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INFERENCE ON STRESS-STRENGTH MODEL FOR A KUMARASWAMY DISTRIBUTION BASED ON HYBRID PROGRESSIVE CENSORED SAMPLE

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Abstract:

- In this paper, we obtain the point and interval estimates of the stress-strength parameter under the hybrid progressive censored scheme, when stress and strength are considered as two independent random variables of Kumaraswamy. We solve the problem in three cases, as follows: First, assuming that stress and strength have different first shape parameters and the common second shape parameter, we obtain maximum likelihood estimation (MLE), approximation maximum likelihood estimation (AMLE) and two Bayesian approximation estimates due to the lack of explicit forms. Also, we construct the asymptotic and highest posterior density (HPD) intervals for R. Moreover, we consider the existence and uniqueness of the MLE. Second, assuming that common second shape parameter is identified, we derive the MLE and exact Bayes estimate of R. Third, assuming that all parameters are unknown and different, we achieve the statistical inference of R, namely MLE, AMLE and Bayesian inference of R. Furthermore, we apply the Monte Carlo simulations for comparing the performance of different methods. Finally, we analyze two data sets for illustrative purposes.

Keywords:

- *stress-strength model; hybrid progressive censored sample; Kumaraswamy distribution; Bayesian inference; Monte Carlo simulation.*

AMS Subject Classification:

- 62F10, 62F15, 62N05.

1. INTRODUCTION

One of the most interesting problems in reliability theory, is inference of the stress-strength parameter, $R = P(X < Y)$. The variables Y and X are known as strength and stress, respectively. In one system, if the applied stress is greater than its strength, as a result the system fails. In statistical science, more attention has been paid to the estimation of R since 1956, beginning with the work of Birnbaum [3]. From that time, estimating the R have been done from the frequentist and Bayesian viewpoints. Recently, some studies about the stress-strength model can be found in Rezaei *et al.* [21], Babayi *et al.* [2], Nadar *et al.* [18] and Kizilaslan and Nadar [8].

Although, in the complete sample case, many authors have been investigated the stress-strength models, they did not pay attention to the censored sample case. Whereas in really applicable situations, for many reasons like financial plane or limited time, the researchers confront censored data.

Among various censoring schemes, Type-I and Type-II are the two most fundamental schemes, which can be explained as follows. We finish the test in a pre-selected time and pre-chosen number of failures, in Type-I and Type-II schemes, respectively. Also, we finish the test at time $T^* = \min\{X_{m:n}, T\}$, where $X_{m:n}$ is the m-th failure times from n items and $T > 0$, in the hybrid scheme, which has been indicated by Epstein [5]. Also, In hybrid scheme, Singh and Goel [24] obtained reliability estimation of modified Weibull distribution. Because in the hybrid scheme, the removal of active units cannot be lost during the test, hybrid progressive (HP) scheme is introduced by Kundu and Joarder [14], which can be described as follows. Let N units be put on the test with censoring scheme $(R_1, ..., R_n)$ and pausing time $T^* = \min\{X_{n:n:N}, T\}$, where $X_{1:n:N} \leq ... \leq X_{n:n:N}$ be a progressive censoring scheme and $T > 0$ is a fixed time. It is obvious that if $X_{n:n:N} < T$ then we finish the test at time $X_{n:n:N}$ and $\{X_{1:n:N}, ..., X_{n:n:N}\}$ is the observed sample. Otherwise, if $X_{J:n:N} < T < X_{J+1:n:N}$ then we finish the test at time T and $\{X_{1:n:N}, ..., X_{J:n:N}\}$ is the observed sample. In symbol, we say that $\{X_{1:n:N}, ..., X_{J:n:N}\}$ is a HP censoring sample with scheme $\{N, n, T, J, R_1, ..., R_J\}$. Recently, some of the authors have studied the stress-strength model and censored data. For example, Shoaee and Khorram considered stress-strength reliability of a two-parameter bathtub-shaped lifetime distribution with respect to progressively censored samples, [22]. Also, they obtained some statistical inference of $R = P(Y < X)$ for Weibull distribution under Type-II progressively hybrid censored data, [23]. Kohansal [9] considered estimation of multicomponent stress-strength reliability for Kumaraswamy distribution under progressive censoring. Rasethuntsa and Nadar [20] studied stress-strength reliability of a non-identical-component-strengths system based on upper record values from the family of Kumaraswamy generalized distributions. Very recently, Maurya and Tripathi [17] derived the reliability estimation in a multicomponent stress strength model for Burr XII distribution under progressive censoring. In addition, Kohansal [10] obtained Bayesian and classical estimation of $R = P(X < Y)$ based on Burr type XII distribution under hybrid progressive censored samples. Kohansal and Rezakhah [12] considered the inference of $R = P(Y < X)$ for two-parameter Rayleigh distribution in terms of progressively censored samples. Ahmadi and Ghafouri [1] obtained the reliability estimation in a multicomponent stress-strength model under generalized half-normal distribution based on progressive Type-II censoring. Furthermore, Kohansal and Shoaee [13] derived Bayesian and classical estimation of reliability in a multicomponent stress-strength model under adaptive hybrid progressive censored data.

Finally, Kohansal and Nadarajah [11] estimated the stress-strength parameter based on Type-II hybrid progressive censored samples for a Kumaraswamy distribution. In this study, based on HP censoring scheme, the reliability parameter $R = P(X < Y)$ is estimated when X and Y are two independent random variables from the Kumaraswamy distribution (KuD). This paper has also some contribution in terms of inference. We consider the different point and interval estimations of R, and all of these estimates are considered in Bayesian and classical viewpoints. Also, we investigate the problem in three different cases, first at the time that X and Y have the unknown common one parameter, secondly when have known common one parameter, and third when they have different unknown parameters. Moreover, as the HP censoring is a general scheme, so we can obtain from it, some cases that are considered (up to now).

KuD with the first and second shape parameters α and λ, respectively, which is denoted by $\text{Ku}(\alpha, \lambda)$, has the probability density function (pdf), cumulative distribution function (cdf) and failure rate function as follows:

$$
\begin{aligned}
f(x) &= \alpha\lambda x^{\lambda-1}(1-x^{\lambda})^{\alpha-1}, && 0 < x < 1, \quad \alpha, \lambda > 0,\\
F(x) &= 1-(1-x^{\lambda})^{\alpha}, && 0 < x < 1, \quad \alpha, \lambda > 0,\\
H(x) &= \frac{\alpha\lambda x^{\lambda-1}}{1-x^{\lambda}}, && 0 < x < 1, \quad \alpha, \lambda > 0,
\end{aligned}
$$

respectively. The probability density and failure rate functions of KuD are presented in Figure 1. KuD has an increasing failure rate function, so the KuD can be used for analyzing the real data sets if the empirical consideration suggests that the failure rate function of the prior distribution is increasing. Moreover, KuD is the very appropriate fit to many natural phenomena, which their outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures, hydrological data, economic data, etc.

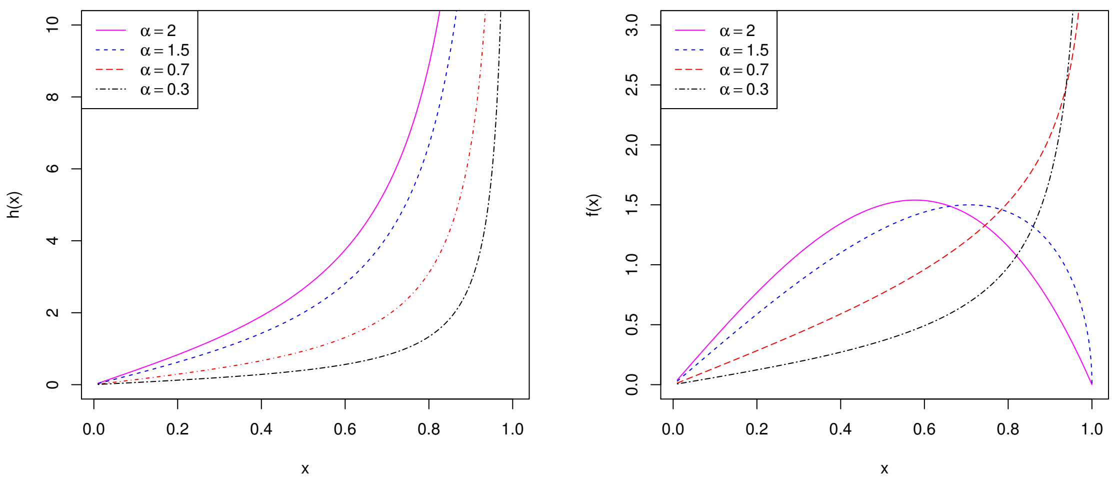

Figure 1: Shape of probability density (right) and failure rate (left) functions of KuD when $\lambda = 2$.

The other parts of this paper are arranged as follows: In Section 2, under the HP censoring scheme, assuming $X \sim \text{Ku}(\alpha, \lambda)$ and $Y \sim \text{Ku}(\beta, \lambda)$, we obtain the point and interval estimates of $R = P(X < Y)$, from the frequentist and Bayesian viewpoints.

More specifically, in Section 2, the existence and uniqueness of MLEs are considered. Because the MLEs of unknown parameters and R cannot be earned in the closed forms, we obtain the AMLEs of parameters and R, which have the explicit forms. In addition, we develop the Bayes estimates of R, by applying Lindley's approximation and MCMC method due to the lack of explicit forms. Moreover, different confidence intervals such as asymptotic and HPD intervals of R are provided. In Section 3, by assuming that the common shape parameter is known, the MLE and exact Bayes estimate of R are earned. Because the assumption studied in Section 2 is quite strong, we consider the statistical inference of R in general case. Accordingly, in Section 4, under the HP censoring scheme, assuming $X \sim \text{Ku}(\alpha, \lambda_1)$ and $Y \sim \text{Ku}(\beta, \lambda_2)$, we provide the MLE, AMLE and Bayes estimate of R, respectively. In Section 5, we give the simulation results and data analysis, and following that we conclude the paper in Section 6.

2. INFERENCE ON R WITH UNKNOWN COMMON λ

2.1. MLE of R

The stress-strength parameter, when X and Y are two independent random variables from $\text{Ku}(\alpha, \lambda)$ and $\text{Ku}(\beta, \lambda)$, respectively, can be obtained simply as $R = P(X < Y) = \frac{\alpha}{\alpha+\beta}$. In this section, under the HP censoring scheme, we derive the MLE of R. Because R is a function of the unknown parameters, consequently at first we obtain the MLEs of α, β, and λ. If $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_m\}$ be two HP censoring samples with censoring schemes $\{N, n, T_1, J_1, R_1, ..., R_{J_1}\}$ and $\{M, m, T_2, J_2, S_1, ..., S_{J_2}\}$, respectively, after that the likelihood function of the unknown parameters α, β and λ can be written as

$$L(\alpha, \beta, \lambda) \propto \left[\prod_{i=1}^{J_1} f(x_i)[1 - F(x_i)]^{R_i}[1 - F(T_1)]^{R^*_{J_1}}\right] \times \left[\prod_{i=1}^{J_2} f(y_j)[1 - F(y_j)]^{S_j}[1 - F(T_2)]^{S^*_{J_2}}\right],$$

where

$$R^*_{J_1} = N - J_1 - \sum_{i=1}^{J_1} R_i, \; S^*_{J_2} = M - J_2 - \sum_{j=1}^{J_2} S_j.$$

The proposed model, in association with the existing ones, has some differences and similarities. About the differences, we notice that it is a general model and some important models can be obtained from it. For example, by setting $T_1 = X_n$ and $T_2 = Y_m$, we derive the likelihood function for $R = P(X < Y)$ in the progressive censoring scheme. Also, by setting $T_1 = X_n, \; R_i = 0 \; (i = 1, ..., n-1), \; R_n = N - n$ and $T_2 = Y_m, \; S_j = 0 \; (j = 1, ..., m-1), \; S_m = M - m$, we obtain the likelihood function for $R = P(X < Y)$ in Type-II censoring scheme. Moreover, by setting $T_1 = X_n, \; R_i = 0 \; (i = 1, ..., n)$ and $T_2 = Y_m, \; S_j = 0 \; (j = 1, ..., m)$, we derive the likelihood function for $R = P(X < Y)$ in complete sample. About the similarities, we identify that most of the censoring schemes have complex computational needs.

The likelihood function, with respect to the observed data can be obtained as:

$$
\begin{aligned}
L(\text{data}|\alpha,\beta,\lambda) \;&\propto\; \alpha^{J_1}\beta^{J_2}\lambda^{J_1+J_2}\left(\prod_{i=1}^{J_1} x_i^{\lambda-1}(1-x_i^{\lambda})^{\alpha(R_i+1)-1}\right)(1-T_1^{\lambda})^{\alpha R^*_{J_1}} \\
&\times \left(\prod_{j=1}^{J_2} y_j^{\lambda-1}(1-y_j^{\lambda})^{\beta(S_j+1)-1}\right)(1-T_2^{\lambda})^{\beta S^*_{J_2}}.
\end{aligned}
$$

Therefore, the log-likelihood function, along with ignoring the constant value, is as:

$$
\begin{aligned}
\ell(\alpha,\beta,\lambda) \;&=\; J_1\log(\alpha) + \sum_{i=1}^{J_1}(\alpha(R_i+1)-1)\log(1-x_i^{\lambda}) + \alpha R^*_{J_1}\log(1-T_1^{\lambda}) \\
&+ J_2\log(\beta) + \sum_{j=1}^{J_2}(\beta(S_j+1)-1)\log(1-y_j^{\lambda}) + \beta S^*_{J_2}\log(1-T_2^{\lambda}) \\
&+ (\lambda-1)\sum_{i=1}^{J_1}\log(x_i) + (\lambda-1)\sum_{j=1}^{J_2}\log(y_j) + (J_1+J_2)\log(\lambda).
\end{aligned}
\tag{2.1}
$$

Consequently, to earn the MLEs of α, β and λ, namely, $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\lambda}$, respectively, we should solve the following equations:

$$
\frac{\partial \ell}{\partial \alpha} = \frac{J_1}{\alpha} + \sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^{\lambda}) + R^*_{J_1}\log(1-T_1^{\lambda}) = 0,
\tag{2.2}
$$

$$
\frac{\partial \ell}{\partial \beta} = \frac{J_2}{\beta} + \sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^{\lambda}) + S^*_{J_2}\log(1-T_2^{\lambda}) = 0,
\tag{2.3}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \lambda} \;&=\; \frac{J_1+J_2}{\lambda} + \sum_{i=1}^{J_1}\log(x_i) - \sum_{i=1}^{J_1}\Big(\alpha(R_i+1)-1\Big)x_i^{\lambda}\frac{\log(x_i)}{1-x_i^{\lambda}} - \alpha R^*_{J_1}T_1^{\lambda}\frac{\log(T_1)}{1-T_1^{\lambda}} \\
&+ \sum_{j=1}^{J_2}\log(y_j) - \sum_{j=1}^{J_2}\Big(\beta(S_j+1)-1\Big)y_j^{\lambda}\frac{\log(y_j)}{1-y_j^{\lambda}} - \beta S^*_{J_2}T_2^{\lambda}\frac{\log(T_2)}{1-T_2^{\lambda}} = 0.
\end{aligned}
\tag{2.4}
$$

From the equations (2.2) and (2.3), we have

$$
\widehat{\alpha}(\lambda) = -J_1\left\{\sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^{\lambda}) + R^*_{J_1}\log(1-T_1^{\lambda})\right\}^{-1},
$$

$$
\widehat{\beta}(\lambda) = -J_2\left\{\sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^{\lambda}) + S^*_{J_2}\log(1-T_2^{\lambda})\right\}^{-1}.
$$

Also, to derive $\widehat{\lambda}$, we apply one numerical method like Newton–Raphson on the equation (2.4). After obtaining the MLEs of α, β, and λ, by the use of the invariance property, the MLE of R can be derived as

$$
\widehat{R}^{\text{MLE}} = \frac{\widehat{\alpha}}{\widehat{\alpha}+\widehat{\beta}}.
\tag{2.5}
$$

2.2. Existence and uniqueness of the MLEs

In this section, we consider the existence and uniqueness of the MLEs.

Theorem 2.1. *The MLEs of the parameters α and β, which were obtained by applying the following equations, are unique:*

$$\widehat{\alpha} = -J_1\left\{\sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^{\lambda}) + R_{J_1}^{*}\log(1-T_1^{\lambda})\right\}^{-1},$$
$$\widehat{\beta} = -J_2\left\{\sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^{\lambda}) + S_{J_2}^{*}\log(1-T_2^{\lambda})\right\}^{-1},$$

and $\widehat{\lambda}$ should be obtained by finding a solution for the following equation:

$$\begin{aligned} G(\lambda) \;=\; & \frac{J_1+J_2}{\lambda} + \sum_{i=1}^{J_1}\log(x_i) - \sum_{i=1}^{J_1}\Big(\widehat{\alpha}(R_i+1)-1\Big)x_i^{\lambda}\frac{\log(x_i)}{1-x_i^{\lambda}} - \widehat{\alpha}R_{J_1}^{*}T_1^{\lambda}\frac{\log(T_1)}{1-T_1^{\lambda}} \\ & + \sum_{j=1}^{J_2}\log(y_j) - \sum_{j=1}^{J_2}\Big(\widehat{\beta}(S_j+1)-1\Big)y_j^{\lambda}\frac{\log(y_j)}{1-y_j^{\lambda}} - \widehat{\beta}S_{J_2}^{*}T_2^{\lambda}\frac{\log(T_2)}{1-T_2^{\lambda}}. \end{aligned}$$

Proof: See Appendix A. □

2.3. AMLE of R

From Section 2.1, we observe that the MLEs of unknown parameters and R cannot be earned in the closed forms. As a result in this section, we obtain the AMLEs of the parameters, which have the explicit forms.

Lemma 2.1. *Let Z' and Z'' be Weibull and Extreme value distributions, in symbols $Z' \sim \mathrm{W}(\alpha,\theta)$ and $Z'' \sim \mathrm{EV}(\mu,\sigma)$, if they have the following cumulative distribution functions, respectively as:*

$$F_{Z'}(z) = 1 - e^{-\frac{x^{\alpha}}{\theta}}, \qquad z>0, \quad \alpha,\theta>0,$$
$$F_{Z''}(z) = 1 - e^{-e^{\frac{x-\mu}{\sigma}}}, \qquad z\in\mathbb{R}, \quad \mu\in\mathbb{R}, \sigma>0.$$

- **(i)** *If $Z \sim \mathrm{Ku}(\alpha,\lambda)$ and $Z' = (-\log(1-Z^{\lambda}))^{\frac{1}{\lambda}}$, then $Z' \sim \mathrm{W}(\lambda,\frac{1}{\alpha})$.*
- **(ii)** *If $Z' \sim \mathrm{W}(\lambda,\frac{1}{\alpha})$ and $Z'' = \log(Z')$, then $Z'' \sim \mathrm{EV}(\mu,\sigma)$, where $\mu = -\frac{1}{\lambda}\log(\alpha)$ and $\sigma = \frac{1}{\lambda}$.*

Proof: Obvious. □

Consider that $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_m\}$ be two HP censoring samples with censoring schemes $\{N, n, T_1, J_1, R_1, ..., R_{J_1}\}$ and $\{M, m, T_2, J_2, S_1, ..., S_{J_2}\}$, respectively and

$$X'_i = (-\log(1 - X_i^\lambda))^{\frac{1}{\lambda}}, \; U_i = \log(X'_i) \;\; \text{and} \;\; Y'_j = (-\log(1 - Y_j^\lambda))^{\frac{1}{\lambda}}, \; V_j = \log(Y'_j).$$

Applying Lemma 2.1, $U_i \sim \text{EV}(\mu_1, \sigma)$ and $V_j \sim \text{EV}(\mu_2, \sigma)$, where

$$\mu_1 = -\frac{1}{\lambda}\log(\alpha), \; \mu_2 = -\frac{1}{\lambda}\log(\beta), \; \text{and } \sigma = \frac{1}{\lambda}.$$

Therefore, in terms of the observed data $\{U_1, ..., U_n\}$ and $\{V_1, ..., V_m\}$, and by ignoring the constant value, the log-likelihood function is as follows:

$$\begin{aligned} \ell^*(\mu_1, \mu_2, \sigma) &= \sum_{i=1}^{J_1} t_i - \sum_{i=1}^{J_1}(R_i + 1)e^{t_i} - R^*_{J_1}e^{\delta_1} \\ &\quad + \sum_{j=1}^{J_2} z_j - \sum_{j=1}^{J_2}(S_j + 1)e^{z_j} - S^*_{J_2}e^{\delta_2} - (J_1 + J_2)\log(\sigma), \end{aligned} \tag{2.6}$$

where

$$t_i = \frac{u_i - \mu_1}{\sigma}, \; z_j = \frac{v_j - \mu_2}{\sigma}, \; \delta_1 = \frac{a_1 - \mu_1}{\sigma}, \; \delta_2 = \frac{a_2 - \mu_2}{\sigma},$$
$$a_1 = \log\left((-\log(1 - T_1^\lambda))^{\frac{1}{\lambda}}\right), \; a_2 = \log\left((-\log(1 - T_2^\lambda))^{\frac{1}{\lambda}}\right).$$

Now by taking derivatives with respect to μ_1, μ_2 and σ from (2.6), we achieve the following equations:

$$\frac{\partial \ell^*}{\partial \mu_1} = -\frac{1}{\sigma}\left[J_1 - \sum_{i=1}^{J_1}(R_i + 1)e^{t_i} - R^*_{J_1}e^{\delta_1}\right] = 0, \tag{2.7}$$

$$\frac{\partial \ell^*}{\partial \mu_2} = -\frac{1}{\sigma}\left[J_2 - \sum_{j=1}^{J_2}(S_j + 1)e^{z_j} - S^*_{J_2}e^{\delta_2}\right] = 0, \tag{2.8}$$

$$\begin{aligned} \frac{\partial \ell^*}{\partial \sigma} &= -\frac{1}{\sigma}\Bigg[J_1 + J_2 + \sum_{i=1}^{J_1} t_i - \sum_{i=1}^{J_1}(R_i + 1)t_i e^{t_i} - R^*_{J_1}\delta_1 e^{\delta_1} \\ &\quad + \sum_{j=1}^{J_2} z_j - \sum_{j=1}^{J_2}(S_j + 1)z_j e^{z_j} - S^*_{J_2}\delta_2 e^{\delta_2}\Bigg] = 0. \end{aligned} \tag{2.9}$$

To obtain the AMLEs of μ_1, μ_2 and σ, let

$$q_i = 1 - \prod_{j=n-i+1}^{n} \frac{j + \sum_{k=n-j+1}^{n} R_k}{j + 1 + \sum_{k=n-j+1}^{n} R_k}, \; i = 1, ..., n, \quad q^*_{J_1} = 1 - \frac{1}{2}(q_{J_1} + q_{J_1+1}),$$

$$\bar{q}_j = 1 - \prod_{i=m-j+1}^{m} \frac{i + \sum_{k=m-i+1}^{m} S_k}{i + 1 + \sum_{k=m-i+1}^{m} S_k}, \; j = 1, ..., m, \quad \bar{q}^*_{J_2} = 1 - \frac{1}{2}(\bar{q}_{J_2} + \bar{q}_{J_2+1}).$$

Also, by expanding the functions e^{t_i}, e^{z_j}, e^{δ_1} and e^{δ_2} in Taylor series around the points

$$\nu_i = \log\big(-\log(1 - q_i)\big), \qquad \bar{\nu}_j = \log\big(-\log(1 - \bar{q}_j)\big),$$
$$\nu^*_{J_1} = \log\big(-\log(1 - q^*_{J_1})\big), \quad \bar{\nu}^*_{J_2} = \log\big(-\log(1 - \bar{q}^*_{J_2})\big),$$

respectively, and keeping the first order derivatives, we have

$$e^{t_i} = \alpha_i + \beta_i t_i, \quad e^{z_j} = \bar{\alpha}_j + \bar{\beta}_j z_j, \quad e^{\delta_1} = \alpha^*_{J_1} + \beta^*_{J_1}\delta_1, \quad e^{\delta_2} = \bar{\alpha}^*_{J_1} + \bar{\beta}^*_{J_2}\delta_2,$$

where

$$\alpha_i = e^{\nu_i}(1-\nu_i), \qquad \beta_i = e^{\nu_i}, \qquad \bar{\alpha}_j = e^{\bar{\nu}_j}(1-\bar{\nu}_j), \qquad \bar{\beta}_j = e^{\bar{\nu}_j},$$
$$\alpha^*_{J_1} = e^{\nu^*_{J_1}}(1-\nu^*_{J_1}), \ \beta^*_{J_1} = e^{\nu^*_{J_1}}, \ \bar{\alpha}^*_{J_2} = e^{\bar{\nu}^*_{J_2}}(1-\bar{\nu}^*_{J_2}), \ \bar{\beta}^*_{J_2} = e^{\bar{\nu}^*_{J_2}}.$$

Now, if we apply the linear approximations in equations (2.7)–(2.9) and solve them, then the AMLEs of μ_1, μ_2, and σ, say $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\sigma}$, respectively, can be resulted from the following equation:

$$\tilde{\mu}_1 = A_1 - \tilde{\sigma}B_1, \qquad \tilde{\mu}_2 = A_2 - \tilde{\sigma}B_2,$$
$$\tilde{\sigma} = \frac{-(D_1+D_2) + \sqrt{(D_1+D_2)^2 + 4(C_1+C_2)(E_1+E_2)}}{2(C_1+C_2)},$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2 are given in details in Appendix B. After deriving $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\sigma}$, the AMLEs of α, β, and λ, say $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\lambda}$, respectively, can be evaluated through

$$\tilde{\alpha} = e^{-\frac{\tilde{\mu}_1}{\tilde{\sigma}}}, \quad \tilde{\beta} = e^{-\frac{\tilde{\mu}_2}{\tilde{\sigma}}}, \quad \tilde{\lambda} = \frac{1}{\tilde{\sigma}}.$$

So, the AMLE of R, namely $\tilde{R}$, is

$$\tilde{R} = \frac{\tilde{\alpha}}{\tilde{\alpha}+\tilde{\beta}}. \tag{2.10}$$

2.4. Asymptotic confidence interval

In this section, we obtain the asymptotic confidence interval of R by the asymptotic distribution of $\widehat{R}$, which was obtained from the asymptotic distribution of $\widehat{\theta} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda})$. We denote the observed Fisher information matrix by $I(\theta) = [I_{ij}] = \left[-\frac{\partial^2 \ell}{\partial\theta_i\,\partial\theta_j}\right]$, $i,j = 1,2,3$. By differentiating from (2.1) for two times with respect to α, β, and λ, the inlines of $I(\theta)$ matrix can be obtained as:

$$I_{11} = \frac{J_1}{\alpha^2}, \qquad I_{22} = \frac{J_2}{\beta^2}, \qquad I_{12} = I_{21} = 0,$$
$$I_{13} = I_{31} = \sum_{i=1}^{J_1}(R_i+1)x_i^{\lambda}\frac{\log(x_i)}{1-x_i^{\lambda}} + R^*_{J_1}T_1^{\lambda}\frac{\log(T_1)}{1-T_1^{\lambda}},$$
$$I_{23} = I_{32} = \sum_{j=1}^{J_2}(S_j+1)y_j^{\lambda}\frac{\log(y_j)}{1-y_j^{\lambda}} + S^*_{J_2}T_2^{\lambda}\frac{\log(T_2)}{1-T_2^{\lambda}},$$
$$I_{33} = \frac{J_1+J_2}{\lambda^2} + \sum_{i=1}^{J_1}\Big(\alpha(R_i+1)-1\Big)x_i^{\lambda}\Big(\frac{\log(x_i)}{1-x_i^{\lambda}}\Big)^2 + \alpha R^*_{J_1}T_1^{\lambda}\Big(\frac{\log(T_1)}{1-T_1^{\lambda}}\Big)^2$$
$$+ \sum_{j=1}^{J_2}\Big(\beta(S_j+1)-1\Big)y_j^{\lambda}\Big(\frac{\log(y_j)}{1-y_j^{\lambda}}\Big)^2 + \beta S^*_{J_2}T_2^{\lambda}\Big(\frac{\log(T_2)}{1-T_2^{\lambda}}\Big)^2.$$

Theorem 2.2. *Let $\widehat{\alpha}, \widehat{\beta}$ and $\widehat{\lambda}$ be the MLEs of α, β, and λ, respectively. So*

$$[(\widehat{\alpha}-\alpha),\ (\widehat{\beta}-\beta),\ (\widehat{\lambda}-\lambda)]^{\mathsf{T}} \xrightarrow{D} N_3(0, \mathbf{I^{-1}}(\alpha,\beta,\lambda)),$$

where $\mathbf{I}(\alpha,\beta,\lambda)$ and $\mathbf{I^{-1}}(\alpha,\beta,\lambda)$ are symmetric matrices and

$$\mathbf{I}(\alpha,\beta,\lambda) = \begin{pmatrix} I_{11} & 0 & I_{13} \\ & I_{22} & I_{23} \\ & & I_{33} \end{pmatrix}, \quad \mathbf{I^{-1}}(\alpha,\beta,\lambda) = \frac{1}{|\mathbf{I}(\alpha,\beta,\lambda)|} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ & & b_{33} \end{pmatrix},$$

in which $|\mathbf{I}(\alpha,\beta,\lambda)| = I_{11}I_{22}I_{33} - I_{11}I_{23}^2 - I_{13}^2 I_{22}$,

$$\begin{aligned} b_{11} &= I_{22}I_{33} - I_{23}^2, & b_{12} &= I_{13}I_{23}, & b_{13} &= -I_{13}I_{22}, \\ b_{22} &= I_{11}I_{33} - I_{13}^2, & b_{23} &= -I_{11}I_{23}, & b_{33} &= I_{11}I_{22}. \end{aligned}$$

Proof: From the asymptotic normality of the MLE, the theorem would be resulted. □

Theorem 2.3. *Let $\widehat{R}^{\mathrm{MLE}}$ be the MLE of R. So,*

$$(\widehat{R}^{\mathrm{MLE}} - R) \xrightarrow{D} N(0, B),$$

where

$$B = \frac{1}{|\mathbf{I}(\alpha,\beta,\lambda)|}\left[(\frac{\partial R}{\partial \alpha})^2 b_{11} + (\frac{\partial R}{\partial \beta})^2 b_{22} + 2(\frac{\partial R}{\partial \alpha})(\frac{\partial R}{\partial \beta}) b_{12} \right]. \tag{2.11}$$

Proof: Using Theorem 2.2 and applying the delta method, the asymptotic distribution of $\widehat{R} = \frac{\widehat{\alpha}}{\widehat{\alpha}+\widehat{\beta}}$ can be obtained as follows:

$$(\widehat{R}^{\mathrm{MLE}} - R) \xrightarrow{D} N(0, B),$$

where $B = \mathbf{b}^{\mathsf{T}}\mathbf{I^{-1}}(\alpha,\beta,\lambda)\mathbf{b}$, with $\mathbf{b} = [\frac{\partial R}{\partial \alpha}, \frac{\partial R}{\partial \beta}, \frac{\partial R}{\partial \lambda}]^{\mathsf{T}} = [\frac{\partial R}{\partial \alpha}, \frac{\partial R}{\partial \beta}, 0]^{\mathsf{T}}$, in which

$$\frac{\partial R}{\partial \alpha} = \frac{\beta}{(\alpha+\beta)^2}, \qquad \frac{\partial R}{\partial \beta} = -\frac{\alpha}{(\alpha+\beta)^2}, \tag{2.12}$$

and $\mathbf{I}^{-1}(\alpha,\beta,\lambda)$ is defined in Theorem 2.2. Therefore, B can be represented as (2.11) and the theorem results. □

Using Theorem 2.3, the asymptotic confidence interval of R can be derived. It is notable that B should be estimated by the MLEs of α, β, and λ. So, a $100(1-\gamma)\%$ asymptotic confidence interval of R can be constructed as

$$(\widehat{R}^{\mathrm{MLE}} - z_{1-\frac{\gamma}{2}}\sqrt{\widehat{B}}, \widehat{R}^{\mathrm{MLE}} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{B}}),$$

where z_γ is 100γ-th percentile of $N(0,1)$.

2.5. Bayes estimation

In this section, under the squared error loss function, we infer the Bayesian estimation and corresponding credible interval of the stress-strength parameter, when $\alpha \sim \Gamma(a_1, b_1)$, $\beta \sim \Gamma(a_2, b_2)$ and $\lambda \sim \Gamma(a_3, b_3)$ are independent random variables. Accordingly, based on the observed censoring samples, the joint posterior density function of α, β and λ are achieved by:

$$\pi(\alpha, \beta, \lambda|\text{data}) = \frac{L(\text{data}|\alpha, \beta, \lambda)\pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\text{data}|\alpha, \beta, \lambda)\pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)d\alpha d\beta d\lambda}, \tag{2.13}$$

where

$$\begin{aligned}
\pi_1(\alpha) &\propto \alpha^{a_1-1}e^{-b_1\alpha}, & \alpha > 0,\ \ a_1, b_1 > 0,\\
\pi_2(\beta) &\propto \beta^{a_2-1}e^{-b_2\beta}, & \beta > 0,\ \ a_2, b_2 > 0,\\
\pi_3(\lambda) &\propto \lambda^{a_3-1}e^{-b_3\lambda}, & \lambda > 0,\ \ a_3, b_3 > 0.
\end{aligned}$$

As we observe from (2.13), the Bayes estimates cannot be derived in the closed-form. Therefore, we approximate them by applying two following methods:

- Lindley's approximation,
- MCMC method.

2.5.1. Lindley's approximation

One of the most applicable numerical methods to approximate the Bayes estimate has been introduced by Lindley in [16]. This method can be described as follows. Let $U(\theta)$ be a function of the parameter value. The Bayes estimate of $U(\theta)$, under the squared error loss function, is

$$\mathbb{E}\big(u(\theta)|\text{data}\big) = \frac{\int u(\theta)e^{Q(\theta)}d\theta}{\int e^{Q(\theta)}d\theta},$$

where $Q(\theta) = \ell(\theta) + \rho(\theta)$, $\ell(\theta)$ and $\rho(\theta)$ are the logarithm of likelihood function and prior density of θ, respectively. Lindley has been approximated $E(u(\theta)|\text{data})$ as

$$\mathbb{E}\big(u(\theta)|\text{data}\big) = u + \frac{1}{2}\sum_i\sum_j(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_p \ell_{ijk}\sigma_{ij}\sigma_{kp}u_p\bigg|_{\theta=\widehat{\theta}},$$

where $\theta = (\theta_1, ..., \theta_m)$, $i, j, k, p = 1, ..., m$, $\widehat{\theta}$ is the MLE of θ, $u = u(\theta)$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$, $\ell_{ijk} = \partial^3\ell/\partial\theta_i\partial\theta_j\partial\theta_k$, $\rho_j = \partial\rho/\partial\theta_j$, and $\sigma_{ij} = (i, j)$-th element in the inverse of matrix $[-\ell_{ij}]$ all calculated at the MLE of parameters.

When we face up to the case of three parameter $\theta = (\theta_1, \theta_2, \theta_3)$, Lindley's approximation conducts to

$$\begin{aligned}
\mathbb{E}(u(\theta)|\text{data}) \ =\ & u + (u_1d_1 + u_2d_2 + u_3d_3 + d_4 + d_5) + \frac{1}{2}\Big[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13})\\
& + \ B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\Big],
\end{aligned} \tag{2.14}$$

which their elements are presented in detail in Appendix C. Therefore, the Bayes estimate of R is

$$
\begin{aligned}
\widehat{R}_{s,k}^{\text{Lin}} &= R + [u_1 d_1 + u_2 d_2 + d_4 + d_5] + \frac{1}{2}\Big[A(u_1\sigma_{11} + u_2\sigma_{12}) \\
&+ B(u_1\sigma_{21} + u_2\sigma_{22}) + C(u_1\sigma_{31} + u_2\sigma_{32})\Big].
\end{aligned}
\tag{2.15}
$$

It should be noted that all parameters are evaluated at $(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda})$, respectively.

As we observe, constructing the HPD credible interval is not possible by using the Lindley's approximation. So, we apply the Markov Chain Monte Carlo (MCMC) method to approximate the Bayes estimate and construct the corresponding HPD credible intervals.

2.5.2. MCMC method

After simplify equation (2.13), we get the posterior pdfs of α, β and λ as:

$$
\begin{aligned}
\alpha|\lambda, \text{data} &\sim \Gamma\Big(J_1 + a_1, b_1 - \sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^{\lambda}) - R_{J_1}^{*}\log(1-T_1^{\lambda})\Big), \\
\beta|\lambda, \text{data} &\sim \Gamma\Big(J_2 + a_2, b_2 - \sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^{\lambda}) - S_{J_2}^{*}\log(1-T_2^{\lambda})\Big), \\
\pi(\lambda|\alpha,\beta,\text{data}) &\propto \left(\prod_{i=1}^{J_1} x_i^{\lambda-1}(1-x_i^{\lambda})^{\alpha(R_i+1)-1}\right)\left(\prod_{j=1}^{J_2} y_j^{\lambda-1}(1-y_j^{\lambda})^{\beta(S_j+1)-1}\right) \\
&\times\ \lambda^{J_1+J_2+a_3-1}e^{-\lambda b_3}(1-T_1^{\lambda})^{\alpha R_{J_1}^{*}}(1-T_2^{\lambda})^{\beta S_{J_2}^{*}}.
\end{aligned}
$$

It is identified that the posterior pdf of λ is not a well known distribution. Therefore, we utilize the Metropolis–Hastings method with normal proposal distribution in order to generate random samples from it. Consequently, the Gibbs sampling algorithm can be proposed as follows:

1. Start with the begin value $(\alpha_{(0)}, \beta_{(0)}, \lambda_{(0)})$.
2. Set $t = 1$.
3. Generate $\lambda_{(t)}$ from $\pi(\lambda|\alpha_{(t-1)}, \beta_{(t-1)}, \text{data})$, using Metropolis–Hastings method.
4. Generate $\alpha_{(t)}$ from $\Gamma\big(J_1 + a_1, b_1 - \sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^{\lambda_{(t-1)}} - R_{J_1}^{*}\log(1-T_1^{\lambda_{(t-1)}})\big)$.
5. Generate $\beta_{(t)}$ from $\Gamma\big(J_2 + a_2, b_2 - \sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^{\lambda_{(t-1)}}) - S_{J_2}^{*}\log(1-T_2^{\lambda_{(t-1)}})\big)$.
6. Calculate $R_t = \frac{\alpha_t}{\alpha_t+\beta_t}$.
7. Set $t = t + 1$.
8. Repeat steps 3–7, for T times.

By applying this algorithm, the Bayes estimate of R, under the squared error loss function is resulted from

$$\widehat{R}^{\text{MC}} = \frac{1}{T-M}\sum_{t=M+1}^{T} R_t, \tag{2.16}$$

where M is the burn-in period. Moreover, a $100(1-\gamma)\%$ HPD credible interval of R can be constructed by applying the method conducted by Chen and Shao [4].

3. INFERENCE ON R WITH KNOWN COMMON λ

3.1. MLE of R

Consider that $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_m\}$ be two HP censoring samples with censoring schemes $\{N, n, T_1, J_1, R_1, ..., R_{J_1}\}$ and $\{M, m, T_2, J_2, S_1, ..., S_{J_2}\}$, respectively. Based on Section 2.1, when the common shape parameter λ is known, the MLE of R can be attained easily by the following equation:

$$\widehat{R}^{\text{MLE}} = \left(1 + \frac{J_2\big(\sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^{\lambda}) + R^*_{J_1}\log(1-T_1^{\lambda})\big)}{J_1\big(\sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^{\lambda}) + S^*_{J_2}\log(1-T_2^{\lambda})\big)}\right)^{-1}. \tag{3.1}$$

In a similar manner to Section 2.4, $(\widehat{R}^{\text{MLE}} - R) \xrightarrow{D} N(0, C)$, where $C = (\frac{\partial R}{\partial \alpha})^2 \frac{1}{I_{11}} + (\frac{\partial R}{\partial \beta})^2 \frac{1}{I_{22}}$, and $\frac{\partial R}{\partial \alpha}$ and $\frac{\partial R}{\partial \beta}$ are indicated in (2.12). Consequently, a $100(1-\gamma)\%$ asymptotic confidence interval for R can be constructed as

$$\Big(\widehat{R}^{\text{MLE}} - z_{1-\frac{\gamma}{2}}\sqrt{\widehat{C}},\ \widehat{R}^{\text{MLE}} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{C}}\Big),$$

where z_γ is 100γ-th percentile of $N(0,1)$.

3.2. Bayes estimation

In this section, we infer the Bayesian estimation and corresponding credible interval of the stress-strength parameter, when $\alpha \sim \Gamma(a_1, b_1)$ and $\beta \sim \Gamma(a_2, b_2)$ are independent random variables. With respect to the observed censoring samples, the joint posterior density function of α and β are given by:

$$\pi(\alpha, \beta|\lambda, \text{data}) = \frac{(V+b_1)^{J_1+a_1}(U+b_2)^{J_2+a_2}}{\Gamma(J_1+a_1)\Gamma(J_2+a_2)}\alpha^{J_1+a_1-1}\beta^{J_2+a_2-1}e^{-\alpha(V+b_1)-\beta(U+b_2)}, \tag{3.2}$$

where

$$V = -\sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^\lambda) - R^*_{J_1}\log(1-T_1^\lambda),$$
$$U = -\sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^\lambda) - S^*_{J_2}\log(1-T_2^\lambda).$$

Under the squared error loss function, for obtaining R Bayes estimate, we solve the following integral:

$$\widehat{R}^{\mathrm{B}} = \int_0^\infty\int_0^\infty \frac{\alpha}{\alpha+\beta}\times\pi(\alpha,\beta|\lambda,\text{data})d\alpha d\beta.$$

Now in this study, we use the idea of Kizilaslan and Nadar [8], and accordingly, obtain the R Bayes estimate as

$$\widehat{R}^{\mathrm{B}} = \begin{cases} \dfrac{(1-z)^{J_1+a_1}(J_1+a_1)}{w}\ {}_2F_1(w, J_1+a_1+1; w+1, z) & \text{if } \ |z|<1, \\[2ex] \dfrac{(J_1+a_1)}{w(1-z)^{J_2+a_2}}\ {}_2F_1(w, J_2+a_2; w+1, \dfrac{z}{1-z}) & \text{if } \ z<-1, \end{cases} \tag{3.3}$$

where $w = J_1+J_2+a_1+a_2$, $z = 1-\dfrac{V+b_1}{U+b_2}$ and

$${}_2F_1(\alpha,\beta;\gamma,z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt,\ |z|<1,$$

is the hypergeometric series, which is quickly evaluated and readily available in standard software like MATLAB. Moreover, we construct a $100(1-\gamma)\%$ Bayesian interval for the stress-strength parameter by (L,U), where L and U are the lower and upper bounds, respectively, which indicate

$$\int_0^L f_R(R)dR = \frac{\gamma}{2},\quad \int_0^U f_R(R)dR = 1-\frac{\gamma}{2}, \tag{3.4}$$

where $f_R(R)$ is the probability density function of R, which obtained from (3.2) as

$$f_R(R) = \frac{(1-z)^{J_1+a_1}R^{J_1+a_1-1}(1-R)^{J_2+a_2-1}(1-Rz)^{-w}}{B(J_1+a_1, J_2+a_2)},\quad 0<R<1.$$

4. ESTIMATION OF R IN GENERAL CASE

4.1. MLE of R

The stress-strength parameter, when X and Y are two independent random variables from $\mathrm{Ku}(\alpha, \lambda_1)$ and $\mathrm{Ku}(\beta, \lambda_2)$, respectively, can be obtained as

$$\begin{aligned} R &= P(X < Y) \\ &= \int_0^1 f_Y(y) F_X(y) dy \\ &= \int_0^1 \beta\lambda_2 y^{\lambda_2 - 1}(1 - y^{\lambda_2})^{\beta-1}\big(1 - (1 - y^{\lambda_1})^{\alpha}\big) dy \\ &= 1 - \int_0^1 \beta\lambda_2 y^{\lambda_2 - 1}(1 - y^{\lambda_2})^{\beta-1}(1 - y^{\lambda_1})^{\alpha} dy. \end{aligned}$$

Assume that $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_m\}$ are two HP censoring samples with censoring schemes $\{N, n, T_1, J_1, R_1, ..., R_{J_1}\}$ and $\{M, m, T_2, J_2, S_1, ..., S_{J_2}\}$, respectively. As a result, the likelihood function of the unknown parameters α, β, λ_1 and λ_2 can be written as

$$\begin{aligned} L(\text{data}|\alpha, \beta, \lambda_1, \lambda_2) \;\propto\; & \alpha^{J_1}\lambda_1^{J_1}\left(\prod_{i=1}^{J_1} x_i^{\lambda_1 - 1}(1 - x_i^{\lambda_1})^{\alpha(R_i+1)-1}\right)(1 - T_1^{\lambda_1})^{\alpha R^*_{J_1}} \\ & \times \beta^{J_2}\lambda_2^{J_2}\left(\prod_{j=1}^{J_2} y_j^{\lambda_2 - 1}(1 - y_j^{\lambda_2})^{\beta(S_j+1)-1}\right)(1 - T_2^{\lambda_2})^{\beta S^*_{J_2}}. \end{aligned}$$

Therefore, the log-likelihood function, along with ignoring the constant value, is as:

$$\begin{aligned} \ell(\alpha, \beta, \lambda_1, \lambda_2) \;=\; & J_1 \log(\alpha\lambda_1) + J_2 \log(\beta\lambda_2) + \sum_{i=1}^{J_1}(\alpha(R_i + 1) - 1) \log(1 - x_i^{\lambda_1}) \\ & + \sum_{j=1}^{J_2}(\beta(S_j + 1) - 1) \log(1 - y_j^{\lambda_2}) + \alpha R^*_{J_1} \log(1 - T_1^{\lambda_1}) \\ & + \beta S^*_{J_2} \log(1 - T_2^{\lambda_2}) + (\lambda_1 - 1)\sum_{i=1}^{J_1} \log(x_i) + (\lambda_2 - 1)\sum_{j=1}^{J_2} \log(y_j). \end{aligned}$$

In a similar manner as Section 2.1, $\widehat{\alpha}$ and $\widehat{\beta}$, respectively, can be obtained from

$$\begin{aligned} \widehat{\alpha}(\lambda_1) &= -J_1\left\{\sum_{i=1}^{J_1}(R_i + 1) \log(1 - x_i^{\lambda_1}) + R^*_{J_1} \log(1 - T_1^{\lambda_1})\right\}^{-1}, \\ \widehat{\beta}(\lambda_2) &= -J_2\left\{\sum_{j=1}^{J_2}(S_j + 1) \log(1 - y_j^{\lambda_2}) + S^*_{J_2} \log(1 - T_2^{\lambda_2})\right\}^{-1}. \end{aligned}$$

Also, to derive $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$, respectively, we apply one numerical method like Newton–Raphson on the following equations:

$$\frac{\partial \ell}{\partial \lambda_1} = \frac{J_1}{\lambda_1} + \sum_{i=1}^{J_1} \log(x_i) - \sum_{i=1}^{J_1} (\alpha(R_i+1)-1) x_i^{\lambda_1} \frac{\log(x_i)}{1-x_i^{\lambda_1}} - \alpha R_{J_1}^* T_1^{\lambda_1} \frac{\log(T_1)}{1-T_1^{\lambda_1}} = 0,$$

$$\frac{\partial \ell}{\partial \lambda_2} = \frac{J_2}{\lambda_2} + \sum_{j=1}^{J_2} \log(y_j) - \sum_{j=1}^{J_2} (\beta(S_j+1)-1) y_j^{\lambda_2} \frac{\log(y_j)}{1-y_j^{\lambda_2}} - \beta S_{J_2}^* T_2^{\lambda_2} \frac{\log(T_2)}{1-T_2^{\lambda_2}} = 0.$$

After obtaining the MLEs of α, β, λ_1, and λ_2, by using the invariance property, the MLE of R can be derived as

$$\widehat{R}^{\text{MLE}} = 1 - \int_0^1 \widehat{\beta\lambda_2} y^{\widehat{\lambda}_2-1}(1-y^{\widehat{\lambda}_2})^{\widehat{\beta}-1}(1-y^{\widehat{\lambda}_1})^{\widehat{\alpha}} dy. \tag{4.1}$$

4.2. AMLE of R

In this section, we obtain AMLE of R. Consider $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_m\}$ are two HP censoring samples with censoring schemes $\{N, n, T_1, J_1, R_1, ..., R_{J_1}\}$ and also by considering $\{M, m, T_2, J_2, S_1, ..., S_{J_2}\}$ from the distributions $\text{Ku}(\alpha, \lambda_1)$ and $\text{Ku}(\beta, \lambda_2)$, respectively, and

$$X_i' = (-\log(1-X_i^{\lambda_1}))^{\frac{1}{\lambda_1}}, U_i = \log(X_i') \text{ and } Y_j' = (-\log(1-Y_j^{\lambda_2}))^{\frac{1}{\lambda_2}}, V_j = \log(Y_j').$$

Based on the observed data $\{U_1, ..., U_n\}$ and $\{V_1, ..., V_m\}$, along with ignoring the constant value, the log-likelihood function is obtained as follows:

$$\begin{aligned} \ell^*(\mu_1, \mu_2, \sigma_1, \sigma_2) &= -J_1 \log(\sigma_1) + \sum_{i=1}^{J_1} t_i - \sum_{i=1}^{J_1} (R_i+1) e^{t_i} - R_{J_1}^* e^{\delta_1} \\ &\quad - J_2 \log(\sigma_2) + \sum_{j=1}^{J_2} z_j - \sum_{j=1}^{J_2} (S_j+1) e^{z_j} - S_{J_2}^* e^{\delta_2}, \end{aligned} \tag{4.2}$$

where

$$t_i = \frac{u_i - \mu_1}{\sigma_1}, \; z_j = \frac{v_j - \mu_2}{\sigma_2}, \; \mu_1 = \frac{-\log(\alpha)}{\lambda_1}, \; \mu_2 = \frac{-\log(\beta)}{\lambda_2},$$

$$\delta_p = \frac{a_p - \mu_p}{\sigma_p}, \; \sigma_p = \frac{1}{\lambda_p}, \; a_p = \log((-\log(1-T_p^{\lambda_p}))^{\frac{1}{\lambda_p}}), \; p = 1, 2.$$

Now by taking derivatives due to μ_1, μ_2, σ_1 and σ_2 from (4.2), we obtain the following equations:

$$\frac{\partial \ell^*}{\partial \mu_1} = -\frac{1}{\sigma_1}\left[J_1 - \sum_{i=1}^{J_1}(R_i+1)e^{t_i} - R^*_{J_1}e^{\delta_1}\right] = 0,$$
$$\frac{\partial \ell^*}{\partial \mu_2} = -\frac{1}{\sigma_2}\left[J_2 - \sum_{j=1}^{J_2}(S_j+1)e^{z_j} - S^*_{J_2}e^{\delta_2}\right] = 0,$$
$$\frac{\partial \ell^*}{\partial \sigma_1} = -\frac{1}{\sigma_1}\left[J_1 + \sum_{i=1}^{J_1}t_i - \sum_{i=1}^{J_1}(R_i+1)t_ie^{t_i} - R^*_{J_1}\delta_1e^{\delta_1}\right] = 0,$$
$$\frac{\partial \ell^*}{\partial \sigma_2} = -\frac{1}{\sigma_2}\left[J_2 + \sum_{j=1}^{J_2}z_j - \sum_{j=1}^{J_2}(S_j+1)z_je^{z_j} - S^*_{J_2}\delta_2e^{\delta_2}\right] = 0.$$

In a similar manner as Section 2.3, we derive the AMLEs of μ_1, μ_2, σ_1 and σ_2, say $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, respectively, by

$$\tilde{\mu}_1 = A_1 - \tilde{\sigma}_1 B_1, \qquad \tilde{\mu}_2 = A_2 - \tilde{\sigma}_2 B_2,$$
$$\tilde{\sigma}_1 = \frac{-D_1 + \sqrt{D_1^2 + 4C_1E_1}}{2C_1}, \quad \tilde{\sigma}_2 = \frac{-D_2 + \sqrt{D_2^2 + 4C_2E_2}}{2C_2},$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2 are given in Section 2.3. After achieving $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\tilde{\sigma}_1$, and $\tilde{\sigma}_2$, the values of $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\tilde{R}$ can be evaluated by $\tilde{\alpha} = e^{-\frac{\tilde{\mu}_1}{\tilde{\sigma}_1}}$, $\tilde{\beta} = e^{-\frac{\tilde{\mu}_2}{\tilde{\sigma}_2}}$, $\tilde{\lambda}_1 = \frac{1}{\tilde{\sigma}_1}$, $\tilde{\lambda}_2 = \frac{1}{\tilde{\sigma}_2}$ and consequently

$$\tilde{R} = 1 - \int_0^1 \tilde{\beta}\tilde{\lambda}_2 y^{\tilde{\lambda}_2 - 1}(1 - y^{\tilde{\lambda}_2})^{\tilde{\beta}-1}(1 - y^{\tilde{\lambda}_1})^{\tilde{\alpha}}dy. \tag{4.3}$$

4.3. Bayes estimation

In this section, under the squared error loss function, we infer the Bayesian estimation and corresponding credible interval of the stress-strength parameter, when the unknown parameters $\alpha \sim \Gamma(a_1, b_1)$, $\beta \sim \Gamma(a_2, b_2)$, $\lambda_1 \sim \Gamma(a_3, b_3)$ and $\lambda_2 \sim \Gamma(a_4, b_4)$ are independent random variables. In a same manner as Section 2.5, as the Bayesian estimation of R has not a closed-form, we approximate it by applying MCMC method. After simplifying the joint posterior density function of the unknown parameters, we get the posterior pdfs of α, β, λ_1 and λ_2 as:

$$\alpha|\lambda_1, \text{data} \sim \Gamma\big(J_1 + a_1, b_1 - \sum_{i=1}^{J_1}(R_i+1)\log(1 - x_i^{\lambda_1}) - R^*_{J_1}\log(1 - T_1^{\lambda_1})\big),$$
$$\beta|\lambda_2, \text{data} \sim \Gamma\big(J_2 + a_2, b_2 - \sum_{j=1}^{J_2}(S_j+1)\log(1 - y_j^{\lambda_2}) - S^*_{J_2}\log(1 - T_2^{\lambda_2})\big),$$
$$\pi(\lambda_1|\alpha, \text{data}) \propto \lambda_1^{J_1+a_3-1}\left(\prod_{i=1}^{J_1} x_i^{\lambda_1-1}(1 - x_i^{\lambda_1})^{\alpha(R_i+1)-1}\right)(1 - T_1^{\lambda_1})^{\alpha R^*_{J_1}}e^{-\lambda_1 b_3}$$
$$\pi(\lambda_2|\beta, \text{data}) \propto \lambda_2^{J_2+a_4-1}\left(\prod_{j=1}^{J_2} y_j^{\lambda_2-1}(1 - y_j^{\lambda_2})^{\beta(S_j+1)-1}\right)(1 - T_2^{\lambda_2})^{\beta S^*_{J_2}}e^{-\lambda_2 b_4}.$$

It is recognized that the posterior pdfs of λ_1 and λ_2 are not well known distributions. So, we utilize the Metropolis–Hastings method with normal proposal distribution for generating random samples from them. Therefore, the Gibbs sampling algorithm can be proposed as follows:

1. Start with the begin value $(\alpha_{(0)},\, \beta_{(0)},\, \lambda_{1(0)},\, \lambda_{2(0)})$.

2. Set $t = 1$.

3. Generate $\lambda_{1(t)}$ from $\pi(\lambda_1|\alpha_{(t-1)}, \text{data})$, using Metropolis–Hastings method.

4. Generate $\lambda_{2(t)}$ from $\pi(\lambda_2|\beta_{(t-1)}, \text{data})$, using Metropolis–Hastings method.

5. Generate $\alpha_{(t)}$ from $\Gamma\big(J_1+a_1, b_1-\sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^{\lambda_{1(t-1)}}-R^*_{J_1}\log(1-T_1^{\lambda_{1(t-1)}})\big)$.

6. Generate $\beta_{(t)}$ from $\Gamma\big(J_2+a_2, b_2-\sum_{j=1}^{J_2}(S_j+1)\log(1-y_j^{\lambda_{2(t-1)}})-S^*_{J_2}\log(1-T_2^{\lambda_{2(t-1)}})\big)$.

7. Calculate $R_t = 1-\int_0^1 \beta_{(t)}\lambda_{2(t)}y^{\lambda_{2(t)}-1}(1-y^{\lambda_{2(t)}})^{\beta_{(t)}-1}(1-y^{\lambda_{1(t)}})^{\alpha_t}dy$.

8. Set $t = t+1$.

9. Repeat steps 3–8, for T times.

Using this algorithm, under the squared error loss function, the R Bayes estimate will be resulted from

$$\widehat{R}^{\text{MC}} = \frac{1}{T-M}\sum_{t=M+1}^{T} R_t, \tag{4.4}$$

where M is the burn-in period. Moreover, a $100(1-\gamma)\%$ HPD credible interval of R can be constructed by applying the method accomplished by Chen and Shao [4].

5. SIMULATION STUDY AND DATA ANALYSIS

In this section, we compare the performance of different methods by Monte Carlo simulations and analyze two real data sets to illustrative aims.

5.1. Numerical experiments and discussions

In this section, we compare the behavior of various estimates by Monte Carlo simulations, under different censoring schemes. The comparison among estimates is accomplished in terms of mean squared errors (MSEs). Also, the comparison of confidence intervals is performed in terms of average lengths and coverage percentages. We apply different schemes, parameters, and hyper parameters to implement the simulation study. All results are reported based on 3000 replications. Also, the nominal level is 0.95 in comparison with the confidence intervals. We utilize the different censoring schemes as:

$$\text{Scheme 1: } R_1 = ... = R_{n-1} = 0,\ R_n = N - n,$$
$$\text{Scheme 2: } R_1 = ... = R_n = \frac{N-n}{n},$$
$$\text{Scheme 3: } R_1 = ... = R_{\frac{n}{2}} = 0,\ R_{\frac{n}{2}+1} = ... = R_n = \frac{2(N-n)}{n}.$$

We can interpret theses schemes as follows. In Scheme 1, the number of removal units at the first, second and so on until reaching the $(n-1)$-th failure times is zero and we remove all $N-n$ units at the n-th failure time. We use Scheme 2 and 3 when $N-n$ to be divisible by n, and n must be an even number. In Scheme 2, the number of removal units at the first, second and so on until reaching the (n)-th failure times is $\frac{N-n}{n}$. In Scheme 3, the number of removal units at the first, second and so on until reaching the $(\frac{n}{2})$-th failure times is zero and the number of removal units at the $\frac{n}{2}+1$, and so on up to the (n)-th failure times is $\frac{2(N-n)}{n}$. All of these schemes are considered for two values of T as 0.7 and 0.9, respectively.

In the First case, by assuming the unknown common shape parameter λ, we choose $\alpha = \beta = \lambda = 2$, without any loss of generality. Also, Bayesian inference are given in terms of three priors as: Prior 1: $a_j = 0,\ b_j = 0,\ j = 1,2,3$, Prior 2: $a_j = 1,\ b_j = 0.1,\ j = 1,2,3$, and Prior 3: $a_j = 2,\ b_j = 0.2,\ j = 1,2,3$. Moreover, we noted that the number of iterations in the MCMC method is $T = 5000$, and the threshold of burn-in is 2000. In this case, we obtained the Biases and MSEs of MLE using (2.5), AMLE using (2.10), Bayes estimates of R through Lindley's approximation and MCMC method using (2.15) and (2.16), respectively. The results are shown in Table 1. Additionally, we derived the asymptotic confidence and HPD credible intervals of R. Theses results are displayed in Table 2. By the above chosen, R was obtained equal to 0.5. Also, using the numerical method, we obtain the mean and variance of R as a random variable. Based on Priors 2 and 3, the variance of R is 0.0833 and 0.05, respectively, and the mean of R is 0.5 for both priors. So we expect that the performance of MSE is the best using Prior 3.

In the second case, by assuming the known common shape parameter λ, we choose $\alpha = \beta = \lambda = 3$, without loss of generality. Also, Bayesian inference are given in terms of three priors as: Prior 4: $a_j = 0,\ b_j = 0,\ j = 1,2$, Prior 5: $a_j = 1,\ b_j = 0.1,\ j = 1,2$, and Prior 6: $a_j = 2,\ b_j = 0.2,\ j = 1,2$. In this case, we obtained the Biases and MSEs of MLE, Bayes estimates and 95% Bayesian intervals of R using (3.1), (3.3) and (3.4), respectively. The results are indicated in Table 3. Similar to the previous case, we expect that the performance of MSE be the best using Prior 6.

In the third case, assuming the different second shape parameters λ_1 and λ_2, we choose $\alpha = \beta = \lambda_1 = \lambda_2 = 2$, without any loss of generality. Also, Bayesian inference are presented based on three priors as: Prior 7: $a_j = 0,\ b_j = 0,\ j = 1,2,3,4$, Prior 8: $a_j = 1,\ b_j = 0.1,\ j = 1,2,3,4$, and Prior 9: $a_j = 2,\ b_j = 0.2,\ j = 1,2,3,4$. Also, we noted that the number of iterations in the MCMC method is $T = 5000$, and the threshold of burn-in is 2000. In this case, we obtained the Biases and MSEs of MLE, AMLE and Bayes estimate by applying MCMC method using (4.1), (4.3) and (4.4), respectively. Also, the results are indicated in Table 4.

Table 1: Biases and MSEs for estimates of R when λ is unknown.

(N,n,T)	C.S	AMLE		MLE		Prior 1				Prior 2				Prior 3			
						MCMC		Lindley		MCMC		Lindley		MCMC		Lindley	
		|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE
(40,10,0.7)	(1,1)	0.0119	0.0179	0.0147	0.0175	0.0147	0.0103	0.0139	0.0172	0.0140	0.0099	0.0157	0.0165	0.0136	0.0095	0.0176	0.0137
	(2,2)	0.0176	0.0220	0.0187	0.0268	0.0153	0.0115	0.0175	0.0215	0.0154	0.0110	0.0178	0.0210	0.0150	0.0107	0.0181	0.0149
	(3,3)	0.0234	0.0175	0.0258	0.0179	0.0213	0.0100	0.0209	0.0161	0.0206	0.0096	0.0242	0.0140	0.0252	0.0093	0.0262	0.0123
	(1,2)	0.1521	0.0614	0.1547	0.0699	0.1383	0.0322	0.1463	0.0524	0.1364	0.0314	0.1633	0.0435	0.1340	0.0305	0.1802	0.0358
(60,10,0.7)	(1,1)	0.0015	0.0209	0.0038	0.0204	0.0047	0.0113	0.0036	0.0199	0.0039	0.0110	0.0071	0.0177	0.0041	0.0106	0.0022	0.0174
	(2,2)	0.0010	0.0195	0.0024	0.0192	0.0026	0.0107	0.0001	0.0184	0.0023	0.0104	0.0009	0.0162	0.0025	0.0101	0.0016	0.0156
	(3,3)	0.0165	0.0178	0.0163	0.0178	0.0170	0.0095	0.0152	0.0169	0.0160	0.0092	0.0160	0.0144	0.0161	0.0092	0.0167	0.0123
	(1,2)	0.0513	0.0532	0.1497	0.0531	0.1418	0.0353	0.1415	0.0526	0.1397	0.0349	0.1617	0.0420	0.1379	0.0346	0.1819	0.0358
(40,20,0.7)	(1,1)	0.0057	0.0086	0.0047	0.0085	0.0058	0.0050	0.0045	0.0076	0.0054	0.0049	0.0048	0.0068	0.0051	0.0049	0.0051	0.0061
	(2,2)	0.0075	0.0089	0.0073	0.0087	0.0062	0.0055	0.0070	0.0077	0.0063	0.0054	0.0071	0.0070	0.0068	0.0052	0.0072	0.0064
	(3,3)	0.0109	0.0063	0.0160	0.0064	0.0148	0.0042	0.0153	0.0055	0.0148	0.0042	0.0157	0.0053	0.0137	0.0040	0.0161	0.0050
	(1,2)	0.2172	0.0485	0.1702	0.0479	0.1647	0.0311	0.1651	0.0395	0.1639	0.0308	0.1753	0.0355	0.1629	0.0304	0.1856	0.0316
(60,20,0.7)	(1,1)	0.0035	0.0082	0.0031	0.0086	0.0023	0.0049	0.0030	0.0077	0.0022	0.0048	0.0033	0.0068	0.0022	0.0048	0.0035	0.0059
	(2,2)	0.0011	0.0083	0.0011	0.0087	0.0010	0.0055	0.0011	0.0082	0.0009	0.0054	0.0010	0.0073	0.0010	0.0053	0.0011	0.0065
	(3,3)	0.0023	0.0114	0.0020	0.0163	0.0017	0.0054	0.0019	0.0073	0.0018	0.0053	0.0021	0.0067	0.0017	0.0052	0.0022	0.0061
	(1,2)	0.1870	0.0466	0.1709	0.0469	0.1709	0.0352	0.1657	0.0426	0.1697	0.0348	0.01774	0.0376	0.1687	0.0345	0.1892	0.0356
(40,10,0.9)	(1,1)	0.0134	0.0172	0.0324	0.0170	0.0292	0.0097	0.0306	0.0161	0.0284	0.0094	0.0347	0.0147	0.0284	0.0092	0.0388	0.0122
	(2,2)	0.0084	0.0188	0.0031	0.0189	0.0054	0.0087	0.0030	0.0168	0.0052	0.0084	0.0058	0.0138	0.0055	0.0083	0.0087	0.0114
	(3,3)	0.0122	0.0146	0.0173	0.0143	0.0146	0.0093	0.0163	0.0133	0.0146	0.0092	0.0176	0.0129	0.0137	0.0088	0.0189	0.0109
	(1,2)	0.1526	0.0509	0.1542	0.0541	0.1384	0.0276	0.1455	0.0454	0.1366	0.0270	0.1629	0.0374	0.1347	0.0262	0.1804	0.0305
(60,10,0.9)	(1,1)	0.0040	0.0201	0.0066	0.0203	0.0022	0.0098	0.0063	0.0195	0.0028	0.0095	0.0068	0.0163	0.0032	0.0093	0.0072	0.0130
	(2,2)	0.0085	0.0189	0.0086	0.0180	0.0053	0.0098	0.0083	0.0179	0.0043	0.0095	0.0064	0.0152	0.0050	0.0092	0.0045	0.0121
	(3,3)	0.0055	0.0145	0.0138	0.0143	0.0113	0.0083	0.0130	0.0139	0.0116	0.0082	0.0138	0.0117	0.0114	0.0078	0.0147	0.0100
	(1,2)	0.1439	0.0499	0.1451	0.0475	0.1293	0.0331	0.1369	0.0453	0.1273	0.0323	0.1540	0.0373	0.1264	0.0314	0.1310	0.0330
(40,20,0.9)	(1,1)	0.0040	0.0084	0.0045	0.0084	0.0036	0.0045	0.0044	0.0074	0.0038	0.0045	0.0046	0.0066	0.0040	0.0044	0.0049	0.0060
	(2,2)	0.0059	0.0062	0.0059	0.0068	0.0044	0.0036	0.0057	0.0052	0.0037	0.0035	0.0059	0.0048	0.0038	0.0034	0.0060	0.0044
	(3,3)	0.0041	0.0048	0.0044	0.0049	0.0052	0.0036	0.0043	0.0046	0.0047	0.0035	0.0040	0.0045	0.0046	0.0034	0.0037	0.0043
	(1,2)	0.1961	0.0415	0.1581	0.0436	0.1476	0.0259	0.1460	0.0321	0.1468	0.0256	0.1551	0.0288	0.1454	0.0252	0.1461	0.0257
(60,20,0.9)	(1,1)	0.0012	0.0081	0.0010	0.0082	0.0009	0.0047	0.0010	0.0076	0.0010	0.0046	0.0011	0.0067	0.0009	0.0045	0.0009	0.0058
	(2,2)	0.0006	0.0081	0.0004	0.0080	0.0007	0.0054	0.0004	0.0079	0.0001	0.0053	0.0006	0.0071	0.0002	0.0052	0.0007	0.0063
	(3,3)	0.0010	0.0079	0.0001	0.0075	0.0004	0.0051	0.0001	0.0069	0.0006	0.0050	0.0003	0.0064	0.0003	0.0048	0.0004	0.0059
	(1,2)	0.1895	0.0462	0.1704	0.0450	0.1705	0.0251	0.1656	0.0422	0.1695	0.0243	0.1774	0.0353	0.1687	0.0240	0.1692	0.0279

Table 2: Average confidence/credible lengths and coverage percentages for estimates of R when λ is unknown.

(N, n, T)	C.S	AMLE		MLE		Prior 1		Prior 2		Prior 3	
		length	C.P	length	C.P	length	C.P	length	C.P	length	C.P
(40,10,0.7)	(1,1)	0.4374	0.8710	0.4197	0.8710	0.4061	0.9000	0.3922	0.9020	0.3791	0.9100
	(2,2)	0.4352	0.8760	0.4216	0.8750	0.4043	0.9020	0.3931	0.9070	0.3787	0.9080
	(3,3)	0.4543	0.8870	0.4270	0.8830	0.4117	0.9030	0.3956	0.9080	0.3813	0.9080
	(1,2)	0.4137	0.8940	0.4009	0.8900	0.3887	0.9010	0.3788	0.9040	0.3669	0.9060
(60,10,0.7)	(1,1)	0.4369	0.8810	0.4242	0.8860	0.4074	0.9120	0.3912	0.9130	0.3831	0.9140
	(2,2)	0.4366	0.8960	0.4221	0.8900	0.4064	0.9080	0.3915	0.9110	0.3806	0.9170
	(3,3)	0.4280	0.9090	0.4209	0.9050	0.4055	0.9200	0.3932	0.9210	0.3799	0.9260
	(1,2)	0.4329	0.9010	0.3987	0.9020	0.3764	0.9200	0.3661	0.9230	0.3605	0.9280
(40,20,0.7)	(1,1)	0.3090	0.9180	0.3055	0.9140	0.3001	0.9350	0.2926	0.9380	0.2888	0.9390
	(2,2)	0.3045	0.9290	0.3082	0.9230	0.3030	0.9340	0.2968	0.9360	0.2903	0.9400
	(3,3)	0.2989	0.9100	0.3148	0.9120	0.3099	0.9350	0.3028	0.9360	0.2947	0.9370
	(1,2)	0.2897	0.9340	0.2877	0.9340	0.2730	0.9360	0.2728	0.9380	0.2690	0.9390
(60,20,0.7)	(1,1)	0.3097	0.9240	0.3051	0.9270	0.2980	0.9310	0.2930	0.9310	0.2874	0.9330
	(2,2)	0.3065	0.9110	0.3049	0.9120	0.2983	0.9310	0.2912	0.9320	0.2888	0.9330
	(3,3)	0.3043	0.9280	0.3066	0.9230	0.3029	0.9360	0.2942	0.9370	0.2903	0.9390
	(1,2)	0.2893	0.9340	0.2807	0.9320	0.2697	0.9380	0.2614	0.9390	0.2599	0.9400
(40,10,0.9)	(1,1)	0.4370	0.8810	0.4135	0.8880	0.4019	0.9150	0.3864	0.9150	0.3780	0.9180
	(2,2)	0.4350	0.8850	0.4152	0.8880	0.4020	0.9150	0.3902	0.9160	0.3783	0.9170
	(3,3)	0.4313	0.8870	0.4250	0.8840	0.4078	0.9180	0.3948	0.9200	0.3810	0.9270
	(1,2)	0.4115	0.9000	0.3988	0.9050	0.3769	0.9150	0.3708	0.9200	0.3612	0.9210
(60,10,0.9)	(1,1)	0.4368	0.9020	0.4200	0.9080	0.4042	0.9290	0.3895	0.9300	0.3796	0.9330
	(2,2)	0.4350	0.8940	0.4219	0.8960	0.4039	0.9220	0.3906	0.9240	0.3804	0.9290
	(3,3)	0.4211	0.8900	0.4187	0.8940	0.4055	0.9210	0.3920	0.9220	0.3778	0.9270
	(1,2)	0.4319	0.9030	0.3957	0.9050	0.3717	0.9300	0.3659	0.9310	0.3567	0.9330
(40,20,0.9)	(1,1)	0.3077	0.9240	0.3040	0.9230	0.2973	0.9390	0.2920	0.9400	0.2848	0.9430
	(2,2)	0.3023	0.9320	0.3038	0.9320	0.2970	0.9390	0.2925	0.9450	0.2845	0.9480
	(3,3)	0.2909	0.9270	0.3028	0.9260	0.2963	0.9390	0.2916	0.9390	0.2844	0.9400
	(1,2)	0.2820	0.9290	0.2863	0.9210	0.2728	0.9420	0.2719	0.9460	0.2682	0.9470
(60,20,0.9)	(1,1)	0.3095	0.9360	0.3046	0.9350	0.2979	0.9400	0.2919	0.9420	0.2866	0.9490
	(2,2)	0.3065	0.9290	0.3029	0.9290	0.2977	0.9390	0.2873	0.9400	0.2859	0.9410
	(3,3)	0.2974	0.9340	0.3040	0.9360	0.2974	0.9420	0.2904	0.9440	0.2857	0.9500
	(1,2)	0.2823	0.9230	0.2767	0.9280	0.2598	0.9390	0.2562	0.9400	0.2558	0.9410

To monitor the convergence of the MCMC method, in the first and third cases, we studied the trace plots for various censoring schemes and parameters. In all cases, the trace plots indicated that the MCMC method is converged. Some of these plots are displayed in Figures 2–5. It is notable that Figures 2 and 3 have considered the problem in the first case (when the common second shape parameter is unknown), and Figures 4 and 5 have considered the problem in the third case (when all parameters are different and unknown), respectively.

Due to the information of Table 1, we observed that the Bayes estimates have the minimum value of MSEs. Also, in Bayesian inference, the informative priors performance was better than non-informative ones and the best performance, in terms of MSE, was belonged to Prior 3. Furthermore, the MCMC method performs better, in comparison with Lindley's approximation. From Table 2, we observed that the HPD credible intervals indicated a better performance compared to the asymptotic confidence intervals. Also, in Bayesian inference, the best performance belonged to Prior 3, namely, the HPD credible intervals based on Prior 3, have the smallest average lengths and largest coverage percentages.

Table 3: Biases, MSEs, Average confidence/credible lengths and coverage percentages for estimates of R when λ is known.

(N,n,T)	C.S	MLE		Asymp. C.I		Prior 4				Prior 5				Prior 6			
						Est.		C.I		Est.		C.I		Est.		C.I	
		|Bias|	MSE	length	C.P	|Bias|	MSE	length	C.P	|Bias|	MSE	length	C.P	|Bias|	MSE	length	C.P
(40,10,0.7)	(1,1)	0.0010	0.0126	0.4189	0.8730	0.0010	0.0109	0.4067	0.9230	0.0008	0.0067	0.3946	0.9240	0.0007	0.0044	0.3822	0.9290
	(2,2)	0.0054	0.0115	0.4171	0.8900	0.0052	0.0105	0.4051	0.9190	0.0040	0.0061	0.3944	0.9250	0.0032	0.0040	0.3819	0.9290
	(3,3)	0.0013	0.0127	0.4197	0.8810	0.0012	0.0117	0.4072	0.9020	0.0008	0.0068	0.3964	0.9050	0.0006	0.0045	0.3825	0.9050
	(1,2)	0.1449	0.0328	0.3819	0.8970	0.1393	0.0315	0.3759	0.9010	0.1117	0.0252	0.3758	0.9030	0.0936	0.0206	0.3687	0.9040
(60,10,0.7)	(1,1)	0.0058	0.0128	0.4190	0.9010	0.0056	0.0117	0.4067	0.9250	0.0045	0.0069	0.3958	0.9260	0.0038	0.0046	0.3816	0.9290
	(2,2)	0.0008	0.0121	0.4175	0.9040	0.0007	0.0111	0.4054	0.9140	0.0005	0.0064	0.3948	0.9250	0.0004	0.0042	0.3818	0.9290
	(3,3)	0.0018	0.0125	0.4177	0.9030	0.0018	0.0115	0.4058	0.9150	0.0014	0.0067	0.3946	0.9220	0.0012	0.0044	0.3818	0.9250
	(1,2)	0.1535	0.0394	0.3805	0.9090	0.1477	0.0379	0.3750	0.9150	0.1188	0.0306	0.3749	0.9180	0.0997	0.0253	0.3676	0.9220
(40,20,0.7)	(1,1)	0.0003	0.0059	0.3024	0.9170	0.0003	0.0056	0.2973	0.9320	0.0002	0.0042	0.2927	0.9340	0.0002	0.0033	0.2864	0.9360
	(2,2)	0.0017	0.0058	0.3079	0.9190	0.0017	0.0056	0.3028	0.9350	0.0014	0.0042	0.2973	0.9360	0.0013	0.0033	0.2916	0.9370
	(3,3)	0.0016	0.0065	0.3174	0.9180	0.0016	0.0062	0.3112	0.9320	0.0013	0.0046	0.3053	0.9350	0.0012	0.0036	0.2988	0.9370
	(1,2)	0.1477	0.0327	0.2802	0.9170	0.1447	0.0304	0.2778	0.9310	0.1280	0.0252	0.2771	0.9330	0.1149	0.0140	0.2748	0.9340
(60,20,0.7)	(1,1)	0.0001	0.0064	0.3020	0.9320	0.0001	0.0061	0.2975	0.9340	0.0006	0.0046	0.2920	0.9350	0.0003	0.0036	0.2867	0.9380
	(2,2)	0.0042	0.0064	0.3030	0.9310	0.0041	0.0061	0.2980	0.9330	0.0036	0.0046	0.2926	0.9340	0.0032	0.0036	0.2873	0.9340
	(3,3)	0.0007	0.0057	0.3070	0.9180	0.0007	0.0055	0.3016	0.9300	0.0006	0.0041	0.2966	0.9320	0.0005	0.0032	0.2900	0.9320
	(1,2)	0.1814	0.0356	0.2626	0.9300	0.1179	0.0331	0.2623	0.9350	0.1592	0.0216	0.2611	0.9360	0.1442	0.0154	0.2600	0.9370
(40,10,0.9)	(1,1)	0.0051	0.0119	0.4154	0.8900	0.0049	0.0116	0.4037	0.9150	0.0036	0.0064	0.3935	0.9190	0.0028	0.0043	0.3816	0.9210
	(2,2)	0.0019	0.0104	0.4160	0.9030	0.0019	0.0095	0.4044	0.9240	0.0013	0.0055	0.3935	0.9250	0.0010	0.0037	0.3814	0.9270
	(3,3)	0.0118	0.0120	0.4173	0.9000	0.0113	0.0110	0.4057	0.9340	0.0086	0.0064	0.3945	0.9350	0.0069	0.0042	0.3822	0.9390
	(1,2)	0.1462	0.0322	0.3790	0.9050	0.1406	0.0299	0.3736	0.9250	0.1127	0.0198	0.3733	0.9290	0.0944	0.0159	0.3675	0.9300
(60,10,0.9)	(1,1)	0.0017	0.0114	0.4182	0.8920	0.0016	0.0105	0.4060	0.9240	0.0011	0.0061	0.3951	0.9260	0.0008	0.0040	0.3815	0.9270
	(2,2)	0.0040	0.0118	0.4170	0.9050	0.0038	0.0108	0.4042	0.9240	0.0028	0.0063	0.3945	0.9250	0.0022	0.0042	0.3817	0.9290
	(3,3)	0.0031	0.0116	0.4173	0.8850	0.0030	0.0106	0.4053	0.9230	0.0025	0.0062	0.3942	0.9240	0.0021	0.0041	0.3817	0.9250
	(1,2)	0.1497	0.0393	0.3766	0.9100	0.1440	0.0378	0.3707	0.9330	0.1156	0.0304	0.3700	0.9340	0.0969	0.0251	0.3664	0.9370
(40,20,0.9)	(1,1)	0.0012	0.0058	0.3019	0.9310	0.0012	0.0056	0.2971	0.9410	0.0010	0.0042	0.2918	0.9440	0.0008	0.0033	0.2862	0.9470
	(2,2)	0.0010	0.0057	0.3028	0.9210	0.0010	0.0055	0.2981	0.9430	0.0008	0.0041	0.2925	0.9440	0.0007	0.0032	0.2869	0.9490
	(3,3)	0.0008	0.0047	0.3031	0.9270	0.0008	0.0045	0.2981	0.9410	0.0002	0.0034	0.2925	0.9430	0.0003	0.0026	0.2870	0.9450
	(1,2)	0.1620	0.0263	0.2685	0.9260	0.1588	0.0253	0.2683	0.9410	0.1415	0.0193	0.2674	0.9430	0.1278	0.0137	0.2669	0.9460
(60,20,0.9)	(1,1)	0.006	0.0063	0.3020	0.9280	0.0005	0.0061	0.2970	0.9440	0.0005	0.0045	0.2919	0.9470	0.0005	0.0035	0.2867	0.9480
	(2,2)	0.0025	0.0057	0.3020	0.9300	0.0024	0.0055	0.2970	0.9440	0.0021	0.0041	0.2921	0.9480	0.0019	0.0032	0.2866	0.9490
	(3,3)	0.0052	0.0055	0.3021	0.9280	0.0051	0.0053	0.2969	0.9430	0.0044	0.0040	0.2922	0.9460	0.0038	0.0031	0.2866	0.9500
	(1,2)	0.1818	0.0346	0.2619	0.9300	0.1782	0.0321	0.2612	0.9460	0.1596	0.0209	0.2606	0.9460	0.1447	0.0148	0.2593	0.9530

Table 4: Biases, MSEs, Average credible lengths and coverage percentages for estimates of R in general case.

(N,n,T)	C.S	AMLE		MLE		Prior 7				Prior 8				Prior 9			
						Est.		C.I		Est.		C.I		Est.		C.I	
		|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	length	C.P	|Bias|	MSE	length	C.P	|Bias|	MSE	length	C.P
(40,10,0.7)	(1,1)	0.0129	0.0234	0.0169	0.0211	0.0140	0.0111	0.4083	0.9000	0.0111	0.0080	0.3933	0.9010	0.0100	0.0060	0.3772	0.9060
	(2,2)	0.0009	0.0183	0.0033	0.0169	0.0007	0.0078	0.4114	0.9120	0.0001	0.0056	0.3948	0.9170	0.0007	0.0081	0.3825	0.9190
	(3,3)	0.0169	0.0182	0.0168	0.0194	0.0152	0.0115	0.4053	0.9000	0.0139	0.0080	0.3944	0.9060	0.0120	0.0060	0.3805	0.9100
	(1,2)	0.1954	0.0464	0.1940	0.0494	0.1395	0.0320	0.3801	0.9110	0.1210	0.0279	0.3729	0.9130	0.1073	0.0245	0.3629	0.9180
(60,10,0.7)	(1,1)	0.0316	0.0171	0.0323	0.0177	0.0095	0.0090	0.4095	0.9190	0.0077	0.0065	0.3948	0.9220	0.0064	0.0048	0.3824	0.9240
	(2,2)	0.0101	0.0223	0.0108	0.0231	0.0046	0.0110	0.4044	0.9050	0.0037	0.0079	0.3936	0.9060	0.0035	0.0060	0.3798	0.9120
	(3,3)	0.0081	0.0152	0.0084	0.0159	0.0073	0.0089	0.4080	0.9220	0.0067	0.0064	0.3905	0.9270	0.0064	0.0048	0.3799	0.9280
	(1,2)	0.2451	0.0639	0.2334	0.0662	0.1413	0.0361	0.3793	0.9200	0.1228	0.0337	0.3746	0.9240	0.1103	0.0297	0.3639	0.9260
(40,20,0.7)	(1,1)	0.0116	0.0077	0.0181	0.0079	0.0093	0.0065	0.2980	0.9370	0.0087	0.0056	0.2912	0.9390	0.0079	0.0047	0.2856	0.9400
	(2,2)	0.0165	0.0135	0.0146	0.0134	0.0041	0.0053	0.3020	0.9330	0.0037	0.0045	0.2975	0.9350	0.0032	0.0038	0.2889	0.9390
	(3,3)	0.0099	0.0171	0.0157	0.0158	0.0139	0.0049	0.3127	0.9320	0.0125	0.0040	0.3039	0.9330	0.0113	0.0034	0.2963	0.9380
	(1,2)	0.1755	0.0374	0.1715	0.0371	0.1583	0.0313	0.2686	0.9340	0.1474	0.0273	0.2666	0.9360	0.1377	0.0239	0.2632	0.9400
(60,20,0.7)	(1,1)	0.0116	0.0098	0.0103	0.0090	0.0074	0.0056	0.2984	0.9350	0.0071	0.0047	0.2930	0.9370	0.0060	0.0040	0.2874	0.9390
	(2,2)	0.0133	0.0105	0.0134	0.0102	0.0085	0.0073	0.2996	0.9320	0.0078	0.0060	0.2920	0.9350	0.0068	0.0051	0.2867	0.9380
	(3,3)	0.0180	0.0130	0.0138	0.0132	0.0127	0.0052	0.3025	0.9340	0.0114	0.0044	0.2950	0.9380	0.0107	0.0037	0.2886	0.9400
	(1,2)	0.2716	0.0517	0.2179	0.0522	0.1801	0.0358	0.2606	0.9310	0.1682	0.0317	0.2600	0.9360	0.1575	0.0282	0.2589	0.9380
(40,10,0.9)	(1,1)	0.0072	0.0121	0.0066	0.0160	0.0076	0.0090	0.4048	0.9230	0.0067	0.0066	0.3922	0.9240	0.0055	0.0050	0.3705	0.9290
	(2,2)	0.0007	0.0137	0.0023	0.0163	0.0006	0.0076	0.4076	0.9150	0.0001	0.0048	0.3940	0.9190	0.0007	0.0051	0.3821	0.9290
	(3,3)	0.0169	0.0160	0.0098	0.0167	0.0123	0.0105	0.4050	0.9210	0.0104	0.0074	0.3935	0.9220	0.0088	0.0054	0.3800	0.9290
	(1,2)	0.1497	0.0362	0.1893	0.0345	0.1285	0.0281	0.3719	0.9330	0.1108	0.0212	0.3532	0.9350	0.0980	0.0167	0.3464	0.9390
(60,10,0.9)	(1,1)	0.0160	0.0138	0.0155	0.0130	0.0039	0.0089	0.4062	0.9310	0.0022	0.0064	0.3946	0.9370	0.0023	0.0040	0.3811	0.9390
	(2,2)	0.0041	0.0220	0.0051	0.0221	0.0014	0.0104	0.4035	0.9300	0.0028	0.0073	0.3930	0.9360	0.0021	0.0051	0.3781	0.9360
	(3,3)	0.0077	0.0142	0.0081	0.0148	0.0073	0.0071	0.4030	0.9210	0.0063	0.0058	0.3811	0.9260	0.0042	0.0039	0.3774	0.9280
	(1,2)	0.2362	0.0636	0.2134	0.0630	0.1299	0.0316	0.3740	0.9150	0.1120	0.0245	0.3668	0.9170	0.0996	0.0198	0.3602	0.9190
(40,20,0.9)	(1,1)	0.0011	0.0073	0.0018	0.0064	0.0004	0.0052	0.2957	0.9400	0.0009	0.0043	0.2901	0.9430	0.0004	0.0037	0.2843	0.9460
	(2,2)	0.0154	0.0107	0.0014	0.0106	0.0029	0.0038	0.2992	0.9390	0.0025	0.0031	0.2919	0.9400	0.0023	0.0026	0.2862	0.9440
	(3,3)	0.0044	0.0146	0.0050	0.0140	0.0060	0.0031	0.2987	0.9410	0.0057	0.0025	0.2931	0.9440	0.0055	0.0021	0.2874	0.9470
	(1,2)	0.1634	0.0339	0.1486	0.0301	0.1559	0.0271	0.2677	0.9400	0.1452	0.0202	0.2654	0.9450	0.1358	0.0158	0.2630	0.9480
(60,20,0.9)	(1,1)	0.0042	0.0089	0.0049	0.0087	0.0043	0.0051	0.2973	0.9420	0.0040	0.0043	0.2923	0.9430	0.0041	0.0036	0.2848	0.9450
	(2,2)	0.0074	0.0082	0.0125	0.0073	0.0071	0.0050	0.2956	0.9400	0.0070	0.0043	0.2892	0.9450	0.0064	0.0036	0.2844	0.9470
	(3,3)	0.0024	0.0079	0.0001	0.0073	0.0001	0.0050	0.2985	0.9390	0.0007	0.0042	0.2923	0.9430	0.0007	0.0035	0.2866	0.9460
	(1,2)	0.2180	0.0514	0.2179	0.0509	0.1729	0.0271	0.2598	0.9400	0.1615	0.0203	0.2577	0.9440	0.1517	0.0161	0.2574	0.9520

As shown in Table 3, we observed that the Bayes estimates have the minimum value of MSEs. Also, in Bayesian inference, the informative priors performed better than non-informative ones and the best performance, in terms of MSE, was belonged to Prior 6. Moreover, we observed that the Bayesian credible intervals have the better performance, in comparison with the asymptotic confidence intervals. Also, in Bayesian inference, the best performance belonged to Prior 6, namely, the Bayesian credible intervals based on Prior 6 have the smallest average lengths and largest coverage percentages.

As we observe from Table 4, the Bayes estimates have the minimum value of MSEs. Also, in Bayesian inference, the informative priors perform better than non-informative ones and the best performance, in terms of MSE, was belonged to Prior 9. Moreover, we observed that HPD credible intervals based on informative priors, indicated better performance compared to non-informative ones.

To tell the truth, from Tables 1, 3 and 4, along by increasing n for fixed N and T, and also with increasing T for fixed N and n, the MSEs of all estimates decrease in all cases. This can be due to the fact in both of the above mentioned cases, some additional information is gathered. Moreover, from Tables 2, 3 and 4, with increasing n for fixed N and T, and also with increasing T for fixed N and n, the average confidence lengths decrease and the associated coverage percentages increase, in all cases.

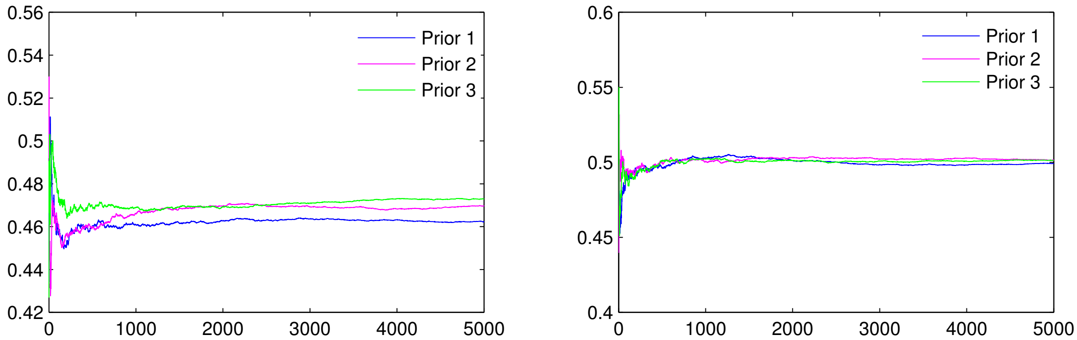

Figure 2: Trace plots with C.S $(1,1)$ (left) and $(3,3)$ (right), for $(N,n,T) = (40,10,0.7)$, in common shape parameter λ.

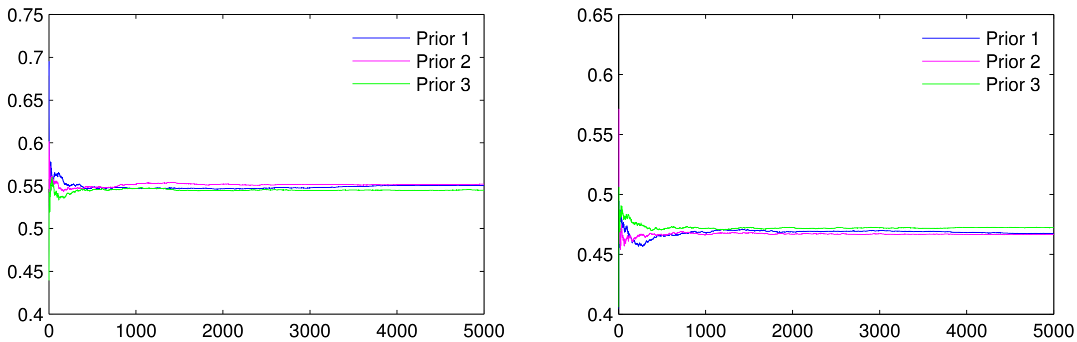

Figure 3: Trace plots with C.S $(2,2)$ (left) and $(3,3)$ (right), for $(N,n,T) = (60,20,0.9)$, in common shape parameter λ.

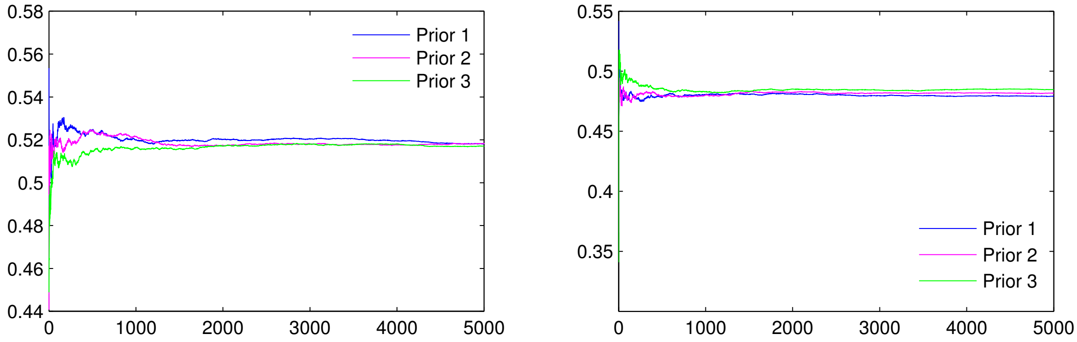

Figure 4: Trace plots with C.S $(1, 3)$ (left) and $(1, 1)$ (right), for $(N, n, T) = (40, 20, 0.7)$, in general case.

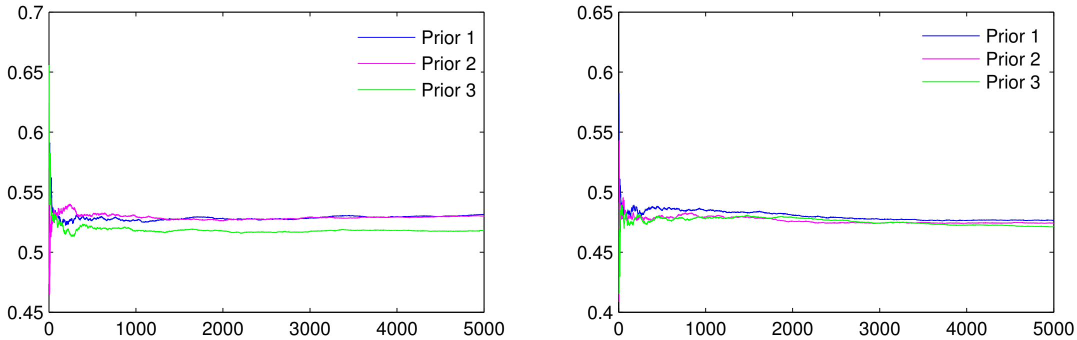

Figure 5: Trace plots with C.S $(2, 3)$ (left) and $(1, 1)$ (right), for $(N, n, T) = (60, 10, 0.9)$, in general case.

5.2. Data analysis

In this section, we analyze two pair of real data set for illustrative proposes.

Example 5.1. In the first example, we use the monthly water capacity of the Shasta reservoir in California, USA, see data in http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA. Some authors such as Sultana *et al.* [25], Kohansal [9], Kizilaslan and Nadar [8], [6] and Nadar *et al.* [19] have been studied this data, previously. From this data, we construct one scenario relating to the excessive drought. In fact, we contract that if the average water capacity in July and August of a same year is more than the water capacity in December, the excessive drought will not occur. With respect to this scenario, we consider the months July, August, and December from 1987 to 2016. So, $X_1, ..., X_{30}$ are the capacity of December and $Y_1, ..., Y_{30}$ are the average capacity of July and August from 1987 to 2016, respectively, and $R = P(X < Y)$ is the probability of non-occurrence of drought. As the range of KuD is $(0, 1)$, all data have been divided by the total capacity of Shasta reservoir, 4552000 acre-feet. This work does not make any change in statistical inference.

At first, we check that the KuD can separately analyze these data sets or not. To fit the KuD, we obtain the initial guess, in the Newton–Raphson method, by using the profile log-likelihood functions, which were indicated in Figure 6. So, we start this method by the starting values 3.45 and 3.65, for X and Y, respectively. By fitting the KuD, for X, $\widehat{\alpha}$, $\widehat{\lambda}$, the Kolmogorov–Smirnov distance and the corresponding p-value are 4.1903, 3.5000, 0.1592 and 0.3916, respectively. Also, for Y, $\widehat{\beta}$, $\widehat{\lambda}$, the Kolmogorov–Smirnov distance and the associated p-value are 3.7828, 3.7700, 0.1218 and 0.7195, respectively. In terms of the p-values, we identify that the KuD provides suitable fits for the data sets. Figures 7 and 8 indicated the empirical distribution functions, PP-plots, and PP-plots with simulated envelope, for X and Y, respectively.

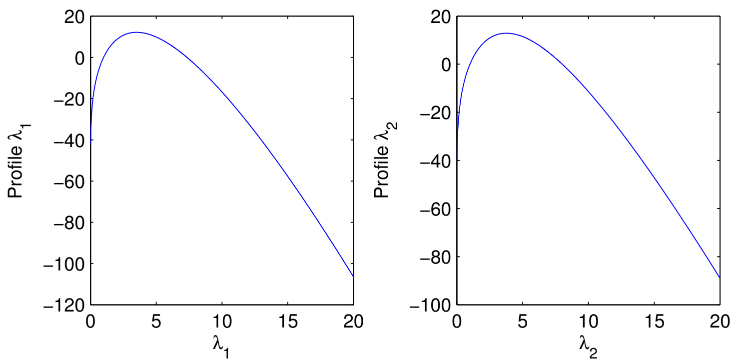

Figure 6: Profile log-likelihood function of λ for X (left) Y (right).

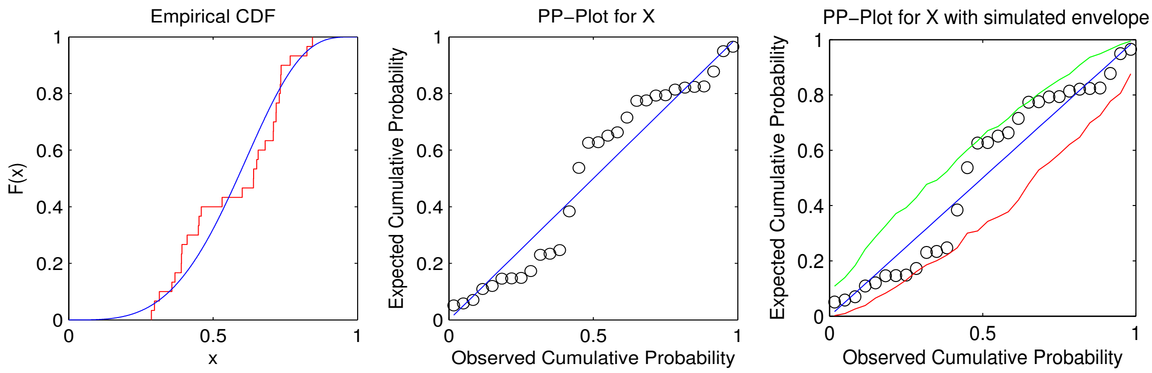

Figure 7: Empirical distribution function (left), PP-plot (center) and PP-plots with simulated envelope (right) for X.

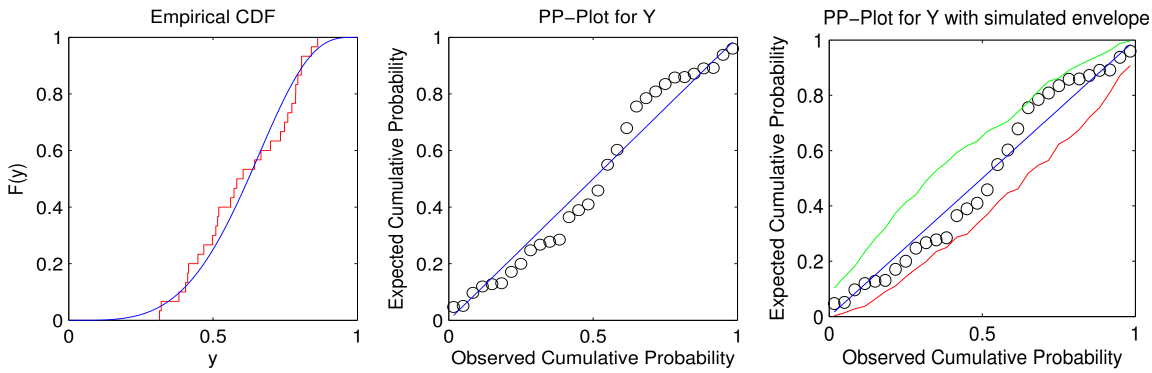

Figure 8: Empirical distribution functions (left)PP-plot (center) and PP plots with simulated envelope (right) for Y.

For the illustrative proposes, we consider two different HP censoring schemes for X and Y as follows:

$$\text{Scheme 1: } [1^{*10}, 0^{*10}],\ T_1 = T_2 = 0.9,$$
$$\text{Scheme 2: } [2^{*10}],\qquad T_1 = T_2 = 0.5.$$

In the first case, when the common shape parameter λ is unknown, for complete data sets, and Schemes 1 and 2, we obtained the ML, AML and Bayes estimates of R with non-informative priors assumption, i.e., $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$ by applying Lindley's approximation and MCMC method. Also, we derived the 95% asymptotic and HPD intervals. The results are listed in Table 5.

Table 5: The ML, AML, Bayes estimates and different confidence/credible intervals of R, in Example 5.1.

		MLE	Asymp. (MLE)	AMLE	Asymp. (AMLE)	Bayes		HPD
						MCMC	Lindley	
λ	Complete	0.5522	(0.4268,0.6776)	0.5641	(0.4403,0.6879)	0.5520	0.5511	(0.4258,0.6707)
	Scheme 1	0.5520	(0.3983,0.7057)	0.5369	(0.3865,0.6927)	0.5523	0.5503	(0.3985,0.7036)
	Scheme 2	0.5723	(0.3563,0.7882)	0.5200	(0.3013,0.7388)	0.5727	0.5673	(0.3530,0.7687)
λ_1, λ_2	Complete	0.5617	—	0.5971	—	0.5647	—	(0.4372,0.6848)
	Scheme 1	0.5533	—	0.5593	—	0.5534	—	(0.3974,0.7027)
	Scheme 2	0.5777	—	0.4899	—	0.5779	—	(0.3501,0.7657)

As we observe, the second shape parameters of two data sets are not exactly same. As a result, in the second case, when the shape parameters λ_1 and λ_2 are different and unknown, for complete data sets, Schemes 1 and 2, we obtained the ML, AML and Bayes estimates of R with non-informative priors assumption, i.e., $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = a_4 = b_4 = 0$, respectively. Also, we derived 95% HPD credible intervals. Theses results are presented in Table 5. By comparing the two schemes, we observed that estimators have smaller standard errors in Scheme 1, compared to Scheme 2, as it was expected. It is notable that the estimation methods which presented a better performance in the simulations are more reliable than the others. So, the results based on the Bayesian estimations and in Bayesian estimation the results obtained by the MCMC method are more preferred, in comparison with the others. Also, we would like to use the HPD credible intervals as the best intervals.

Example 5.2. In the second example, we use the lifetime data for insulation specimens. The length of time was observed until each specimen failed or "broke down". Also, the results for seven groups of specimens, tested at voltages ranging from 26 to 38 kilovolts (kV) were presented. We consider the data sets for 34 kV and 36 kV, reported in Lawless [15], as the strength and stress variables, respectively. Therefore, the parameter $R = P(X < Y)$ can be investigated as the probability of insulation resistance. For the same reason as it was earlier explained in Example 5.1, we have converted all data between 0 and 1. Recently, Kizilaslan and Nadar [7] considered this data set.

At first, we must check that the KuD can analyze these data sets, separately. By fitting the KuD, for X, $\widehat{\alpha}$, $\widehat{\lambda}$, the Kolmogorov–Smirnov distance and the corresponding p-value are

9.7733, 0.84, 0.2103 and 0.4592, respectively. Also, for Y, $\widehat{\beta}$, $\widehat{\lambda}$, the Kolmogorov–Smirnov distance and the associated p-value are 0.8963, 0.3736, 0.2756 and 0.0911, respectively. In terms of the p-values, we observe that the KuD provides suitable fits for the data sets.

For the illustrative proposes, we consider the HP censoring scheme as Scheme 3: $[1^{*5}, 0^{*5}]$, $T_1 = 0.1$ and $[1^{*9}, 0^{*1}]$, $T_2 = 0.2$ for X and Y, respectively.

In the first case, when the common shape parameter λ is unknown, for complete data sets and Scheme 3, we obtained the ML, AML, and Bayes estimates of R with non-informative priors assumption, i.e., $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$ by applying Lindley's approximation and MCMC method. Also, we derived the 95% asymptotic and HPD intervals. These obtained results are listed in Table 6.

As indicated, the second shape parameters of two data sets are not similar. So, when the shape parameters λ_1 and λ_2 are different and unknown, for complete data sets, Schemes 1 and 2, we obtained the ML, AML and Bayes estimates of R with non-informative priors assumption, i.e., $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = a_4 = b_4 = 0$. Also, we derived 95% HPD credible intervals. These results are given in Table 6.

Table 6: The ML, AML, Bayes estimates and different confidence/credible intervals of R, in Example 5.2.

		MLE	Asymp. (MLE)	AMLE	Asymp. (AMLE)	Bayes		HPD
						MCMC	Lindley	
λ	Complete	0.8007	(0.6763,0.9252)	0.7034	(0.5944,0.8619)	0.8016	0.7892	(0.6798,0.8938)
	Scheme 3	0.6368	(0.4151,0.8614)	0.6739	(0.4131,0.9048)	0.6326	0.6273	(0.3851,0.8183)
λ_1, λ_2	Complete	0.7127	—	0.6058	—	0.7252	—	(0.5979,0.8360)
	Scheme 3	0.6371	—	0.6760	—	0.6351	—	(0.3989,0.8234)

To see a motivation based on real data set that presents the need for the new methodology, we consider the progressive scheme, one of the most applicable censoring scheme, for this data set. Comparison between two methodologies (HP and progressive schemes) is performed by obtaining the values of Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan–Quinn information criterion (HQC). We have shown the results in Table 7. From Table 7, by ignoring minor differences, we see that the new methodology (results based on HP scheme) is better than the previous one (results based on the progressive scheme.)

Table 7: AIC, BIC and HQC in comparison of two methodology, in Example 5.2.

		HP				Progressive			
		MLE	AMLE	Lindley	MCMC	MLE	AMLE	Lindley	MCMC
λ	AIC	−42.9761	−37.8133	−42.9575	−42.9762	−42.0119	−37.5979	−42.0107	−42.0119
	BIC	−40.4765	−35.3137	−40.4579	−40.4765	−39.0247	−34.6107	−39.0235	−39.0249
	HQC	−42.7276	−37.5648	−42.7090	−42.7277	−41.4288	−37.0148	−41.4276	−41.4289
λ_1, λ_2	AIC	−41.0665	−37.4275	—	−41.0906	−40.0246	−35.5724	—	−40.0373
	BIC	−37.7337	−34.0946	—	−37.7678	−36.0417	−31.5895	—	−36.0544
	HQC	−40.7352	−37.0962	—	−40.7694	−39.2471	−34.7949	—	−39.2597

6. CONCLUSION

In this paper, we obtain different estimates of the stress-strength parameter, under the hybrid progressive censored scheme, at the time that stress and strength are considered as two independent Kumaraswamy random variables. The problem is going to be solved in three cases. First, when $X \sim \mathrm{Ku}(\alpha, \lambda)$ and $Y \sim \mathrm{Ku}(\beta, \lambda)$, we derive ML, AML and two approximated Bayes estimates by applying Lindley's approximation and MCMC method, due to the lack of explicit forms. Also, we consider the existence and uniqueness of the MLE and construct the asymptotic and HPD intervals for R. Second, when the common second shape parameter, λ, is known, we obtain the MLE and exact Bayes estimate of R. Third, in general case, when $X \sim \mathrm{Ku}(\alpha, \lambda_1)$ and $Y \sim \mathrm{Ku}(\beta, \lambda_2)$, we provide ML, AML and Bayesian inferences of R, respectively.

From the simulation results, which were obtained using the Monte Carlo method, in point estimates, we observed that the Bayes estimates have the minimum value of MSEs. Also, in Bayesian inference, the informative priors perform better than non-informative ones. Furthermore, the MCMC method performs better than Lindley's approximation. In interval estimates, we observed that the HPD credible intervals have a better performance in comparison with the asymptotic confidence intervals. Also, in Bayesian inference, the HPD credible intervals based on informative priors have the smallest average lengths and largest coverage percentages.

A. APPENDIX

Proof of Theorem 2.1: By a simple method, we can rewrite $G(\lambda)$ as:

$$G(\lambda) = \frac{J_1}{\lambda} + G_1(\lambda) + J_1 \frac{G_2(\lambda)}{G_3(\lambda)} + \frac{J_2}{\lambda} + H_1(\lambda) + J_2 \frac{H_2(\lambda)}{H_3(\lambda)},$$

where

$$G_1(\lambda) = \sum_{i=1}^{J_1} \frac{\log(x_i)}{1 - x_i^{\lambda}}, \quad G_2(\lambda) = \sum_{i=1}^{J_1} (R_i + 1) x_i^{\lambda} \frac{\log(x_i)}{1 - x_i^{\lambda}} + R_{J_1}^* T_1^{\lambda} \frac{\log(T_1)}{1 - T_1^{\lambda}},$$

$$G_3(\lambda) = \sum_{i=1}^{J_1} (R_i + 1) \log(1 - x_i^{\lambda}) + R_{J_1}^* \log(1 - T_1^{\lambda}),$$

$$H_1(\lambda) = \sum_{j=1}^{J_2} \frac{\log(y_j)}{1 - y_j^{\lambda}}, \quad H_2(\lambda) = \sum_{j=1}^{J_2} (S_j + 1) y_j^{\lambda} \frac{\log(y_j)}{1 - y_j^{\lambda}} + S_{J_2}^* T_2^{\lambda} \frac{\log(T_2)}{1 - T_2^{\lambda}},$$

$$H_3(\lambda) = \sum_{j=1}^{J_2} (S_j + 1) \log(1 - y_j^{\lambda}) + S_{J_2}^* \log(1 - T_2^{\lambda}).$$

We observe that $\lim_{\lambda \to 0^+} G(\lambda) = \infty$ and $\lim_{\lambda \to \infty} G(\lambda) < 0$. Consequently, $G(\lambda)$ has at least one root in $(0, \infty)$ by the intermediate value theorem. So, it is enough to show that $G'(\lambda) < 0$.

We can obtain $G'(\lambda)$, after accomplishing some steps, as:

$$\begin{aligned} G'(\lambda) &= -\frac{1}{\lambda^2}\left\{G_4(\lambda) - J_1\frac{G_3(\lambda)G_5(\lambda) + \big(G_2(\lambda)\big)^2}{\big(G_3(\lambda)\big)^2}\right\} \\ &\quad -\frac{1}{\lambda^2}\left\{H_4(\lambda) - J_2\frac{H_3(\lambda)H_5(\lambda) + \big(H_2(\lambda)\big)^2}{\big(H_3(\lambda)\big)^2}\right\}, \end{aligned}$$

where

$$\begin{aligned} G_4(\lambda) &= J_1 - \sum_{i=1}^{J_1} x_i^\lambda\left(\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\right)^2, \quad H_4(\lambda) = J_2 - \sum_{j=1}^{J_2} y_j^\lambda\left(\frac{\log(y_j^\lambda)}{1-y_j^\lambda}\right)^2, \\ G_5(\lambda) &= \sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\left(\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\right)^2 + R_{J_1}^*T_1^\lambda\left(\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right)^2, \\ H_5(\lambda) &= \sum_{j=1}^{J_2}(S_j+1)y_j^\lambda\left(\frac{\log(y_j^\lambda)}{1-y_j^\lambda}\right)^2 + S_{J_2}^*T_2^\lambda\left(\frac{\log(T_2^\lambda)}{1-T_2^\lambda}\right)^2. \end{aligned}$$

It can be observed that $G_4(\lambda) > 0$, as $f(x) = x\big(\frac{\log(x)}{1-x}\big)^2$, so $f(x) < 1$ for $x \in (0,1)$. Moreover,

$$\begin{aligned} (G_2(\lambda))^2 &= \left(\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\right)^2 + \left(R_{J_1}^*T_1^\lambda\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right)^2 \\ &\quad +2\left(\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\right)\left(R_{J_1}^*T_1^\lambda\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right) \\ &\le \left(\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\right)\left(\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\left(\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\right)^2\right) + \left(R_{J_1}^*T_1^\lambda\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right)^2 \\ &\quad +\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\left(R_{J_1}^*T_1^\lambda\left(\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right)^2\right) + \sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\big(R_{J_1}^*T_1^\lambda\big) \\ &\le \left(-\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\log(1-x_i^\lambda)\right)\left(\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\left(\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\right)^2\right) \\ &\quad +R_{J_1}^*T_1^\lambda\left(\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right)^2\big(-R_{J_1}^*\log(1-T_1^\lambda)\big) \\ &\quad -\sum_{i=1}^{J_1}(R_i+1)\log(1-x_i^\lambda)\left(R_{J_1}^*T_1^\lambda\left(\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right)^2\right) \\ &\quad +\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\big(-R_{J_1}^*\log(1-T_1^\lambda)\big) \\ &= \left[-\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\log(1-x_i^\lambda) - R_{J_1}^*\log(1-T_1^\lambda)\right] \\ &\quad \times\left[\sum_{i=1}^{J_1}(R_i+1)x_i^\lambda\left(\frac{\log(x_i^\lambda)}{1-x_i^\lambda}\right)^2 + R_{J_1}^*T_1^\lambda\left(\frac{\log(T_1^\lambda)}{1-T_1^\lambda}\right)^2\right] = -G_3(\lambda)G_5(\lambda). \end{aligned}$$

The above equations have been obtained by applying the Cauchy–Schwarz inequality and $x < -\log(1-x)$, $x \in (0,1)$. Consequently, $G'(\lambda) < 0$ and the proof is completed. □

B. APPENDIX

We compute $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\sigma}$ at

$$
\begin{aligned}
A_1 &= \frac{\sum_{i=1}^{J_1}(R_i+1)\beta_i u_i + R^*_{J_1}\beta^*_{J_1}a_1}{\sum_{i=1}^{J_1}(R_i+1)\beta_i + R^*_{J_1}\beta^*_{J_1}}, \quad B_1 = \frac{\sum_{i=1}^{J_1}\alpha_i - \sum_{i=1}^{J_1}R_i(1-\alpha_i) - R^*_{J_1}(1-\alpha^*_{J_1})}{\sum_{i=1}^{J_1}(R_i+1)\beta_i + R^*_{J_1}\beta^*_{J_1}},\\
A_2 &= \frac{\sum_{j=1}^{J_2}(S_j+1)\bar{\beta}_j v_j + S^*_{J_2}\bar{\beta}^*_{J_2}a_2}{\sum_{j=1}^{J_2}(S_j+1)\bar{\beta}_j + S^*_{J_2}\bar{\beta}^*_{J_2}}, \quad B_2 = \frac{\sum_{j=1}^{J_2}\bar{\alpha}_j - \sum_{j=1}^{J_2}S_j(1-\bar{\alpha}_j) - S^*_{J_2}(1-\bar{\alpha}^*_{J_2})}{\sum_{j=1}^{J_2}(S_j+1)\bar{\beta}_j + S^*_{J_2}\bar{\beta}^*_{J_2}},\\
D_1 &= \sum_{i=1}^{J_1}\alpha_i u_i - A_1B_1\left(\sum_{i=1}^{J_1}(R_i+1)\beta_i + R^*_{J_1}\beta^*_{J_1}\right) - \sum_{i=1}^{J_1}R_i u_i(1-\alpha_i)\\
&\quad - R^*_{J_1}(1-\alpha^*_{J_1})a_1, \qquad C_1 = J_1,\\
D_2 &= \sum_{j=1}^{J_2}\bar{\alpha}_j v_j - A_2B_2\left(\sum_{j=1}^{J_2}(S_j+1)\bar{\beta}_j + S^*_{J_2}\bar{\beta}^*_{J_2}\right) - \sum_{j=1}^{J_2}S_j v_j(1-\bar{\alpha}_j)\\
&\quad - S^*_{J_2}(1-\bar{\alpha}^*_{J_2})a_2, \qquad C_2 = J_2,\\
E_1 &= \sum_{i=1}^{J_1}(R_i+1)\beta_i(u_i-A_1)^2 + R^*_{J_1}\beta^*_{J_1}(a_1-A_1)^2,\\
E_2 &= \sum_{j=1}^{J_2}(S_j+1)\bar{\beta}_j(v_j-A_2)^2 + S^*_{J_2}\bar{\beta}^*_{J_2}(a_2-A_2)^2.
\end{aligned}
$$

C. APPENDIX.

For three parameters case, we compute (2.14) at $\widehat{\theta} = (\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3)$, where

$$
\begin{aligned}
d_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \quad i = 1,2,3,\\
d_4 &= u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},\\
d_5 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}),\\
A &= \ell_{111}\sigma_{11} + 2\ell_{121}\sigma_{12} + 2\ell_{131}\sigma_{13} + 2\ell_{231}\sigma_{23} + \ell_{221}\sigma_{22} + \ell_{331}\sigma_{33},\\
B &= \ell_{112}\sigma_{11} + 2\ell_{122}\sigma_{12} + 2\ell_{132}\sigma_{13} + 2\ell_{232}\sigma_{23} + \ell_{222}\sigma_{22} + \ell_{332}\sigma_{33},\\
C &= \ell_{113}\sigma_{11} + 2\ell_{123}\sigma_{12} + 2\ell_{133}\sigma_{13} + 2\ell_{233}\sigma_{23} + \ell_{223}\sigma_{22} + \ell_{333}\sigma_{33}.
\end{aligned}
$$

In our case, for $(\theta_1, \theta_2, \theta_3) \equiv (\alpha, \beta, \lambda)$ and $u = R = \frac{\alpha}{\alpha+\beta}$, we have

$$\rho_1 = \frac{a_1 - 1}{\alpha} - b_1, \quad \rho_2 = \frac{a_2 - 1}{\beta} - b_2, \quad \rho_3 = \frac{a_3 - 1}{\lambda} - b_3,$$

$$\ell_{11} = -\frac{J_1}{\alpha^2}, \qquad \ell_{22} = -\frac{J_2}{\beta^2}, \qquad \ell_{12} = \ell_{21} = 0,$$

$$\ell_{13} = \ell_{31} = -\sum_{i=1}^{J_1} (R_i + 1) x_i^\lambda \frac{\log(x_i)}{1 - x_i^\lambda} - R_{J_1}^* T_1^\lambda \frac{\log(T_1)}{1 - T_1^\lambda},$$

$$\ell_{23} = \ell_{32} = -\sum_{j=1}^{J_2} (S_j + 1) y_j^\lambda \frac{\log(y_j)}{1 - y_j^\lambda} - S_{J_2}^* T_2^\lambda \frac{\log(T_2)}{1 - T_2^\lambda},$$

$$\ell_{33} = -\frac{J_1 + J_2}{\lambda^2} - \sum_{i=1}^{J_1} \left(\alpha(R_i + 1) - 1\right) x_i^\lambda \left(\frac{\log(x_i)}{1 - x_i^\lambda}\right)^2 - \alpha R_{J_1}^* T_1^\lambda \left(\frac{\log(T_1)}{1 - T_1^\lambda}\right)^2$$
$$- \sum_{j=1}^{J_2} \left(\beta(S_j + 1) - 1\right) y_j^\lambda \left(\frac{\log(y_j)}{1 - y_j^\lambda}\right)^2 - \beta S_{J_2}^* T_2^\lambda \left(\frac{\log(T_2)}{1 - T_2^\lambda}\right)^2,$$

σ_{ij}, $i, j = 1, 2, 3$ are obtained using ℓ_{ij}, $i, j = 1, 2, 3$ and

$$\ell_{111} = \frac{2J_1}{\alpha^3}, \qquad \ell_{222} = \frac{2J_2}{\beta^3}$$

$$\ell_{133} = \ell_{331} = \ell_{313} = -\sum_{i=1}^{J_1} (R_i + 1) x_i^\lambda \left(\frac{\log(x_i)}{1 - x_i^\lambda}\right)^2 - R_{J_1}^* T_1^\lambda \left(\frac{\log(T_1)}{1 - T_1^\lambda}\right)^2,$$

$$\ell_{233} = \ell_{332} = \ell_{323} = -\sum_{j=1}^{J_2} (S_j + 1) y_j^\lambda \left(\frac{\log(y_j)}{1 - y_j^\lambda}\right)^2 - S_{J_2}^* T_2^\lambda \left(\frac{\log(T_2)}{1 - T_2^\lambda}\right)^2,$$

$$\ell_{333} = \frac{2(J_1 + J_2)}{\lambda^3} - \sum_{i=1}^{J_1} \left(\alpha(R_i + 1) - 1\right) x_i^\lambda (1 + x_i^\lambda) \left(\frac{\log(x_i)}{1 - x_i^\lambda}\right)^3$$
$$- \sum_{j=1}^{J_2} \left(\beta(S_j + 1) - 1\right) y_j^\lambda (1 + y_j^\lambda) \left(\frac{\log(y_j)}{1 - y_j^\lambda}\right)^3 - \alpha R_{J_1}^* T_1^\lambda (1 + T_1^\lambda) \left(\frac{\log(T_1)}{1 - T_1^\lambda}\right)^3$$
$$- \beta S_{J_2}^* T_2^\lambda (1 + T_2^\lambda) \left(\frac{\log(T_2)}{1 - T_2^\lambda}\right)^3,$$

and other $\ell_{ijk} = 0$. Moreover, $u_3 = u_{i3} = 0$, $i = 1, 2, 3$, and u_1, u_2 are given in (2.12). Also, $u_{11} = \frac{-2\beta}{(\alpha+\beta)^3}$, $u_{12} = u_{21} = \frac{\alpha-\beta}{(\alpha+\beta)^3}$, $u_{22} = \frac{2\alpha}{(\alpha+\beta)^3}$. So,

$$d_4 = u_{12}\sigma_{12}, \qquad d_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}),$$

$$A = \ell_{111}\sigma_{11} + \ell_{331}\sigma_{33}, \quad B = \ell_{222}\sigma_{22} + \ell_{332}\sigma_{33}, \quad C = 2\ell_{133}\sigma_{13} + 2\ell_{233}\sigma_{23} + \ell_{333}\sigma_{33}.$$

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