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PRODUCTION PROCESSES WITH DIFFERENT LEVELS OF RISK: ADDRESSING THE REPLACEMENT OPTION

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Abstract:

- It is often found that a company has the opportunity to change its original production process to a different one. Here we consider two different situations: in the first one, the new production process may lead to larger losses (in case the demand decreases), but may also lead to larger profits (in case the demand increases); so we increase the risk. In the second one, the opposite holds (and we decrease the risk).
  We derive the optimal replacement strategy, and we study the impact of the drift and the volatility in the decision. Afterward, we include the option to exit the market and we compare both situations (with and without the exit option), concluding that in case the investment is in a less risky process, the impact of the exit option is different, depending on a relation involving the costs and the drift parameters.

Keywords:

- *risk management; replacement policies; exit decision; real options; optimal stopping.*

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1. INTRODUCTION

Last decades have been characterized by relevant changes in the global macroeconomic scenario. Naturally, this has affected the way managers make decisions about the future of firms. Real options theory provides efficient tools to analyze what is the best decision to make, taking into account all the options faced by the firm. In particular, we can highlight the following options/problems: the strategy options (see Huisman and Kort (2003) [11], Pawlina and Kort (2006) [18], and Brealey *et al.* (2012) [3]), valuation of options for real assets (see Dixit and Pindyck (1994) [6]), investment options (see Bjerksund and Ekern (1990) [2], Dixit and Pindyck (1994) [6], Majd and Pindyck (1987) [13], and McDonald and Siegel (1985) [14]), technology adoption problem (see Farzin *et al.* (1998) [8], and Hagspiel *et al.* (2016) [10]), or abandonment problem (see Brennan and Schwartz (1985) [4], and Myers and Majd (2001) [15]).

During the last economic crisis, many firms felt the need to adjust their production process, to face declining markets and to avoid large losses. An example of a strategy used by decision-makers to decrease the costs associated with the production process is the layoff. Companies like Merck, Yahoo, General Electric, Xerox, Pratt & Whitney, Goldman Sachs, Whirlpool, Bank of America, Alcoa and Coca-Cola implemented layoff periods, to reduce costs and face adverse market conditions[1]. The main goal of firms adopting this type of strategies is to reduce the risk of having large losses by adopting a production process which results in a "flat payoff function": the profits would not be very large if the demand is large but in case the demand decreases, the firm faces also small losses; the resulting is a sort of a compromise situation. The idea of flat payoff function is already used in Decision Theory, with a similar meaning; see, for instance, Pannell (2006) [17]. On the other side of the scale, you also find companies with a more *aggressive* behavior, meaning, in this particular framework, that the firm adapts its production process in order to obtain large profits for high levels of demand, even if the losses may be large, for small levels of demand. There are, of course, many strategies which lead to intermediate payoff functions: between the flat payoff functions (less risky, in terms of potential losses) and the more aggressive ones (more risky, in terms of potential losses).

The changes in the profit function may be due to several causes, such as technology innovation or improvement in the production process. Indeed, nowadays, companies face many challenges, as the markets are very competitive, and technology innovations can radically change the costs and profits. Technology innovation may change the production costs, as it gets more advanced, prices drop and products get better. However, it can exist some drawbacks, such as the costs associated with the technological process or even a chance that the switch to the new technology does not lead to positive profits but leads to losses, due to declining markets, for instance. These challenges amplify with the large uncertainty that is inherent to the market, as Ward *et al.* (1995) [20] refer.

There are several examples of such a situation. One of such examples occurs in the area of IT (information technology), the decision of where and when to allocate resources to IT programs is risky, as although there are many positive outcomes, the executives struggle with the massive costs and high uncertainty. According to Clemons and Weber (1990) [5],

[1] Uchitelle, L. (2008, October 26). U.S. layoffs increase as businesses confront the crisis, *The New York Times*. Retrieved from http://www.nytimes.com/2008/10/26/business/worldbusiness/26iht-layoffs.1.17246245.html

IT can confer advantage under appropriate conditions, and equally important, even when it fails to confer advantage, it may still prove crucial. The same authors mention the case of Manufacturers Hanover, that in the early 80's invested 300 million dollars in a telecommunication network. The actual volumes reached only 50% of the estimates, well below the capacity, and leading to massive losses, as they could not recover the system cost. See Benaroch (2002) [1].

Another such example is the present situation of ASML: the largest supplier in the world of photolithography systems for the semiconductor industry. ASML is one of the 8th foreign companies that have sales of at least 1 billion dollars in South Korea. Recent investments in the next-generation technologies have allowed ASML to reduce their potential costs by 30% or 40%. But a serious flare-up between North and South Korea would cause a huge disruption to commerce. And if operations in the country were suspended or set back for a long time due to the destruction of facilities, that would disrupt the supply chain of companies around the world. And ASML, which is vulnerable to this situation, would then face major losses.[2] Therefore investments in this area of the planet, although lead to potentially large profits, also may lead the massive losses.

The last example that we provide is related to the use of statistical process control (SPC) charts to monitor quality. Control charts are used to keep a process in statistical control, where the output quality is at a target level; the design of the control chart is usually known as economic design (see Lorenzen and Vance (1986) [12]). But the implementation of statistical control can be quite expensive, as Nembhard *et al.* (2002) [16] refer. But, on the other hand, if a control chart is not used, the manufacturer may not be aware that the system is producing low-quality parts. And this may have a cost, as these products may be returned, with extra replacement costs. Therefore the choice between implementing a production scheme with or without a rigorous statistical control is a relevant decision in terms of profits and losses, and the decision must take into account the dynamics in the market conditions.

These examples show a common feature: firms have the opportunity to change their production systems, due to several reasons, but when deciding about it they need to balance between potential losses and gains, as these investments do not lead only to larger profits. Our main objective is to study the time at which the firm should optimally change its production system. Reporting to the literature of real options, this problem falls into the category of single-switch or replacement problems, a problem that is crucial from the management viewpoint. We will be mainly concerned with the implications of adjusting the current production process in a risky or less risky way, where we use the following interpretation:

- The risk increases if when compared with the current profit, the gains of the firm increase when the demand is sufficiently high, but the losses also increase in case the demand is not sufficiently high;
- The risk decreases if, when compared with the current profit, the losses of the firm decrease when the demand reaches sufficiently small levels, but the gains also decrease in case the demand becomes sufficiently high.

Throughout the paper, we use the terms *replacement* and *investing* indistinctly, in the sense that they both mean that the firm will change its original production process (leading to a profit function Π_1) by a different production process (leading to a profit function Π_2).

[2] Wong, S. and Miller, L.J. (2017, August 20). These are the most vulnerable foreign companies in Korea, *Bloomberg Politics*. Retrieved from https://www.bloomberg.com/news/articles/2017-08-20/in-shadow-of-red-line-companies-with-a-lot-to-lose-in-korea

Besides the option to change its production process, a firm may still decide to abandon the market, in case the conditions are no longer favorable in terms of its profits. Therefore we also analyze the situation where after the investment in the second production process, the firm may decide to exit. Moreover, we compare the impact of the abandonment option in the invest moment in the second production market and, as we will see, this impact depends on whether the firm intends to increase the risk or not, and the relation between the involved costs and the parameters of the demand process. Here we assume that abandonment only happens after investing in the second production process, which is equivalent to say that abandonment out of the first production process is equally costly as first investing in the second production process and then abandon. This assumption is also considered in chapter 7 of Dixit and Pindyck (1994) [6].

The rest of the paper is organized as follows: in Section 2 we describe the model, along with some considerations about the economical meaning; in Section 3 we present the Hamilton–Jacobi–Bellman equation for the optimization problem. In Section 4 we derive the solution of the problem and in Section 5 we present comparative statics results. Finally, in Section 6 we consider the option to abandon the market, after investing in the second production process. The proofs of the propositions and corollaries can be found in Appendix A.

2. MODEL

In this paper, we consider a firm that produces an established product in a stochastic environment, which is characterized by the stochastic demand process $X = \{X_t : t \geq 0\}$, defined on a complete filtered space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Moreover, we assume that X follows a geometric Brownian motion, solution of the stochastic differential equation:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t \,,$$

where $X_0 = x$, $\mu \in \mathbb{R}$ is the drift, the volatility is equal to $\sigma > 0$, and $\{W_t : t \geq 0\}$ is a Brownian motion.

Currently, the profit of the firm is Π_1, that depends on X, and the firm has the option to change its profit function to Π_2, but staying in the same market (and thus the uncertainty process, X, does not change its dynamics as a consequence of this change). If the firm decides to materialize this option at time τ, then its value is given by

$$\begin{aligned} J(x,\tau) &= E_x\left[\int_0^{\tau} e^{-\gamma s}\,\Pi_1(X_s)\,ds \;-\; e^{-\gamma\tau}R + \int_{\tau}^{\infty} e^{-\gamma s}\,\Pi_2(X_s)\,ds\right] \\ &= E_x\left[\int_0^{\tau} e^{-\gamma s}\,\Pi_1(X_s)\,ds \;+\; \int_{\tau}^{\infty} e^{-\gamma s}\Big(\Pi_2(X_s) - \gamma R\Big)\,ds\right], \end{aligned}$$

where $R \geq 0$ is the cost of adjusting its production process, $\gamma > 0$ is the interest rate and E_x represents the conditional expectation when $X_0 = x$. Defining by $\mathcal{S}$ the set of all admissible $\{\mathcal{F}_t\}$-stopping times, we are looking for the right moment of changing the production process. Thus, we define the value function $\mathcal{V}$, given by:

$$\mathcal{V}(x) = \sup_{\tau \in S} J(x,\tau) = J(x,\tau^*)\,. \tag{2.1}$$

If in the problem (2.1), one has $\Pi_1(x) \leq \Pi_2(x) - \gamma R$, for all $x > 0$, then the decision is trivial: $\tau^* = 0$, and therefore the firm must change immediately. On the other hand, when $\Pi_2(x) - \gamma R \leq \Pi_1(x)$, for all $x > 0$, then the decision is also trivial: $\tau^* = \infty$, and therefore the firm never takes the decision to invest in the second production process. However, the most interesting situation is illustrated in Figure 1. In fact, assuming that there is $c > 0$, such that $\Pi_1(c) = \Pi_2(c) - \gamma R \equiv d > 0$, then, we have the following situations:

- **a)** $\Pi_1(x) < \Pi_2(x) - \gamma R$ if and only if $x > c$. In this case, for lower values of the demand process, Π_1 leads to larger profits or lower losses than Π_2, whereas for large values of demand, Π_2 is more profitable. For this reason, we say that in this situation the *risk increases*, when we switch from Π_1 to Π_2;
- **b)** $\Pi_1(x) > \Pi_2(x) - \gamma R$ if and only if $x > c$. Then Π_2 leads to smaller losses/smaller earnings in case the demand decreases/increases, when compared with Π_1. For this reason, we say that in this situation the *risk decreases*.

Here, we use isoelastic profit functions, with some constant linear factor:

$$\Pi_i(x) = a_i \, x^{\theta_i} - b_i \,, \qquad \text{with} \quad \theta_i \geq 1 \,, \;\; a_i, b_i \geq 0 \,,$$

where θ_i is the elasticity coefficient and b_i denotes a fixed cost.

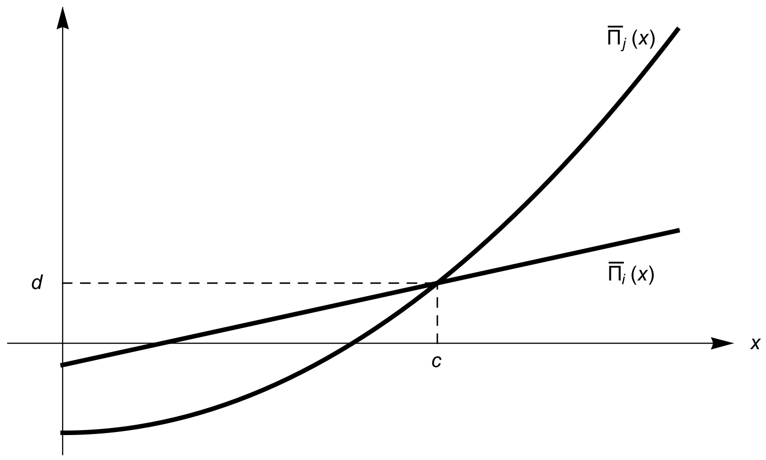

Figure 1: Representation of the functions $\overline{\Pi}_s$, with $s = i, j$ and $i \neq j \in \{1, 2\}$, where $\overline{\Pi}_s(x) = \Pi_1(x)$ if $s = 1$ or $\overline{\Pi}_s(x) = \Pi_2(x) - \gamma R$, if $s = 2$, for all $x > 0$.

Additionally, we will discuss how the option to abandon definitely the market after the replacement influences the value of the firm as well as the economic mechanisms behind the decisions. Then, the problem can be re-stated as follows:

$$\begin{aligned} \mathcal{W}(x) &= \sup_{\tau_1 \leq \tau_2 \in \mathcal{S}} E\left[\int_0^{\tau_1} e^{-\gamma s}\, \Pi_1(X_s)\, ds \;-\; e^{-\gamma \tau_1} R \;+\; \int_{\tau_1}^{\tau_2} e^{-\gamma s}\, \Pi_2(X_s)\, ds \;-\; e^{-\gamma \tau_2} S\right] \\ &\equiv \sup_{\tau_1 \leq \tau_2 \in \mathcal{S}} I(x, \tau_1, \tau_2)\,, \end{aligned} \tag{2.2}$$

where τ_1 is the time to replace Π_1 by Π_2, and τ_2 is the time to abandon the market. In (2.2), S represents the abandonment cost, when S is positive (meaning that the firm needs to pay to abandon the market) or a salvage value/disinvestment subsidy, when S is negative (meaning that the firm receives money upon the exit of the market).

In order to have a well-posed problem, in the sense that the next integrability condition holds:

$$E_x\left[\int_0^\infty e^{-\gamma s}\left|\Pi_i(X_s)\right| ds\right] < \infty\,, \qquad \text{for} \quad i=1,2\,, \tag{2.3}$$

we assume the following relation on the parameters:

$$\gamma > \frac{\sigma^2}{2}(\theta_i-1)\,\theta_i+\theta_i\,\mu \equiv \mu_{\theta_i}\,, \qquad \text{for} \quad i=1,2\,.$$

See Guerra *et al.* (2016) [9] for further mathematical explanations about the integrability condition (2.3). Additionally, for (γ,μ,σ) fixed, let β_1 and β_2 denote the two roots of the quadratic equation

$$\gamma = \frac{\sigma^2}{2}(y-1)\,y+\mu\,y\,,$$

with $\beta_1<0<\beta_2$. We notice that the condition (2.3) implies that $\beta_2>\theta>1$.

Although the natural economic modeling of this problem relies on the set of parameters (r,μ,σ), it can be, equivalently, modeled by using the set of parameters (β_1,β_2,σ), since

$$\gamma = -\frac{\sigma^2}{2}\,\beta_1\,\beta_2 \qquad \text{and} \qquad \mu = \frac{\sigma^2}{2}(1-\beta_1-\beta_2)\,.$$

For future reference, we note that the functions $(\mu,\sigma)\to\beta_i(\mu,\sigma)$, with $i=1,2$, are such that the function $\beta_1(\cdot,\sigma)$, $\beta_2(\cdot,\sigma)$ and $\beta_2(\mu,\cdot)$ [3] are decreasing, while $\beta_1(\mu,\cdot)$ is increasing. This follows in view of the following derivatives:

$$\frac{\partial\beta_i}{\partial\sigma} = (-1)^{i+1}\frac{\sigma\,\beta_i\,(\beta_i-1)}{\sqrt{\left(\mu-\frac{1}{2}\,\sigma^2\right)^2+2\,\sigma^2\gamma}} \qquad \text{and} \qquad \frac{\partial\beta_i}{\partial\mu} = (-1)^{i+1}\frac{\beta_i}{\sqrt{\left(\mu-\frac{1}{2}\,\sigma^2\right)^2+2\,\sigma^2\gamma}}\,.$$

3. HAMILTON–JACOBI–BELLMAN EQUATIONS

In this section, we introduce the HJB equations that lead to the solution of the optimization problems. We start by noticing that for the replacement problem, we may write the functional J as follows:

$$\begin{aligned} J(x,\tau) &= E_x\left[\int_0^\tau e^{-\gamma s}\Big(\Pi_1(X_s)-\Pi_2(X_s)+\gamma R\Big)\,ds\right] + E_x\left[\int_0^\infty e^{-\gamma s}\Big(\Pi_2(X_s)-\gamma R\Big)\,ds\right] \\ &= E_x\left[\int_0^\tau e^{-\gamma s}\Big(a_1X_s^{\theta_1}-a_2X_s^{\theta_2}-b+\gamma R\Big)\,ds\right] + a_2\,\frac{x^{\theta_2}}{\gamma-\mu_2}-\frac{b_2+\gamma R}{\gamma}\,, \end{aligned}$$

[3] Along this paper, we use $f(\cdot,y)$ to denote the function f as a function of the first variable, keeping the second fixed and equal to y.

for every $(x,\tau) \in]0,\infty[\times\mathcal{S}$, with $b = b_1 - b_2$.[4] Then, for all $x > 0$,

$$\mathcal{V}(x) \;=\; V(x) + a_2\,\frac{x^{\theta_2}}{\gamma-\mu_2} - \frac{b_2+\gamma R}{\gamma}\,, \tag{3.1}$$

with

$$V(x) \;=\; \sup_{\tau\in S} E_x\left[\int_0^{\tau} e^{-\gamma s}\Big(a_1X_s^{\theta_1} - a_2X_s^{\theta_2} - b + \gamma R\Big)\,ds\right], \tag{3.2}$$

and the remaining part of the right-hand side of Equation (3.1) representing the net present value associated to the second production process. Thus, henceforward, we will be concerned about the optimal stopping problem defined in (3.2).

In light of the classical Theory of Optimal Stopping (see, for instance, Peskir and Shiryaev (2006) [19]), V satisfies the HJB equation:

$$\min\left\{\gamma v(x) - \mu x v'(x) - \frac{\sigma^2}{2}x^2v''(x) - \Big(\Pi_1(x) - \Pi_2(x) + \gamma R\Big),\;\; v(x)\right\} \;=\; 0\,.$$

From this equation, it follows that $V(x) \geq 0$, for $x > 0$. Additionally, if there is $x_0 > 0$ such that $V(x_0) > 0$, then V should satisfy the ODE

$$\gamma v(x) - \mu x v'(x) - \frac{\sigma^2}{2}x^2v''(x) - \Big(\Pi_1(x) - \Pi_2(x) + \gamma R\Big) \;=\; 0\,, \tag{3.3}$$

in the set $\big\{x > 0\colon |x - x_0| < \epsilon\big\}$, for some $\epsilon > 0$. Equation (3.3) is an Euler–Cauchy differential equation and admits as solution the function

$$v(x) \;=\; Ax^{\beta_1} + Bx^{\beta_2} + \alpha x^{\theta_1} - \beta x^{\theta_2} - \frac{b}{\gamma} + R\,, \tag{3.4}$$

with

$$\alpha = \frac{a_1}{\gamma - \mu_{\theta_1}} \qquad \text{and} \qquad \beta = \frac{a_2}{\gamma-\mu_{\theta_2}}\,,$$

for every $A, B \in \mathbb{R}$.

When we consider the exit option after investing in the second process production process, one may see that standard arguments (see, for instance, Duckworth and Zervos (2000) [7]) allow us to get an equivalent expression to (2.2), that is:

$$\begin{aligned}\mathcal{W}(x) \;&=\; \sup_{\tau_1\in\mathcal{S}} E\left[\int_0^{\tau_1} e^{-\gamma s}\Big(\Pi_1(X_s) + \gamma R + \gamma S\Big)\,ds \;+\; e^{-\gamma\tau_1}\tilde{W}(X_{\tau_1})\right] - R - S \\ &\equiv \sup_{\tau_1\leq\tau_2\in\mathcal{S}} \tilde{I}(\tau_1,\tau_2,x) - R - S\,,\end{aligned} \tag{3.5}$$

where

$$\tilde{W}(x) \;=\; \sup_{\tau\in\mathcal{S}} E\left[\int_0^{\tau} e^{-\gamma s}\Big(\Pi_2(X_s) + \gamma S\Big)\,ds\right].$$

Thus the corresponding HJB equation is the following:

$$\min\left\{\gamma w(x) - \mu x w'(x) - \frac{\sigma^2}{2}x^2w''(x) - \Pi_1(x) - \gamma(R+S),\;\; w(x) - \tilde{W}(x)\right\} \;=\; 0$$

where, in its turn, $\tilde{W}$ is a solution of the HJB equation corresponding to the exit problem:

$$\min\left\{\gamma\tilde{W}(x) - \mu x\tilde{W}'(x) - \frac{\sigma^2}{2}x^2\tilde{W}''(x) - \Pi_2(x) - \gamma S,\;\; \tilde{W}(x)\right\} \;=\; 0\,.$$

[4] From now on we will use the notation $a = a_1 - a_2$.

4. THE REPLACEMENT OPTION

In this section we present the solution to the problem (3.2), assuming that $\theta_i = 1$ and $\theta_j = \theta \geq 1$, for $i \neq j \in \{1, 2\}$. With this assumption, we may derive analytical expressions for the relevant quantities. Recall that $c > 0$ is such that $\Pi_1(c) - \Pi_2(c) + \gamma R = 0$; moreover, we let $d = \Pi_1(c) = \Pi_2(c) - \gamma R$.

To solve the problem (3.2), we need to use the smooth pasting conditions in order to find the unknown terms of (3.4), and its domain. Therefore we need to propose a continuation region. In fact, depending on the sign of $\Pi_1 - \Pi_2 + \gamma R$, the geometry of the problem is different and, consequently, the continuation region is also distinct.

In case of the increasing risk, we expect that the continuation region is of the form $\mathcal{C} = \{x > 0 \colon x < \delta\}$, with $\delta \geq c$, as in that case one should only invest in the more risky production process when the demand is high (and higher than c, because for $x < c$, $\Pi_2(x) - \gamma R - \Pi_1(x) < 0$). But if the risk decreases, then we expect the continuation region to be $\mathcal{C} = \{x > 0 \colon x > \zeta\}$, with $\zeta \leq c$, since in that case the replacement should be undertaken when the levels of demand are low. Therefore we need to study the two cases separately, as we present in the next sections.

4.1. INCREASING RISK

Here, we assume that the profit functions Π_1 and Π_2 are given by:

$$\Pi_1(x) = a_1 x - b_1 \quad \text{and} \quad \Pi_2(x) = a_2 x^\theta - b_2 \,, \tag{4.1}$$

with $\theta > 1$, and

$$b_1 \leq b_2 + \gamma R \,. \tag{4.2}$$

Note that this inequality may be interpreted as follows: the fixed cost when using Π_1 must be lower than or equal to the sum of the investment rate cost plus the fixed cost of using Π_2. If this condition does not hold, then replacement would be optimal right away (i.e., the optimal time would be zero).

Proposition 4.1. *Let Π_i, with $i = 1, 2$ be given by (4.1). Then, the solution of (3.2) is as follows:*

$$V(x) = \begin{cases} B x^{\beta_2} + \dfrac{a_1}{\gamma - \mu}\, x - \dfrac{a_2}{\gamma - \mu_\theta}\, x^\theta - \dfrac{b - \gamma R}{\gamma} \,, & x < \delta \,, \\ 0 \,, & x \geq \delta \,, \end{cases} \tag{4.3}$$

where B is given by

$$B = \left(\frac{a_2\, \delta^\theta}{\gamma - \mu_\theta} - \frac{a_1 \delta}{\gamma - \mu} + \frac{b - \gamma R}{\gamma} \right) \delta^{-\beta_2} \geq 0 \,. \tag{4.4}$$

Additionally, δ is the unique positive solution to

$$f(x) \;:=\; \frac{a_2\,(\beta_2-\theta)}{\gamma-\mu_\theta}\,x^\theta - \frac{a_1(\beta_2-1)}{\gamma-\mu}\,x + \beta_2\,\frac{b-\gamma R}{\gamma} \;=\; 0\,, \tag{4.5}$$

and verifies $\delta \geq c$. The result remains true when $\theta = 1$, $a_1 < a_2$ and $b_1 < b_2 + \gamma R$.

Taking into account the explanations provided in Section 3, it follows that

$$\mathcal{V}(x) \;=\; \begin{cases} Bx^{\beta_2} + \dfrac{a_1}{\gamma-\mu}\,x - \dfrac{b_1}{\gamma}\,, & x < \delta\,, \\[2ex] \dfrac{a_2}{\gamma-\mu_\theta}\,x^\theta - \dfrac{b_2+\gamma R}{\gamma}\,, & x \geq \delta\,, \end{cases}$$

which means that for large levels of demand ($x > \delta$) it is always optimal to switch from the actual production process to the new one. This reinforce the idea that this type of strategy may be useful in markets that are in expansion. While the terms $\frac{a_1}{\gamma-\mu} - \frac{b_1}{\gamma}$ and $\frac{a_2}{\gamma-\mu_\theta} - \frac{b_2}{\gamma}$ represent the net present value associated to the first and second production process, respectively, the term Bx^{β_2} gives the value associated with the replacement option when the current value of the demand is x.

Corollary 4.1. *If $\tilde{b} \equiv b - \gamma R = 0$, then the replacement threshold δ can be explicitly given by:*

$$\delta_0 \;\equiv\; \delta\Big|_{\tilde{b}=0} \;=\; \sqrt[\theta-1]{\frac{a_1}{a_2}\,\frac{\beta_2-1}{\gamma-\mu}\,\frac{\gamma-\mu_\theta}{\beta_2-\theta}} \;=\; \sqrt[\theta-1]{\frac{a_1}{a_2}\,\frac{\beta_1-\theta}{\beta_1-1}}\,.$$

If $\theta = 1$, $a_1 < a_2$ and $b_1 < b_2 + \gamma R$, then δ can be explicitly given by:

$$\delta\Big|_{\theta=1} \;=\; \frac{b-\gamma R}{a}\left(\frac{\gamma-\mu}{\gamma}\,\frac{\beta_2}{\beta_2-1}\right) \;=\; \frac{b-\gamma R}{a}\left(1-\frac{1}{\beta_1}\right) \;\geq\; \frac{b-\gamma R}{a} \;=\; c\,.$$

For future reference, one can note that δ_0 is a lower bound to δ since the function $\tilde{b} \to \delta(\tilde{b})$ is decreasing and consequently $\delta_0 \leq \delta$. Indeed, in light of the calculations presented in the proof of Lemma A.1, we get

$$\frac{\partial\delta}{\partial\tilde{b}}(\tilde{b}) \;=\; -\frac{\beta_2}{\gamma}\,f'(\delta) \;<\; 0\,.$$

4.2. DECREASING RISK

Consider now the case:

$$\Pi_1(x) = a_1 x^\theta - b_1 \qquad \text{and} \qquad \Pi_2(x) = a_2 x - b_2\,, \tag{4.6}$$

with $\theta > 1$, and

$$b_1 \geq b_2 + \gamma R\,.$$

Similarly to the previous situation, the interpretation of this condition is also clear. In order to have a non-trivial problem, we need to impose that the fixed cost associated with Π_1 is larger than the investment cost rate plus the fixed cost of Π_2. Otherwise, replacement would never be optimal and we would have the optimal time equal to ∞.

Proposition 4.2. *The value function defined by (3.2) is given by:*

$$V(x) = \begin{cases} 0\,, & x < \zeta\,, \\ Ax^{\beta_1} + \dfrac{a_1}{\gamma-\mu_\theta}\,x^\theta - \dfrac{a_2}{\gamma-\mu}\,x - \dfrac{b-\gamma R}{\gamma}\,, & x \geq \zeta\,, \end{cases} \tag{4.7}$$

where A is given by

$$A = \left(\frac{a_2\zeta}{\gamma-\mu} - \frac{a_1\zeta^\theta}{\gamma-\mu_\theta} + \frac{b-\gamma R}{\gamma}\right)\zeta^{-\beta_1} \geq 0 \tag{4.8}$$

and ζ is the unique positive solution to

$$g(x) := \frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}\,x^\theta - \frac{a_2\,(1-\beta_1)}{\gamma-\mu}\,x + \beta_1\frac{b-\gamma R}{\gamma} = 0\,, \tag{4.9}$$

and verifies $\zeta \leq c$. The result remains true when $\theta = 1$, $a_1 > a_2$ and $b_1 > b_2 + \gamma R$.

In this case, the value function $\mathcal{V}$ is given by

$$\mathcal{V}(x) = \begin{cases} \dfrac{a_2}{\gamma-\mu}\,x - \dfrac{b_2+\gamma R}{\gamma}\,, & x < \zeta\,, \\ Ax^{\beta_1} + \dfrac{a_1}{\gamma-\mu_\theta}\,x^\theta - \dfrac{b_1}{\gamma}\,, & x \geq \zeta\,, \end{cases}$$

and, consequently, it is always optimal to reduce the risk associated with the production process when the demand is sufficiently small ($x < \zeta$). This strategy may be very useful in declining markets, since it allows the firm to protect itself against the possibility of having large losses. The term Ax^{β_1} represents the value of the replacement option when the current value is $x > \zeta$; otherwise is zero.

Corollary 4.2. *If $\tilde{b} \equiv b - \gamma R = 0$, then, the replacement threshold ζ can be given by:*

$$\zeta_0 \equiv \zeta\Big|_{\tilde{b}=0} = \sqrt[\theta-1]{\frac{a_2}{a_1}\,\frac{1-\beta_1}{\gamma-\mu}\,\frac{\gamma-\mu_\theta}{\theta-\beta_1}} = \sqrt[\theta-1]{\frac{a_2}{a_1}\,\frac{\beta_2-\theta}{\beta_2-1}}\,. \tag{4.10}$$

If $\theta = 1$, $a_1 > a_2$ and $b_1 > b_2 + \gamma R$, then, the replacement threshold ζ can be given by:

$$\zeta\Big|_{\theta=1} = \frac{b-\gamma R}{a}\left(\frac{\gamma-\mu}{\gamma}\,\frac{\beta_1}{\beta_1-1}\right) = \frac{b-\gamma R}{a}\left(1-\frac{1}{\beta_2}\right) \leq \frac{b-\gamma R}{a} = c\,. \tag{4.11}$$

For future reference, we note that the function $\tilde{b} \to \zeta(\tilde{b})$ admits the derivative

$$\frac{\partial\zeta}{\partial\tilde{b}}(\tilde{b}) = -\frac{\beta_1}{\gamma}\,g'(\zeta) > 0\,,$$

which means that $\zeta \geq \zeta_0$.

5. COMPARATIVE STATICS

In this section, we assess the impact of changing the demand parameters μ and σ on the decision strategy. We expect that this behavior depends on whether the replacement leads to higher or lower risks. We also analyze the effect of increasing /decreasing, even more, the risk. This will be analyzed by studying the movement of the respective threshold when a_i is replaced by $a_i + \Delta$ and $b_i \equiv b_i(a_i) = a_i c - d$ is replaced by $b_i(a_i + \Delta)$, where i is such that $\Pi_i(x; a_i) = a_i x - b_i(a_i)$. This is illustrated in Figure 2.

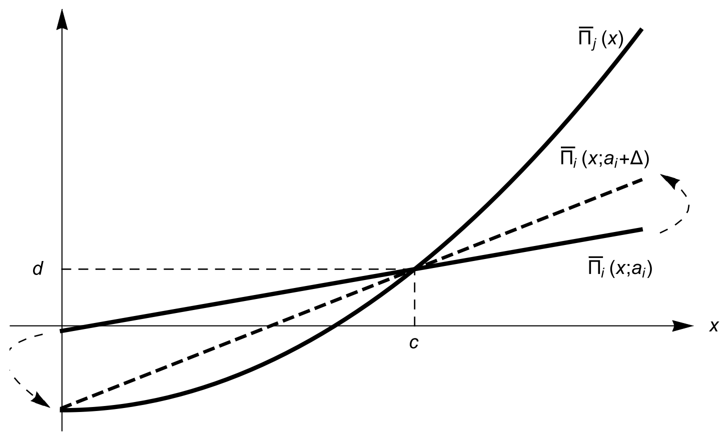

Figure 2: Representation of the functions $\overline{\Pi}_j(x) = a_j x^\theta - \bar{b}_j$ and the function $\overline{\Pi}_i(x; a) = a x^1 - \bar{b}_i(a)$, when $a = a_i$ and $a = a_i + \Delta$, $b_s(a)$, with $s = i, j$ and $i \neq j \in \{1, 2\}$, verifies $\bar{b}_s(a) = b_1(a)$ if $s = 1$ or $\bar{b}_s(a) = b_2(a) + \gamma R$ if $s = 2$.

In Proposition 5.1 we show that when the market becomes more uncertain, the firm waits longer until makes the decision of adjusting the production process. This is coherent with the classical Theory of Real Options, which postulates that more uncertainty postpones decisions. Furthermore, when the market becomes more attractive, i.e., the trend associated with the demand process increases, the decision of replacing the production process reacts in two ways: if the firm intends to increase the risk then it anticipates the decision, otherwise, it postpones the decision.

Proposition 5.1. *Let δ and ζ be implicitly defined by Equations (4.5) and (4.9). Then, the functions $(\mu, \sigma^2) \to \delta(\mu, \sigma^2)$ and $(\mu, \sigma^2) \to \zeta(\mu, \sigma^2)$ are such that*

$$\frac{\partial \delta}{\partial \mu}(\mu, \sigma) \leq 0 \quad \text{and} \quad \frac{\partial \zeta}{\partial \mu}(\mu, \sigma) \leq 0\,,$$

$$\frac{\partial \delta}{\partial \sigma}(\mu, \sigma) \geq 0 \quad \text{and} \quad \frac{\partial \zeta}{\partial \sigma}(\mu, \sigma) \leq 0\,.$$

First of all, we materialize the situation described in Figure 2 by setting that one of the following situations happen: (a) $i = 1$ and $j = 2$ or (b) $j = 1$ and $i = 2$. In the situation (a), changing a_i to $a_i + \Delta$ makes the scenario of adjusting the production process less risky than the original one. Consequently, when we decrease the slope of Π_1, the replacement is even riskier. In the case (b) by changing a_i to $a_i + \Delta$, the second production process becomes a bit riskier, and, consequently, such adjustment would be more contained in terms of gains and losses. Therefore, we can say that all the process of adjustment comes riskier.

We prove that for $\theta > 1$, the riskier the replacement process the later is made the decision of replacement. Note that in the case $\theta = 1$ (i.e., both Π_1 and Π_2 are linear functions), changing the risk does not have any impact on the thresholds, as in this case both δ and ϵ depend on a_1, a_2, b_1 and b_2 through c, which we assume to be constant.

Proposition 5.2. *Let δ and ζ be implicitly defined by Equations (4.5) and (4.9). Then the functions $(a_1, b_1) \to \delta(a_1, b_1)$ and $(a_2, b_2) \to \zeta(a_2, b_2)$ are such that*

$$\frac{\partial \delta}{\partial a_1}\big(a_1, a_1 c - d; \theta\big) < 0\,, \quad \text{and} \quad \frac{\partial \zeta}{\partial a_2}\big(a_2, a_2 c - d; \theta\big) < 0 \quad \text{for all } \theta > 1\,,$$

$$\frac{\partial \delta}{\partial a_1}\big(a_1, a_1 c - d; \theta{=}1\big) = 0\,, \quad \text{and} \quad \frac{\partial \zeta}{\partial a_2}\big(a_2, a_2 c - d; \theta{=}1\big) = 0\,.$$

6. THE EFFECT OF THE EXIT OPTION

In this section we discuss how the abandonment option may influence the replacement decision. We denote by α the exit threshold, and thus, once the firm invests in the second production process, the firm stays active as long as the demand is above α; then it abandons the market. To avoid trivial problems we assume that

$$b_2 > \gamma S\,, \tag{6.1}$$

which means that the abandonment problem is not trivial, in the sense that the time to abandon is finite, as the fixed cost (in the second production process) is larger than the exit rate cost.

For future reference, assuming that Π_2 is such that $\Pi_2(x) = a_2 x^{\theta_2} - b_2$, with $\theta_2 \geq 1$, then

$$\tilde{W}(x) = \begin{cases} 0\,, & x \leq \alpha\,, \\ \tilde{A} x^{\beta_1} + \dfrac{a_2}{\gamma - \mu_{\theta_2}} x^{\theta_2} - \dfrac{b_2 - \gamma S}{\gamma}\,, & x > \alpha\,, \end{cases}$$

where

$$\tilde{A} = -\frac{1}{\beta_1} \frac{a_2}{\gamma - \mu_{\theta_2}} \alpha^{\theta_2 - \beta_1} > 0 \quad \text{and} \quad \alpha = \sqrt[\theta_2]{\frac{b_2 - \gamma S}{a_2}\left(1 - \frac{\theta_2}{\beta_2}\right)}\,. \tag{6.2}$$

These results follow in light of the Propositions 4.1 and 4.2 presented in the previous section. Additionally, the firm postpones the exit decision when either the uncertainty or the drift of the demand process increase. One can obtain such conclusions noticing that the function $(\mu, \sigma) \to \alpha(\mu, \sigma)$ verifies

$$\frac{\partial \alpha}{\partial \eta}(\mu, \sigma) = \frac{\alpha^{1-\theta_2}}{\beta_2^2} \frac{b_2 - \gamma S}{a_2} \frac{\partial \beta_2}{\partial \eta} < 0\,, \quad \text{with} \quad \eta = \mu, \sigma\,.$$

Next we analyze separately the two cases: increasing and decreasing risk.

6.1. Increasing risk

In this section we consider the framework described in Section 4.1. In addition to the conditions (4.2) and (6.1), we also assume that

$$b_1 \leq \gamma S + \gamma R\,,$$

which means that the replacement followed by the abandonment is more costly than the fixed cost in the less risky production process, and therefore the time to invest is strictly positive.

The optimal strategy is depicted in Figure 3, and should be interpreted as follows: the firm stays in the first production process as long as the demand is below $\tilde{\delta}$. Then, as soon as it reaches this value, the firm replaces the production process, investing in the risky one. If the demand decreases below α, the firm exits the market.

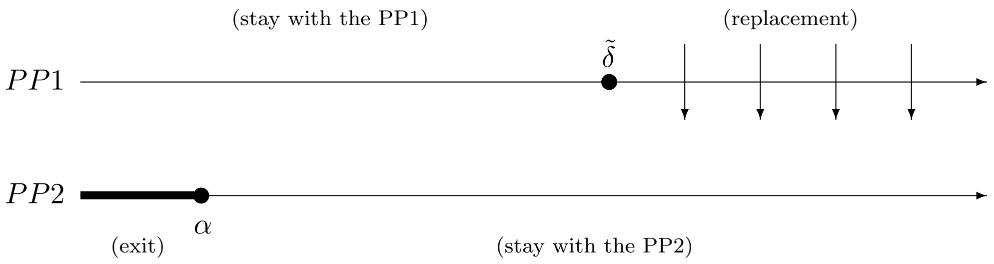

Figure 3: Replacement and abandonment strategy, when investing in the risky market.

Note that in this case the firm will stay in production after replacement for a strictly positive time, as $\tilde{\delta} > \alpha$. Thus, the value function is such that

$$\text{(6.3)} \qquad \mathcal{W}(x) = \begin{cases} \tilde{B}x^{\beta_2} + \dfrac{a_1}{\gamma-\mu}\,x - \dfrac{b_1}{\gamma}\,, & x < \tilde{\delta}\,, \\[2ex] \tilde{A}x^{\beta_1} + \dfrac{a_2}{\gamma-\mu_\theta}\,x^\theta - \dfrac{b_2}{\gamma} - R\,, & x \geq \tilde{\delta}\,, \end{cases}$$

where $\tilde{A}$ is as in Equation (6.2), when we assume that $\theta_2 = \theta$, and $\tilde{B}$ is given by

$$\text{(6.4)} \qquad \tilde{B} = \left(\tilde{A}\tilde{\delta}^{\beta_1} + \frac{a_2\tilde{\delta}^\theta}{\gamma-\mu_\theta} - \frac{a_1\tilde{\delta}}{\gamma-\mu} + \frac{b}{\gamma} - R\right)\tilde{\delta}^{-\beta_2}\,.$$

Additionally, $\tilde{\delta}$ satisfies the following equation

$$\text{(6.5)} \qquad h(x) := \tilde{A}(\beta_2-\beta_1)\,x^{\beta_1} + \frac{a_2(\beta_2-\theta)}{\gamma-\mu_\theta}\,x^\theta - \frac{a_1(\beta_2-1)}{\gamma-\mu}\,x + \beta_2\frac{b-\gamma R}{\gamma} = 0\,.$$

As in Section 4, the terms $\frac{a_1}{\gamma-\mu}x - \frac{b_1}{\gamma}$ and $\frac{a_2}{\gamma-\mu_\theta}x^\theta - \frac{b_2}{\gamma}$ represent the net present value associated with the first and second production processes, respectively. Additionally, the terms $\tilde{B}x^{\beta_2}$ and $\tilde{A}x^{\beta_1}$ represent, respectively, the value added by the replacement and exit options when the demand is x.

Proposition 6.1. *Let Π_i, with $i = 1, 2$ be given by (4.1) and $\tilde{A}$ and α be defined as in Equation (6.2) by setting that $\theta_2 = \theta$. Then, the solution of (3.5), $\mathcal{W}$, is given by (6.3), with $\tilde{B} > 0$ given by (6.4). Additionally, $\tilde{\delta}$ is the unique positive solution to the Equation (6.5) satisfying $\tilde{\delta} > \alpha$.*

In the next proposition we discuss the influence of the exit option in the investment threshold. As expected, in the case we invest in a more risky production process, the decision is anticipated in case we still have the option to abandon the market. The proof of next proposition is trivial since $\tilde{A}(\beta_2 - \beta_1)\, x^{\beta_1} > 0$.

Proposition 6.2. *Let δ be the unique positive solution to Equation (4.5) and $\tilde{\delta}$ is the unique solution of Equation (6.5) such that $\tilde{\delta} > \alpha$. Then, $\tilde{\delta} < \delta$.*

Additionally, we can say that, as it holds when there is no option to abandon the market, a risky scenario, in the sense that a_1 is replaced by $a_1 - \Delta$ and $b_i \equiv b_i(a_i) = a_i c - d$ is replaced by $b_i(a_i - \Delta)$, postpones the replacement decision, when compared with the initial situation. The proof of this result follows in light of the proof of Proposition 5.2.

Proposition 6.3. *Let $\tilde{\delta}$ be implicitly defined by Equation (6.5). Then, the function $(a_1, b_1) \rightarrow \tilde{\delta}(a_1, b_1)$ is such that*

$$\frac{\partial \tilde{\delta}}{\partial a_1}\big(a_1,\, a_1 c - d\big) \,<\, 0\,.$$

The following table presents a numerical example which illustrates that although both replacement thresholds (δ, without the abandonment option, and $\tilde{\delta}$, with the abandonment option) increase with risk, the pace is not the same: $\tilde{\delta}$ increases faster with increasing risk (here measured by Δ) than δ.

Table 1: Thresholds δ and $\tilde{\delta}$ considering the parameters: $\mu = 0.001$, $\sigma^2 = 0.005$, $\gamma = 0.01$, $a_1 = 1$, $b_1 = 1$, $a_2 = 1$, $b_2 = 10$, $\theta = 2$, $R = 10$, $S = 110$.

Δ	$\tilde{\delta}(a_1 - \Delta)$	$\delta(a_1 - \Delta)$	$\delta(a_1 - \Delta) - \tilde{\delta}(a_1 - \Delta)$
0	5.046	5.171	0.125
0.1	5.194	5.311	0.117
0.2	5.342	5.451	0.109
0.3	5.488	5.591	0.103

6.2. Decreasing risk

In this section we consider the set up introduced in Section 4.2. From conditions $b_1 \geq b_2 + \gamma R$ and $b_2 > \gamma S$, trivially follows that:

$$b_1 \geq \gamma R + \gamma S\,.$$

This condition means that replacement occurs in finite time, as the fixed cost rate, before replacement, is larger than the total cost of replacement and abandonment.

We find two different strategies according to the value of the replacement cost. On the one hand, when R is sufficiently large, meaning that $R > R^*$, where

$$R^* \equiv \frac{1}{\beta_1}\left(a_1 \frac{\theta - \beta_1}{\gamma - \mu_\theta}\,\alpha^\theta + \beta_1 \frac{b - \gamma S}{\gamma}\right), \tag{6.6}$$

the optimal strategy is depicted in Figure 4. In this case the value function takes the form

$$\mathcal{W}(x) = \begin{cases} -R - S\,, & x \le \tilde{\zeta}\,, \\ \tilde{A}_1 x^{\beta_1} + \dfrac{a_1}{\gamma - \mu_\theta}\,x^\theta - \dfrac{b_1}{\gamma}\,, & x > \tilde{\zeta}\,, \end{cases} \tag{6.7}$$

where

$$\tilde{A}_1 = -\frac{1}{\beta_1}\,\frac{a_1}{\gamma - \mu}\,\tilde{\zeta}^{\theta - \beta_1} \quad \text{and} \quad \tilde{\zeta} = \sqrt[\theta]{\frac{b_1 - \gamma(R+S)}{a_1}\left(1 - \frac{1}{\beta_2}\right)}\,. \tag{6.8}$$

Here, $\tilde{A}_1 x^{-\beta_1}$ represents the value of the abandonment option. Therefore, the firm produces using the first production process for large values of the demand, as long as they are above $\tilde{\zeta}$. Once the demand hits $\tilde{\zeta}$, it decides to abandon the market, paying a sunk cost equal to $R + S$. In this case, the firm does not actually produce with the second production process, as the time elapsed between replacement and abandonment is zero.

Figure 4: Abandonment strategy, when investing in the less risky market.

On the other hand, when $R < R^*$, the firm decides either to replace its production process by a second one when the demand reaches any level in $]\alpha, \zeta]$, or to abandon the market when the demand is smaller than or equal to the level α. The optimal strategy, in this case, is depicted in Figure 5. Furthermore, the value function is given by

$$\mathcal{W}(x) = \begin{cases} -R - S\,, & x \le \alpha\,, \\ \tilde{A} x^{\beta_1} + \dfrac{a_2}{\gamma - \mu}\,x - \dfrac{b_2}{\gamma} - R\,, & \alpha < x \le \zeta\,, \\ \tilde{A}_2 x^{\beta_1} + \dfrac{a_1}{\gamma - \mu_\theta}\,x^\theta - \dfrac{b_1}{\gamma}\,, & x > \zeta\,, \end{cases} \tag{6.9}$$

where $\tilde{A}$ and α are defined in Equation (6.2) by setting that $\theta_2 = 1$, and

$$\tilde{A}_2 - \tilde{A} = \frac{1}{\beta_1}\left(\frac{a_2}{\gamma - \mu}\,\zeta - \frac{a_1 \theta}{\gamma - \mu_\theta}\,\zeta^\theta\right)\zeta^{-\beta_1} \tag{6.10}$$

and ζ is the unique solution to the equation (4.9).

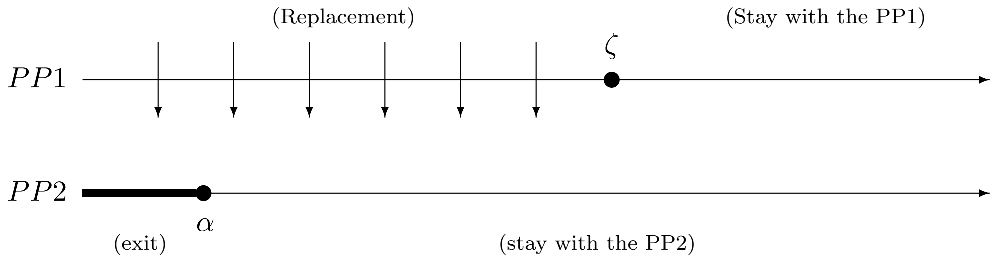

Figure 5: Replacement and abandonment strategy, when investing in the less risky market.

We start by noting that $\tilde{A}x^{\beta_1}$ represents the value of the abandonment option, while $\tilde{A}_2 x^{\beta_1}$ represents the value of the replacement option when the demand is x. This representation is coherent with the classical theory since we are assuming that the exit option is only available after the replacement. Therefore in each moment until the scrapping, it is only possible to make one decision.

Proposition 6.4. *Let Π_i, with $i = 1, 2$ be given by (4.6) and $\tilde{A}$ and α be defined as in Equation (6.2) by setting that $\theta_2 = 1$. Then, the solution of (3.5) is as follows:*

- *When $R \geq R^*$, the value function, $\mathcal{W}$, is given by (6.7), and $\tilde{A}_1 > 0$ given by (6.8);*
- *When $R < R^*$, the value function, $\mathcal{W}$, is given by (6.9), and $\tilde{A}_2 > 0$ given by (6.10). Additionally, ζ is the unique positive solution to Equation (4.9) satisfying $\zeta > \alpha$.*

Finally, we study the impact of changing the drift and/or the volatility in the parameter R^*. As we show in the next proposition, the situation depicted in Figure 5 is more likely to occur than the situation depicted in Figure 4 with increasing the drift or the volatility.

Proposition 6.5. *Consider $R^*(\mu, \sigma) \equiv R^*$, with R^* defined as in (6.6). Then the functions $R^*(\cdot, \sigma)$ and $R^*(\mu, \cdot)$ are both decreasing.*

7. CONCLUSION

This paper considers the problem of a producing firm that has the option to replace its current production process by a riskier / less risky one. The concept of risk here considered relies on the structure of the running payoff function, as described before.

Our main result is that the time until the decision of replacement increases when the risk associated with the replacement option increases. Additionally, if the firm evaluates the replacement option taking into account the abandonment option, then its decision regarding replacement is anticipated. But not only the timing changes, but also there is a clear change in the structure of the values of the economic indicator that lead to the decision. In fact, if, on the one hand, when we increase the level of risk of the alternative production process the replacement is optimal for large levels of the economic indicator, on the other hand, if we decrease, the replacement is optimal for small levels of the economic indicator.

A. APPENDIX – Proofs

Unless otherwise stated, we assume, without loss of generality, that $R = 0$.

A.1. Section 4

Before we prove Propositions 4.1 and 4.2, we state an auxiliary lemma, which will simplify the proof of this proposition.

Lemma A.1. *Equation (4.5) (resp., (4.9)) has a unique solution, δ (resp., ζ).*

Proof: To prove that δ is the unique root of Equation (4.5), we calculate f':

$$f'(x) = \frac{a_2\,\theta(\beta_2-\theta)}{\gamma-\mu_\theta}\,x^{\theta-1} - \frac{a_1(\beta_2-1)}{\gamma-\mu}\,. \tag{A.1}$$

Then, $f'(x) \geq 0$, for $x \in [x_1, \infty[$, where x_1 is the unique zero of $f'(x)$, given by

$$x_1 = \left(\frac{a_1}{\theta\,a_2}\,\frac{\beta_2-1}{\gamma-\mu}\,\frac{\gamma-\mu_\theta}{\beta_2-\theta}\right)^{\frac{1}{\theta-1}}.$$

Furthermore, as

$$f(0) = \frac{\beta_2\,b}{\gamma} \leq 0 \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty\,,$$

we conclude that there is a unique positive solution to the equation $f(x) = 0$, denoted by δ.

To prove that there is a unique positive solution ζ to Equation (4.9), we can follow the same strategy. For future reference, we note that $g'(x) \geq 0$ for $x \in [x_2, \infty[$, where x_2 is the unique zero of $g'(x)$. Furthermore, as

$$g(0) = \frac{\beta_1 b}{\gamma} \leq 0 \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = \infty\,,$$

we conclude that there is a unique positive solution to the equation $g(x) = 0$, denoted by ζ. □

Proof of Proposition 4.1: To find the parameter B and the threshold δ we use the smooth pasting conditions

$$\frac{a_1}{\gamma-\mu}\,\delta - \frac{a_2}{\gamma-\mu_\theta}\,\delta^\theta + B\delta^{\beta_2} - \frac{b}{\gamma} = 0\,,$$

$$\frac{a_1}{\gamma-\mu} - \frac{a_2}{\gamma-\mu_\theta}\,\theta\,\delta^{\theta-1} + \beta_2\,B\,\delta^{\beta_2-1} = 0\,.$$

Consequently, we obtain B given by (4.4) and δ as a solution to Equation (4.5). Additionally, Lemma A.1 states that δ is the unique solution to Equation (4.5).

To prove that the function V defined by (4.3) satisfies the HJB equation, we need to prove the following relationships:

$$\gamma V(x) - \mu x V'(x) - \frac{\sigma^2}{2} x^2 V''(x) - \Pi_1(x) + \Pi_2(x) \geq 0 \,, \qquad \text{for all} \quad x \geq \delta \,, \tag{A.2}$$

$$V(x) \geq 0 \,, \qquad \text{for all} \quad x \leq \delta \,. \tag{A.3}$$

First, we note that the inequality in (A.2) may be written as

$$f_1(x) := \Pi_1(x) - \Pi_2(x) \leq 0 \,, \qquad \text{for all} \quad x \geq \delta \,, \tag{A.4}$$

as for $x \geq \delta$, $V = 0$. Since $f_1'(x) = a_1 - a_2 \theta x^{\theta-1}$, then f_1 is increasing for $x < \left(\frac{a_2}{a_1 \theta}\right)^{\frac{1}{\theta-1}}$ and decreasing for $x > \left(\frac{a_2}{a_1 \theta}\right)^{\frac{1}{\theta-1}}$. Taking into account that

$$f_1(0) = -b \geq 0 \qquad \text{and} \qquad \lim_{x\to+\infty} f_1(x) = -\infty \,,$$

then (A.4) holds true if and only if

$$\Pi_1(\delta) - \Pi_2(\delta) \leq 0 \,. \tag{A.5}$$

To prove this, we note that

$$\Pi_1(\delta) - \Pi_2(\delta) \;=\; -\frac{1}{2}\,\sigma^2 (\delta)^2\, V''(\delta) \,,$$

where the equality follows because $\gamma V(\delta) - \mu \delta V'(\delta) - \frac{\sigma^2}{2} \zeta^2 V''(\delta) - \Pi_1(\delta) + \Pi_2(\delta) = 0$ and the smooth pasting conditions. Additionally, we can calculate

$$V''(\delta) \;=\; \beta_2 (\beta_2 - 1) B \delta^{\beta_2 - 2} - \frac{a_2 \theta (\theta - 1)}{\gamma - \mu_\theta} \,,$$

which, combined with the smooth pasting conditions, allow us to obtain

$$\begin{aligned} -\frac{1}{2}\,\sigma^2 \delta^2 V''(\delta) \;&=\; -\frac{1}{2}\,\sigma^2 \left[\frac{a_2 (\beta_2 - \theta)\, \theta}{\gamma - \mu_\theta}\, \delta^\theta - \frac{a_1(\beta_2 - 1)}{\gamma - \mu}\, \delta \right] \\ &<\; -\frac{1}{2}\,\sigma^2 \left[\frac{a_2 (\beta_2 - \theta)}{\gamma - \mu_\theta}\, \delta^\theta - \frac{a_1(\beta_2 - 1)}{\gamma - \mu}\, \delta \right] \;=\; \frac{1}{2}\,\sigma^2\, \frac{\beta_2 b}{\gamma} \;\leq\; 0 \,. \end{aligned}$$

This proves (A.5) and allow us to conclude that

$$\delta \geq c \,.$$

Finally, to prove the inequality in (A.3), we notice that, in light of the relationship $f(\delta) = 0$, the parameter B can be written as

$$B \;=\; -\frac{1}{\beta_2} \left[\frac{a_1}{\gamma - \mu}\, \delta - \frac{a_2 \theta}{\gamma - \mu_\theta}\, \delta^\theta \right] \delta^{-\beta_2} \,.$$

Now, calculating the derivative of the function $f_2(x) := -\frac{1}{\beta_2} \left(\frac{a_1}{\gamma-\mu}\, x - \frac{a_2 \theta}{\gamma - \mu_\theta}\, x^\theta \right)$, we obtain $f_2'(x) = -\frac{1}{\beta_2} \left(\frac{a_1}{\gamma-\mu} - \frac{a_2 \theta^2}{\gamma-\mu_\theta}\, x^{\theta-1} \right)$. Consequently, the function f_2 is increasing for $x \in]\delta_1, \infty[$, where δ_1 is the unique positive root of f_2'. Combining this with the fact that $f_2(\delta_0) = \beta_2 \frac{a_1 \delta_0}{\gamma-\mu} \frac{\theta-1}{\beta_2-\theta} > 0$, then $\delta_1 < \delta_0 \leq \delta$, and, consequently, $B \geq 0$.

Taking into account Equation (4.3) and the smooth pasting conditions, we have that

$$V(0) = -\frac{b}{\gamma} > 0 \quad \text{and} \quad V(\delta) = V'(\delta) = 0\,. \tag{A.6}$$

Additionally,

$$V'(x) \;=\; \beta_2 B x^{\beta_2 - 1} + \frac{a_1}{\gamma - \mu} - \frac{a_2\,\theta}{\gamma - \mu}\, x^{\theta - 1}\,,$$

and, consequently, $V'(0) = \frac{a_1}{\gamma-\mu} > 0$. Since

$$V''(x) \;=\; \left(\beta_2\,(\beta_2 - 1)\,B x^{\beta_2 - \theta} - \frac{a_2\,\theta(\theta - 1)}{\gamma - \mu_\theta}\right) x^{\theta - 2}\,,$$

then V' is decreasing for all $x \in \,]0, \delta_2[$ and increasing for all $x \in \,]\delta_2, \infty[$, where δ_2 is the unique positive root of the equation $V''(x) = 0$. This means that one of two situations may happen: (1) $V'(x) > 0$ for all $x \in \,]0, \delta[$ or (2) $V'(x) > 0$ for all $x \in \,]0, \delta_2[$ and $V'(x) < 0$ for all $x \in \,]\delta_2, \delta[$. The situation (1) cannot happen in light of (A.6). Naturally, this implies that $V \geq 0$. □

Proof of Proposition 4.2: In order to determine values for A and ζ, we use the smooth pasting conditions

$$\frac{a_1 \zeta^{\theta}}{\gamma - \mu_\theta} - \frac{a_2\,\zeta}{\gamma - \mu} + A \zeta^{\beta_1} - \frac{b}{\gamma} \;=\; 0\,,$$

$$\frac{a_1\,\theta}{\gamma - \mu_\theta}\,\zeta^{\theta - 1} - \frac{a_2}{\gamma - \mu} + \beta_1 A \zeta^{\beta_1 - 1} \;=\; 0\,,$$

which allow us to obtain the parameter A, as defined in (4.8), and ζ as a solution to Equation (4.9). Additionally, Lemma A.1 states that ζ is the unique solution to Equation (4.9).

To prove that the function V defined by (4.7) satisfies the HJB equation, we need to prove the following relationships:

$$\gamma V(x) - \mu x V'(x) - \frac{\sigma^2}{2}\, x^2 V''(x) - \Pi_1(x) + \Pi_2(x) \geq 0\,, \qquad \text{for all} \quad x \leq \zeta\,, \tag{A.7}$$

$$V(x) \geq 0\,, \qquad \text{for all} \quad x \geq \zeta\,. \tag{A.8}$$

First, we note that the inequality in (A.7) can be written as

$$\Pi_1(x) - \Pi_2(x) \leq 0\,, \qquad \text{for all} \quad x \leq \zeta\,. \tag{A.9}$$

In fact, a similar argument to the one used to prove the inequality in (A.2) proves that the inequality in (A.9) is satisfied. Additionally, we get that

$$\zeta \leq c\,.$$

To prove the inequality in (A.8), we note that, in light of the relationship $g(\zeta) = 0$, the parameter A can be written as

$$A \;=\; -\frac{1}{\beta_1}\left(\frac{a_1\,\theta}{\gamma - \mu_\theta}\,\zeta^{\theta} - \frac{a_2}{\gamma - \mu}\,\zeta\right)\zeta^{-\beta_1}\,. \tag{A.10}$$

Additionally, $V(\zeta) = 0$ in light of the smooth pasting conditions. Taking into account Equation (A.10), we can calculate

$$V'(x) = a_1\theta \frac{x^{\theta-1}}{\gamma-\mu_\theta} - \frac{a_2}{\gamma-\mu} + A_1\beta_1 x^{\beta_1-1}$$
$$= \frac{a_1\theta}{\gamma-\mu_\theta} x^{\theta-1} - \frac{a_2}{\gamma-\mu} - \left(\frac{a_1\theta}{\gamma-\mu_\theta}\zeta^{\theta-1} - \frac{a_2}{\gamma-\mu}\right)\frac{x^{\beta-1}}{\zeta^{\beta_1-1}} \geq 0,$$

where the last inequality follows from:

$$x \to \frac{a_1\theta}{\gamma-\mu_\theta} x^{\theta-1} - \frac{a_2}{\gamma-\mu} \text{ is an increasing function, and } \frac{x^{\beta-1}}{\zeta^{\beta_1-1}} \leq 1 \text{ for all } x \geq \zeta.$$

As a consequence, the inequality (A.8) holds true, because V is increasing. To finish the proof, we just need to verify that $A > 0$. Consider the function

$$g_1(x) := \frac{a_1\theta}{\gamma-\mu_\theta} x^\theta - \frac{a_2}{\gamma-\mu} x. \tag{A.11}$$

Taking into account that $g_1'(x) := \frac{a_1\theta^2}{\gamma-\mu_\theta} x^{\theta-1} - \frac{a_2}{\gamma-\mu}$, then g_1 is increasing in $]\zeta_1,\infty[$, where ζ_1 is the unique positive root of g_1'. The results follow in light of the facts:

$$g(0) = 0, \qquad g(\zeta_0) = \zeta_0 \frac{a_2}{\gamma-\mu}\frac{\beta_1(1-\theta)}{\theta-\beta_1} > 0 \quad \text{and} \quad \zeta \geq \zeta_0. \qquad \square$$

A.2. Section 5

Proof of Proposition 5.1: By using the Implicit Function Theorem, we obtain that

$$\frac{\partial\delta}{\partial\mu}(\mu) = -\frac{\partial f}{\partial\mu}(\delta;\mu)\left(\frac{\partial f}{\partial\delta}\right)^{-1}(\delta;\mu) \quad \text{and} \quad \frac{\partial\zeta}{\partial\mu}(\mu) = -\frac{\partial g}{\partial\mu}(\zeta;\mu)\left(\frac{\partial g}{\partial\zeta}\right)^{-1}(\zeta;\mu).$$

Taking into account Lemma A.1, we note that $\frac{\partial f}{\partial\delta}(\delta;\mu) > 0$ and $\frac{\partial g}{\partial\delta}(\delta;\mu) > 0$, and consequently, we just need to study the sign of $\frac{\partial f}{\partial\mu}(\delta;\mu)$ and $\frac{\partial g}{\partial\mu}(\delta;\mu)$. Taking into account the smooth pasting conditions we get, after some simplifications,

$$\frac{\partial f}{\partial\mu}(\delta;\mu) = \frac{a_2\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial\beta_2}{\partial\mu} + \frac{\beta_2-\theta}{\gamma-\mu_\theta}\right)\delta^\theta - \frac{a_1}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial\beta_2}{\partial\mu} + \frac{\beta_2-1}{\gamma-\mu}\right)\delta := p_1(\delta;\theta),$$
$$\frac{\partial g}{\partial\mu}(\zeta;\mu) = \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} + \frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)\zeta^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} - \frac{1-\beta_1}{\gamma-\mu}\right)\zeta := p_2(\zeta;\theta).$$

Assume for now that (i) $\theta = 1$, $a_1 < a_2$ and $b_1 < b_2$ and (ii) $\theta = 1$, $a_1 > a_2$ and and $b_1 > b_2$. Then, we can calculate explicitly the following derivatives (see Corollaries 4.1 and 4.2):

$$\text{(i)} \quad \frac{\partial\delta}{\partial\mu}(\mu) = \frac{b}{a}\frac{1}{\gamma}\frac{\frac{\partial\beta_2}{\partial\mu}}{(\beta_2-1)^2} \leq 0 \quad \text{and} \quad \text{(ii)} \quad \frac{\partial\zeta}{\partial\mu}(\mu) = \frac{b}{a}\frac{1}{\gamma}\frac{\frac{\partial\beta_1}{\partial\mu}}{(\beta_1-1)^2} \leq 0.$$

Combining these derivatives with the expressions of $\frac{\partial f}{\partial\mu}(\delta;\mu)$ and $\frac{\partial g}{\partial\mu}(\zeta;\mu)$, it is easy to note that

$$\frac{1}{\beta_2}\frac{\partial\beta_2}{\partial\mu} + \frac{\beta_2-1}{\gamma-\mu} \geq 0 \quad \text{and} \quad \frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} - \frac{1-\beta_1}{\gamma-\mu} \leq 0. \tag{A.12}$$

Indeed, the previous inequalities do not depend on a_1 neither on a_2, and thus this means that the result is true for every $a_1, a_2 > 0$. Additionally, returning to the case $\theta > 1$, it is a matter of calculations to see that $\frac{\partial \delta_0}{\partial \mu}(\mu) \leq 0$. In fact, this implies that $0 \leq \frac{\partial f}{\partial \mu}(\delta_0; \mu)$, and, consequently,

$$\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - \theta}{\gamma - \mu_\theta} \geq 0\,.$$

Furthermore, noticing that $\frac{\theta-\beta_1}{\gamma-\mu_\theta} = \frac{\theta-\beta_1}{-\frac{1}{2}\sigma^2(\theta-\beta_1)(\theta-\beta_2)} = \frac{2}{\sigma^2}\frac{1}{(\beta_2-\theta)}$ and that

$$\frac{\partial}{\partial \theta}\left(\frac{\theta-\beta_1}{\gamma-\mu_\theta}\right) = \frac{2}{\sigma^2}\frac{1}{(\beta_2-\theta)^2} > 0\,,$$

we obtain:

$$-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{(\theta-\beta_1)}{\gamma-\mu_\theta} \geq -\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{(1-\beta_1)}{\gamma-\mu} \geq 0\,.$$

Note now that the function $p_1(x;\theta)$ has two roots: $x = 0$ and $x = a^*$, where a^* is its unique positive root. Additionally, it is a matter of calculations to see that there is a unique b^* such that $\frac{\partial p_1}{\partial x}(b^*;\theta) = 0$ and that, in light of Equation (A.12), $\frac{\partial p_1}{\partial x}(0;\theta) < 0$. Therefore, $p_1(x;\theta)$ is increasing for all $x > b^*$, and decreasing for all $x < b^*$, and, consequently, $0 \leq p_1(\delta_0;\theta) \leq p_1(\delta,\theta)$, since $\frac{\partial \delta}{\partial b} < 0$. Finally, we can observe that $p_2(x,\theta) = 0$ if and only if $x = 0$ and $x = c^* > 0$, where

$$c^* = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}\frac{\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} + \frac{1-\beta_1}{\gamma-\mu}\right)}{\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} + \frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)}} \leq \zeta_0 \sqrt[\theta-1]{\frac{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} + \frac{1-\beta_1}{\gamma-\mu}}{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} + \frac{\theta-\beta_1}{\gamma-\mu_\theta}}} \leq \zeta_0\,.$$

Furthermore, $\frac{\partial p_2}{\partial x}(x;\theta) < 0$ for all $x < d^*$ and $\frac{\partial p_2}{\partial x}(x;\theta) > 0$ for all $x > d^*$, where d^* is the unique root of the function $x \to \frac{\partial p_2}{\partial x}(x;\theta)$. Therefore, $p_2(x;\theta)$ is an increasing function in x, for a fixed θ, if $x \geq d^*$, with $c^* > d^*$. Combining this with the roots to the equation $p_2(x;\theta)$ we get that $\zeta > \zeta_0 \geq c^* > d^*$ and, consequently,

$$0 = p_2(c^*;\theta) \leq p(\zeta;\theta)\,,$$

which concludes this part of the proof.

To finish the proof, we use the Implicit Function Theorem

$$\frac{\partial \delta}{\partial \sigma}(\sigma) = -\frac{\partial f}{\partial \sigma}(\delta;\sigma)\left(\frac{\partial f}{\partial \delta}\right)^{-1}(\delta;\sigma) \quad \text{and} \quad \frac{\partial \zeta}{\partial \sigma}(\sigma) = -\frac{\partial g}{\partial \sigma}(\zeta;\sigma)\left(\frac{\partial g}{\partial \zeta}\right)^{-1}(\zeta;\sigma)\,.$$

From the previous considerations, one just need to discuss the signs of $\frac{\partial f}{\partial \sigma}(\delta;\sigma)$ and $\frac{\partial g}{\partial \sigma}(\delta;\sigma)$. By using the smooth pasting conditions, one can prove that

$$\frac{\partial f}{\partial \sigma}(\delta;\sigma) = \frac{a_2\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial\beta_2}{\partial\sigma} + \frac{(\beta_2-\theta)\,\sigma(\theta-1)}{\gamma-\mu_\theta}\right)\delta^\theta - \frac{a_1}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial\beta_2}{\partial\sigma}\right)\delta := q_1(\delta)\,,$$

$$\frac{\partial g}{\partial \sigma}(\zeta;\sigma) = \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\sigma} + \frac{(\theta-\beta_1)\,\sigma(\theta-1)}{\gamma-\mu_\theta}\right)\zeta^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\sigma}\right)\zeta := q_2(\zeta)\,.$$

To show that $\frac{\partial f}{\partial \sigma} \leq 0$, we note that, since $\beta_2 \geq 0$ and $\frac{\partial \beta_2}{\partial \sigma} \leq 0$, then $-\frac{a_1}{\gamma-\mu}\frac{1}{\beta_2}\frac{\partial\beta_2}{\partial\sigma} \geq 0$. Assuming now that $b = 0$, then $\delta = \delta_0$, it is a matter of calculations to see that $\frac{\partial \delta_0}{\partial \sigma} \geq 0$.

Consequently, $0 \geq q(\delta_0)$, thus $\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\,\sigma(\theta-1)}{\gamma-\mu_\theta} \leq 0$. Trivial calculations allow us to conclude that $q_1(x)$ is decreasing for all $x > e^*$ and increasing for all $x < e^*$, where e^* is the unique positive root of the function $x \to q'(x)$. Since there is $x^* > 0$ such that $x = 0$ and $x = x^* > 0$ are the unique non-negative roots of the function $x \to q(x)$, it follows that $0 \geq q(\delta_0) \geq q(\delta)$.

Now, one can note that $q_2(x) = 0$ if and only if $x = 0$ and $x = m^* > 0$, where

$$m^* \;=\; \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\,\frac{\gamma-\mu_\theta}{\gamma-\mu}\,\frac{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\sigma}}{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\sigma}+\frac{(\theta-\beta_1)\,\sigma(\theta-1)}{\gamma-\mu_\theta}}} \;<\; \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\,\frac{\gamma-\mu_\theta}{\gamma-\mu}} \;<\; \zeta_0 \;<\; \zeta\,.$$

The first inequality follows because $\frac{\partial\beta_1}{\partial\sigma} > 0$. Moreover, calculating the derivative of q_2, in order to x, we can conclude that $q(x)$ is increasing for $x \geq n^*$, where n^* is such that ${q_2}'(n^*) = 0$. Combining all these facts we have

$$0 \,=\, q_2(m^*) \,\leq\, q_2(\zeta)\,,$$

which ends the proof. □

Proof of Proposition 5.2: We will focus our attention in the case $\theta > 1$. To prove Proposition 5.2 we note that

$$f(x;a_1) \;=\; \frac{a_2\,(\beta_2-\theta)}{\gamma-\mu_\theta}\,x^\theta - \frac{a_1(\beta_2-1)}{\gamma-\mu} - \frac{\beta_2\,\big(b_1(a_1)-b_2\big)}{\gamma}\,,$$

then,

$$f(x;a_1{+}\Delta) - f(x;a_1) \;=\; \Delta\left(\frac{\beta_2}{\gamma}\,c - \frac{\beta_2-1}{\gamma-\mu}\,x\right).$$

Therefore, $f(x;a_1{+}\Delta) > f(x;a_1)$ for every $x < \tilde{y}$, where

$$\tilde{y} \,=\, c\,\frac{\beta_2}{\gamma}\,\frac{\gamma-\mu}{1-\beta_2}\,.$$

Note that

$$\begin{aligned} f(\tilde{y}) \;&=\; a_2\,c^\theta\left(\frac{\beta_2-\theta}{\gamma-\mu_\theta}\left(\frac{\beta_2}{\gamma}\,\frac{\gamma-\mu}{\beta_2-1}\right)^{\theta} - \frac{\beta_2}{\gamma}\right) \\ &=\; a_2\,c^\theta\left(\frac{1}{\sigma^2/2\,(\theta-\beta_1)}\left(1-\frac{1}{\beta_1}\right)^{\theta} + \frac{1}{\beta_1\sigma^2/2}\right) \\ &=\; a_2\,c^\theta\,\frac{1}{\sigma^2/2\,(\theta-\beta_1)}\left(\left(1-\frac{1}{\beta_1}\right)^{\theta} - \left(1-\frac{\theta}{\beta_1}\right)\right), \end{aligned}$$

where we have used the following relationships:

$$\frac{\beta_2}{\gamma}\,\frac{\gamma-\mu}{\beta_2-1} \,=\, \frac{\beta_1(\beta_2-1)+1-\beta_2}{\beta_1(\beta_2-1)} > 1 \qquad\text{and}\qquad \gamma-\mu_\theta \,=\, -\frac{\sigma^2}{2}\,(\theta-\beta_1)\,(\theta-\beta_2)\,.$$

To determine the sign of $f(\tilde{y})$, we define the function

$$(\theta;\beta_1,\sigma_2) \;\to\; n(\theta;\beta_1,\sigma_2) \;=\; \left(1-\frac{1}{\beta_1}\right)^{\theta} - \left(1-\frac{\theta}{\beta_1}\right).$$

Then, taking into account that

$$\text{(A.13)} \qquad \frac{\partial n}{\partial \beta_1}(\theta;\beta_1,\sigma^2) \;=\; \frac{\theta}{\beta_1^2}\left(\left(1-\frac{1}{\beta_1}\right)^{\theta-1}-1\right) \;>\; 0$$

and

$$\text{(A.14)} \qquad \lim_{\beta_1\to-\infty} n(\theta;\beta_1,\sigma^2) \;=\; 0\,,$$

it follows that $f(\tilde{y}) > 0$. Consequently, $f(x;a_1+\Delta) - f(x;a_1) > 0$ for all $x < \tilde{y}$, which implies that $\delta\big(a_1+\Delta,\, b(a_1+\Delta)\big) < \delta\big(a_1, b(a_1)\big)$. The case $\theta = 1$ follows in light of similar arguments, using the relation $n(1;\beta_1,\sigma^2) = 0$.

To finish the proof, we can apply the same type of arguments to the function g. In fact, $g(x;a_2+\Delta) > g(x;a_2)$ for every $x < \tilde{x}$, where

$$\tilde{x} \;=\; -c\,\frac{\beta_1}{\gamma}\,\frac{\gamma-\mu}{1-\beta}\,.$$

Taking into account that

$$g(\tilde{x}) \;=\; a_1\,c^{\theta}\,\frac{1}{\sigma^2/2\,(\beta_2-\theta)}\left(\left(1-\frac{1}{\beta_2}\right)^{\theta}-\left(1-\frac{\theta}{\beta_2}\right)\right),$$

by using similar arguments to the previous ones, we get that $g(\tilde{x}) > 0$. □

A.3. Section 6

Before we start the proofs, we note that the value function may be re-written as follows:

$$\mathcal{W}(x) \;=\; \sup_{\tau_1\le\tau_2\in\mathcal{S}} \tilde{I}(\tau_1,\tau_2,x) - R - S\,,$$

where $\tilde{I}$ is defined as in (2.2). Therefore, throughout this section we will use the following notation:

$$H(x) \equiv \sup_{\tau_1\le\tau_2\in\mathcal{S}} \tilde{I}(\tau_1,\tau_2,x)\,.$$

Additionally, we consider $R \ge 0$.

Lemma A.2. *Equation (6.5) has a unique solution $\tilde{\delta}$, which satisfies $\tilde{\delta} > \alpha$, where α is defined as in Equation (6.2) by setting that $\theta_2 = \theta$.*

Proof: To prove that $\tilde{\delta}$ is the unique root of Equation (6.5) satisfying $\tilde{\delta} > \alpha$, we calculate h'':

$$h''(x) \;=\; \tilde{A}(\beta_2-\beta_1)\,\beta_1(\beta_1-1)\,x^{\beta_1-2} + \frac{a_2\,\theta\,(\theta-1)\,(\beta_2-\theta)}{\gamma-\mu_\theta}\,x^{\theta-2} \;>\; 0\,.$$

Taking into account that

$$\lim_{x\to0^+} h(x) \;=\; \lim_{x\to+\infty} h(x) \;=\; +\infty\,,$$

the result follows in light of the calculations:

$$h(\alpha) = -\frac{a_1(\beta_2-1)}{\gamma-\mu} + \beta_2 \frac{b_1-\gamma R-\gamma S}{\gamma} < 0\,,$$

where we have used the smooth pasting conditions (used to obtain $\tilde{A}$ and α):

$$\tilde{A}\alpha^{\beta_1} + \frac{a_2}{\gamma-\mu_\theta}\,\alpha^\theta\,\frac{b_2-\gamma E}{\gamma} = 0\,,$$

$$\tilde{A}\beta_1\alpha^{\beta_1} + \frac{a_2\,\theta}{\gamma-\mu_\theta}\,\alpha^\theta = 0\,. \qquad \square$$

Proof of Proposition 6.1: The parameters $\tilde{B}$ and $\tilde{\delta}$ may be obtained by using the smooth pasting conditions:

$$\tilde{B}\,\tilde{\delta}^{\beta_2} + \frac{a_1}{\gamma-\mu}\,\tilde{\delta} - \frac{b_1-\gamma R-\gamma S}{\gamma} = \tilde{A}\,\tilde{\delta}_1^{\beta} + \frac{a_2}{\gamma-\mu_\theta}\,\tilde{\delta}^\theta - \frac{b_2-\gamma S}{\gamma}\,,$$

$$\tilde{B}\,\beta_2\,\tilde{\delta}^{\beta_2-1} + \frac{a_1}{\gamma-\mu} = \tilde{A}\,\beta_1\,\tilde{\delta}^{\beta_1-1} + \frac{a_2\,\theta}{\gamma-\mu_\theta}\,\tilde{\delta}^{\theta-1}\,.$$

Moreover, in light of Lemma A.2, $\tilde{\delta}$ is the unique positive solution Equation (6.5) satisfying the condition $\tilde{\delta} > \alpha$.

To prove that $\mathcal{W}$, where $\mathcal{W}$ is defined by (6.7), is the solution to the optimal stopping problem (2.2), we need to verify that the function $H(x) = \mathcal{W}(x) + S + R$ satisfies the following inequalities:

$$\text{(A.15)} \qquad \gamma\,H(x) - \mu\,x\,H'(x) - \frac{\sigma^2}{2}\,x^2\,H''(x) - \Big(\Pi_1(x) + \gamma R + \gamma S\Big) \geq 0\,, \qquad \text{for all} \quad x \geq \tilde{\delta}\,,$$

$$\text{(A.16)} \qquad H(x) \geq \tilde{W}(x)\,, \qquad \text{for all} \quad x \leq \tilde{\delta}\,.$$

First of all, we note that (A.15) can be written as

$$\Pi_1(x) - \Pi_2(x) + \gamma R \leq 0$$

because $H(x) = \tilde{W}(x)$ and

$$\text{(A.17)} \qquad \gamma\,\tilde{W}(x) - \mu\,x\,\tilde{W}'(x) - \frac{\sigma^2}{2}\,x^2\,\tilde{W}''(x) - \Pi_2(x) - \gamma S = 0$$

for $x \geq \tilde{\delta}$. Since the function $x \to \Pi_1(x) - \Pi_2(x) + \gamma R$ is increasing for $x < \left(\frac{a_1}{a_2\,\theta}\right)^{\frac{1}{\theta-1}}$ and decreasing for $x > \left(\frac{a_1}{a_2\,\theta}\right)^{\frac{1}{\theta-1}}$, we just need to prove that $\Pi_1(\tilde{\delta}) - \Pi_2(\tilde{\delta}) + \gamma R \leq 0$. Now, combining Equation (A.17) with

$$\gamma\,H(x) - \mu\,x\,H'(x) - \frac{\sigma^2}{2}\,x^2\,H''(x) - \Big(\Pi_1(x) + \gamma R + \gamma S\Big) = 0\,,$$

we obtain the following equality:

$$-\frac{\sigma^2}{2}\,\tilde{\delta}^2\,\Big(H''(\tilde{\delta}) - \tilde{W}''(\tilde{\delta})\Big) = \Pi_1(\tilde{\delta}) - \Pi_2(\tilde{\delta}) + \gamma R\,.$$

It is a matter of calculations to see that

$$H''(\tilde{\delta}) - \tilde{W}''(\tilde{\delta}) = h'(\tilde{\delta})\,\tilde{\delta}^{-1} > 0\,,$$

where h' is the derivative of h, defined in (6.5), and the last inequality follows in light of the calculations in the proof of Lemma A.2.

To prove the inequality (A.16), we note that the function $x \to H(x)$ is increasing if $B > 0$. In case $B > 0$, as $H(0) = -\frac{b_1-\gamma R-\gamma S}{\gamma} \geq 0$, this proves that $H(x) \geq \tilde{W}(x)$ for all $x \leq \alpha$. To see that $\alpha \leq x \leq \tilde{\delta}$, we note that $\tilde{\delta}$ is the unique solution to the equation $H(x) = \tilde{W}(x)$. Therefore, since $H(\alpha) - \tilde{W}(\alpha) = H(\alpha) > H(0) \geq 0$, the result is straightforward.

To see that $B > 0$, one can see that

$$B\,\tilde{\delta}^{\beta_2} = \frac{\tilde{\delta}}{\beta_2}\left(\tilde{A}\beta_1\tilde{\delta}^{\beta_1-1} + \frac{a_2\,\theta}{\gamma-\mu_\theta}\,\tilde{\delta}^{\theta-1} - \frac{a_1}{\gamma-\mu}\right).$$

Additionally, the function $x \to \tilde{A}\beta_1 x^{\beta_1-1} + \frac{a_2\,\theta}{\gamma-\mu_\theta}\,x^{\theta-1} - \frac{a_1}{\gamma-\mu}$ is increasing and crosses zero once. By using the smooth pasting conditions (used to obtain $\tilde{A}$ and α), we get

$$\tilde{A}\beta_1\alpha^{\beta_1-1} + \frac{a_2\,\theta}{\gamma-\mu_\theta}\,\alpha^{\theta-1} - \frac{a_1}{\gamma-\mu} = -\frac{a_1}{\gamma-\mu} < 0\,.$$

Let $\tilde{x}$ be such that $\tilde{A}\beta_1\tilde{x}^{\beta_1-1} + \frac{a_2\,\theta}{\gamma-\mu_\theta}\,\tilde{x}^{\theta-1} - \frac{a_1}{\gamma-\mu} = 0$. Then

$$h(\tilde{x}) = \beta_2\left(A(1-\beta_1)\,\tilde{x}^{\beta_1} + \frac{a_2(1-\theta)}{\gamma-\mu_\theta}\,\tilde{x}^{\theta} + \frac{b-\gamma R}{\gamma}\right) \equiv \tilde{h}(\tilde{x})\,.$$

Once again, due to the smooth pasting condition, $\tilde{h}(\alpha) = 0$, and

$$\tilde{h}'(x) = \beta_2\left(A(1-\beta_1)\,\beta_1\tilde{x}^{\beta_1-1} + \frac{a_2(1-\theta)\,\theta}{\gamma-\mu_\theta}\,x^{\theta-1}\right) < 0\,.$$

It follows that $h(\tilde{x}) < 0$, and therefore $\tilde{x} < \tilde{\delta}$. Consequently $B > 0$. □

Proof of Proposition 6.4: We start by noticing that, since the terminal cost is $\tilde{W}(x)$, as one can see through (3.5), the smooth pasting conditions are different according to $\zeta > \alpha$ or $\zeta \leq \alpha$. Let g be defined as in (4.9). Then it is a matter of calculations to see that

$$g(\alpha) = a_1\,\frac{\theta-\beta_1}{\gamma-\mu_\theta}\,\alpha^{\theta} + \beta_1\,\frac{b_1-\gamma S-\gamma R}{\gamma}\,.$$

Taking into account the analysis made in Lemma A.1, $\zeta > \alpha \Leftrightarrow g(\alpha) < 0$, which means that

$$R < R^* \equiv \frac{1}{\beta_1}\left(a_1\,\frac{\theta-\beta_1}{\gamma-\mu_\theta}\,\alpha^{\theta} + \beta_1\,\frac{b-\gamma S}{\gamma}\right).$$

The proof of Proposition 6.4 when $R \geq R^*$ follows in light of the arguments used in the proof of Proposition 4.2. From now on, we will treat the case $R < R^*$.

By using the smooth pasting conditions, we obtain the following equations

$$\tilde{A}_2\,\zeta^{\beta_1} + \frac{a_1}{\gamma-\mu_\theta}\,\zeta^{\theta} - \frac{b_1-\gamma S-\gamma R}{\gamma} = \tilde{A}\zeta^{\beta_1} + \frac{a_2}{\gamma-\mu}\,\zeta - \frac{b_2-\gamma S}{\gamma}\,,$$

$$\tilde{A}_2\,\beta_1\zeta^{\beta_1-1} + \frac{a_1\theta}{\gamma-\mu_\theta}\,\zeta^{\theta-1} = \tilde{A}\beta_1\zeta^{\beta_1-1} + \frac{a_2}{\gamma-\mu}\,.$$

These equations allow us to obtain the expression of $\tilde{A}_2$ as in (6.10) and Equation (4.9). In light of Lemma A.1, there is a unique solution ζ to Equation (A.1) and $\zeta < c$.

To prove that $\mathcal{W}$, defined by (6.9), is the solution of the optimal stopping problem (2.2), we need to verify that $H(x) = \mathcal{W}(x) + S + R$ satisfies the following inequalities:

$$\text{(A.18)} \qquad \gamma H(x) - \mu x H'(x) - \frac{\sigma^2}{2} x^2 H''(x) - \Big(\Pi_1(x) + \gamma R + \gamma S\Big) \geq 0\,, \qquad \text{for all} \quad x \leq \zeta\,,$$

$$\text{(A.19)} \qquad H(x) \geq \tilde{W}\,, \qquad \text{for all} \quad x \geq \zeta\,.$$

In order to prove the inequality (A.18), we start by noting that, for $x < \alpha$, we can write this equation as

$$\text{(A.20)} \qquad \Pi_1(x) + \gamma\,(S+R) \leq 0 \qquad \text{for all} \quad x \leq \alpha\,.$$

Since $\Pi_1(0) + \gamma\,(S+R) = -b_1 + \gamma\,(S+R) < 0$ and Π_1 is an increasing function, it follows that (A.20) holds true if and only if $\Pi_1(\alpha) + \gamma\,(S+R) \leq 0$. This is true because

$$0 \,\geq\, g(\alpha) \,=\, -\frac{\beta_1}{\gamma}\left(a_1 \frac{\theta-\beta_1}{\gamma-\mu_\theta}\,\frac{\gamma}{(-\beta_1)}\,\alpha^\theta - \big(b_1 - \gamma S - \gamma R\big)\right) \,\geq\, -\frac{\beta_1}{\gamma}\Big(\Pi_1(\alpha) + \gamma\,(S+R)\Big),$$

where the last inequality follows in light of the fact $\frac{\theta-\beta_1}{\gamma-\mu_\theta}\,\frac{\gamma}{(-\beta_1)} > 1$. For $\alpha < x < \zeta$, we use a similar argument to the one used in the proof of Proposition 6.1. Therefore, the inequality (A.18) can be written as

$$\gamma\,\tilde{W}(x) - \mu x\,\tilde{W}'(x) - \frac{\sigma^2}{2}\,x^2\,\tilde{W}''(x) - \Big(\Pi_1(x) + \gamma R + \gamma S\Big) \,=\, \Pi_2(x) - \Pi_1(x) - \gamma R\,,$$

which means that we just need to show that $\Pi_2(x) - \Pi_1(x) - \gamma R \geq 0$, for all $\alpha < x \leq \zeta$. We can easily prove that the function $x \to \Pi_2(x) - \Pi_1(x) - \gamma R$ increases for $x < \frac{a_1}{\theta a_2}$ and decreases for $x > \frac{a_1}{\theta a_2}$. Combining this with the fact that $\Pi_2(0) - \Pi_1(0) - \gamma R = b - \gamma R \geq 0$, we need to prove that $\Pi_2(\zeta) - \Pi_1(\zeta) - \gamma R \geq 0$, which is true in light of Proposition 4.2.

To prove the inequality (A.19), we note that

$$H(x) - \tilde{W}(x) \,=\, (\tilde{A}_2 - \tilde{A})\,x^{\beta_1} + \frac{a_1}{\gamma-\mu_\theta}\,x^\theta - \frac{a_2}{\gamma-\mu}\,x - \frac{b-\gamma R}{\gamma}\,, \qquad H(\zeta) - \tilde{W}(\zeta) = 0\,,$$

and

$$H'(x) - \tilde{W}'(x) \,=\, \left(\frac{a_2}{\gamma-\mu} - \frac{a_1\theta}{\gamma-\mu_\theta}\,\zeta^{\theta-1}\right)\left(\frac{x}{\zeta}\right)^{\beta_1-1} - \left(\frac{a_2}{\gamma-\mu} - \frac{a_1\theta}{\gamma-\mu_\theta}\,x^{\theta-1}\right).$$

Taking into account the proof of Proposition 4.2, we have that

$$\frac{a_2}{\gamma-\mu} - \frac{a_1\theta}{\gamma-\mu_\theta}\,\zeta^{\theta-1} \,=\, \beta_1 A_1\,\zeta^{\beta_1-1} \,<\, 0\,.$$

Since A_1 is defined in (4.8) and verifies $A_1 > 0$, the result follows because $\left(\frac{x}{\zeta}\right)^{\beta_1-1} < 1$, for all $x > \zeta$, and the function $x \to \frac{a_2}{\gamma-\mu} - \frac{a_1\theta}{\gamma-\mu_\theta}\,x^{\theta-1}$ is decreasing. Additionally, we can conclude that $\tilde{A}_2 > 0$. □

Proof of Proposition 6.5: Noticing that R^* can be written as

$$R^*(\mu,\sigma) \,=\, \frac{a_1}{\gamma^2}\left(\frac{b_2-\gamma S}{\gamma}\right)^\theta \frac{1}{\frac{\theta}{\beta_2}-1}\left(1-\frac{1}{\beta_2}\right)^\theta + \frac{b-\gamma S}{\gamma}\,,$$

the result follow in light of the following calculations:

$$\frac{\partial}{\partial\eta}\left(\frac{1}{\frac{\theta}{\beta_2}-1}\left(1-\frac{1}{\beta_2}\right)^\theta\right) \,=\, \beta_2\left(1-\frac{1}{\beta_2}\right)^{\theta-1}\frac{\theta-1}{(\theta-\beta_2)^2}\,\frac{\theta}{\beta_2^2}\,\frac{\partial\beta_2}{\partial\eta} \,<\, 0\,.$$

□

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