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On Some Stationary INAR(1) Processes with Compound Poisson Distributions

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Abstract:

- Aly and Bouzar ([2]) used the backward approach in presence of the binomial thinning operator to construct underdispersed stationary first-order autoregressive integer-valued (INAR (1)) processes. The present paper is to be seen as a continuation of their work. The focus of this paper is on the development of stationary INAR (1) processes with discrete compound Poisson innovations. We expand on some recent results obtained by several authors for these processes. A number of theoretical results are established and then used to develop stationary INAR (1) processes with compound Poisson innovations with finite mean. We apply our results to obtain in detail important distributional properties of the new models when the innovation follows the Polya–Aeppli distribution, the non-central Polya–Aeppli distribution, the negative binomial distribution, the noncentral negative binomial distribution, the Poisson–Lindley distribution, the Euler-type distribution and the Euler distribution.

Keywords:

- *integer-valued time series; the binomial thinning operator; infinite divisibility; Euler distribution.*

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1. INTRODUCTION

Assume that X is a $\mathbb{Z}_+$-valued random variable (rv) and $\alpha \in (0,1)$. The binomial thinning operator (Steutel and van Harn ([22])) of X, denoted by $\alpha \odot X$, is defined by

$$\alpha \odot X = \sum_{i=1}^{X} Y_i, \tag{1.1}$$

where $\{Y_i\}$ is a sequence of independent identically distributed (iid) Bernoulli(α) ($Ber(\alpha)$) rv's independent of X. The operation $\odot$ acts as the analogue of the standard multiplication used in standard time series models.

The main results of this paper use the two facts below without further reference. For α and β in $(0,1)$,

$$\alpha \odot (\beta \odot X) \stackrel{d}{=} \beta \odot (\alpha \odot X) \stackrel{d}{=} (\alpha\beta) \odot X$$

and for X and Y independent $\mathbb{Z}_+$-valued rv's,

$$\alpha \odot (X + Y) \stackrel{d}{=} \alpha \odot X + \alpha \odot Y.$$

Assume that $\{\varepsilon_t\}$ is a sequence of iid $\mathbb{Z}_+$-valued rv's. A sequence $\{X_t\}$ of $\mathbb{Z}_+$-valued rv's is said to be an INAR (1) process if

$$X_t = \alpha \odot X_{t-1} + \varepsilon_t \qquad (t \geq 1), \tag{1.2}$$

where $\{\varepsilon_t\}$ is the innovation sequence and α is the coefficient of the process. The binomial thinning $\alpha \odot X_{t-1}$ in (1.2) is performed independently for each t. More precisely, we assume the existence of an array $(Y_{i,t},\ i \geq 1,\ t \geq 0)$ of iid $Ber(\alpha)$ rv's, independent of $\{\varepsilon_t\}$, such that

$$\alpha \odot X_{t-1} = \sum_{i=1}^{X_{t-1}} Y_{i,t-1}.$$

Let $\varphi_{X_t}(z)$ be the pgf of X_t of (1.2) and $\Psi(z)$ be the pgf ε_t. Then we have by (1.2)

$$\varphi_{X_t}(z) = \varphi_{X_{t-1}}(1 - \alpha + \alpha z)\Psi(z).$$

If one further assumes that $\{X_t\}$ is stationary with $\varphi_X(z)$ as the pgf of its marginal distribution, then the following functional equation holds

$$\varphi_X(z) = \varphi_X(1 - \alpha + \alpha z)\Psi(z). \tag{1.3}$$

It is a well known result that if $\alpha \in (0,1)$ and $\varphi_X(z)$ and $\Psi(z)$ are pgf's that satisfy (1.3), then there exists a stationary INAR (1) process $\{X_t\}$ on some probability space such that $\varphi_X(z)$ and $\Psi(z)$ are respectively the pgf of its marginal distribution and the pgf of its innovation sequence $\{\varepsilon_t\}$.

In the backward approach, one starts out with the pgf $\Psi(z)$ of the innovation sequence and solve (1.3) for the pgf $\varphi_X(\cdot)$ of the marginal distribution of the INAR (1) process. In this case

$$\varphi_X(z) = \lim_{n \longrightarrow \infty} \prod_{i=0}^{n} \Psi(1 - \alpha^i + \alpha^i z),$$

provided that the limit exists and is a pgf (see [2]).

The main focus of the present paper is on the development of stationary INAR (1) models driven by (1.2) with an infinitely divisible (Compound Poisson) innovation whose mean is finite. In Section 2, we prove a number of basic results in the context of the backward approach for these models. The results of Section 2 are used in Sections 3–9 to obtain in detail key distributional properties of the marginal distributions of some important INAR (1) processes. We discuss models whose innovations follow the Polya–Aeppli distribution, the non-central Polya–Aeppli distribution, the negative binomial distribution, the noncentral negative binomial distribution, the Poisson–Lindley distribution, and the Euler-type and Euler distributions.

The above INAR (1) models are necessarily overdispersed. An example of a data set which is empirically overdispersed is presented and analyzed in [4]. This data set gives the monthly claim counts by workers in the heavy manufacturing industry who were collecting benefits due to a burn related injury. The same data set was further analyzed in [23] and [18] and shown to have an INAR (1)-like autocorrelation structure. Another example of an overdispersed data set was introduced in [11] and was further analyzed in [12]. This data set involves the number of publications produced by Ph.D. biochemists. Several examples of underdispersed data sets are reported and analyzed in [20].

In the rest of this paper we will assume that $\alpha \in (0,1)$ and $\overline{a} = 1 - a$ for $a \in (0,1)$. We will also use the notation $\mu_r^{(u)} (\kappa_r^{(u)})$ and $\mu_{[r]}^{(u)} (\kappa_{[r]}^{(u)})$ to designate the r-th moment (cumulant) and the r-th factorial moment (factorial cumulant) of the pmf $\{u_r\}$, respectively.

The backward approach rests heavily on the following important result found in [2].

Theorem 1.1. *Assume that $\Psi'(1) < \infty$. The function*

$$\varphi(z) = \prod_{i=0}^{\infty} \Psi(1 - \alpha^i + \alpha^i z) \tag{1.4}$$

is a pgf. Moreover, the convergence of the infinite product is uniform over the interval $[0,1]$ and $\varphi(z)$ satisfies (1.3).

2. PROCESSES WITH COMPOUND POISSON INNOVATIONS

2.1. Basic Results

We start out by specializing Theorem 1.1 to infinitely divisible distributions with finite mean. Recall that a distribution on $\mathbb{Z}_+$ is infinitely divisible if and only if it is a discrete compound Poisson distribution with pgf

$$\Psi(z) = \exp\{\lambda(H(z) - 1)\}, \tag{2.1}$$

for some $\lambda > 0$ and some unique pgf $H(z) = \sum_{r=1}^{\infty} h_r z^r$ with pmf $\{h_r\}$ and $H(0) = h_0 = 0$. We will refer to such distributions as $DCP(\lambda, H)$ distributions.

First, we need a lemma.

Lemma 2.1. *Assume that $\Psi(z)$ is the pgf of a $DCP(\lambda, H)$ distribution. Then for each $i \geq 0, \Psi(1-\alpha^i+\alpha^i z)$ is the pgf of a $DCP(\lambda_i', H_i)$ distribution which is described below:*

(i) *For every $i \geq 0$,*

$$\lambda_i' = \lambda m_i, \quad m_i = 1 - H(1-\alpha^i), \tag{2.2}$$

and

$$H_i(z) = 1 - \frac{1}{m_i}\big(1 - H(1-\alpha^i+\alpha^i z)\big). \tag{2.3}$$

(ii) *The pmf $\{h_r^{(i)}\}$ with pgf $H_i(z)$ is*

$$h_r^{(i)} = \frac{\alpha^{ir}}{m_i}\sum_{n=r}^{\infty}\binom{n}{r}(1-\alpha^i)^{n-r}h_n \quad (r \geq 1). \tag{2.4}$$

Note that $H_0(z) = H(z)$ and $\{h_r^{(0)}\} = \{h_r\}$.

(iii) *If the factorial moment generating function (fmgf) $H(1+t)$ of the pmf $\{h_r\}$ exists for $|t| < \rho_0$ for some $\rho_0 > 0$, then for every $i \geq 0$, the pmf $\{h_r^{(i)}\}$ has finite factorial moments $\{\mu_{[r]}^{(h^{(i)})}\}$ for all $r \geq 1$, and*

$$\mu_{[r]}^{(h^{(i)})} = \frac{\alpha^{ir}}{m_i}\mu_{[r]}^{(h)}. \tag{2.5}$$

Proof: By (2.1), we have $\ln\Psi(1-\alpha^i+\alpha^i z) = \lambda(H(1-\alpha^i+\alpha^i z) - 1)$, $i \geq 0$, which can be rewritten as

$$\ln\Psi(1-\alpha^i+\alpha^i z) = \lambda(1-H(1-\alpha^i))\Big(\frac{H(1-\alpha^i+\alpha^i z) - H(1-\alpha^i)}{1-H(1-\alpha^i)} - 1\Big).$$

Letting m_i and λ_i' be as in (2.2), we have

$$\ln\Psi(1-\alpha^i+\alpha^i z) = \lambda_i'\Big(\frac{H(1-\alpha^i+\alpha^i z) + m_i - 1}{m_i} - 1\Big),$$

which leads to (2.3). The identity $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}$ implies

$$H(1-\alpha^i+\alpha^i z) - H(1-\alpha^i) = \sum_{r=1}^{\infty}\Big(\sum_{n=r}^{\infty}\binom{n}{r}\alpha^{ir}(1-\alpha^i)^{n-r}h_n\Big)z^r.$$

Hence, $H_i(z)$ is the pgf of $\{h_r^{(i)}\}$ of (2.4). This establishes (i)–(ii). To prove (iii), we note that since the fmgf $H(1+t)$ of the pmf $\{h_r\}$ exists, then $\{h_r\}$ has finite factorial moments $\mu_{[r]}^{(h)}$ of all orders $r \geq 1$. It follows by equation (1.274), p. 59, in [6] and (2.3) that

$$H_i(1+t) = 1 + \frac{1}{m_i}\sum_{r=1}^{\infty}\mu_{[r]}^{(h)}\alpha^{ir}\frac{t^r}{r!} \qquad (|t| < \rho_0), \tag{2.6}$$

which in turn leads to (2.5). □

Next, we study the pgf $\varphi(\cdot)$ of (1.4) when $\Psi(z)$ is the pgf of a $DCP(\lambda, H)$ distribution.

Theorem 2.1. *Let $\varphi(\cdot)$ and $\Psi(\cdot)$ be as in (1.4). If $\Psi(z)$ is the pgf of a $DCP(\lambda, H)$ distribution with $\Psi'(1) < \infty$, then the following assertions hold:*

(i) *$\varphi(z)$ is the pgf of the infinite convolution of the distributions $(DCP(\lambda m_i, H_i),\ i \geq 0)$, as described in Lemma 2.1.*

(ii) *$\varphi(z)$ is the pgf of a $DCP(\widetilde{\lambda}, G)$ distribution, where*

$$\widetilde{\lambda} = \lambda M > 0, \quad M = \sum_{i=0}^{\infty} m_i = \sum_{i=0}^{\infty}(1 - H(1 - \alpha^i)), \tag{2.7}$$

and

$$G(z) = \sum_{i=0}^{\infty} \frac{m_i}{M} H_i(z) \quad (G(0) = 0). \tag{2.8}$$

Moreover, the pmf $\{g_r\}$ with pgf $G(z)$ is the infinite countable mixture

$$g_r = \sum_{i=0}^{\infty} \frac{m_i}{M} h_r^{(i)} \quad (r \geq 1), \tag{2.9}$$

with $\left(\{h_r^{(i)}\}, i \geq 0\right)$ of (2.4) and mixing probabilities $\left(\frac{m_i}{M}, i \geq 0\right)$.

Proof: By Theorem 1.1, $\varphi(z)$ is a pgf. Part (i) follows directly from Lemma 2.1. To prove (ii), first we note $\Psi(z)$ is the pgf of an infinitely divisible distribution. Therefore, there exists a pgf $\Psi_n(z)$ such that $\Psi(z) = (\Psi_n(z))^n$ for every $n \geq 1$. Since $\Psi'(z) = n(\Psi_n(z))^{n-1}\Psi_n'(z)$ and $\Psi'(1) < \infty$, we have $\Psi_n'(1) < \infty$. Applying Theorem 1.1 to Ψ_n, it follows that $\prod_{i=0}^{\infty} \Psi_n(1 - \alpha^i + \alpha^i z)$ is a pgf. Note that

$$\varphi(z) = \prod_{i=0}^{\infty} \Psi(1 - \alpha^i + \alpha^i z) = \left\{\prod_{i=0}^{\infty} \Psi_n(1 - \alpha^i + \alpha^i z)\right\}^n \quad (n \geq 1).$$

Hence, $\varphi(z)$ is the the pgf of an infinitely divisible distribution, or a $DCP(\widetilde{\lambda}, G)$ distribution for some $\widetilde{\lambda} > 0$ and pgf $G(z)$. We have by Theorem 1.1 and (2.1)

$$\varphi(z) = \prod_{i=0}^{\infty} \Psi(1 - \alpha^i + \alpha^i z) = \exp\left\{\lambda \sum_{i=0}^{\infty}(H(1 - \alpha^i + \alpha^i z) - 1)\right\}.$$

It is clear that $\varphi'(1) < \infty$ implies $H'(1) < \infty$. Let $Q_H(z) = \frac{1 - H(z)}{1 - z}$ $(z \neq 1)$ be the generating function of the tail probabilities $q_r = \sum_{i=r+1}^{\infty} h_i$ of $\{h_r\}$. It follows that $1 - H(1 - \alpha^i + \alpha^i z) \leq \alpha^i H'(1)$ (recall $Q_H(1) = H'(1)$) and thus $\sum_{i=0}^{\infty}(1 - H(1 - \alpha^i + \alpha^i z))$ converges uniformly over $[0, 1]$. This implies that $M = \sum_{i=0}^{\infty} m_i < \infty$ (see (2.2)). The fact that $\widetilde{\lambda} = \lambda M$ follows by setting $z = 0$ in the equation $\lambda \sum_{i=0}^{\infty}(H(1 - \alpha^i + \alpha^i z) - 1) = \widetilde{\lambda}(G(z) - 1)$. Solving for $G(z)$ and using (2.3) leads to (2.8) and (2.9) follows from (2.4) and (2.8). □

The following result is a direct consequence of Theorem 2.1 and equation (9.43), p. 390, in [6], for infinitely divisible distributions.

Corollary 2.1. *Under the assumptions and notation of Theorem 2.1, the pmf* $\{p_r\}$ *with pgf* $\varphi(z)$ *can be derived via the recurrence formula*

$$(2.10)\qquad (r+1)p_{r+1} = \lambda \sum_{j=0}^{r} (r+1-j) g_{r+1-j} p_j \quad \textit{with} \quad p_0 = e^{-\lambda M} \quad (r \geq 0).$$

Remark 2.1. A distribution on $\mathbb{Z}_+$ with pgf $\Psi(z)$ is discrete self-decomposable (DSD) (cf. Steutel and van Harn [22]) if for any $\beta \in (0,1)$,

$$(2.11)\qquad \Psi(z) = \Psi(1-\beta+\beta z)\Psi_\beta(z),$$

for some pgf $\Psi_\beta(z)$. If $\Psi(z)$ is the pgf of a DSD distribution with finite mean, then $\varphi(z)$ of (1.4) is the pgf of a DSD distribution. Indeed, basic properties of infinite products and the fact that $\Psi'_\beta(1) < \infty$ lead to

$$\varphi(z) = \varphi(1-\beta+\beta z) \prod_{i=0}^{\infty} \Psi_\beta(1-\alpha^i+\alpha^i z).$$

We conclude by Theorem 1.1 applied to $\Psi_\beta(z)$ that $\prod_{i=0}^{\infty} \Psi_\beta(1-\alpha^i+\alpha^i z)$ is a pgf.

We proceed to discuss the case of INAR (1) processes with a $DCP(\lambda, H)$ innovation. We will add to results obtained in [18], [19] and [24]. These papers deal mainly with $DCP(\lambda, H)$ innovation when the compounding distribution has a pgf of the form $H(z) = \sum_{i=1}^{n} h_i z^i$, $n < \infty$. For example, on page 355 in [24], it is mentioned, quoting, "Let (X_t) be a stationary $CP_\infty - INAR(1)$ process. In general, a closed-form expression for the observations' pmf is not available". In addition, on page 624 in [19], it is mentioned that "the structural implications of Theorem 2.1 can be extended to the case of compound Poisson arrival distributions with an infinite compounding structure. The stationary distribution in this general case is again compound Poisson distributed with infinite compounding structure. However, a way to explicitly calculate the stationary distribution in this case is not known".

The next result asserts the existence of a stationary INAR (1) process whose innovation is DCP with infinite compounding structure. It is a consequence of Theorem 2.1 and the standard result on the existence of stationary INAR (1) processes recalled in the introduction. The proof is omitted.

Theorem 2.2. *Any* $DCP(\lambda, H)$ *distribution with pgf* $\Psi(z)$ *of (2.1) such that* $H'(1) < \infty$ *gives rise to a stationary INAR (1) process* $\{X_t\}$ *defined on some probability space and driven by equation (1.2). Its innovation has pgf* $\Psi(z)$ *and its marginal distribution is the* $DCP(\widetilde{\lambda}, G)$ *distribution described by (2.7)–(2.10).*

Next, we list key distributional properties of a stationary INAR (1) process $\{X_t\}$ with a $DCP(\lambda, H)$ innovation:

1. The 1-step transition probabilities of $\{X_t\}$ are given by

$$(2.12)\qquad P(X_t = k | X_{t-1} = l) = \sum_{j=0}^{\min(l,k)} \binom{l}{j} \alpha^j (1-\alpha)^{l-j} f_{k-j},$$

where

$$f_x = P(\varepsilon = x) = \begin{cases} e^{-\lambda}, & \text{if } x = 0, \\ \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} h_x^{*n}, & \text{if } x > 0, \end{cases} \tag{2.13}$$

and $\{h_x^{*n}\}$ is the n-fold convolution of the pmf $\{h_r\}$ with pgf $H(z)$. Similarly to (2.10), f_x can be obtained by the recurrence formula

$$(x+1) f_{x+1} = \lambda \sum_{j=0}^{x} (x+1-j) h_{x+1-j} f_j \,, \quad \text{with} \quad f_0 = e^{-\lambda} \quad (x \geq 0). \tag{2.14}$$

2. The k-step-ahead version of (1.2) for $k \geq 1$ is given by

$$X_{t+k} \overset{d}{=} \alpha^k \odot X_t + \sum_{j=1}^{k} \alpha^{j-1} \odot \varepsilon_{t+k-j+1}. \tag{2.15}$$

Consequently, the conditional pgf of X_{t+k} given X_t satisfies

$$\varphi_{X_{t+k}|X_t}(z) = \Big(1 - \alpha^k + \alpha^k z\Big)^{X_t} \times \prod_{i=0}^{k-1} \Psi(1 - \alpha^i + \alpha^i z). \tag{2.16}$$

3. It follows by Lemma 2.1 and (2.16) that the conditional distribution of X_{t+k} given $X_t = n$ results from the convolution of a binomial distribution, $Bin(n, \alpha^k)$, and the distributions $(DCP(\lambda m_i, H_i),\ 0 \leq i \leq k-1)$ with characteristics (2.2)–(2.4).

4. Assume the fmgf $H(1+t)$ of the pmf $\{h_r\}$ exists for $|t| < \rho_0$ for some $\rho_0 > 0$. By Lemma 2.1-(iii), the fmgf $H_i(1+t)$ of the pmf $\{h_r^{(i)}\}$ admits the representation (2.6), for every $i \geq 0$ and $|t| < \rho_0$. Using (2.8) and a standard argument, one can show that $G(1+t) = \sum_{i=0}^{\infty} \frac{m_i}{M} H_i(1+t)$ converges uniformly in the interval $|t| \leq \rho$ for every $0 < \rho < \rho_0$. Therefore, by Weierstrass Theorem, p. 430 in [8], we have

$$G(1+t) = 1 + \sum_{r=1}^{\infty} \left[\sum_{i=0}^{\infty} \frac{m_i}{M} \mu_{[r]}^{(h^{(i)})} \right] \frac{t^r}{r!} \qquad (|t| < \rho_0),$$

which implies

$$\mu_{[r]}^{(g)} = \sum_{i=0}^{\infty} \frac{m_i}{M} \mu_{[r]}^{(h^{(i)})}. \tag{2.17}$$

By (2.5), (2.17) and equation (1.246), p. 53, in [6], the factorial moments and the moments of $\{g_r\}$ are

$$\mu_{[r]}^{(g)} = \frac{\mu_{[r]}^{(h)}}{M(1-\alpha^r)} \quad \text{and} \quad \mu_r^{(g)} = \frac{1}{M} \sum_{j=1}^{r} S(r,j) \frac{\mu_{[j]}^{(h)}}{1-\alpha^j} \qquad (r \geq 1), \tag{2.18}$$

where $\{S(r,j)\}$ are the Stirling numbers of the second kind defined as

$$S(r,j) = \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^r \qquad (S(0,0) = 1,\ S(0,k) = S(r,0) = 0).$$

5. By (2.18), equations (9.49), p. 391, and (1.257), p. 55, in [6], the factorial cumulants and cumulants of X_t are:

$$\kappa_{[r]}^{(p)} = \frac{\lambda}{1-\alpha^r}\mu_{[r]}^{(h)} \quad \text{and} \quad \kappa_r^{(p)} = \lambda \sum_{j=1}^{r} S(r,j)\frac{\mu_{[j]}^{(h)}}{1-\alpha^j} \qquad (r \geq 1). \tag{2.19}$$

6. The first and second cumulants of a pmf are its mean and variance, respectively. The mean $\mu_1^{(p)}$ and the variance $(\sigma^{(p)})^2$ of X_t follow from the above formulas:

$$\mu_1^{(p)} = \frac{\lambda\mu_1^{(h)}}{1-\alpha} \quad \text{and} \quad (\sigma^{(p)})^2 = \frac{\lambda(\mu_2^{(h)} + \alpha\mu_1^{(h)})}{1-\alpha^2}. \tag{2.20}$$

7. The moments and factorial moments of X_t can be computed recursively by a formula in [21] for the former and equation (1.244) in [6] for the latter:

$$\mu_r^{(p)} = \sum_{i=0}^{r-1} \binom{r-1}{i} \kappa_{r-i}^{(p)}\mu_i^{(p)} \quad \text{and} \quad \mu_{[r]}^{(p)} = \sum_{j=0}^{r} s(r,j)\mu_j^{(p)}, \tag{2.21}$$

where $\{s(r,j)\}$ are the Stirling numbers of the first kind satisfying the recurrence relation

$$s(r+1,j) = s(r,j-1) - r s(r,j) \qquad (s(n,0)=0,\ s(1,1)=1).$$

We note that the moments and factorial moments of the marginal distributions of the INAR (1) models we introduce here are only obtainable through (2.21). Except for a couple of instances, we will make no further reference to these moments.

2.2. Processes whose innovations are convolutions of *DCP* distributions

We consider stationary INAR (1) processes whose innovation is the finite convolution of *DCP* distributions with finite means.

Let ν be a positive integer. We assume throughout the section that $(\widetilde{H}_k, 1 \leq k \leq \nu)$ is a collection of pgf's such that $\widetilde{H}_k(0) = 0$, $\widetilde{H}_k'(1) < \infty$ and $(\lambda_k, 1 \leq k \leq \nu)$ are positive constants. We denote by $\{h_r^{(k)}\}$ the pmf of $\widetilde{H}_k(z)$.

Lemma 2.2. *Let $\Psi_k(z)$ be the pgf of a $DCP(\lambda_k, \widetilde{H}_k)$ distribution, $1 \leq k \leq \nu$. The following assertions hold:*

- **(i)** *The convolution of the $DCP(\lambda_k, \widetilde{H}_k)$ distributions, $1 \leq k \leq \nu$, is $DCP(\lambda, H)$, where*

$$\lambda = \sum_{k=1}^{\nu} \lambda_k \quad \text{and} \quad H(z) = \sum_{k=1}^{\nu} \frac{\lambda_k}{\lambda}\widetilde{H}_k(z). \tag{2.22}$$

- **(ii)** *For each $k = 1, 2, ..., \nu$, $\Psi_k(1-\alpha^i+\alpha^i z)$ is the pgf of a $DCP(\lambda_k m_i^{(k)}, \widetilde{H}_{ki}(z))$ distribution, where $m_i^{(k)} = 1 - \widetilde{H}_k(1-\alpha^i)$ and $\widetilde{H}_{ki}(z)$ is the pgf of a pmf we denote $\{h_r^{(k,i)}\}$, with $\widetilde{H}_{ki}(0) = 0$ and $\widetilde{H}_{ki}'(1) < \infty$.*

(iii) *$\Psi(1-\alpha^i+\alpha^i z)$ is the pgf of a $DCP(\lambda m_i, H_i)$ distribution, where $m_i = 1 - H(1-\alpha^i) = \sum_{k=1}^{\nu} \frac{\lambda_k}{\lambda} m_i^{(k)}$, with λ and H of (2.22), and*

$$H_i(z) = \sum_{k=1}^{\nu} \frac{\lambda_k m_i^{(k)}}{\lambda m_i} \widetilde{H}_{ki}(z) \quad \text{and} \quad h_r^{(i)} = \sum_{k=1}^{\nu} \frac{\lambda_k m_i^{(k)}}{\lambda m_i} h_r^{(k,i)} \quad (r \geq 1). \tag{2.23}$$

(iv) *For every $i \geq 0$, the $DCP(\lambda m_i, H_i)$ distribution admits the following representation, with $\lambda_i^{(k)} = \lambda_k m_i^{(k)}$ $(1 \leq k \leq \nu)$,*

$$DCP(\lambda m_i, H_i) \sim DCP(\lambda_i^{(1)}, \widetilde{H}_{1i}) * DCP(\lambda_i^{(2)}, \widetilde{H}_{2i}) * \cdots * DCP(\lambda_i^{(\nu)}, \widetilde{H}_{\nu i}). \tag{2.24}$$

Proof: (i) is clear and (ii) follows from Lemma 2.1. For (iii), m_i follows from (2.22) by Theorem 2.1. We have by (i) $\Psi_k(1-\alpha^i+\alpha^i z) = \exp\{\lambda_k m_i^{(k)}(\widetilde{H}_{ki}(z)-1)\}$, which implies

$$\varphi(z) = \exp\left\{\sum_{k=1}^{\nu} \lambda_k m_i^{(k)} (\widetilde{H}_{ki}(z) - 1)\right\} = \exp\left\{\left(\sum_{k=1}^{\nu} \lambda_k m_i^{(k)} \widetilde{H}_{ki}(z)\right) - \lambda m_i\right\}$$

and (2.23), as $\sum_{k=1}^{\nu} \frac{\lambda_k m_i^{(k)}}{\lambda m_i} = 1$. (iv) follows from (iii) and (2.23). □

Next, we present key distributional properties of a stationary INAR(1) with an innovation that is the convolution of DCP distributions. The proofs are omitted as the results are a direct consequence of Lemma 2.2 and Theorem 2.1.

Theorem 2.3. *Let $\{X_t\}$ be a stationary INAR(1) process driven by (1.2) with the $DCP(\lambda, H)$ innovation that results from the convolution of the $DCP(\lambda_k, \widetilde{H}_k)$ distributions, $1 \leq k \leq \nu$ (as described in Lemma 2.2). Let $M_k = \sum_{i=0}^{\infty} m_i^{(k)}$, $1 \leq k \leq \nu$. The following assertions hold:*

(i) *The marginal distribution of $\{X_t\}$ is the infinite convolution of the sequence of distributions $\big(DCP(\lambda m_i, H_i), i \geq 0\big)$ with the representation (2.24).*

(ii) *The marginal distribution of $\{X_t\}$ is $DCP(\widetilde{\lambda}, G)$, where*

$$M = \sum_{k=1}^{\nu} \frac{\lambda_k}{\lambda} M_k; \quad \widetilde{\lambda} = \lambda M = \sum_{k=1}^{\nu} \lambda_k M_k \tag{2.25}$$

and $G(z)$ admits the representation (2.8).

(iii) *The pmf $\{g_r\}$ is the infinite mixture of the pmf's $(\{h_r^{(i)}\}, i \geq 0)$ of (2.23) with mixing probabilities $(\frac{m_i}{M},\ i \geq 0)$.*

We discuss additional properties of the process $\{X_t\}$ of Theorem 2.3.

The 1-step transition probabilities of $\{X_t\}$ can be obtained from equations (2.12)–(2.14). By (2.16), the conditional distribution of X_{t+k} given $X_t = n$ results from the convolution of a $Bin(n, \alpha^k)$ distribution and the distributions $\big(DCP(\lambda m_i, H_i), 0 \leq i \leq k-1\big)$ of (2.24).

If we assume that for each $k = 1, 2, ..., \nu$, the fmgf $\widetilde{H}_k(1+t)$ of the pmf $\{h_r^{(k)}\}$ exists for $|t| < \rho_0^{(k)}$ for some $\rho_0^{(k)} > 0$, then it is easily seen that the fmgf $H(1+t)$ of (2.22) exists for $|t| < \min_{1\le k\le\nu} \rho_0^{(k)}$. It follows by Lemma 2.1-(iii), Theorem 2.1, and (2.18) applied to λ and $H(z)$ of (2.22) that the r-th factorial moment of $\{g_r\}$ is

$$\mu_{[r]}^{(g)} = \frac{1}{M(1-\alpha^r)} \sum_{k=1}^{\nu} \frac{\lambda_k}{\lambda} \mu_{[r]}^{(h^{(k)})}. \tag{2.26}$$

By (2.19), the factorial cumulants and the cumulants of X_t are (for $r \ge 1$)

$$\kappa_{[r]}^{(p)} = \frac{1}{1-\alpha^r} \sum_{k=1}^{\nu} \lambda_k \mu_{[r]}^{(h^{(k)})} \quad \text{and} \quad \kappa_r^{(p)} = \sum_{k=1}^{\nu} \lambda_k \left[\sum_{j=1}^{r} \frac{S(r,j)}{1-\alpha^j} \mu_{[j]}^{(h^{(k)})} \right]. \tag{2.27}$$

The mean and variance of X_t can be obtained from (2.20). We omit the details.

3. PROCESSES WITH POLYA-AEPPLI INNOVATIONS

A $\mathbb{Z}_+$-valued random variable with pgf $\Psi(z) = \exp\left(-\lambda \frac{1-z}{1-\theta z}\right)$ and pmf

$$f_r = \begin{cases} e^{-\lambda}, & \text{if } r = 0, \\ e^{-\lambda}\theta^r \displaystyle\sum_{j=1}^{r} \binom{r-1}{j-1} \frac{(\lambda\overline{\theta}/\theta)^j}{j!}, & \text{if } r > 0, \end{cases} \tag{3.1}$$

is said to have a Polya–Aeppli (or Poisson Geometric) distribution ($PA(\lambda, \theta)$) with parameters (λ, θ), $\lambda > 0$ and $\theta \in (0, 1)$. The $PA(\lambda, \theta)$ is $DCP(\lambda, H)$, where $H(z)$ is the pgf of the shifted geometric ($Geo_1(\theta)$) distribution with pmf $\{h_r\}$:

$$H(z) = \frac{\overline{\theta} z}{1-\theta z} \qquad \text{and} \qquad h_r = \overline{\theta}\theta^{r-1} \quad (r \ge 1). \tag{3.2}$$

Theorem 3.1. *Let $\{X_t\}$ be a stationary INAR (1) process with a $PA(\lambda, \theta)$ innovation. The following assertions hold:*

- **(i)** *The sequence $\{m_i\}$ of (2.2) satisfies*

$$m_i = \frac{\alpha^i}{1-\theta(1-\alpha^i)} \quad \text{and} \quad 0 < m_i \le 1 \quad (i \ge 0).$$

- **(ii)** *The pmf $\{h_r^{(i)}\}$ of (2.4), $i \ge 0$, is a $Geo_1(m_i\theta)$ distribution, and*

$$DCP(\lambda m_i, H_i) \sim PA(\lambda m_i, m_i\theta) \qquad (i \ge 0). \tag{3.3}$$

- **(iii)** *The distribution of $\{X_t\}$ is the infinite convolution of the $PA(\lambda m_i, m_i\theta)$ distributions $(i \ge 0)$.*
- **(iv)** *The distribution of $\{X_t\}$ is $DCP(\widetilde{\lambda}, G)$, where $\widetilde{\lambda} = \lambda M$, $M = \sum_{i=0}^{\infty} m_i$, and G is the pgf of the infinite mixture of $Geo_1(m_i\theta)$ distributions with respective mixing probabilities $\frac{m_i}{M}$, $i \ge 0$.*

Proof: Part (i) and the first part of (ii) follow from Lemma 2.1, (2.4), (3.2), and the result $(1-t)^{-r-1} = \sum_{n=r}^{\infty} \binom{n}{r} t^{n-r}$. In turn, the first part of (ii) implies (3.3). Part (iii) ensues from Theorem 2.1-(i). Part (iv) is a direct consequence of Theorem 2.1. □

We state some additional properties of the process $\{X_t\}$ of Theorem 3.1.

The 1-step transition probability of $\{X_t\}$ can be computed from (2.12)–(2.14) with $P(\varepsilon = x) = f_x$ of (3.1). By (2.16) and (3.3), the conditional distribution of X_{t+k} given $X_t = n$ arises as the convolution of a $Bin(n, \alpha^k)$ distribution and the $PA(\lambda m_i, m_i\theta)$ distributions, $0 \le i \le k-1$.

The fmgf $H(1+t)$ of the $Geo_1(\theta)$ distribution with pmf $\{h_r\}$ of (3.2) exists for $|t| < \overline{\theta}/\theta$. Its power series expansion yields the factorial moments of $\{h_r\}$,

$$\mu_{[r]}^{(h)} = \frac{r!}{\theta}(\theta/\overline{\theta})^r \qquad (r \ge 1). \tag{3.4}$$

Formulas for the moments of $\{g_r\}$ and the cumulants, mean and variance of X_t are given below. They are derived from (2.18)–(2.20) and (3.4):

$$\mu_{[r]}^{(g)} = \frac{r!(\theta/\overline{\theta})^r}{M\theta(1-\alpha^r)} \quad \text{and} \quad \mu_r^{(g)} = \frac{1}{M\theta}\sum_{j=1}^{r} S(r,j)\frac{j!(\theta/\overline{\theta})^j}{1-\alpha^j},$$

$$\kappa_{[r]}^{(p)} = \frac{\lambda r!(\theta/\overline{\theta})^r}{\theta(1-\alpha^r)} \quad \text{and} \quad \kappa_r^{(p)} = \frac{\lambda}{\theta}\sum_{j=1}^{r} S(r,j)\frac{j!(\theta/\overline{\theta})^j}{1-\alpha^j},$$

and

$$\mu_1^{(p)} = \frac{\lambda}{\overline{\alpha}\overline{\theta}} \quad \text{and} \quad (\sigma^{(p)})^2 = \frac{\lambda(2-\overline{\alpha}\overline{\theta})}{(1-\alpha^2)\overline{\theta}^2}.$$

Remark 3.1.

(i) The $PA(\lambda, 0)$ distribution is Poisson (λ) and the corresponding stationary INAR(1) process simplifies to the Poisson $(\frac{\lambda}{\overline{\alpha}})$ INAR (1) process discussed in [1], [13], and [14].

(ii) One can extend the model discussed in this section to INAR (1) processes whose innovations are finite convolutions of Polya–Aeppli distributions. The extension can be established in fairly straightforward fashion by combining the results in this section with those in Subsection 2.2.

4. PROCESSES WITH NONCENTRAL POLYA-AEPPLI INNOVATIONS

A noncentral Polya–Aeppli distribution ($NPA(\lambda_1, \lambda_2, \theta)$) with parameters $\lambda_1, \lambda_2 > 0$ and $\theta \in (0,1)$, as introduced in [9], results from the convolution of a Poisson(λ_1) distribution and a $PA(\lambda_2, \theta)$ distribution. Its pmf is

$$f_r = \begin{cases} e^{-\lambda}, & \text{if } r = 0, \\ e^{-\lambda}\theta^r \displaystyle\sum_{j=1}^{r} \frac{1}{j!}\Big(\sum_{k=0}^{j} \binom{j}{k}\binom{r-j+k-1}{k-1}(\lambda_2\overline{\theta}/\theta)^k(\lambda_1/\theta)^{j-k}\Big), & \text{if } r > 0. \end{cases} \tag{4.1}$$

An $NPA(\lambda_1, \lambda_2, \theta)$ distribution is $DCP(\lambda, H)$, where $\lambda = \lambda_1 + \lambda_2$ and $H(z)$ is the pgf of a mixture of a Dirac measure δ_1 sitting at 1, i.e., $\delta_1(\{1\}) = 1$, and a $Geo_1(\theta)$ distribution, with respective mixing probabilities λ_1/λ and λ_2/λ, or

$$(4.2) \qquad H(z) = \frac{\lambda_1}{\lambda} z + \frac{\lambda_2}{\lambda} \frac{\overline{\theta} z}{1 - \theta z}, \quad h_1 = \frac{\lambda_1 + \overline{\theta}\lambda_2}{\lambda} \quad \text{and} \quad h_r = \frac{\lambda_2}{\lambda} \overline{\theta} \theta^{r-1} \quad (r \geq 2).$$

Theorem 4.1. *Let $\{X_t\}$ be a stationary INAR(1) process with an $NPA(\lambda_1, \lambda_2, \theta)$ innovation. The following assertions hold:*

(i) *The sequence $\{m_i\}$ of (2.2) satisfies*

$$m_i = \frac{\lambda_1}{\lambda} \cdot \alpha^i + \frac{\lambda_2}{\lambda} \cdot \frac{\alpha^i}{1 - \theta(1 - \alpha^i)} \quad \text{and} \quad 0 < m_i \leq 1 \quad (i \geq 0).$$

(ii) *The pmf $\{h_r^{(i)}\}$ of (2.4), $i \geq 0$, is a mixture of a Dirac measure δ_1 sitting at 1 and a $Geo_1(\beta_i)$ distribution, with mixing probabilities b_{i1} and b_{i2}, where*

$$\beta_i = \frac{\theta \alpha^i}{1 - \theta(1 - \alpha^i)}, \qquad b_{i1} = \frac{\lambda_1 \alpha^i}{\lambda m_i}, \qquad b_{i2} = \frac{\lambda_2}{\lambda m_i} \frac{\alpha^i}{1 - \theta(1 - \alpha^i)},$$

$$h_1^{(i)} = 1 - b_{i2}\beta_i \qquad \text{and} \qquad h_r^{(i)} = b_{i2} \overline{\beta_i} \beta_i^{\,r-1} \quad (r \geq 2).$$

Moreover,

$$(4.3) \qquad DCP(\lambda m_i, H_i) \sim NPA(\lambda_1 \alpha^i, \lambda_2 \beta_i/\theta, \beta_i) \qquad (i \geq 0).$$

(iii) *The marginal distribution of $\{X_t\}$ is the infinite convolution of the $NPA(\lambda_1 \alpha^i, \lambda_2 \beta_i/\theta, \beta_i)$ distributions ($i \geq 0$).*

(iv) *The marginal distribution of $\{X_t\}$ is $DCP(\widetilde{\lambda}, G)$, where $\widetilde{\lambda} = \lambda M$, $M = \frac{\lambda_1}{\lambda(1-\alpha)} + \frac{\lambda_2}{\lambda \theta} \sum_{i=0}^{\infty} \beta_i$ and G is the pgf of the infinite countable mixture of the sequence of pmf's ($\{h_r^{(i)}\}, i \geq 0$), described in (ii) above, with respective mixing probabilities ($\frac{m_i}{M}$, $i \geq 0$).*

Proof: Parts (i) and (ii) follow essentially from (3.3), (4.2), Lemma 2.2, and Theorem 2.3 (for $k = 2$). Part (iii) ensues from Theorem 2.1-(i) and part (iv) is a direct consequence of Theorem 2.1-(ii). □

We obtain additional properties of the process $\{X_t\}$ of Theorem 4.1.

The 1-step transition probability of $\{X_t\}$ is obtained from (2.12)–(2.14) with $P(\varepsilon = x) = f_x$ of (4.1). By (2.16), Lemma 2.1, and Theorem 4.1-(ii), the conditional distribution of X_{t+k} given $X_t = n$ is the convolution of a $Bin(n, \alpha^k)$ distribution and the $NPA(\lambda_1 \alpha^i, \lambda_2 \beta_i/\theta, \beta_i)$ distributions ($0 \leq i \leq k - 1$).

The fcmgf $H(1 + t)$ of the pmf $\{h_n\}$ of (4.2) exists for $|t| < \overline{\theta}/\theta$. Its power series expansion, (2.18) and (3.4), lead to the factorial moments of $\{g_r\}$:

$$\mu_{[r]}^{(g)} = \begin{cases} \dfrac{1}{\lambda M(1 - \alpha)} \big(\lambda_1 + \lambda_2/\overline{\theta}\big), & \text{if } r = 1, \\[2ex] \dfrac{1}{\lambda M(1 - \alpha^r)} \big(\lambda_2 r!/\theta\big) \big(\theta/\overline{\theta}\big)^r, & \text{if } r \geq 2. \end{cases}$$

Factorial cumulants and cumulants of X_t follow from (2.19):

$$\kappa_{[r]}^{(p)} = \begin{cases} \dfrac{1}{1-\alpha}(\lambda_1 + \lambda_2/\overline{\theta}), & \text{if } r = 1, \\ \dfrac{1}{1-\alpha^r}\big(\lambda_2 r!/\theta\big)\big(\theta/\overline{\theta}\big)^r, & \text{if } r \geq 2, \end{cases}$$

and

$$\kappa_r^{(p)} = \frac{\lambda_1\overline{\theta} + \lambda_2}{\overline{\alpha}\overline{\theta}} + \frac{\lambda_2}{\theta}\sum_{j=2}^{r} S(r,j)\frac{j!\big(\theta/\overline{\theta}\big)^j}{1-\alpha^j}.$$

By (2.20), the mean and variance of X_t are

$$\mu_1^{(p)} = \frac{\lambda_1\overline{\theta} + \lambda_2}{\overline{\alpha}\overline{\theta}} \quad \text{and} \quad (\sigma^{(p)})^2 = \frac{\lambda_1\overline{\theta}^2(1+\alpha) + \lambda_2(2-\overline{\alpha}\overline{\theta})}{(1-\alpha^2)\overline{\theta}^2}.$$

5. PROCESSES WITH NEGATIVE BINOMIAL INNOVATIONS

The negative binomial (NB) distribution with parameters $s > 0$ and $\theta \in (0,1)$, denoted by $NB(s,\theta)$), has pgf and pmf

$$\Psi(z) = \left\{\frac{\overline{\theta}}{1-\theta z}\right\}^s \quad \text{and} \quad f_r = \binom{s+r-1}{r}\overline{\theta}^s\theta^r \quad (r \geq 0). \tag{5.1}$$

The $NB(s,\theta)$ distribution is $DCP(\lambda, H)$, where $\lambda = -s\ln\overline{\theta}$ and $H(z)$ is the pgf of the logarithmic distribution with pmf $\{h_r\}$ described below:

$$H(z) = \frac{\ln(1-\theta z)}{\ln\overline{\theta}} \quad \text{and} \quad h_r = -\frac{\theta^r}{n\ln\overline{\theta}}, \quad (r \geq 1). \tag{5.2}$$

Theorem 5.1. *Let $\{X_t\}$ be a stationary INAR(1) process with an $NB(s,\theta)$ innovation. The following assertions hold:*

(i) *The sequence $\{m_i\}$ of (2.2) is*

$$m_i = \frac{\ln(1-\tilde{\theta}_i)}{\ln\overline{\theta}} \quad \text{with} \quad \tilde{\theta}_i = \frac{\theta\alpha^i}{1-\theta(1-\alpha^i)} \quad (i \geq 0). \tag{5.3}$$

Note $0 < \tilde{\theta}_i \leq \theta$ and $0 < m_i \leq 1$ $(i \geq 0)$. Moreover,

$$M = \sum_{i=0}^{\infty} m_i = \frac{\ln p(\alpha,\theta)}{\ln\overline{\theta}}, \quad \text{where} \quad p(\alpha,\theta) = \prod_{i=0}^{\infty}(1-\tilde{\theta}_i).$$

(ii) *The pmf $\{h_r^{(i)}\}$ of (2.4), $i \geq 0$, is logarithmic$(\tilde{\theta}_i)$ (cf. (5.2)) and*

$$DCP(\lambda m_i, H_i) \sim NB(s,\tilde{\theta}_i) \quad (i \geq 0). \tag{5.4}$$

(iii) *The marginal distribution of $\{X_t\}$ is the infinite convolution of the $NB(s,\tilde{\theta}_i)$ distributions, $i \geq 0$.*

(iv) *The marginal distribution of $\{X_t\}$ is $DCP(\widetilde{\lambda}, G)$, where $\widetilde{\lambda} = -s\ln p(\alpha,\theta)$ and G is the pgf of an infinite countable mixture of logarithmic$(\tilde{\theta}_i)$ distributions with mixing probabilities $\left(\frac{\ln(1-\tilde{\theta}_i)}{\ln p(\alpha,\theta)},\, i \geq 0\right)$.*

Proof: By (5.2), $m_i = 1 - H(1-\alpha^i) = (\ln\overline{\theta} - \ln(1-\theta(1-\alpha^i))/\ln\overline{\theta}$, which implies (5.3), since $1-\tilde{\theta}_i = \overline{\theta}/(1-\theta(1-\alpha^i))$. Thus (i) holds. Straightforward calculations show that

$$H_i(z) = 1 - \frac{1}{m_i}\bigl(1 - H(1-\alpha^i+\alpha^i z)\bigr) = \frac{\ln(1-\tilde{\theta}_i z)}{\ln(1-\tilde{\theta}_i)},$$

where $H(z)$ is as in (5.2). This establishes the first part of (ii), which in turn implies (5.4). Clearly, (iii) follows from Theorem 2.1-(i). Part (iv) is a direct consequence of (i)–(ii) and Theorem 2.1-(ii). □

We give additional proprerties of the process $\{X_t\}$ of Theorem 5.1.

The 1-step transition probability of $\{X_t\}$ can be computed from (2.12)–(2.14) with $P(\varepsilon = x) = f_x$ of (5.1). By (2.16), Lemma 2.1, and Theorem 5.1 (i)–(ii), the conditional distribution of X_{t+k} given $X_t = n$ results from the convolution of a $Bin(n,\alpha^k)$ distribution and the $NB(s,\tilde{\theta}_i)$ distributions $(0 \le i \le k-1)$.

The fmgf $H(1+t)$ of the logarithmic(θ) distribution with pgf $H(z)$ and pmf $\{h_r\}$ of (5.2) exists for $|t| < \overline{\theta}/\theta$. The factorial moments of $\{h_r\}$ are given by (see equation 7.11, p. 305, in [6])

$$\mu_{[r]}^{(h)} = -\frac{(r-1)!\bigl(\theta/\overline{\theta}\bigr)^r}{\ln\overline{\theta}} \qquad (r \ge 1). \tag{5.5}$$

Formulas for the moments of $\{g_r\}$ and the cumulants, mean and variance of X_t are given below. They are derived from (2.18)–(2.20) and (5.5):

$$\mu_{[r]}^{(g)} = -\frac{(r-1)!\bigl(\theta/\overline{\theta}\bigr)^r}{(1-\alpha^r)\ln p(\alpha,\theta)} \quad \text{and} \quad \mu_r^{(g)} = -\frac{1}{\ln p(\alpha,\theta)}\sum_{j=1}^{r} S(r,j)\frac{(j-1)!(\theta/\overline{\theta})^j}{1-\alpha^j},$$

$$\kappa_{[r]}^{(p)} = \frac{s(r-1)!\bigl(\theta/\overline{\theta}\bigr)^r}{1-\alpha^r} \quad \text{and} \quad \kappa_r^{(p)} = s\sum_{j=1}^{r} S(r,j)\frac{(j-1)!(\theta/\overline{\theta})^j}{1-\alpha^j},$$

$$\mu_1^{(p)} = \frac{s\theta}{\overline{\alpha}\overline{\theta}} \quad \text{and} \quad (\sigma^{(p)})^2 = \frac{s\theta(1+\alpha\overline{\theta})}{(1-\alpha^2)\overline{\theta}^2}.$$

Remark 5.1.

(i) Note that the special case of $s = 1$ of Theorem 5.1 covers the important special case of the unshifted geometric(θ), or $Geo_0(\theta)$, innovation. These results can be seen as extensions of some of the work in [5].

(ii) One can extend the model discussed in this section to INAR (1) processes whose innovations are finite convolutions of negative binomial distributions. The extension can be established in fairly straightforward fashion by combining the results in this section with those in Subsection 2.2.

6. PROCESSES WITH NONCENTRAL NEGATIVE BINOMIAL INNOVATIONS

Assume that $\theta \in (0,1)$, $s > 0$ and $\lambda_2 > 0$. Ong and Lee ([16]) introduced the noncentral NB distribution, $NNB(\lambda_2, s, \theta)$, as the mixture of $NB(v, \theta)$ distributions, where v is a value of the random variable $V = Y + s$ and Y is Poisson(λ_2). The pgf of $NNB(\lambda_2, s, \theta)$ is $\Psi(z) = \left(\frac{\overline{\theta}}{1-\theta z}\right)^s \exp\left(-\lambda_2 \frac{1-z}{1-\theta z}\right)$, and

$$f_r = \begin{cases} \overline{\theta}^s e^{-\lambda_2}, & \text{if } r = 0, \\ e^{-\lambda_2}\theta^r \overline{\theta}^s \displaystyle\sum_{k=0}^{r}\sum_{j=1}^{k} \binom{k-1}{j-1}\binom{s+r-k-1}{r-k} \frac{\lambda_2(\overline{\theta}/\theta)^j}{j!}, & \text{if } r > 0. \end{cases} \tag{6.1}$$

The $NNB(\lambda_2, s, \theta)$ distribution is the convolution of an $NB(s, \theta)$ distribution and a $PA(\lambda_2, \theta)$ distribution. Hence, by Lemma 2.2 (for k=2), $NNB(\lambda_2, s, \theta) \sim DCP(\lambda, H)$, where $\lambda = \lambda_2 - s \ln \overline{\theta} > 0$ and

$$H(z) = \frac{1}{\lambda}\left(-s\ln(1-\theta z) + \lambda_2 \frac{\overline{\theta} z}{1-\theta z}\right) \quad \text{and} \quad h_r = \frac{\theta^r}{\lambda}\left(\frac{s}{r} + \lambda_2 \frac{\overline{\theta}}{\theta}\right) \quad (r \geq 1). \tag{6.2}$$

We note that $\{h_r\}$ is a mixture of a logarithmic(θ) distribution and a $Geo_1(\theta)$ distribution with respective mixing probabilities $-s \ln \overline{\theta}/\lambda$ and λ_2/λ.

Theorem 6.1. *Let $\{X_t\}$ be a stationary INAR(1) process with an $NNB(\lambda_2, s, \theta)$ innovation of (6.1)–(6.2). Let*

$$\tilde{\theta}_i = \frac{\theta\alpha^i}{1-\theta(1-\alpha^i)} \qquad \text{and} \qquad p(\alpha, \theta) = \prod_{i=0}^{\infty}(1-\tilde{\theta}_i).$$

The following assertions hold:

(i) *For $\{m_i\}$ of (2.2) we have*

$$m_i = \frac{1}{\lambda}\left(-s\ln(1-\tilde{\theta}_i) + \lambda_2 \frac{\tilde{\theta}_i}{\theta}\right) \quad \text{and} \quad M = \frac{1}{\lambda}\left(-s\ln p(\alpha,\theta) + \frac{\lambda_2}{\theta}\sum_{i=0}^{\infty}\tilde{\theta}_i\right).$$

(ii) *The pmf $\{h_r^{(i)}\}$ of (2.4), $i \geq 0$, is a mixture of a logarithmic($\tilde{\theta}_i$) distribution and a $Geo_1(\tilde{\theta}_i)$ distribution, with respective mixing probabilities $b_{i1} = \big(-s\ln(1-\tilde{\theta}_i)/(\lambda m_i)\big)$ and $b_{i2} = (\lambda_2\tilde{\theta}_i)/(\lambda m_i \theta)$. Moreover,*

$$DCP(\lambda m_i, H_i) \sim NB(s, \tilde{\theta}_i) * PA\left(\lambda_2 \frac{\tilde{\theta}_i}{\theta}, \tilde{\theta}_i\right). \tag{6.3}$$

(iii) *The marginal distribution of $\{X_t\}$ is the infinite convolution of the $\big(DCP(\lambda m_i, H_i),\ i \geq 0\big)$ of (6.3).*

(iv) *The marginal distribution of $\{X_t\}$ is $DCP(\widetilde{\lambda}, G)$, where $\widetilde{\lambda} = \lambda M$ and G is the pgf of an infinite countable mixture of the sequence of pmf's $(\{h_r^{(i)}\}, i \geq 0)$ (described in (ii) above) with mixing probabilities $\big(\frac{m_i}{M},\ i \geq 0\big)$.*

Proof: The proof is similar to that of Theorem 4.1. The results follow from Lemma 2.2, Theorem 2.3 (with $k = 2$), Theorem 3.1 and Theorem 5.1. We omit the details. □

We give some additional properties of the process $\{X_t\}$ of Theorem 6.1.

The 1-step transition probability of $\{X_t\}$ can be computed from (2.12)–(2.14) with $P(\varepsilon = x) = f_x$ of (6.1). By (2.16), the conditional distribution of X_{t+k} given $X_t = n$ results from the convolution of a $Bin(n, \alpha^k)$ distribution and the distributions $\big(DCP(\lambda m_i, H_i),\ 0 \le i \le k-1\big)$ of (6.3).

As a mixture of a logarithmic(θ) distribution and a $Geo_1(\theta)$ distribution, the pmf $\{h_r\}$ of (6.2) has a finite fmgf $H(1+t)$ for $|t| < \overline{\theta}/\theta$. Therefore, the factorial moments of $\{g_r\}$ are, by (2.26), (3.4) and (5.5),

$$\mu^{(g)}_{[r]} = \frac{(r-1)!(\theta/\overline{\theta})^r}{\lambda M \theta (1-\alpha^r)} (s\theta + \lambda_2 r).$$

Combining (2.27) with the moment and cumulant formulas derived in Section 6 yields the factorial cumulants and the cumulants of X_t:

$$\kappa^{(p)}_{[r]} = \frac{(r-1)!(\theta/\overline{\theta})^r}{\theta(1-\alpha^r)} (s\theta + \lambda_2 r) \tag{6.4}$$

and

$$\kappa^{(p)}_{r} = \frac{1}{\theta} \sum_{j=1}^{r} S(r,j) \frac{(j-1)!(\theta/\overline{\theta})^j}{1-\alpha^j} (s\theta + \lambda_2 j). \tag{6.5}$$

By (2.20), the mean and variance of $\{X_t\}$ are

$$\mu^{(p)}_1 = \frac{\lambda_2 + s\theta}{\overline{\alpha}\overline{\theta}} \qquad \text{and} \qquad (\sigma^{(p)})^2 = \frac{\lambda_2(2-\overline{\alpha}\overline{\theta}) + s\theta(1+\alpha\overline{\theta})}{(1-\alpha^2)\overline{\theta}}.$$

7. PROCESSES WITH POISSON-LINDLEY INNOVATIONS

In this section, we revisit the INAR (1) model with Poisson–Lindley innovation introduced in [10] (see also [17]) and expand on their results. The Poisson–Lindley distribution ($PL(\phi)$) with parameter $\phi > 0$ is the mixture of a $Geo_1(\frac{1}{1+\phi})$ distribution and an $NB(2, \frac{1}{1+\phi})$ distribution with respective mixing probabilities $\frac{\phi}{1+\phi}$ and $\frac{1}{1+\phi}$. Its pgf and pmf are

$$\Psi(z) = \frac{\phi^2}{1+\phi} \cdot \frac{2+\phi-z}{(1+\phi-z)^2} \qquad \text{and} \qquad f_r = \frac{\phi^2}{(1+\phi)^{r+2}} \left(1 + \frac{r+1}{1+\phi}\right) \quad (r \ge 0). \tag{7.1}$$

For additional details and references on the $PL(\phi)$ distribution, we refer to [15]. A $PL(\phi)$ distribution is $DCP(\lambda, H)$ with

$$\lambda = \ln\left[\frac{(1+\phi)^3}{\phi^2(2+\phi)}\right], \qquad H(z) = 1 + \frac{1}{\lambda}\ln\left[\frac{\phi^2(2+\phi-z)}{(1+\phi)(1+\phi-z)^2}\right], \tag{7.2}$$

and

$$h_r = \frac{1}{\lambda r}\left(\frac{2}{(1+\phi)^r} - \frac{1}{(2+\phi)^r}\right) \quad (r \geq 1). \tag{7.3}$$

We introduce the Modified Poisson–Lindley distribution ($MPL(\phi,\beta)$) with parameters $\phi > 0$ and $\beta \in (0,1]$ as the distribution of $\beta \odot X$, where $X \sim PL(\phi)$. The pgf of the $MPL(\phi,\beta)$ distribution is $\Psi(1-\beta+\beta z)$, with $\Psi(z)$ of (7.1). Note that, $MPL(\phi,1) \sim PL(\phi)$.

Lemma 7.1. *An $MPL(\phi,\beta)$ distribution arises as a mixture of a $Geo_1(\beta/(\beta+\phi))$ distribution and an $NB(2,\beta/(\beta+\phi))$ distribution with resp. mixing probabilities $\frac{\phi}{1+\phi}$ and $\frac{1}{1+\phi}$. Moreover, $MPL(\phi,\beta) \sim DCP(\lambda_\beta, H_\beta)$, where*

$$\lambda_\beta = \ln\left[\frac{(1+\phi)(\beta+\phi)^2}{\phi^2(1+\beta+\phi)}\right], \qquad H_\beta(z) = 1 + \frac{1}{\lambda_\beta}\ln\left[\frac{\phi^2(1+\beta+\phi-\beta z)}{(1+\phi)(\beta+\phi-\beta z)^2}\right]. \tag{7.4}$$

Moreover, the pmf $\{h_r^{(\beta)}\}$ of $H_\beta(z)$ is

$$h_r^{(\beta)} = \frac{1}{\lambda_\beta r}\left[2\left(\frac{\beta}{\beta+\phi}\right)^r - \left(\frac{\beta}{1+\beta+\phi}\right)^r\right] \qquad (r \geq 1). \tag{7.5}$$

Proof: If X is $Geo_1(1/(1+\phi))$ (resp. $NB(2,1/(1+\phi))$, then $\beta \odot X$ is $Geo_1(\beta/(\beta+\phi))$ (resp. $NB(2,\beta/(\beta+\phi))$. By (7.1), we obtain

$$\Psi(1-\beta+\beta z) = \frac{\phi^2}{1+\phi} \cdot \frac{1+\beta+\phi-\beta z}{(\beta+\phi-\beta z)^2}.$$

A standard argument leads to the representation

$$\Psi(1-\beta+\beta z) = \exp\bigl\{\lambda_\beta(H_\beta - 1)\bigr\},$$

where λ_β and H_β and its pmf are as in (7.4)–(7.5). □

Theorem 7.1. *Let $\{X_t\}$ be a stationary INAR(1) process with a $PL(\phi)$ innovation with characteristics (7.1)–(7.3). The following assertions hold:*

- **(i)** *For every $i \geq 0$,*

$$m_i = \frac{1}{\lambda}\ln\left[\frac{(1+\phi)(\phi+\alpha^i)^2}{\phi^2(1+\phi+\alpha^i)}\right] \qquad \text{and} \qquad M = \frac{1}{\lambda}\ln\prod_{i=0}^{\infty}(1+a_i),$$

where $a_i = \dfrac{\alpha^i(\phi^2+2\phi+\alpha^i\phi+\alpha^i)}{\phi^2(1+\phi+\alpha^i)}$.

(**ii**) *The pmf* $\{h_r^{(i)}\}$ *of* (2.4), $i \geq 0$, *is given in* (7.5) *with* $\beta = \alpha^i$ *and* $\lambda_\beta = \lambda m_i$, *and*

$$DCP(\lambda m_i, H_i) \sim MPL(\phi, \alpha^i). \tag{7.6}$$

(**iii**) *The marginal distribution of* $\{X_t\}$ *is the infinite convolution of the distributions* $\big(MPL(\phi, \alpha^i),\ i \geq 0\big)$.

(**iv**) *The marginal distribution of* $\{X_t\}$ *is* $DCP(\widetilde{\lambda}, G)$, *where* $\widetilde{\lambda} = \ln \prod_{i=0}^{\infty}(1 + a_i)$, *and* G *is the pgf of the infinite countable mixture of the pmf's* $\big(\{h_r^{(i)}\},\ i \geq 0\big)$ *with respective mixing probabilities* $\big(\frac{m_i}{M},\ i \geq 0\big)$.

Proof: (i) follows from Lemma 2.1, (7.1)–(7.2), and the formula $M = \sum_{i=0}^{\infty} m_i$. Part (ii) is a direct consequence of Lemma 9.1 by setting $\beta = \alpha^i$. Part (iii) and (iv) result from (ii) and Theorem 2.1-(ii), respectively. □

We give additional properties of the process $\{X_t\}$ of Theorem 7.1.

The 1-step transition probability of $\{X_t\}$ can be computed from (2.12)–(2.14) with $P(\varepsilon = x) = f_x$ of (7.1). By (2.16) and Theorem 7.1-(ii), the conditional distribution of X_{t+k} given $X_t = n$ results from the convolution of a $Bin(n, \alpha^k)$ distribution and the $MPL(\phi, \alpha^i)$ distributions, $0 \leq i \leq k-1$.

The fcmgf $H(1+t)$ of the pmf $\{h_r\}$ of (7.2)–(7.3) exists for $|t| < \phi/2$. Its power series expansion yields the factorial moments of $\{h_r\}$:

$$\mu_{[r]}^{(h)} = \frac{(r-1)!}{\lambda}\left(\frac{2}{\phi^r} - \frac{1}{(1+\phi)^r}\right).$$

Formulas for the factorial moment of $\{g_r\}$ and the cumulants, mean and variance of X_t are given below. They are derived from (2.18)–(2.20):

$$\mu_{[r]}^{(g)} = \frac{(r-1)!}{\lambda M(1-\alpha^r)}\left(\frac{2}{\phi^r} - \frac{1}{(1+\phi)^r}\right),$$

$$\kappa_{[r]}^{(p)} = \frac{(r-1)!}{(1-\alpha^r)}\left(\frac{2}{\phi^r} - \frac{1}{(1+\phi)^r}\right) \quad \text{and} \quad \kappa_r^{(p)} = \sum_{j=1}^{r} S(r,j)\frac{(j-1)!}{(1-\alpha^j)}\left(\frac{2}{\phi^j} - \frac{1}{(1+\phi)^j}\right),$$

and

$$\mu_1^{(p)} = \frac{2+\phi}{\overline{\alpha}\phi(1+\phi)} \qquad \text{and} \qquad (\sigma^{(p)})^2 = \frac{(1+\alpha)\phi^3 + (4+3\alpha)\phi^2 + 2(3+\alpha)\phi + 2}{(1-\alpha^2)\phi^2(1+\phi)^2}.$$

8. PROCESSES WITH EULER-TYPE INNOVATIONS

Let $l(0,1)$ be the set of sequences $\Theta = (\theta_k, k \geq 0)$ such that $\theta_k \in (0,1)$ for every $k \geq 0$ and

$$\sum_{k=0}^{\infty} \frac{\theta_k}{1-\theta_k} < \infty. \tag{8.1}$$

Define

$$S_r(\Theta) = \sum_{k=0}^{\infty} \theta_k^r \quad \text{and} \quad T_r(\Theta) = \sum_{k=0}^{\infty} \left(\frac{\theta_k}{1-\theta_k}\right)^r \qquad (r \geq 1). \tag{8.2}$$

Note that the condition (8.1) implies $S_r(\Theta) < \infty$ and $T_r(\Theta) < \infty$ for all $r \geq 1$.

A $\mathbb{Z}_+$-valued rv is said to have an Euler-type distribution ($Euler - T(\Theta)$), $\Theta \in l(0,1)$, if it is an infinite convolution of $Geo_0(\theta_k)$ rv's. Its pgf is

$$\Psi(z) = \prod_{k=0}^{\infty} \left(\frac{1-\theta_k}{1-\theta_k z}\right). \tag{8.3}$$

We gather a few basic properties of an $Euler - T(\Theta)$ distribution.

Lemma 8.1. *Let $\{q_r\}$ be the pmf of an $Euler - T(\Theta)$ for some $\Theta \in l(0,1)$. The following assertions hold:*

(i) *$\{q_r\}$ is the pmf of a $DCP(\lambda, H)$ with*

$$\lambda = \sum_{k=0}^{\infty} (-\ln(1-\theta_k)) \quad \text{and} \quad H(z) = \sum_{k=0}^{\infty} \frac{-\ln(1-\theta_k)}{\lambda} H_k(z), \tag{8.4}$$

where, for each $k \geq 0$, $H_k(z)$ is the pgf of a logarithmic(θ_k) distribution. The pmf $\{h_r\}$ with pgf $H(z)$ is an infinite countable mixture of logarithmic(θ_k) distributions ($k \geq 0$) with respective mixing probabilities $\left(\frac{-\ln(1-\theta_k)}{\lambda},\ k \geq 0\right)$, or $h_r = S_r(\Theta)/(\lambda r)$, $r \geq 1$.

(ii) *$\{q_r\}$ satisfies the following recurrence relation:*

$$(r+1) q_{r+1} = \sum_{k=0}^{r} q_k S_{r+1-k}(\Theta) \quad \text{and} \quad q_0 = \prod_{k=0}^{\infty} (1-\theta_k). \tag{8.5}$$

(iii) *There exists $0 < \rho_0 \leq 1$ such that the fcmgf $H(1+t)$ of the pmf $\{h_r\}$ of part (i) is finite for $|t| < \rho_0$. Consequently, $\{h_r\}$ has finite factorial moments of all orders:*

$$\mu_{[r]}^{(h)} = \frac{(r-1)!}{\lambda} T_r(\Theta) \qquad (r \geq 1). \tag{8.6}$$

(iv) *$\{q_r\}$ has finite factorial cumulants of all orders:*

$$\kappa_{[r]}^{(q)} = (r-1)! T_r(\Theta) \qquad (r \geq 1). \tag{8.7}$$

Proof: Since $-\ln(1-x) \sim x$, as $x \to 0$, the two infinite series with respective positive summands $-\ln(1-\theta_k)$ and $-\ln(1-\theta_k z)$, $z \in (0,1)$, are convergent. Therefore, $\ln \Psi(z) = \sum_{k=0}^{\infty} \ln(1-\theta_k) - \sum_{j=0}^{\infty} \ln(1-\theta_k z)$. Letting λ be as in (8.4), we have

$$\ln \Psi(z) = \lambda \left(-1 + \sum_{k=0}^{\infty} \frac{-\ln(1-\theta_k)}{\lambda} \frac{\ln(1-\theta_k z)}{\ln(1-\theta_k)} \right).$$

The function $H_k(z) = \frac{\ln(1-\theta_k z)}{\ln(1-\theta_k)}$ is the pgf of a logarithmic(θ_k) for each $k \geq 0$ (see (5.2)). Therefore, $\ln \Psi(z) = \lambda(H(z) - 1)$, with $H(z)$ of (8.4). Again by (8.4), $\{h_r\}$ is an infinite countable mixture of logarithmic(θ_k) distributions with the stated mixing probabilities. We have by (8.4) and (5.2)

$$h_r = \sum_{k=0}^{\infty} \frac{-\ln(1-\theta_k)}{\lambda} \frac{\theta_k^r}{-r\ln(1-\theta_k)} \qquad (r \geq 1),$$

which establishes (i), via (8.2). Note that $q_0 = e^{-\lambda}$ and, similarly to (2.10), q_r satisfies the recurrence formula (8.5). We now prove (iii). By (8.1), there exists $k_0 > 1$ such that $\theta_k/\overline{\theta_k} < 1$ for $k \geq k_0$. Therefore, $\inf_{k \geq k_0} \overline{\theta_k}/\theta_k \geq 1$. Let $\rho_0 = \min\bigl(1, \min_{0 \leq k < k_0} \overline{\theta_k}/\theta_k\bigr)$. Since $\rho_0 \leq \overline{\theta_k}/\theta_k$ for every $k \geq 0$, the fmgf $H_k(1+t)$ of the logarithmic(θ_k) distribution exists for $|t| < \rho_0$. We have by (5.5) and equation (1.274), p. 59, in [6],

$$H_k(1+t) = 1 + \sum_{r=1}^{\infty} \frac{(r-1)!\bigl(\theta_k/\overline{\theta_k}\bigr)^r}{-\ln \overline{\theta_k}} \frac{t^r}{r!} \qquad (|t| < \rho_0).$$

A standard argument shows that $H(1+t) = \sum_{k=0}^{\infty} \frac{-\ln \overline{\theta_k}}{\lambda} H_k(1+t)$ converges uniformly over the interval $|t| \leq \rho$ for every $0 < \rho < \rho_0$. By Weierstrass Theorem, p. 430, in [8], we have

$$H(1+t) = 1 + \sum_{r=1}^{\infty} \left[\sum_{k=0}^{\infty} \frac{-\ln \overline{\theta_k}}{\lambda} \frac{(r-1)!\bigl(\theta_k/\overline{\theta_k}\bigr)^r}{-\ln \overline{\theta_k}} \right] \frac{t^r}{r!} \qquad (|t| < \rho_0),$$

which implies (8.6). Finally, by equation 9.49, p. 391, in [6], we have $\kappa_{[r]}^{(q)} = \lambda \mu_{[r]}^{(h)}$ which leads to (8.7). □

One can conclude from (8.7) and (2.21) that an $Euler-T(\Theta)$ has finite moments $\{\mu_r^{(q)}\}$ of all orders, and thus finite factorial moments $\{\mu_{[r]}^{(q)}\}$ of all orders.

Theorem 8.1. *Let $\{X_t\}$ be a stationary INAR(1) process with an $Euler-T(\Theta)$ innovation for some $\Theta \in l(0,1)$. For $i, k \geq 0$, let*

$$\theta_i^{(k)} = \frac{\theta_k \alpha^i}{1-\theta_k(1-\alpha^i)} \quad \text{and} \quad p_i(\alpha,\Theta) = \prod_{k=0}^{\infty} \Bigl(1 + \frac{\theta_k \alpha^i}{1-\theta_k}\Bigr). \tag{8.8}$$

The following assertions hold:

(i) *The sequence $\{m_i\}$ of (2.2) is*

$$m_i = \frac{1}{\lambda} \sum_{k=0}^{\infty} (-\ln(1-\theta_i^{(k)})) = \frac{1}{\lambda} \ln p_i(\alpha,\Theta) \quad (i \geq 0). \tag{8.9}$$

Note that $0 < \theta_i^{(k)} \leq \theta_k$ and $0 < m_i \leq 1$. Moreover,

$$M = \sum_{i=0}^{\infty} m_i = \frac{1}{\lambda} \ln \left[\prod_{i=0}^{\infty} p_i(\alpha,\Theta) \right]. \tag{8.10}$$

(ii) *The pmf $\{h_r^{(i)}\}$ of (2.4), $i \geq 0$, is an infinite countable mixture of logarithmic($\theta_i^{(k)}$) distributions, $k \geq 0$, with mixing probabilities $\Bigl(\frac{-\ln(1-\theta_i^{(k)})}{p_i(\alpha,\Theta)}, k \geq 0\Bigr)$, and*

$$DCP(\lambda m_i, H_i) \sim Euler-T(\Theta_i), \qquad \Theta_i = (\theta_i^{(k)}, k \geq 0). \tag{8.11}$$

(iii) *The marginal distribution of $\{X_t\}$ is the infinite convolution of the $Euler - T(\Theta_i)$ distributions $(i \geq 0)$ of (8.11).*

(iv) *The marginal distribution of $\{X_t\}$ is $DCP(\widetilde{\lambda}, G)$, where $\widetilde{\lambda} = \ln\Big[\prod_{i=0}^{\infty} p_i(\alpha, \Theta)\Big]$ and G is the pgf of an infinite countable mixture of the pmf's $(h_r^{(i)}, i \geq 0)$ of (ii) with mixing probabilities $\Big(\ln p_i(\alpha, \Theta) \Big/ \ln\Big[\prod_{j=0}^{\infty} p_j(\alpha, \Theta)\Big], i \geq 0\Big)$.*

Proof: For (i), we have by (8.4),

$$m_i \;=\; 1 - H(1-\alpha^i) \;=\; \sum_{k=0}^{\infty} \frac{-\ln(1-\theta_k)}{\lambda}(1 - H_k(1-\alpha^i)).$$

Since $H_k(z)$ is the pgf of a logarithmic(θ_k) distribution, it follows that $1 - H_k(1-\alpha^i) = \frac{\ln(1-\theta_i^{(k)})}{\ln(1-\theta_k)}$, from which we deduce the first equation in (8.9). The second equation as well as (8.10) are easily seen to hold. The convergence of the infinite products in part (i) stems from $\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\theta_k \alpha^i}{1-\theta_k} < \infty$. This leads to

$$1 - H_k(1-\alpha^i + \alpha^i z) = \frac{\ln(1-\theta_i^{(k)})}{\ln(1-\theta_k)}(1 - H_{ki}(z)),$$

where $H_{ki}(z)$ is the pgf of a logarithmic$(\theta_i^{(k)})$. We conclude by (2.3) and (8.4)

$$H_i(z) = \sum_{k=0}^{\infty} \frac{-\ln(1-\theta_i^{(k)})}{\lambda m_i} H_{ki}(z). \tag{8.12}$$

Now, by (5.2),

$$h_r^{(i)} = \sum_{k=1}^{\infty} \frac{-\ln(1-\theta_i^{(k)})}{\lambda m_i} \frac{[\theta_i^{(k)}]^r}{-r\ln(1-\theta_i^{(k)})} = \frac{S_r(\Theta_i)}{r p_i(\alpha, \theta)}.$$

which proves the first part of (ii). Let $\Psi(z)$ be as in (8.3). By Lemma 2.1, (8.9) and (8.12), the pgf, $\Psi(1-\alpha^i+\alpha^i z)$, of $DCP(\lambda m_i, H_i)$ is shown to be

$$\Psi(1-\alpha^i+\alpha^i z) = \exp\big\{\lambda m_i(H_i(z) - 1)\big\} = \prod_{k=0}^{\infty}\left(\frac{1-\theta_i^{(k)}}{1-\theta_i^{(k)} z}\right).$$

It is easily seen that $\Theta_i = (\theta_i^{(k)}, k \geq 0)$ belongs to $l(0,1)$. Therefore, (8.11) holds, thus completing the proof of (ii). Part (iii) follows from (8.11) and Theorem 2.1-(i). Part (iv) is a direct consequence of (i)–(ii) and Theorem 2.1-(ii). □

We discuss additional properties of the process $\{X_t\}$ of Theorem 8.1.

The 1-step transition probability of $\{X_t\}$ can be computed from (2.12)–(2.14) where the probabilities $P(\varepsilon = x) = q_x$, $x \geq 0$, can be obtained using (8.5). By (2.16), Lemma 8.1, and Theorem 8.1 (i)–(ii), the conditional distribution of X_{t+k} given $X_t = n$ arises as the convolution of a $Bin(n, \alpha^k)$ distribution and the $Euler - T(\Theta_i)$ distributions $(0 \leq i \leq k-1)$ of (8.11).

Formulas for the moments of $\{g_r\}$ and the factorial moments, mean and variance of X_t are obtained from (2.18)–(2.20) and (8.6):

$$\mu_{[r]}^{(g)} = \frac{(r-1)!}{\lambda M(1-\alpha^r)} T_r(\Theta) \qquad \text{and} \qquad \mu_r^{(g)} = \frac{1}{\lambda M} \sum_{j=1}^{r} S(r,j) \frac{(j-1)!}{(1-\alpha^j)} T_j(\Theta),$$

$$\kappa_{[r]}^{(p)} = \frac{(r-1)!}{(1-\alpha^r)} T_r(\Theta) \qquad \text{and} \qquad \kappa_r^{(p)} = \sum_{j=1}^{r} S(r,j) \frac{(j-1)!}{(1-\alpha^j)} T_j(\Theta).,$$

and

$$\mu_1^{(p)} = \frac{T_1(\Theta)}{1-\alpha} \qquad \text{and} \qquad (\sigma^{(p)})^2 = \frac{(1+\alpha)T_1(\Theta) + T_2(\Theta)}{1-\alpha^2}.$$

9. PROCESSES WITH EULER INNOVATIONS

The Euler distribution ($Euler(\eta, q)$) introduced by Benkherouf and Bather ([3]) (see [6]) is an $Euler-T(\Theta)$ distribution with $\Theta = (\eta q^k, k \geq 0)$ for $0 < \eta < 1$ and $0 < q < 1$. An application of the ratio test shows that indeed $\Theta \in l(0,1)$. We also note that $S_r(\Theta) = \frac{\eta^r}{1-q^r}$, $r \geq 1$. We use the notation $T_r(\eta, q)$ in lieu of $T_r(\Theta)$.

We recall a few basic properties of the $Euler(\eta, q)$ distribution (cf., for example, [7]). Its pmf $\{q_x\}$ is

$$q_0 = \prod_{j=0}^{\infty} (1 - \eta q^j) \qquad \text{and} \qquad q_x = \frac{\eta^x}{\prod_{l=1}^{x} (1-q^l)} q_0 \quad (x \geq 1). \tag{9.1}$$

Its mean and variance are

$$\mu = \sum_{x=0}^{\infty} \frac{\eta q^x}{1 - \eta q^x} \qquad \text{and} \qquad \sigma^2 = \sum_{x=0}^{\infty} \frac{\eta q^x}{(1-\eta q^x)^2}.$$

The following result is known. We refer to Lemma 8.1 for convenience.

The $Euler(\eta, q)$ distribution is $DCP(\lambda, H)$ with $\lambda = -\ln\left(\prod_{k=0}^{\infty}(1-\eta q^k)\right)$ and $H(z)$ is the pgf of an infinite countable mixture of logarithmic(ηq^k) distributions, $k \geq 0$, with respective mixing probabilities $\left(\frac{-\ln(1-\eta q^k)}{\lambda}, k \geq 0\right)$. Its pmf is $h_r = \eta^k/(\lambda k(1-q^k))$, $r \geq 1$.

The main result of the section is stated without proof as it is a particular case of Theorem 8.1.

Theorem 9.1. *Let $\{X_t\}$ be a stationary INAR(1) process with an $Euler(\eta, q)$ innovation for some η, $q \in (0,1)$. For $i, k \geq 0$, let*

$$\theta_i^{(k)} = \frac{\eta q^k \alpha^i}{1 - \eta q^k (1-\alpha^i)} \qquad \text{and} \qquad p_i(\alpha, \eta, q) = \prod_{k=0}^{\infty} \left(1 + \frac{\eta q^k \alpha^i}{1 - \eta q^k}\right). \tag{9.2}$$

The following assertions hold:

(i) *The sequence* $\{m_i\}$ *of* (2.2) *and* $M = \sum_{i=0}^{\infty} m_i$ *are as follows:*

$$(9.3) \qquad m_i = \frac{1}{\lambda} \ln p_i(\alpha,\eta,q) \quad \textit{and} \quad M = \frac{1}{\lambda} \ln\left[\prod_{i=0}^{\infty} p_i(\alpha,\eta,q)\right].$$

Note that $0 < \theta_i^{(k)} \le \eta q^k$ *and* $0 < m_i \le 1$ $(i \ge 0)$*.*

(ii) *The pmf* $\{h_r^{(i)}\}$ *of* (2.4)*,* $i \ge 0$*, is an infinite countable mixture of logarithmic*$(\theta_i^{(k)})$ *distributions,* $k \ge 0$*, with mixing probabilities* $\left(\frac{-\ln(1-\theta_i^{(k)})}{p_i(\alpha,\eta,q)},\, k \ge 0\right)$*, and*

$$(9.4) \qquad DCP(\lambda m_i, H_i) \sim Euler\!-\!T(\Theta_i), \qquad \Theta_i = (\theta_i^{(k)},\, k \ge 0).$$

(iii) *The marginal distribution of* $\{X_t\}$ *is the infinite convolution of the* $Euler\!-\!T(\Theta_i)$ *distributions* $(i \ge 0)$ *of* (9.4)*.*

(iv) *The marginal distribution of* $\{X_t\}$ *is* $DCP(\widetilde{\lambda}, G)$*, where* $\widetilde{\lambda} = \ln\left[\prod_{i=0}^{\infty} p_i(\alpha,\Theta)\right]$ *and* G *is the pgf of an infinite countable mixture of the pmf's* $(h_r^{(i)},\, i \ge 0)$ *of (ii) with mixing probabilities* $\left(\ln p_i(\alpha,\eta,q) / \ln\left[\prod_{j=0}^{\infty} p_j(\alpha,\eta,q)\right],\, i \ge 0\right)$*.*

Additional properties of the process $\{X_t\}$ of Theorem 9.1 are given next.

The 1-step transition probability of $\{X_t\}$ can be computed from (2.12)–(2.14) where the probabilities $P(\varepsilon = x) = q_x$, $x \ge 0$, are as in (9.1). By (2.16) and Theorem 8.1 (i)–(ii), the conditional distribution of X_{t+k} given $X_t = n$ arises as from the convolution of a $Bin(n, \alpha^k)$ distribution and the $Euler\!-\!T(\Theta_i)$ distributions $(0 \le i \le k-1)$ of (9.4).

Formulas for the moments of $\{g_r\}$ and the factorial moments, mean and variance of X_t are as follows:

$$\mu_{[r]}^{(g)} = \frac{(r-1)!}{\lambda M (1-\alpha^r)} T_r(\alpha,\eta,q) \quad \text{and} \quad \mu_r^{(g)} = \frac{1}{\lambda M} \sum_{j=1}^{r} S(r,j) \frac{(j-1)!}{(1-\alpha^j)} T_j(\alpha,\eta,q),$$

$$\kappa_{[r]}^{(p)} = \frac{(r-1)!}{(1-\alpha^r)} T_r(\alpha,\eta,q) \quad \text{and} \quad \kappa_r^{(p)} = \sum_{j=1}^{r} S(r,j) \frac{(j-1)!}{(1-\alpha^j)} T_j(\alpha,\eta,q),$$

and

$$\mu_1^{(p)} = \frac{T_1(\alpha,\eta,q)}{1-\alpha} \quad \text{and} \quad (\sigma^{(p)})^2 = \frac{(1+\alpha) T_1(\alpha,\eta,q) + T_2(\alpha,\eta,q)}{1-\alpha^2}.$$

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