstract:

- In this paper, a generalized matrix variate gamma distribution, which includes a trace function in the kernel of the density, is introduced. Some important statistical properties including Laplace transform, distributions of functions, expected value of the determinant and expected value of zonal polynomial of the generalized gamma matrix are derived. Further, by using the distribution of the inverse of this newly defined generalized gamma matrix as the prior for the characteristic matrix of a matrix variate normal distribution, a new generalized matrix t type family of distributions is generated. Some important statistical characteristics of this family are also exhibited.

Keywords:

- *Bayes estimator; generalized matrix gamma distribution; generalized matrix t distribution; zonal polynomial.*

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- 62E05, 62H99.

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1. INTRODUCTION

In 1964, Lukacs and Laha defined the matrix variate gamma (MG) distribution. In multivariate statistical analysis, the MG distribution has been the subject of considerable interest, study, and applications for many years. For example, the Wishart distribution, which is the distribution of the sample variance covariance matrix when sampling from a multivariate normal distribution, is a special case of the MG distribution. Applications of the MG distribution have included: damping modeling (Adhikari [1]); models for stochastic upscaling for inelastic material behavior from limited experimental data (Das and Ghanem [7], [8]); models for fusion yield [15]; models for uncertainty quantification (Pascual and Adhikari [31]); characterizing the distribution of anisotropic micro-structural environments with diffusion-weighted imaging (Scherrer *et al.* [32]); models for magnetic tractography (Chamberland *et al.* [4]); models for diffusion compartment imaging (Scherrer *et al.* [33]); models for image classification (Luo *et al.* [24]); models for accurate signal reconstruction (Jian *et al.* [22], Bates *et al.* [3]). Two recent applications of the Wishart distribution can be found in Arashi *et al.* [2] and Ferreira *et al.* [13].

However, generalizations of the MG distribution have been neglected and there is no account on this matter in the literature. The only extension that we are aware of is the inverted matrix variate gamma distribution due to Iranmanesh *et al.* [17]: if $\boldsymbol{X}$ has the MG distribution then $\boldsymbol{X}^{-1}$ has the inverted matrix variate gamma distribution. A generalization of the MG distribution must contain the MG distribution as a particular case. See also Iranmanesh *et al.* [18] and references there in for more details.

The goal of this paper is to give the first generalization to the MG distribution, where its kernel includes zonal polynomials (Takemura [34]). The generalization proposed has two shape parameters. One of the shape parameters acts on the determinant of the data while the other acts on the trace of the data. The MG distribution has only one shape parameter acting on the determinant of the data. The proposed generalization can be more flexible for data modeling:

- **i**) if both trace and determinant are significant (that is, the empirical distribution of the data has significant patterns involving both the trace and determinant);
- **ii**) if trace is significant but determinant is not (that is, the empirical distribution of the data has significant patterns involving only the trace);
- **iii**) if trace is more significant than determinant is (that is, the empirical distribution of the data has more significant patterns involving the trace).

For our purpose, we first provide the reader with some preliminary definitions and lemmas. Most of the following definitions and results can be found in Gupta and Nagar [14], Muirhead [27], and Mathai [26].

2. PRELIMINARIES

In this section we state certain well known definitions and results. These results will be used in subsequent sections.

Let $\boldsymbol{A} = (a_{ij})$ be a $p \times p$ matrix. Then, $\boldsymbol{A}'$ denotes the transpose of $\boldsymbol{A}$; $\operatorname{tr}(\boldsymbol{A}) = a_{11} + \cdots + a_{pp}$; $\operatorname{etr}(\boldsymbol{A}) = \exp(\operatorname{tr}(\boldsymbol{A}))$; $\det(\boldsymbol{A})$ = determinant of $\boldsymbol{A}$; norm of $\boldsymbol{A} = \|\boldsymbol{A}\|$ = maximum of absolute values of eigenvalues of the matrix $\boldsymbol{A}$; $\boldsymbol{A}^{1/2}$ denotes a symmetric positive definite square root of $\boldsymbol{A}$; $\boldsymbol{A} > \boldsymbol{0}$ means that $\boldsymbol{A}$ is symmetric positive definite and $\boldsymbol{0} < \boldsymbol{A} < \boldsymbol{I}_p$ means that the matrices $\boldsymbol{A}$ and $\boldsymbol{I}_p - \boldsymbol{A}$ are symmetric positive definite. The multivariate gamma function which is frequently used in multivariate statistical analysis is defined by

$$\begin{aligned}\Gamma_p(a) &= \int_{\boldsymbol{X}>\boldsymbol{0}} \operatorname{etr}(-\boldsymbol{X}) \det(\boldsymbol{X})^{a-(p+1)/2}\, d\boldsymbol{X} \\ &= \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(a - \frac{i-1}{2}\right), \qquad \operatorname{Re}(a) > \frac{p-1}{2}\,. \end{aligned} \tag{2.1}$$

Let $C_\kappa(\boldsymbol{X})$ be the zonal polynomial of $p \times p$ complex symmetric matrix $\boldsymbol{X}$ corresponding to the ordered partition $\kappa = (k_1, ..., k_p)\,, k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$ and $\sum_\kappa$ denotes summation over all partitions κ of k. The generalized hypergeometric coefficient $(a)_\kappa$ used above is defined by

$$(a)_\kappa = \prod_{i=1}^{p} \left(a - \frac{i-1}{2}\right)_{k_i},$$

where $(a)_r = a(a+1)\cdots(a+r-1)$, $r = 1, 2, ...$ with $(a)_0 = 1$.

Lemma 2.1. *Let $\boldsymbol{Z}$ be a complex symmetric $p \times p$ matrix with $\operatorname{Re}(\boldsymbol{Z}) > \boldsymbol{0}$, and let $\boldsymbol{Y}$ be a symmetric $p \times p$ matrix. Then, for $\operatorname{Re}(a) > (p-1)/2$, we have*

$$\int_{\boldsymbol{X}>\boldsymbol{0}} \operatorname{etr}(-\boldsymbol{X}\boldsymbol{Z}) (\det \boldsymbol{X})^{a-(p+1)/2} C_\kappa(\boldsymbol{X}\boldsymbol{Y})\, d\boldsymbol{X} = (a)_\kappa\, \Gamma_p(a)\, (\det \boldsymbol{Z})^{-a} C_\kappa\left(\boldsymbol{Y}\boldsymbol{Z}^{-1}\right). \tag{2.2}$$

Lemma 2.2. *Let $\boldsymbol{Z}$ be a complex symmetric $p \times p$ matrix with $\operatorname{Re}(\boldsymbol{Z}) > \boldsymbol{0}$, and let $\boldsymbol{Y}$ be a symmetric $p \times p$ matrix. Then, for $\operatorname{Re} > (p-1)/2$, we have*

$$\int_{\boldsymbol{X}>\boldsymbol{0}} \operatorname{etr}(-\boldsymbol{X}\boldsymbol{Z}) (\det \boldsymbol{X})^{a-(p+1)/2} \left[\operatorname{tr}(\boldsymbol{X}\boldsymbol{Y})\right]^k d\boldsymbol{X} = \Gamma_p(a)\, (\det \boldsymbol{Z})^{-a} \sum_\kappa (a)_\kappa C_\kappa(\boldsymbol{Y}\boldsymbol{Z}^{-1})\,. \tag{2.3}$$

For $\boldsymbol{Z} = \boldsymbol{Y}$ in (2.3), we get

$$\begin{aligned}\int_{\boldsymbol{X}>\boldsymbol{0}} \operatorname{etr}(-\boldsymbol{X}\boldsymbol{Y}) (\det \boldsymbol{X})^{a-(p+1)/2} \left[\operatorname{tr}(\boldsymbol{X}\boldsymbol{Y})\right]^k d\boldsymbol{X} &= \Gamma_p(a)\, (\det \boldsymbol{Y})^{-a} \sum_\kappa (a)_\kappa C_\kappa(\boldsymbol{I}_p) \\ &= \Gamma_p(a)\, (ap)_k\, (\det \boldsymbol{Y})^{-a}\,. \end{aligned} \tag{2.4}$$

The above result was derived by Khatri [23].

Davis [9, 10] introduced a class of polynomials $C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{Y})$ of $p\times p$ symmetric matrix arguments $\boldsymbol{X}$ and $\boldsymbol{Y}$, which are invariant under the transformation $(\boldsymbol{X},\boldsymbol{Y})\to(\boldsymbol{HXH}',\boldsymbol{HYH}')$, $\boldsymbol{H}\in O(p)$. For properties and applications of invariant polynomials we refer to Davis [9, 10], Chikuse [5] and Nagar and Gupta [28]. Let κ, λ, ϕ and ρ be ordered partitions of non-negative integers k, ℓ, $f=k+\ell$ and r, respectively, into not more than p parts. Then

$$C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{X})=\theta_\phi^{\kappa,\lambda}C_\phi(\boldsymbol{X}),\qquad \theta_\phi^{\kappa,\lambda}=\frac{C_\phi^{\kappa,\lambda}(\boldsymbol{I}_p,\boldsymbol{I}_p)}{C_\phi(\boldsymbol{I}_p)}, \tag{2.5}$$

$$C_\phi^{\kappa,\lambda}(\boldsymbol{X},I_p)=\theta_\phi^{\kappa,\lambda}\frac{C_\phi(\boldsymbol{I}_p)\,C_\kappa(\boldsymbol{X})}{C_\kappa(\boldsymbol{I}_p)},\qquad C_\phi^{\kappa,\lambda}(I_p,\boldsymbol{Y})=\theta_\phi^{\kappa,\lambda}\frac{C_\phi(\boldsymbol{I}_p)\,C_\lambda(\boldsymbol{X})}{C_\lambda(\boldsymbol{I}_p)}, \tag{2.6}$$

$$C_\kappa^{\kappa,0}(\boldsymbol{X},\boldsymbol{Y})\equiv C_\kappa(\boldsymbol{X}),\qquad C_\lambda^{0,\lambda}(\boldsymbol{X},\boldsymbol{Y})\equiv C_\lambda(\boldsymbol{Y})$$

and

$$C_\kappa(\boldsymbol{X})\,C_\lambda(\boldsymbol{Y})=\sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{Y}), \tag{2.7}$$

where $\phi\in\kappa\cdot\lambda$ signifies that irreducible representation of $Gl(p,R)$ indexed by 2ϕ, occurs in the decomposition of the Kronecker product $2\kappa\otimes2\lambda$ of the irreducible representations indexed by 2κ and 2λ. Further,

$$\begin{aligned}\int_{\boldsymbol{R}>\boldsymbol{0}}\operatorname{etr}(-\boldsymbol{CR})\,\det(\boldsymbol{R})^{t-(p+1)/2}\,C_\phi^{\kappa,\lambda}(\boldsymbol{ARA}',\boldsymbol{BRB}')\,d\boldsymbol{R}&=\\ =\Gamma_p(t,\phi)\,\det(\boldsymbol{C})^{-t}\,C_\phi^{\kappa,\lambda}&(\boldsymbol{AC}^{-1}\boldsymbol{A}',\boldsymbol{BC}^{-1}\boldsymbol{B}'),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\int_{\boldsymbol{0}}^{\boldsymbol{I}_p}\det(\boldsymbol{R})^{t-(p+1)/2}\,\det(\boldsymbol{I}_p-\boldsymbol{R})^{u-(p+1)/2}\,C_\phi^{\kappa,\lambda}(\boldsymbol{R},\boldsymbol{I}_p-\boldsymbol{R})\,d\boldsymbol{R}&=\\ =\frac{\Gamma_p(t,\kappa)\,\Gamma_p(u,\lambda)}{\Gamma_p(t+u,\phi)}\,\theta_\phi^{\kappa,\lambda}C_\phi(\boldsymbol{I}_p)&\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\int_{\boldsymbol{0}}^{\boldsymbol{I}_p}\det(\boldsymbol{R})^{t-(p+1)/2}\,\det(\boldsymbol{I}_p-\boldsymbol{R})^{u-(p+1)/2}\,C_\phi^{\kappa,\lambda}(\boldsymbol{AR},\boldsymbol{BR})\,d\boldsymbol{R}&=\\ =\frac{\Gamma_p(t,\phi)\,\Gamma_p(u)}{\Gamma_p(t+u,\phi)}\,C_\phi^{\kappa,\lambda}(\boldsymbol{A},\boldsymbol{B})&.\end{aligned} \tag{2.10}$$

In expressions (2.8), (2.9) and (2.10), $\Gamma_p(a,\rho)$ is defined by

$$\Gamma_p(a,\rho)=(a)_\rho\,\Gamma_p(a). \tag{2.11}$$

Note that $\Gamma_p(a,0)=\Gamma_p(a)$, which is the multivariate gamma function.

Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $p \times p$ symmetric matrices. Then

$$\int_{\boldsymbol{H}\in O(p)} C_\phi^{\kappa,\lambda}\big(\boldsymbol{A}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H}, \boldsymbol{B}\boldsymbol{H}'\boldsymbol{Y}\boldsymbol{H}\big)\,[d\boldsymbol{H}] \;=\; \frac{C_\phi^{\kappa,\lambda}(\boldsymbol{A},\boldsymbol{B})\, C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{Y})}{\theta_\phi^{\kappa,\lambda}\, C_\phi(\boldsymbol{I}_p)}\,, \tag{2.12}$$

where $[d\boldsymbol{H}]$ is the unit invariant Haar measure. The above result is a generalization of Davis [10, Eq. 5.4] and is due to Díaz-García [11]. Finally, using (2.9) and (2.12), it is straightforward to see that

$$\begin{aligned} &\int_{\boldsymbol{0}}^{\boldsymbol{I}_p} \det(\boldsymbol{R})^{t-(p+1)/2} \det(\boldsymbol{I}_p-\boldsymbol{R})^{u-(p+1)/2}\, C_\phi^{\kappa,\lambda}\big(\boldsymbol{A}\boldsymbol{R}, \boldsymbol{B}\,(\boldsymbol{I}_p-\boldsymbol{R})\big)\, d\boldsymbol{R} \;= \\ &\qquad\qquad = \frac{\Gamma_p(t,\kappa)\,\Gamma_p(u,\lambda)}{\Gamma_p(t+u,\phi)}\, C_\phi^{\kappa,\lambda}(\boldsymbol{A},\boldsymbol{B})\,. \end{aligned} \tag{2.13}$$

Definition 2.1. The $n \times p$ random matrix $\boldsymbol{X}$ is said to have a matrix variate normal distribution with $n \times p$ mean matrix $\boldsymbol{M}$ and $np \times np$ covariance matrix $\boldsymbol{\Omega}\otimes\boldsymbol{\Sigma}$, denoted by $\boldsymbol{X} \sim N_{n,p}\,(\boldsymbol{M}, \boldsymbol{\Omega}\otimes\boldsymbol{\Sigma})$, if its probability density function (p.d.f) is given by (Gupta and Nagar [14])

$$(2\pi)^{-np/2} \det(\boldsymbol{\Omega})^{-p/2} \det(\boldsymbol{\Sigma})^{-n/2} \exp\left\{-\frac{1}{2}\,\mathrm{tr}\big[\boldsymbol{\Omega}^{-1}(\boldsymbol{X}-\boldsymbol{M})\,\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{M})'\big]\right\}, \qquad \boldsymbol{X}\in\mathbb{R}^{n\times p}, \quad \boldsymbol{M}\in\mathbb{R}^{n\times p}\,,$$

where $\boldsymbol{\Sigma}(p\times p) > \boldsymbol{0}$ and $\boldsymbol{\Omega}(n\times n) > \boldsymbol{0}$.

If $\boldsymbol{X} \sim N_{n,p}\,(\boldsymbol{M}, \boldsymbol{\Omega}\otimes\boldsymbol{\Sigma})$, then the characteristic function of $\boldsymbol{X}$ is

$$\begin{aligned} \phi_{\boldsymbol{X}}(\boldsymbol{Z}) &= E\big[\exp\big(\mathrm{tr}\,(\iota\,\boldsymbol{Z}'\boldsymbol{X})\big)\big] \\ &= \exp\left[\mathrm{tr}\left(\iota\,\boldsymbol{Z}'\boldsymbol{M} - \frac{1}{2}\,\boldsymbol{Z}'\boldsymbol{\Omega}\boldsymbol{Z}\boldsymbol{\Sigma}\right)\right], \qquad \boldsymbol{Z}\in\mathbb{R}^{n\times p}\,, \quad \iota=\sqrt{-1}\,. \end{aligned}$$

The present paper has been organized in the following sections. In Section 3, a new generalized matrix gamma (GMG) distribution has been defined. Some important properties of this newly defined distribution are given in Section 4. In Section 5, using the conditioning approach for the matrix variate normal distribution, a new matrix t type family of distributions is introduced. Some important statistical characteristics of this family are studied in Section 6. A Bayesian application is given in Section 7. The paper is concluded in Section 8.

3. GENERALIZED MATRIX GAMMA DISTRIBUTION

Recently, Nagar *et al.* [30] defined a generalized matrix variate gamma distribution by generalizing the multivariate gamma function. We also refer to Nagar *et al.* [29] for further generalizations. In this paper, by incorporating an additional factor in the p.d.f, we give a generalization of the matrix variate gamma distribution (Das and Dey [6], Iranmanesh *et al.* [17]).

In the following we provide the reader with the definition of the generalized matrix variate gamma distribution.

Definition 3.1. A random symmetric matrix $\boldsymbol{X}$ of order p is said to have a generalized matrix gamma (GMG) distribution with parameters α, β, k, $\boldsymbol{\Sigma}$ and $\boldsymbol{U}$, denoted by $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$, if its p.d.f is given by

$$C(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U}) \operatorname{etr}\left(-\frac{1}{\beta} \boldsymbol{\Sigma}^{-1} \boldsymbol{X}\right) \det(\boldsymbol{X})^{\alpha-(p+1)/2} \left[\operatorname{tr}(\boldsymbol{X}\boldsymbol{U})\right]^k, \qquad \boldsymbol{X} > \boldsymbol{0}, \tag{3.1}$$

where $\alpha > (p-1)/2$, $\beta > 0$, $\boldsymbol{\Sigma} > \boldsymbol{0}$, $\boldsymbol{U} > \boldsymbol{0}$, $k \in \mathbb{N}_0$ and $C(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$ is the normalizing constant.

By integrating the p.d.f of $\boldsymbol{X}$ over its support set, the normalizing constant $C(\alpha,\beta,k,\boldsymbol{\Sigma},\boldsymbol{U})$ can be evaluated as

$$\begin{aligned} \left[C(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})\right]^{-1} &= \int_{\boldsymbol{X}>\boldsymbol{0}} \operatorname{etr}\left(-\frac{1}{\beta} \boldsymbol{\Sigma}^{-1} \boldsymbol{X}\right) \det(\boldsymbol{X})^{\alpha-(p+1)/2} \left[\operatorname{tr}(\boldsymbol{X}\boldsymbol{U})\right]^k d\boldsymbol{X} \\ &= \beta^{p\alpha+k}\, \Gamma_p(\alpha) \det(\boldsymbol{\Sigma})^{\alpha} \sum_{\kappa} (\alpha)_{\kappa} C_{\kappa}(\boldsymbol{U}\boldsymbol{\Sigma}), \end{aligned} \tag{3.2}$$

where the last line has been obtained by using (2.3).

The distribution given by the p.d.f (3.1) is a generalization of the matrix variate gamma distribution (Das and Dey [6], Iranmanesh *et al.* [17]). For $\boldsymbol{U} = \boldsymbol{\Sigma}^{-1}$, the p.d.f in (3.1) simplifies to

$$\frac{\operatorname{etr}\left(-\boldsymbol{\Sigma}^{-1}\boldsymbol{X}/\beta\right) \det(\boldsymbol{X})^{\alpha-(p+1)/2} \left[\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\right)\right]^k}{\beta^{\alpha p+k} (\alpha p)_k\, \Gamma_p(\alpha) \det(\boldsymbol{\Sigma})^{\alpha}}, \qquad \boldsymbol{X} > \boldsymbol{0}. \tag{3.3}$$

Further, for $\boldsymbol{U} = \boldsymbol{0}$ or $k = 0$ the p.d.f (3.1) reduces to the matrix variate gamma p.d.f given by

$$\frac{\operatorname{etr}\left(-\boldsymbol{\Sigma}^{-1}\boldsymbol{X}/\beta\right) \det(\boldsymbol{X})^{\alpha-(p+1)/2}}{\beta^{\alpha p}\, \Gamma_p(\alpha) \det(\boldsymbol{\Sigma})^{\alpha}}, \qquad \boldsymbol{X} > \boldsymbol{0}. \tag{3.4}$$

By suitably choosing β we can derive a number of special cases of (3.3). If we choose $\alpha = n/2$ and $\beta = 2$, then $\boldsymbol{X}$ has a generalized Wishart distribution with p.d.f

$$\frac{\operatorname{etr}\left(-\boldsymbol{\Sigma}^{-1}\boldsymbol{X}/2\right) \det(\boldsymbol{X})^{n/2-(p+1)/2} \left[\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\right)\right]^k}{2^{np/2+k}\, \Gamma_p(n/2)\, (np/2)_k \det(\boldsymbol{\Sigma})^{n/2}}, \qquad \boldsymbol{X} > \boldsymbol{0}. \tag{3.5}$$

Note that n is a positive integer, generally considered as the sample size. If we choose $\boldsymbol{\Sigma} = \boldsymbol{I}_p$, $\beta = 2$ and $p = 1$ in (3.3), then the scalar variable X follows a chi-square distribution with $n + 2k$ degrees of freedom. Further, if we take $p = 1$ and $\beta = 1$ in (3.3), then the scalar variable X follows a univariate gamma distribution with shape parameter $\alpha + k$ and scale parameter σ. Finally, for $\boldsymbol{\Sigma} = \boldsymbol{I}_p$ and $p = 1$, the scalar variable X follows a univariate gamma distribution with shape parameter $\alpha + k$ and scale parameter β.

Definition 3.2. If $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$ then $\boldsymbol{X}^{-1}$ is said to have an inverted generalized matrix gamma (IGMG) distribution with parameters α, β, k, $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{U}$, denoted by $\boldsymbol{X}^{-1} \sim \mathrm{IGMG}_p\left(\alpha, \beta, k, \boldsymbol{\Sigma}^{-1}, \boldsymbol{U}\right)$.

In the following theorem, the p.d.f of the IGMG distribution is derived.

Proposition 3.1. *Let* $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$. *Then,* $\boldsymbol{Y} = \boldsymbol{X}^{-1} \sim \mathrm{IGMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}^{-1}, \boldsymbol{U})$ *has the p.d.f given by*

$$C\big(\alpha, \beta, k, \boldsymbol{\Sigma}^{-1}, \boldsymbol{U}\big)\ \mathrm{etr}\bigg(-\frac{1}{\beta}\,\boldsymbol{\Sigma}\boldsymbol{Y}^{-1}\bigg)\,\det(\boldsymbol{Y})^{-\alpha-(p+1)/2}\,\big[\mathrm{tr}\big(\boldsymbol{Y}^{-1}\boldsymbol{U}\big)\big]^{k}, \qquad \boldsymbol{Y} > \boldsymbol{0}, \tag{3.6}$$

where $\alpha > (p-1)/2$, $\beta > 0$, $\boldsymbol{\Sigma} > \boldsymbol{0}$, $\boldsymbol{U} > \boldsymbol{0}$, $k \in \mathbb{N}_0$ *and* $C\left(\alpha, \beta, k, \boldsymbol{\Sigma}^{-1}, \boldsymbol{U}\right)$ *is the normalizing constant.*

Proof: The proof follows from the fact that the Jacobian of the transformation $\boldsymbol{Y} = \boldsymbol{X}^{-1}$ is given by $J\left(\boldsymbol{X} \to \boldsymbol{Y}\right) = \det(\boldsymbol{Y})^{-(p+1)}$. □

By taking $\boldsymbol{U} = \boldsymbol{\Sigma}$, $\alpha = n/2$ and $\beta = 2$ in (3.6), the inverted generalized Wishart p.d.f can be obtained as

$$\frac{\mathrm{etr}\big(-\boldsymbol{\Sigma}\boldsymbol{Y}^{-1}/2\big)\,\det(\boldsymbol{Y})^{-n/2-(p+1)/2}\,\big[\mathrm{tr}\big(\boldsymbol{\Sigma}\boldsymbol{Y}^{-1}\big)\big]^{k}}{2^{np/2+k}\,\Gamma_p(n/2)\,(np/2)_k\,\det(\boldsymbol{\Sigma})^{-n/2}}, \qquad \boldsymbol{Y} > \boldsymbol{0}. \tag{3.7}$$

4. PROPERTIES OF GMG AND IGMG DISTRIBUTIONS

In this section, various properties of the GMG and IGMG distributions are derived.

Proposition 4.1. *Let* $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$. *Then, the Laplace transform of* $\boldsymbol{X}$ *is*

$$\varphi_{\boldsymbol{X}}(\boldsymbol{T}) \;=\; \det(\boldsymbol{I}_p + \beta\boldsymbol{\Sigma}\boldsymbol{T})^{-\alpha}\,\frac{\sum_{\kappa}(\alpha)_\kappa\,C_\kappa\Big(\boldsymbol{U}\big(\beta\boldsymbol{T} + \boldsymbol{\Sigma}^{-1}\big)^{-1}\Big)}{\sum_{\kappa}(\alpha)_\kappa\,C_\kappa(\boldsymbol{U}\boldsymbol{\Sigma})}, \tag{4.1}$$

where $\boldsymbol{T}$ *is a complex symmetric matrix of order* p *with* $\mathrm{Re}(\boldsymbol{T}) > \boldsymbol{0}$.

Proof: By definition, we have

$$\begin{aligned}\varphi_{\boldsymbol{X}}(\boldsymbol{T}) \;&=\; E\big[\exp\big(-\,\mathrm{tr}(\boldsymbol{T}\boldsymbol{X})\big)\big] \\ &=\; C\left(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U}\right)\int_{\boldsymbol{X}>\boldsymbol{0}} \mathrm{etr}\bigg[-\boldsymbol{X}\bigg(\boldsymbol{T} + \frac{1}{\beta}\,\boldsymbol{\Sigma}^{-1}\bigg)\bigg]\,\det(\boldsymbol{X})^{\alpha-(p+1)/2}\,\big[\mathrm{tr}(\boldsymbol{X}\boldsymbol{U})\big]^{k}\,d\boldsymbol{X}.\end{aligned}$$

Now, evaluating the above integral by using (3.2) and simplifying the resulting expression, we get the desired result. □

Corollary 4.0.1. *Let $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$. Then the characteristic function of $\boldsymbol{X}$ is*

$$\psi_{\boldsymbol{X}}(\boldsymbol{T}) \;=\; \det(\boldsymbol{I}_p - \iota\,\beta\boldsymbol{\Sigma}\boldsymbol{T})^{-\alpha}\, \frac{\sum_\kappa (\alpha)_\kappa\, C_\kappa\Big(\boldsymbol{U}\boldsymbol{\Sigma}\,(\boldsymbol{I}_p - \iota\,\beta\boldsymbol{T}\boldsymbol{\Sigma})^{-1}\Big)}{\sum_\kappa (\alpha)_\kappa\, C_\kappa(\boldsymbol{U}\boldsymbol{\Sigma})}\,, \tag{4.2}$$

where $\iota = \sqrt{-1}$, $\boldsymbol{T}$ is a symmetric positive definite matrix of order p with $\boldsymbol{T} = ((1+\delta_{ij})\,t_{ij}/2)$ and δ_{ij} is the Kronecker delta.

Proposition 4.2. *If $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$, then for a $p \times p$ non-singular constant matrix $\boldsymbol{A}$, we have*

$$\boldsymbol{A}\boldsymbol{X}\boldsymbol{A}' \sim \mathrm{GMG}_p\Big(\alpha, \beta, k, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}', \big(\boldsymbol{A}^{-1}\big)'\,\boldsymbol{U}\boldsymbol{A}^{-1}\Big)\,.$$

Proposition 4.3. *Let $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$. Then*

$$E\big[\det(\boldsymbol{X})^h\big] \;=\; \det(\beta\boldsymbol{\Sigma})^h\, \frac{\Gamma_p(\alpha+h)}{\Gamma_p(\alpha)}\, \frac{\sum_\kappa (\alpha+h)_\kappa\, C_\kappa(\boldsymbol{U}\boldsymbol{\Sigma})}{\sum_\kappa (\alpha)_\kappa\, C_\kappa(\boldsymbol{U}\boldsymbol{\Sigma})}\,.$$

Proof: By definition

$$\begin{aligned}
E\big[\det(\boldsymbol{X})^h\big] \;&=\; C(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U}) \int_{\boldsymbol{X}>\boldsymbol{0}} \mathrm{etr}\Big(-\frac{1}{\beta}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\Big)\det(\boldsymbol{X})^{\alpha+h-(p+1)/2}\,\big[\mathrm{tr}(\boldsymbol{X}\boldsymbol{U})\big]^k\, d\boldsymbol{X}\\
&=\; \frac{C(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})}{C(\alpha+h, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})}\,.
\end{aligned}$$

Now, simplification of the above expression yields the desired result. □

Proposition 4.4. *If $\boldsymbol{X} \sim \mathrm{GMG}_p\big(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}^{-1}\big)$. Then*

$$E\big[\det(\boldsymbol{X})^h\big] \;=\; \det(\beta\boldsymbol{\Sigma})^h\, \frac{\Gamma_p(\alpha+h)}{\Gamma_p(\alpha)}\, \frac{(\alpha p + hp)_k}{(\alpha p)_k}\,.$$

In order to find the expectation of the trace of a GMG random matrix, it is useful to find the expectation of zonal polynomials, which is derived below.

Theorem 4.1. *Let $\boldsymbol{X} \sim \mathrm{GMG}_p(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$ and $\boldsymbol{B}$ be a constant symmetric matrix of order p. Then*

$$E\big[C_\tau(\boldsymbol{X}\boldsymbol{B})\big] \;=\; C(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})\,\beta^{p\alpha+t+k}\det(\boldsymbol{\Sigma})^\alpha \sum_\kappa \sum_{\phi\in\kappa\cdot\tau} \theta_\phi^{\kappa,\tau}\,\Gamma_p(\alpha,\phi)\,C_\phi^{\kappa,\tau}(\boldsymbol{U}\boldsymbol{\Sigma}, \boldsymbol{B}\boldsymbol{\Sigma})\,.$$

Proof: By definition, we have

$$\begin{aligned}
E\big[C_\tau(\boldsymbol{X}\boldsymbol{B})\big] \;&=\; C(\alpha, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})\\
&\quad\times \int_{\boldsymbol{X}>\boldsymbol{0}} \mathrm{etr}\Big(-\frac{1}{\beta}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\Big)\det(\boldsymbol{X})^{\alpha-(p+1)/2}\,\big[\mathrm{tr}(\boldsymbol{X}\boldsymbol{U})\big]^k\, C_\tau(\boldsymbol{X}\boldsymbol{B})\, d\boldsymbol{X}\,.
\end{aligned}$$

Now, writing

$$\begin{aligned}\big[\mathrm{tr}(\boldsymbol{XU})\big]^k C_\tau(\boldsymbol{XB}) &= \sum_\kappa C_\kappa(\boldsymbol{XU})\, C_\tau(\boldsymbol{XB}) \\ &= \sum_\kappa \sum_{\phi\in\kappa\cdot\tau} \theta_\phi^{\kappa,\tau}\, C_\phi^{\kappa,\tau}(\boldsymbol{XU},\boldsymbol{XB})\,,\end{aligned}$$

where we have used (2.7), and integrating $\boldsymbol{X}$ by using (2.8), we obtain

$$\begin{aligned}E\big[C_\tau(\boldsymbol{XB})\big] &= C(\alpha,\beta,k,\boldsymbol{\Sigma},\boldsymbol{U}) \sum_\kappa \sum_{\phi\in\kappa\cdot\tau} \theta_\phi^{\kappa,\tau} \\ &\qquad \times \int_{\boldsymbol{X}>\boldsymbol{0}} \mathrm{etr}\Big(-\frac{1}{\beta}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\Big)\,\det(\boldsymbol{X})^{\alpha-(p+1)/2}\, C_\phi^{\kappa,\tau}(\boldsymbol{XU},\boldsymbol{XB})\; d\boldsymbol{X} \\ &= C(\alpha,\beta,k,\boldsymbol{\Sigma},\boldsymbol{U})\,\det(\beta\boldsymbol{\Sigma})^\alpha \sum_\kappa \sum_{\phi\in\kappa\cdot\tau} \theta_\phi^{\kappa,\tau}\,\Gamma_p(\alpha,\phi)\, C_\phi^{\kappa,\tau}(\beta\boldsymbol{U\Sigma},\beta\boldsymbol{B\Sigma})\,.\end{aligned}$$

Now, the result follows from the fact that

$$C_\phi^{\kappa,\tau}(\beta\boldsymbol{U\Sigma},\beta\boldsymbol{B\Sigma}) = \beta^{k+t}\, C_\phi^{\kappa,\tau}(\boldsymbol{U\Sigma},\boldsymbol{B\Sigma})\,. \qquad \square$$

Theorem 4.2. *Let $\boldsymbol{Y}\sim \mathrm{IGMG}_p(\alpha,\beta,k,\boldsymbol{\Psi},\boldsymbol{U})$. Then, the Laplace transform of $\boldsymbol{Y}$ is given by*

$$\varphi_{\boldsymbol{Y}}(\boldsymbol{T}) = C\big(\alpha,\beta,k,\boldsymbol{\Psi}^{-1},\boldsymbol{U}\big)\,\det(\boldsymbol{T})^\alpha \left[\frac{d^k}{dz^k} B_\alpha\Big(\boldsymbol{T}\big(\beta^{-1}\boldsymbol{\Psi}-z\boldsymbol{U}\big)\Big)\right]_{z=0}, \tag{4.3}$$

where $\boldsymbol{T}$ is a complex symmetric matrix of order p with $\mathrm{Re}(\boldsymbol{T})>\boldsymbol{0}$ and $B_\delta(\cdot)$ is the Bessel function of matrix argument (Herz [16]) given by

$$B_\delta(\boldsymbol{WZ}) = \det(\boldsymbol{W})^{-\delta} \int_{\boldsymbol{S}>\boldsymbol{0}} \det(\boldsymbol{S})^{\delta-(p+1)/2}\,\mathrm{etr}\big(-\boldsymbol{SZ}-\boldsymbol{S}^{-1}\boldsymbol{W}\big)\; d\boldsymbol{S}\,. \tag{4.4}$$

Proof: The Laplace transform of $\boldsymbol{Y}$, denoted by $\varphi_{\boldsymbol{Y}}(\boldsymbol{T})$ can be derived as

$$\begin{aligned}\varphi_{\boldsymbol{Y}}(\boldsymbol{T}) &= C\big(\alpha,\beta,k,\boldsymbol{\Psi}^{-1},\boldsymbol{U}\big) \\ &\qquad \times \int_{\boldsymbol{Y}>\boldsymbol{0}} \mathrm{etr}(-\boldsymbol{TY})\,\mathrm{etr}\Big(-\frac{1}{\beta}\,\boldsymbol{\Psi Y}^{-1}\Big)\,\det(\boldsymbol{Y})^{-\alpha-(p+1)/2}\,\big[\mathrm{tr}\,\big(\boldsymbol{Y}^{-1}\boldsymbol{U}\big)\big]^k\; d\boldsymbol{Y}\,.\end{aligned} \tag{4.5}$$

Note that we can write

$$\big[\mathrm{tr}\,\big(\boldsymbol{Y}^{-1}\boldsymbol{U}\big)\big]^k = \left[\frac{d^k}{dz^k}\exp\big[z\,\mathrm{tr}\,\big(\boldsymbol{Y}^{-1}\boldsymbol{U}\big)\big]\right]_{z=0}. \tag{4.6}$$

Now, substituting (4.6) in (4.5), we have

$$\begin{aligned}\varphi_{\boldsymbol{Y}}(\boldsymbol{T}) &= C\big(\alpha,\beta,k,\boldsymbol{\Psi}^{-1},\boldsymbol{U}\big) \\ &\qquad \times \left[\frac{d^k}{dz^k}\int_{\boldsymbol{Y}>\boldsymbol{0}} \mathrm{etr}(-\boldsymbol{TY})\,\mathrm{etr}\big[-\big(\beta^{-1}\boldsymbol{\Psi}-z\boldsymbol{U}\big)\,\boldsymbol{Y}^{-1}\big]\,\det(\boldsymbol{Y})^{-\alpha-(p+1)/2}\; d\boldsymbol{Y}\right]_{z=0} \\ &= C\big(\alpha,\beta,k,\boldsymbol{\Psi}^{-1},\boldsymbol{U}\big) \\ &\qquad \times \left[\frac{d^k}{dz^k}\int_{\boldsymbol{Y}>\boldsymbol{0}} \mathrm{etr}\big[-\boldsymbol{TY}^{-1}-\big(\beta^{-1}\boldsymbol{\Psi}-z\boldsymbol{U}\big)\,\boldsymbol{Y}\big]\,\det(\boldsymbol{Y})^{\alpha-(p+1)/2}\; d\boldsymbol{Y}\right]_{z=0}.\end{aligned} \tag{4.7}$$

Finally, using (4.4) in (4.7), we get the desired result. $\square$

Proposition 4.5. *Let* $\boldsymbol{Y} \sim \mathrm{IGMG}_p(\alpha, \beta, k, \boldsymbol{\Psi}, \boldsymbol{U})$. *Then*

$$E\big[\det(\boldsymbol{Y})^h\big] = \frac{\det(\boldsymbol{\Psi})^h \, \Gamma_p(\alpha-h)}{\beta^{ph}\, \Gamma_p(\alpha)} \, \frac{\sum_\kappa (\alpha-h)_\kappa \, C_\kappa\big(\boldsymbol{U}\boldsymbol{\Psi}^{-1}\big)}{\sum_\kappa (\alpha)_\kappa \, C_\kappa(\boldsymbol{U}\boldsymbol{\Psi}^{-1})}\,, \qquad \mathrm{Re}(\alpha-h) > \frac{p-1}{2}\,.$$

Proof: By definition,

$$\begin{aligned} E\big[\det(\boldsymbol{Y})^h\big] &= C\big(\alpha, \beta, k, \boldsymbol{\Psi}^{-1}, \boldsymbol{U}\big) \\ &\quad \times \int_{\boldsymbol{Y}>\boldsymbol{0}} \mathrm{etr}\left(-\beta^{-1}\boldsymbol{\Psi}\boldsymbol{Y}^{-1}\right) \det(\boldsymbol{Y})^{-(\alpha-h)-(p+1)/2} \big[\mathrm{tr}\left(\boldsymbol{Y}^{-1}\boldsymbol{U}\right)\big]^k \, d\boldsymbol{Y} \\ &= \frac{C\big(\alpha, \beta, k, \boldsymbol{\Psi}^{-1}, \boldsymbol{U}\big)}{C\left(\alpha-h, \beta, k, \boldsymbol{\Psi}^{-1}, \boldsymbol{U}\right)}\,, \qquad \mathrm{Re}(\alpha-h) > \frac{p-1}{2}\,. \end{aligned}$$

Now, the desired result is obtained by simplifying the above expression. □

Proposition 4.6. *Let* $\boldsymbol{Y} \sim \mathrm{IGMG}_p(\alpha, \beta, k, \boldsymbol{\Psi}, \boldsymbol{U})$ *and* $\boldsymbol{A}$ *be a constant symmetric matrix of order* p. *Then* $\boldsymbol{A}\boldsymbol{Y}\boldsymbol{A}' \sim \mathrm{IGMG}_p(\alpha, \beta, k, \boldsymbol{A}\boldsymbol{\Psi}\boldsymbol{A}', \boldsymbol{A}\boldsymbol{U}\boldsymbol{A}')$.

Proof: The Jacobian of the transformation $\boldsymbol{Z} = \boldsymbol{A}\boldsymbol{Y}\boldsymbol{A}'$ is $J(\boldsymbol{Y} \to \boldsymbol{Z}) = \det(\boldsymbol{A})^{-(p+1)}$. Substituting appropriately in the p.d.f of $\boldsymbol{Y}$, we get the desired result. □

Theorem 4.3. *Let the* $p \times p$ *random symmetric matrices* $\boldsymbol{X}_1$ *and* $\boldsymbol{X}_2$ *be independent,* $\boldsymbol{X}_1 \sim \mathrm{GMG}_p(\alpha_1, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U})$ *and* $\boldsymbol{X}_2 \sim \mathrm{GMG}_p(\alpha_2, \beta, l, \boldsymbol{\Sigma}, \boldsymbol{U})$. *Define* $\boldsymbol{R} = (\boldsymbol{X}_1 + \boldsymbol{X}_2)^{-1/2} \times \boldsymbol{X}_1 (\boldsymbol{X}_1 + \boldsymbol{X}_2)^{-1/2}$ *and* $\boldsymbol{S} = \boldsymbol{X}_1 + \boldsymbol{X}_2$. *The p.d.f of* $\boldsymbol{S}$ *is given by*

$$\begin{aligned} &C(\alpha_1, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U}) \; C(\alpha_2, \beta, l, \boldsymbol{\Sigma}, \boldsymbol{U}) \; \mathrm{etr}\Big[-(\beta\boldsymbol{\Sigma})^{-1}\boldsymbol{S}\Big] \det(\boldsymbol{S})^{\alpha_1+\alpha_2-(p+1)/2} \times \\ &\qquad \times \sum_\kappa \sum_\lambda \sum_{\phi \in \kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda} \frac{\Gamma_p(\alpha_1, \kappa) \; \Gamma_p(\alpha_1, \lambda)}{\Gamma_p(\alpha_1+\alpha_2, \phi)} \, C_\phi^{\kappa,\lambda}(\boldsymbol{S}\boldsymbol{U}, \boldsymbol{S}\boldsymbol{U}) \,, \qquad \boldsymbol{S} > \boldsymbol{0}\,. \end{aligned}$$

Further, for $\boldsymbol{U} = \boldsymbol{I}_p$, *the p.d.f of* $\boldsymbol{R}$ *is given by*

$$\begin{aligned} &C(\alpha_1, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{I}_p) \; C(\alpha_2, \beta, l, \boldsymbol{\Sigma}, \boldsymbol{I}_p) \; \det(\beta\boldsymbol{\Sigma})^{\alpha_1+\alpha_2} \det(\boldsymbol{R})^{\alpha_1-(p+1)/2} \det(\boldsymbol{I}_p - \boldsymbol{R})^{\alpha_2-(p+1)/2} \times \\ &\qquad \times \sum_\kappa \sum_\lambda \sum_{\phi \in \kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda} \, \Gamma_p(\alpha_1+\alpha_2, \phi) \; C_\phi^{\kappa,\lambda}\big(\beta\boldsymbol{\Sigma}\boldsymbol{R}, \beta\boldsymbol{\Sigma}(\boldsymbol{I}_p - \boldsymbol{R})\big), \quad \boldsymbol{0} < \boldsymbol{R} < \boldsymbol{I}_p\,. \end{aligned}$$

Proof: The joint p.d.f of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ is given by

$$\begin{aligned} &C(\alpha_1, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U}) \; C(\alpha_2, \beta, l, \boldsymbol{\Sigma}, \boldsymbol{U}) \; \mathrm{etr}\Big[-(\beta\boldsymbol{\Sigma})^{-1}(\boldsymbol{X}_1 + \boldsymbol{X}_2)\Big] \times \\ &\qquad \times \det(\boldsymbol{X}_1)^{\alpha_1-(p+1)/2} \det(\boldsymbol{X}_2)^{\alpha_2-(p+1)/2} \big[\mathrm{tr}(\boldsymbol{X}_1\boldsymbol{U})\big]^k \big[\mathrm{tr}(\boldsymbol{X}_2\boldsymbol{U})\big]^l, \qquad \boldsymbol{X}_1 > \boldsymbol{0}, \; \boldsymbol{X}_2 > \boldsymbol{0}\,. \end{aligned}$$

Transforming $\boldsymbol{R} = (\boldsymbol{X}_1 + \boldsymbol{X}_2)^{-1/2} \boldsymbol{X}_1 (\boldsymbol{X}_1 + \boldsymbol{X}_2)^{-1/2}$ and $\boldsymbol{S} = \boldsymbol{X}_1 + \boldsymbol{X}_2$ with the Jacobian $J(\boldsymbol{X}_1, \boldsymbol{X}_2 \to \boldsymbol{R}, \boldsymbol{S}) = \det(\boldsymbol{S})^{(p+1)/2}$ in the joint p.d.f of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, the joint p.d.f of $\boldsymbol{R}$ and $\boldsymbol{S}$ can be derived as

$$\begin{aligned} &C(\alpha_1, \beta, k, \boldsymbol{\Sigma}, \boldsymbol{U}) \; C(\alpha_2, \beta, l, \boldsymbol{\Sigma}, \boldsymbol{U}) \; \mathrm{etr}\Big[-(\beta\boldsymbol{\Sigma})^{-1}\boldsymbol{S}\Big] \det(\boldsymbol{S})^{\alpha_1+\alpha_2-(p+1)/2} \det(\boldsymbol{R})^{\alpha_1-(p+1)/2} \times \\ &\qquad \times \det(\boldsymbol{I}_p - \boldsymbol{R})^{\alpha_2-(p+1)/2} \big[\mathrm{tr}\big(\boldsymbol{S}^{1/2}\boldsymbol{R}\boldsymbol{S}^{1/2}\boldsymbol{U}\big)\big]^k \big[\mathrm{tr}\big(\boldsymbol{S}^{1/2}(\boldsymbol{I}_p - \boldsymbol{R})\boldsymbol{S}^{1/2}\boldsymbol{U}\big)\big]^l, \end{aligned} \tag{4.8}$$

where $\boldsymbol{S} > \mathbf{0}$ and $\mathbf{0} < \boldsymbol{R} < \boldsymbol{I}_p$. Now, writing

$$\left[\operatorname{tr}\left(\boldsymbol{S}^{1/2}\boldsymbol{R}\boldsymbol{S}^{1/2}\boldsymbol{U}\right)\right]^k \left[\operatorname{tr}\left(\boldsymbol{S}^{1/2}\left(\boldsymbol{I}_p-\boldsymbol{R}\right)\boldsymbol{S}^{1/2}\boldsymbol{U}\right)\right]^l = \\ = \sum_\kappa \sum_\lambda \sum_{\phi\in\kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda}\, C_\phi^{\kappa,\lambda}\left(\boldsymbol{S}^{1/2}\boldsymbol{U}\boldsymbol{S}^{1/2}\boldsymbol{R},\, \boldsymbol{S}^{1/2}\boldsymbol{U}\boldsymbol{S}^{1/2}\left(\boldsymbol{I}_p-\boldsymbol{R}\right)\right)$$

in (4.8), the joint p.d.f of $\boldsymbol{R}$ and $\boldsymbol{S}$ can be re-written as

$$\begin{aligned} &C(\alpha_1,\beta,k,\boldsymbol{\Sigma},\boldsymbol{U})\, C(\alpha_2,\beta,l,\boldsymbol{\Sigma},\boldsymbol{U})\, \operatorname{etr}\left[-(\beta\boldsymbol{\Sigma})^{-1}\boldsymbol{S}\right] \det(\boldsymbol{S})^{\alpha_1+\alpha_2-(p+1)/2} \times \\ &\quad\times \det(\boldsymbol{R})^{\alpha_1-(p+1)/2}\det(\boldsymbol{I}_p-\boldsymbol{R})^{\alpha_2-(p+1)/2} \\ &\quad\times \sum_\kappa \sum_\lambda \sum_{\phi\in\kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda}\, C_\phi^{\kappa,\lambda}\left(\boldsymbol{S}^{1/2}\boldsymbol{U}\boldsymbol{S}^{1/2}\boldsymbol{R},\, \boldsymbol{S}^{1/2}\boldsymbol{U}\boldsymbol{S}^{1/2}\left(\boldsymbol{I}_p-\boldsymbol{R}\right)\right), \end{aligned} \tag{4.9}$$

where $\boldsymbol{S} > \mathbf{0}$ and $\mathbf{0} < \boldsymbol{R} < \boldsymbol{I}_p$. Finally, integrating the above expression with respect to $\boldsymbol{R}$ by using (2.13), we get the p.d.f of $\boldsymbol{S}$. Further, substituting $\boldsymbol{U} = \boldsymbol{I}_p$ in the above expression and integrating $\boldsymbol{S}$ by using (2.8), we get the p.d.f of $\boldsymbol{R}$. □

5. FAMILY OF GENERALIZED MATRIX VARIATE *t*-DISTRIBUTIONS

In this section, a new family of matrix variate t distributions is introduced. This distribution will be useful in Bayesian analysis.

Definition 5.1. The $n \times p$ random matrix $\boldsymbol{T}$ is said to have a generalized matrix variate t distribution (GMT) with parameters $\boldsymbol{M} \in \mathbb{R}^{n\times p}$, $\boldsymbol{\Psi}(p\times p) > \mathbf{0}$, $\boldsymbol{\Omega}(n\times n) > \mathbf{0}$, $\boldsymbol{U}(p\times p) > \mathbf{0}$, $\alpha > (p-1)/2$, $\beta > 0$, $\kappa = (k_1, ..., k_p)$, $k_1 \geq \cdots \geq k_p \geq 0$, if its p.d.f is given by

$$\begin{aligned} &\frac{\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Psi})^{-n/2}\,\Gamma_p(\alpha+n/2)}{\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa(\boldsymbol{U}\boldsymbol{\Psi}^{-1})}\left(\frac{\beta}{2\pi}\right)^{np/2} \times \\ &\quad\times \det\left(\boldsymbol{I}_n + \frac{\beta}{2}\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})\,\boldsymbol{\Psi}^{-1}(\boldsymbol{T}-\boldsymbol{M})'\right)^{-(\alpha+n/2)} \\ &\quad\times \sum_\kappa\left(\alpha+\frac{n}{2}\right)_\kappa C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Psi}+\frac{\beta}{2}(\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})\right)^{-1}\right), \qquad \boldsymbol{T}\in\mathbb{R}^{n\times p}. \end{aligned} \tag{5.1}$$

We shall use the notation $\boldsymbol{T} \sim \mathrm{GMT}_{n,p}(\alpha,\beta,k,\boldsymbol{M},\boldsymbol{\Omega},\boldsymbol{\Psi},\boldsymbol{U})$.

For $\beta = 2$, $\alpha = (m+p-1)/2$ and $k = 0$, the GMT distribution simplifies to the matrix variate t distribution (Gupta and Nagar [14]). Further, for $k = 0$, the GMT simplifies to the generalized matrix variate t distribution defined by Iranmanesh *et al.* [19].

Theorem 5.1. *Let* $\boldsymbol{X}|\boldsymbol{\Sigma} \sim N_{n,p}(\mathbf{0},\boldsymbol{\Omega}\otimes\boldsymbol{\Sigma})$ *and* $\boldsymbol{\Sigma} \sim \mathrm{IGMG}_p(\alpha,\beta,k,\boldsymbol{\Psi},\boldsymbol{U})$. *Then,* $\boldsymbol{X} \sim \mathrm{GMT}_{n,p}(\alpha,\beta,k,\mathbf{0},\boldsymbol{\Omega},\boldsymbol{\Psi},\boldsymbol{U})$.

Proof: Let $g(\boldsymbol{X}|\boldsymbol{\Sigma})$ be the conditional p.d.f of $\boldsymbol{X}$ given $\boldsymbol{\Sigma}$. Further, let $h(\boldsymbol{\Sigma})$ be the marginal p.d.f of $\boldsymbol{\Sigma}$. Then, using conditional method, we find the marginal p.d.f of $\boldsymbol{X}$ as

$$f(\boldsymbol{X}) = \int_{\boldsymbol{\Sigma}>\boldsymbol{0}} g(\boldsymbol{X}|\boldsymbol{\Sigma})\, h(\boldsymbol{\Sigma})\; d\boldsymbol{\Sigma}.$$

Now, substituting for $g(\boldsymbol{X}|\boldsymbol{\Sigma})$ and $h(\boldsymbol{\Sigma})$ above, we get the marginal p.d.f of $\boldsymbol{X}$ as

$$\begin{aligned} f(\boldsymbol{X}) &= (2\pi)^{-np/2}\det(\boldsymbol{\Omega})^{-p/2} C\left(\alpha,\beta,k,\boldsymbol{\Psi}^{-1},\boldsymbol{U}\right)\\ &\times \int_{\boldsymbol{\Sigma}>\boldsymbol{0}} \operatorname{etr}\left[-\frac{1}{\beta}\left(\boldsymbol{\Psi}+\frac{\beta}{2}\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}\right)\boldsymbol{\Sigma}^{-1}\right]\det(\boldsymbol{\Sigma})^{-\alpha-(n+p+1)/2}\left[\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{U}\right)\right]^{k}\, d\boldsymbol{\Sigma}. \end{aligned}$$

Further, substituting $\boldsymbol{\Sigma}^{-1} = \boldsymbol{Z}$ with the Jacobian $J(\boldsymbol{\Sigma}\to\boldsymbol{Z}) = \det(\boldsymbol{Z})^{-(p+1)}$ in the above integral and using (3.2), we get

$$f(\boldsymbol{X}) = (2\pi)^{-np/2}\det(\boldsymbol{\Omega})^{-p/2}\frac{C\left(\alpha,\beta,k,\boldsymbol{\Psi}^{-1},\boldsymbol{U}\right)}{C\left(\alpha+n/2,\beta,k,\left(\boldsymbol{\Psi}+\beta\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}/2\right)^{-1},\boldsymbol{U}\right)}.$$

Finally, simplifying the above expression, we get

$$\begin{aligned} &\frac{\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Psi})^{-n/2}\Gamma_p(\alpha+n/2)}{\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Psi}^{-1}\right)}\left(\frac{\beta}{2\pi}\right)^{np/2}\det\left(\boldsymbol{I}_n+\frac{\beta}{2}\boldsymbol{\Omega}^{-1}\boldsymbol{X}\boldsymbol{\Psi}^{-1}\boldsymbol{X}'\right)^{-(\alpha+n/2)}\times\\ &\qquad\times\sum_\kappa\left(\alpha+\frac{n}{2}\right)_\kappa C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Psi}+\frac{\beta}{2}\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}\right)^{-1}\right),\quad \boldsymbol{X}\in\mathbb{R}^{n\times p}, \end{aligned}$$

which is the desired result. □

Next, in Corollary 5.1.1, Corollary 5.1.2 and Theorem 5.2, we give three different variations of the above theorem.

Corollary 5.1.1. *Let* $\boldsymbol{Y}|\boldsymbol{\Sigma}\sim N_{p,n}\left(\boldsymbol{0},\boldsymbol{\Sigma}\otimes\boldsymbol{\Omega}\right)$ *and* $\boldsymbol{\Sigma}\sim \mathrm{IGMG}_p(\alpha,\beta,k,\boldsymbol{\Psi},\boldsymbol{U})$. *Then, the marginal p.d.f of* $\boldsymbol{Y}$ *is given by*

$$\begin{aligned} &\frac{\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Psi})^{-n/2}\Gamma_p(\alpha+n/2)}{\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Psi}^{-1}\right)}\left(\frac{\beta}{2\pi}\right)^{np/2}\det\left(\boldsymbol{I}_p+\frac{\beta}{2}\boldsymbol{\Psi}^{-1}\boldsymbol{Y}\boldsymbol{\Omega}^{-1}\boldsymbol{Y}'\right)^{-(\alpha+n/2)}\times\\ &\qquad\times\sum_\kappa\left(\alpha+\frac{n}{2}\right)_\kappa C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Psi}+\frac{\beta}{2}\boldsymbol{Y}\boldsymbol{\Omega}^{-1}\boldsymbol{Y}'\right)^{-1}\right),\quad \boldsymbol{Y}\in\mathbb{R}^{p\times n}. \end{aligned}$$

Proof: Take $\boldsymbol{Y} = \boldsymbol{X}'$ in Theorem 5.1. □

Corollary 5.1.2. *Let* $\boldsymbol{X}|\boldsymbol{\Omega}\sim N_{n,p}\left(0,\boldsymbol{\Omega}\otimes\boldsymbol{\Sigma}\right)$ *and* $\boldsymbol{\Omega}\sim \mathrm{IGMG}_n(\alpha,\beta,k,\boldsymbol{\Psi},\boldsymbol{U})$. *Then, the marginal p.d.f of* $\boldsymbol{X}$ *is*

$$\begin{aligned} &\frac{\det(\boldsymbol{\Sigma})^{-n/2}\det(\boldsymbol{\Psi})^{-p/2}\Gamma_n(\alpha+p/2)}{\Gamma_n(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Psi}^{-1}\right)}\left(\frac{\beta}{2\pi}\right)^{np/2}\det\left(\boldsymbol{I}_n+\frac{\beta}{2}\boldsymbol{\Psi}^{-1}\boldsymbol{X}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'\right)^{-(\alpha+p/2)}\times\\ &\qquad\times\sum_\kappa\left(\alpha+\frac{p}{2}\right)_\kappa C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Psi}+\frac{\beta}{2}\boldsymbol{X}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'\right)^{-1}\right),\quad \boldsymbol{X}\in\mathbb{R}^{n\times p}. \end{aligned}$$

Proof: This result can be obtained from Corollary 5.1.1. □

Theorem 5.2. *Let* $\boldsymbol{Y}|\boldsymbol{\Omega} \sim N_{p,n}(0, \boldsymbol{\Sigma} \otimes \boldsymbol{\Omega})$ *and* $\boldsymbol{\Omega} \sim \mathrm{IGMG}_n(\alpha, \beta, k, \boldsymbol{\Psi}, \boldsymbol{U})$. *Then,* $\boldsymbol{Y} \sim \mathrm{GMT}_{p,n}(\alpha, \beta, k, \boldsymbol{0}, \boldsymbol{\Sigma}, \boldsymbol{\Psi}, \boldsymbol{U})$.

Proof: Let $g(\boldsymbol{Y}|\boldsymbol{\Omega})$ be the conditional p.d.f of $\boldsymbol{Y}$ given $\boldsymbol{\Omega}$. Further, let $h(\boldsymbol{\Omega})$ be the marginal p.d.f of $\boldsymbol{\Omega}$. Then, using conditional method, we find the marginal p.d.f of $\boldsymbol{Y}$ as

$$f_{\boldsymbol{Y}}(\boldsymbol{Y}) = \int_{\boldsymbol{\Omega}>\boldsymbol{0}} g(\boldsymbol{Y}|\boldsymbol{\Omega}) h(\boldsymbol{\Omega})\ d\boldsymbol{\Omega}.$$

Now, substituting for $g(\boldsymbol{Y}|\boldsymbol{\Omega})$ and $h(\boldsymbol{\Omega})$ above, we get the marginal p.d.f of $\boldsymbol{Y}$ as

$$\begin{aligned} f_{\boldsymbol{Y}}(\boldsymbol{Y}) &= (2\pi)^{-np/2} \det(\boldsymbol{\Sigma})^{-n/2} C\left(\alpha, \beta, k, \boldsymbol{\Psi}^{-1}, \boldsymbol{U}\right) \\ &\quad \times \int_{\boldsymbol{\Omega}>\boldsymbol{0}} \mathrm{etr}\left[-\frac{1}{\beta}\left(\boldsymbol{\Psi} + \frac{\beta}{2}\boldsymbol{Y}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}\right)\boldsymbol{\Omega}^{-1}\right] \det(\boldsymbol{\Omega})^{-\alpha-(n+p+1)/2} \left[\mathrm{tr}\left(\boldsymbol{\Omega}^{-1}\boldsymbol{U}\right)\right]^k\ d\boldsymbol{\Omega}. \end{aligned}$$

Further, substituting $\boldsymbol{\Omega}^{-1} = \boldsymbol{Z}$ with the Jacobian $J(\boldsymbol{\Omega} \rightarrow \boldsymbol{Z}) = \det(\boldsymbol{Z})^{-(p+1)}$ in the above integral and using (3.2), we get

$$f_{\boldsymbol{Y}}(\boldsymbol{Y}) = (2\pi)^{-np/2} \det(\boldsymbol{\Sigma})^{-n/2} \frac{C\left(\alpha, \beta, k, \boldsymbol{\Psi}^{-1}, \boldsymbol{U}\right)}{C\left(\alpha + p/2, \beta, k, \left(\boldsymbol{\Psi} + \beta\boldsymbol{Y}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}/2\right)^{-1}, \boldsymbol{U}\right)}.$$

Finally, simplifying the above expression, we get the desired result. □

6. SOME PROPERTIES OF THE GMT FAMILY OF DISTRIBUTIONS

In this section, various properties of the GMT distribution are derived.

Proposition 6.1. *Let* $\boldsymbol{T} \sim \mathrm{GMT}_{n,p}(\alpha, \beta, k, \boldsymbol{M}, \boldsymbol{\Omega}, \boldsymbol{\Psi}, \boldsymbol{U})$. *Let* $\boldsymbol{A}(n \times n)$ *and* $\boldsymbol{B}(p \times p)$ *be constant nonsingular matrices. Then,* $\boldsymbol{ATB} \sim \mathrm{GMT}_{n,p}(\alpha, \beta, k, \boldsymbol{AMB}, \boldsymbol{A\Omega A}', \boldsymbol{B}'\boldsymbol{\Psi B}, \boldsymbol{B}'\boldsymbol{UB})$.

Proof: Transforming $\boldsymbol{W} = \boldsymbol{ATB}$, with the Jacobian $J(\boldsymbol{T} \rightarrow \boldsymbol{W}) = \det(\boldsymbol{A})^{-p} \det(\boldsymbol{B})^{-n}$, in the p.d.f (5.1) of $\boldsymbol{T}$, and simplifying the resulting expression, we get the result. □

Corollary 6.0.1. *If* $\boldsymbol{T} \sim \mathrm{GMT}_{n,p}(\alpha, \beta, k, \boldsymbol{M}, \boldsymbol{\Omega}, \boldsymbol{\Psi}, \boldsymbol{U})$, *then*

$$\boldsymbol{\Omega}^{-1/2}\boldsymbol{TB} \sim \mathrm{GMT}_{n,p}\left(\alpha, \beta, k, \boldsymbol{\Omega}^{-1/2}\boldsymbol{MB}, \boldsymbol{I}_n, \boldsymbol{B}'\boldsymbol{\Psi B}, \boldsymbol{B}'\boldsymbol{UB}\right),$$

$$\boldsymbol{AT\Psi}^{-1/2} \sim \mathrm{GMT}_{n,p}\left(\alpha, \beta, k, \boldsymbol{AM\Psi}^{-1/2}, \boldsymbol{A\Omega A}', \boldsymbol{I}_p, \boldsymbol{\Psi}^{-1/2}\boldsymbol{U\Psi}^{-1/2}\right)$$

and

$$\boldsymbol{\Omega}^{-1/2}\boldsymbol{T\Psi}^{-1/2} \sim \mathrm{GMT}_{n,p}\left(\alpha, \beta, k, \boldsymbol{\Omega}^{-1/2}\boldsymbol{M\Psi}^{-1/2}, \boldsymbol{I}_n, \boldsymbol{I}_p, \boldsymbol{\Psi}^{-1/2}\boldsymbol{U\Psi}^{-1/2}\right).$$

Proposition 6.2. *If $\boldsymbol{T} \sim \mathrm{GMT}_{n,p}(\alpha,\beta,k,\boldsymbol{M},\boldsymbol{\Omega},\boldsymbol{\Psi},\boldsymbol{U})$, then for $n \geq p$, the p.d.f of $\boldsymbol{Z} = (\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})$ is given by*

$$\frac{\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Psi})^{-n/2}\Gamma_p(\alpha+n/2)}{\Gamma_p(n/2)\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Psi}^{-1}\right)}\left(\frac{\beta}{2}\right)^{np/2}\det(\boldsymbol{Z})^{(n-p-1)/2}\times$$

$$\times\det\left(\boldsymbol{I}_p+\frac{\beta}{2}\boldsymbol{\Psi}^{-1}\boldsymbol{Z}\right)^{-(\alpha+n/2)}\sum_\kappa\left(\alpha+\frac{n}{2}\right)_\kappa C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Psi}+\frac{\beta}{2}\boldsymbol{Z}\right)^{-1}\right),\quad \boldsymbol{Z}>\boldsymbol{0}. \tag{6.1}$$

Proof: The p.d.f of $\boldsymbol{Z}$ is given by

$$\frac{\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Psi})^{-n/2}\Gamma_p(\alpha+n/2)}{\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Psi}^{-1}\right)}\left(\frac{\beta}{2\pi}\right)^{np/2}\sum_\kappa\left(\alpha+\frac{n}{2}\right)_\kappa\times$$

$$\times\int_{(\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})=\boldsymbol{Z}}\det\left(\boldsymbol{I}_p+\frac{\beta}{2}\boldsymbol{\Psi}^{-1}(\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})\right)^{-(\alpha+n/2)}$$

$$\times C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Psi}+\frac{\beta}{2}(\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})\right)^{-1}\right)d\boldsymbol{Z},\quad \boldsymbol{Z}>\boldsymbol{0}.$$

Now, evaluating the above integral by using Theorem 1.4.10 of Gupta and Nagar [14], we get the result. □

The following result is a generalization of the work of Dickey [12].

Theorem 6.1. *Let $\boldsymbol{X} \sim N_{n,p}(0,\boldsymbol{\Omega}\otimes\boldsymbol{I}_p)$, independent of $\boldsymbol{S} \sim \mathrm{GMG}_p\left(\alpha,\beta,k,\boldsymbol{\Lambda}^{-1},\boldsymbol{U}\right)$. Define $\boldsymbol{T} = \boldsymbol{X}\boldsymbol{S}^{-1/2}+\boldsymbol{M}$, where $\boldsymbol{M}$ is an $n\times p$ constant matrix and $\boldsymbol{S}^{1/2}\left(\boldsymbol{S}^{1/2}\right)' = \boldsymbol{S}$. Then, the p.d.f of $\boldsymbol{T}$ is given by*

$$\frac{\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Lambda})^{-n/2}\Gamma_p(\alpha+n/2)}{\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\right)}\left(\frac{\beta}{2\pi}\right)^{np/2}\times$$

$$\times\det\left(\boldsymbol{I}_p+\frac{\beta}{2}\boldsymbol{\Lambda}^{-1}(\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})\right)^{-(\alpha+n/2)}$$

$$\times\sum_\kappa\left(\alpha+\frac{n}{2}\right)_\kappa C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Lambda}+\frac{\beta}{2}(\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})\right)^{-1}\right),\quad \boldsymbol{T}\in\mathbb{R}^{n\times p}.$$

Proof: The joint p.d.f of $\boldsymbol{X}$ and $\boldsymbol{S}$ is given by

$$\frac{(2\pi)^{-np/2}\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Lambda})^{\alpha}}{\beta^{p\alpha+k}\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\right)}\det(\boldsymbol{S})^{\alpha-(p+1)/2}\left[\operatorname{tr}(\boldsymbol{S}\boldsymbol{U})\right]^k\times$$

$$\times\exp\left[-\operatorname{tr}\left(\frac{1}{\beta}\boldsymbol{\Lambda}\boldsymbol{S}+\frac{1}{2}\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}\right)\right],\quad \boldsymbol{S}>\boldsymbol{0},\quad \boldsymbol{X}\in\mathbb{R}^{n\times p}.$$

Now, let $\boldsymbol{T} = \boldsymbol{X}\boldsymbol{S}^{-1/2}+\boldsymbol{M}$. The Jacobian of this transformation is $J(\boldsymbol{X}\to\boldsymbol{T}) = \det(\boldsymbol{S})^{n/2}$. Substituting for $\boldsymbol{X}$ in terms of $\boldsymbol{T}$ in the joint p.d.f of $\boldsymbol{X}$ and $\boldsymbol{S}$, and multiplying the resulting expression by $J(\boldsymbol{X}\to\boldsymbol{T})$, we get the joint p.d.f of $\boldsymbol{T}$ and $\boldsymbol{S}$ as

$$\frac{(2\pi)^{-np/2}\det(\boldsymbol{\Omega})^{-p/2}\det(\boldsymbol{\Lambda})^{\alpha}}{\beta^{p\alpha+k}\Gamma_p(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\right)}\det(\boldsymbol{S})^{\alpha+n/2-(p+1)/2}\left[\operatorname{tr}(\boldsymbol{S}\boldsymbol{U})\right]^k\times$$

$$\times\operatorname{etr}\left[-\frac{1}{\beta}\left(\boldsymbol{\Lambda}+\frac{\beta}{2}(\boldsymbol{T}-\boldsymbol{M})'\boldsymbol{\Omega}^{-1}(\boldsymbol{T}-\boldsymbol{M})\right)\boldsymbol{S}\right],\quad \boldsymbol{S}>\boldsymbol{0},\quad \boldsymbol{T}\in\mathbb{R}^{n\times p}.$$

Now, integrating out $\boldsymbol{S}$ by using (3.2) and simplifying the resulting expression the p.d.f of $\boldsymbol{T}$ is obtained. □

Theorem 6.2. *Let $\boldsymbol{X} \sim N_{n,p}(0, \boldsymbol{I}_n \otimes \boldsymbol{\Sigma})$, independent of $\boldsymbol{S} \sim \mathrm{GMG}_n\left(\alpha, \beta, k, \boldsymbol{\Lambda}^{-1}, \boldsymbol{U}\right)$. Define $\boldsymbol{T} = \left(\boldsymbol{S}^{-1/2}\right)' \boldsymbol{X} + \boldsymbol{M}$, where $\boldsymbol{M}$ is an $n \times p$ constant matrix and $\boldsymbol{S}^{1/2}\left(\boldsymbol{S}^{1/2}\right)' = \boldsymbol{S}$. Then, the p.d.f of $\boldsymbol{T}$ is*

$$\frac{\det(\boldsymbol{\Sigma})^{-n/2}\det(\boldsymbol{\Lambda})^{-p/2}\Gamma_n(\alpha+p/2)}{\Gamma_n(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\right)}\left(\frac{\beta}{2\pi}\right)^{np/2}\times$$
$$\times\det\left(\boldsymbol{I}_n + \frac{\beta}{2}\boldsymbol{\Lambda}^{-1}(\boldsymbol{T}-\boldsymbol{M})\boldsymbol{\Sigma}^{-1}(\boldsymbol{T}-\boldsymbol{M})'\right)^{-(\alpha+p/2)}$$
$$\times\sum_\kappa\left(\alpha+\frac{p}{2}\right)_\kappa C_\kappa\left(\boldsymbol{U}\left(\boldsymbol{\Lambda}+\frac{\beta}{2}(\boldsymbol{T}-\boldsymbol{M})\boldsymbol{\Sigma}^{-1}(\boldsymbol{T}-\boldsymbol{M})'\right)^{-1}\right), \qquad \boldsymbol{T}\in\mathbb{R}^{n\times p}.$$

Proof: The joint p.d.f of $\boldsymbol{X}$ and $\boldsymbol{S}$ is given by

$$\frac{(2\pi)^{-np/2}\det(\boldsymbol{\Sigma})^{-n/2}\det(\boldsymbol{\Lambda})^{\alpha}}{\beta^{n\alpha+k}\Gamma_n(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\right)}\det(\boldsymbol{S})^{\alpha-(n+1)/2}\left[\mathrm{tr}(\boldsymbol{S}\boldsymbol{U})\right]^k\times$$
$$\times\exp\left[-\mathrm{tr}\left(\frac{1}{\beta}\boldsymbol{\Lambda}\boldsymbol{S}+\frac{1}{2}\boldsymbol{X}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'\right)\right], \quad \boldsymbol{S}>\boldsymbol{0}, \quad \boldsymbol{X}\in\mathbb{R}^{n\times p}.$$

Now, let $\boldsymbol{T} = \left(\boldsymbol{S}^{-1/2}\right)'\boldsymbol{X}+\boldsymbol{M}$. The Jacobian of the transformation is $J(\boldsymbol{X}\to\boldsymbol{T}) = \det(\boldsymbol{S})^{p/2}$. Substituting for $\boldsymbol{X}$ in terms of $\boldsymbol{T}$ in the joint p.d.f of $\boldsymbol{X}$ and $\boldsymbol{S}$, and multiplying the resulting expression by $J(\boldsymbol{X}\to\boldsymbol{T})$, we get the joint p.d.f of $\boldsymbol{T}$ and $\boldsymbol{S}$ as

$$\frac{(2\pi)^{-np/2}\det(\boldsymbol{\Sigma})^{-n/2}\det(\boldsymbol{\Lambda})^{\alpha}}{\beta^{n\alpha+k}\Gamma_n(\alpha)\sum_\kappa(\alpha)_\kappa C_\kappa\left(\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\right)}\det(\boldsymbol{S})^{\alpha+p/2-(n+1)/2}\left[\mathrm{tr}(\boldsymbol{S}\boldsymbol{U})\right]^k\times$$
$$\times\,\mathrm{etr}\left[-\left(\frac{1}{\beta}\boldsymbol{\Lambda}+\frac{1}{2}(\boldsymbol{T}-\boldsymbol{M})\boldsymbol{\Sigma}^{-1}(\boldsymbol{T}-\boldsymbol{M})'\right)\boldsymbol{S}\right], \quad \boldsymbol{S}>\boldsymbol{0}, \quad \boldsymbol{X}\in\mathbb{R}^{n\times p}.$$

Now, integrating out $\boldsymbol{S}$ by using (3.2) and simplifying the resulting expression, the p.d.f of $\boldsymbol{T}$ is obtained. □

7. APPLICATIONS IN BAYESIAN ANALYSIS

As in Iranmanesh *et al.* [17], consider the Kullback-Leibler divergence loss (KLDL) function $\log\left(\frac{\pi(\boldsymbol{A}|\boldsymbol{D})}{\pi(\boldsymbol{\Sigma}|\boldsymbol{D})}\right)$ with the posterior expected loss function

$$\rho(\boldsymbol{\Sigma},\boldsymbol{A}) = E\left[\log\left(\frac{\pi(\boldsymbol{A}|\boldsymbol{D})}{\pi(\boldsymbol{\Sigma}|\boldsymbol{D})}\right)\right].$$

One may use the inverted generalized matrix gamma distribution as a prior distribution in Bayesian context. It is straightforward to prove that posterior distributions are IGMG. They are stated in Propositions 7.1 and 7.2 without proof.

Proposition 7.1. *Let* $\boldsymbol{X}|\boldsymbol{\Sigma} \sim N_{n,p}(0, \boldsymbol{\Omega} \otimes \boldsymbol{\Sigma})$. *Further suppose that the prior distribution of* $\boldsymbol{\Sigma}$ *is IGMG with parameters* $(\alpha, \beta, k, \boldsymbol{\Psi}, \boldsymbol{U})$. *Then, the posterior distribution of* $\boldsymbol{\Sigma}$ *is IGMG with parameters* $\left(\alpha + n/2, \beta, k, \left(\boldsymbol{\Psi} + \beta \boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}/2\right)^{-1}, \boldsymbol{U}\right)$. *That is, the posterior p.d.f of* $\boldsymbol{\Sigma}$ *is*

$$\pi(\boldsymbol{\Sigma}|\boldsymbol{X}) = C\left(\alpha + \frac{n}{2}, \beta, k, \left(\boldsymbol{\Psi} + \frac{\beta}{2}\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}\right)^{-1}, \boldsymbol{U}\right)$$
$$\times \operatorname{etr}\left[-\frac{1}{\beta}\left(\boldsymbol{\Psi} + \frac{\beta}{2}\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}\right)\boldsymbol{\Sigma}^{-1}\right] \det(\boldsymbol{\Sigma})^{-\alpha-(n+p+1)/2}\left[\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{U}\right)\right]^k, \quad \boldsymbol{\Sigma} > \boldsymbol{0}.$$

Proposition 7.2. *Let* $\boldsymbol{X}|\boldsymbol{\Sigma} \sim N_{n,p}(0, \boldsymbol{\Omega} \otimes \boldsymbol{\Sigma})$. *Further suppose that the prior distribution of* $\boldsymbol{\Omega}$ *is IGMG with parameters* $(\alpha, \beta, k, \boldsymbol{\Psi}, \boldsymbol{U})$. *Then, the posterior distribution of* $\boldsymbol{\Omega}$ *is IGMG with parameters* $\left(\alpha + p/2, \beta, k, \left(\boldsymbol{\Psi} + \beta \boldsymbol{X}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'/2\right)^{-1}, \boldsymbol{U}\right)$. *That is, the posterior p.d.f of* $\boldsymbol{\Omega}$ *is*

$$\pi(\boldsymbol{\Omega}|\boldsymbol{X}) = C\left(\alpha + \frac{p}{2}, \beta, k, \left(\boldsymbol{\Psi} + \frac{\beta}{2}\boldsymbol{X}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'\right)^{-1}, \boldsymbol{U}\right)$$
$$\times \operatorname{etr}\left[-\frac{1}{\beta}\left(\boldsymbol{\Psi} + \frac{\beta}{2}\boldsymbol{X}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'\right)\boldsymbol{\Omega}^{-1}\right] \det(\boldsymbol{\Omega})^{-\alpha-(n+p+1)/2}\left[\operatorname{tr}\left(\boldsymbol{\Omega}^{-1}\boldsymbol{U}\right)\right]^k, \quad \boldsymbol{\Omega} > \boldsymbol{0}.$$

By definition, the Bayes estimator of $\boldsymbol{\Sigma}$, under the KLDL function, is given by $\widehat{\boldsymbol{\Sigma}} = \arg\max_{\boldsymbol{\Sigma}} \pi(\boldsymbol{\Sigma}|\boldsymbol{X})$. Iranmanesh *et al.* [19] have shown that

$$\widehat{\boldsymbol{\Sigma}} = [\alpha + n/2 + (p+1)/2]^{-1}\left(\frac{1}{2}\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X} + \frac{1}{\beta}\boldsymbol{\Psi}\right)$$

for the special case $k = 0$.

8. CONCLUSION

In this paper, a generalized matrix variate gamma distribution has been introduced. The corresponding inverted matrix variate gamma distribution has also been derived. By making use of this newly defined matrix variate distribution as the prior for the characteristic matrix of a matrix variate normal distribution, using conditioning approach, a family of generalized matrix variate t distributions has also been defined.

A future work is to consider estimation of the newly introduced matrix variate distributions. One issue is that the new distributions are over parameterized; that is, there is parameter redundancy. This can be accounted for numerically by constrained maximization of the log likelihood. For example, if the data follow the overparameterized p.d.f $ab\exp(-abx)$ then the log likelihood can be maximized using the constraint $ab = c$. Usually, partial derivatives of the log likelihood are not required for evaluating maximum likelihood estimates numerically.

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