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CONCOMITANTS OF ORDER STATISTICS AND RECORD VALUES FROM ITERATED FGM TYPE BIVARIATE-GENERALIZED EXPONENTIAL DISTRIBUTION

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Abstract:

- We introduce the successive iterations in the original FGM type bivariate-generalized exponential distribution. Some distributional properties of concomitants of order statistics as well as record values for this family are studied. Recurrence relations between single as well as product moments of concomitants are obtained. Finally, we give some different applications of this study.

Keywords:

- *concomitants; order statistics; record values; generalized exponential distribution; iterated FGM family.*

AMS Subject Classification:

- 62B10, 62G30.

1. INTRODUCTION

Ordered random variables (rv's) have attracted many researchers due to their applicability in many practical areas, like order statistics (os's) and record values. Both of os's and record values are used extensively in statistical models and inference, where they describe rv's arranged in order magnitude. The os's occur as a natural choice when dealing with floods, drought, earthquakes, etc. Also, record values arise naturally in many real life applications involving data related to sport, weather, and life testing studies. Actually, there is strong relation between the os and the record value models. For example, the record values provide the information about the maximum (minimum) value among all previously recorded observations, for more detail, see Arnold *et al.* [5] (1998). The concept of concomitant os's, also called induced os's, is related to the ordering bivariate rv's. The concomitant os's arise when one sorts the members of a random sample according to corresponding values of an other random sample. The term concomitant of os's was first induced and applied extensively by David [13] (1973). According to Hanif [22] (2007) in collecting any data for an observation, several characteristics are often recorded, some of them are considered as primary and others can be observed from the primary data automatically. The latter one is called concomitant, for more detail see David and Nagaraja [14, 15] (1998, 2003). The most important use of concomitants of record values arises in experiments, in which specified characteristic's measurements of an individual are made sequentially. Moreover, only values that exceed or fall below the current extreme value are recorded, so that only observations are bivariate record values, i.e., records and their concomitants. Some properties of concomitants of record values are discussed by Ahsanullah [1] (2009) and Ahsanullah and Shakil [2] (2013). Clearly, both concomitants of os's and record values are strongly relevant with a bivariate data that has a common bivariate distribution function (df). One of the most useful and popular bivariate df is the so-called Farlie–Gumbel–Morgenstern (FGM). The FGM df is defined by $H(x,y) = F_X(x)F_Y(y)[1+\alpha F_X(x)F_Y(y)]$, where F_X and F_Y are the marginals df's, while F_X and F_Y are the survival function of F_X and F_Y, respectively, and $-1 \le \alpha \le 1$. The FGM distribution is a flexible family useful in applications provided that the correlation between the variables is not too large. It can be utilized for arbitrary continuous marginals. The FGM df was originally introduced by Morgenstern [29] (1956) for Cauchy marginals. In 1960 Gumbel [20] investigated the same structure for exponential marginals. Also, in 1960, Farlie [18], in connection with his investigations of the correlation coefficient, suggested a generalization of the bivariate form studied by Morgenstern and Gumbel. Huang and Kotz [25] (1984) used successive iterations in the original FGM distribution to increase the correlation between components. As a particular case, the bivariate FGM with a single iteration is defined by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)\Big[1+\lambda \bar{F}_X(x)\bar{F}_Y(y) + \gamma F_X(x)F_Y(y)\bar{F}_X(x)\bar{F}_Y(y)\Big], \tag{1.1}$$

denoted by FGM(λ,γ). The corresponding probability density functions (pdf) is given by:

$$\begin{aligned} &f_{X,Y}(x,y) = \\ &= f_X(x)f_Y(y)\Big[1+\lambda\big(1-2F_X(x)\big)\big(1-2F_Y(y)\big)+\gamma F_X(x)F_Y(y)\big(2-3F_X(x)\big)\big(2-3F_Y(y)\big)\Big], \end{aligned} \tag{1.2}$$

where $F_X(x)$ and $F_Y(y)$ are df's, while $f_X(x)$ and $f_Y(y)$ are the pdf's of the rv's X and Y, respectively. When the two marginals $F_X(x)$ and $F_Y(y)$ are continuous, Huang and Kotz [25] (1984) showed that the natural parameter space Ω (the admissible set of the parameters λ

and γ that makes $F_{X,Y}(x,y)$ is a df) is convex, where $\Omega = \big\{(\lambda,\gamma)\colon -1 \le \lambda \le 1;\ \lambda+\gamma \ge -1;\ \gamma \le \frac{3-\lambda+\sqrt{9-6\lambda-3\lambda^2}}{2}\big\}$. Moreover, when the marginals are uniform then, the correlation coefficient is $\rho = \frac{\lambda}{3} + \frac{\gamma}{12}$ (cf. Huang and Kotz [26], 1999). Finally, the maximal correlation coefficient attained for this family is $\max\rho = 0.434$ versus $\max\rho = \frac{1}{3} = 0.333$ achieved for $\lambda = 1$ in the original FGM version. This fact gives a satisfactory motivation to deal with the model FGM(λ,γ) rather than the classical model FGM. The model FGM(λ,γ) provides a very general expression of a bivariate distribution from which members can be derived by substituting expressions of any desired set of marginal distributions. On the other hand, since both the bivariate df's and density are given in terms of marginals, it is easy to generate a random sample from the model FGM(λ,γ). Thus members of this family can be used in simulation studies. Moreover, a number of properties results from the simple analytic form of the model FGM(λ,γ), for example, rv's having a FGM(λ,γ) are exchangeable whenever the marginal distributions are identical. Also, the model FGM(λ,γ) is closed with respect to monotonic increasing functions of rv's. Moreover, the system is closed with respect to mixtures of bivariate FGM(λ,γ) df's having the same marginal distributions. the bivariate FGM(λ,γ) df's are specially suited to data situations describing weak dependence between the rv's X and Y. Measures of dependence vary over a larger range than for the classical FGM df's.

In this paper, we study the family FGM(λ,γ), with generalized exponential (GE) marginals. The generalized exponential distribution (GE), a most attractive generalization of the exponential distribution, introduced by Gupta and Kundu [21] (1999), has widespread interest and applications, e.g., it can be used quite effectively in analyzing many lifetime data, particularly in place of two-parameter gamma and two-parameter Weibull distributions. Many authors studied various properties of the GE, see for example, Ahsanullah *et al.* [3] (2013) and AL-Hussaini and Ahsanullah [4] (2015).

A continuous rv is said to be has the GE with scale parameter $\theta > 0$ and shape parameter $\alpha > 0$ (denoted by GE$(\theta;\alpha)$), if the df and the corresponding pdf are given, for $x > 0$, respectively, by

$$F_X(x) = \big(1 - \exp(-\theta x)\big)^{\alpha}$$

and

$$f_X(x) = \alpha\theta\big(1 - \exp(-\theta x)\big)^{\alpha-1} \exp(-\theta x). \tag{1.3}$$

Gupta and Kundu [21] (1999) showed that the k-th moment of GE$(\theta;\alpha)$ is

$$\mu_k = \frac{\alpha k!}{\theta^k} \sum_{i=0}^{\aleph(\alpha-1)} \frac{(-1)^i}{(i+1)^{k+1}} \binom{\alpha-1}{i},$$

where $\aleph(x) = \infty$, if x is non-integer and $\aleph(x) = x$, if x is integer. Furthermore, the mean, variance and moment generating function of GE$(\theta;\alpha)$ are given by $\mu_1 = \mathrm{E}(X) = \frac{B(\alpha)}{\theta}$, $\mathrm{Var}(X) = \frac{C(\alpha)}{\theta^2}$ and $M_X(t) = \alpha\beta(\alpha, 1-\frac{t}{\theta})$, respectively, where $B(\alpha) = \Psi(\alpha+1) - \Psi(1)$, $C(\alpha) = \Psi'(1) - \Psi'(\alpha+1)$, $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Psi(\cdot)$ is the digamma function, while $\Psi'(\cdot)$ is its derivation ($\Psi'(\cdot)$ is known as the trigamma function). Tahmasebi and Jafari [38] (2015) studied some properties of the classical FGM type bivariate GE df. Moreover, Tahmasebi and Jafari [38] (2015) studied some distributional properties of concomitants of os's as well as record values of this df.

In this paper, the result of Tahmasebi and Jafari [38] (2015) is extended to FGM(λ, γ) family with two marginals F_X and F_Y, where $X \sim \mathrm{GE}(\theta_1; \alpha_1)$ and $Y \sim \mathrm{GE}(\theta_2; \alpha_2)$ (denoted by FGM$(\lambda, \gamma\colon \theta_1, \alpha_1;\, \theta_2, \alpha_2)$). Moreover, some new results, which were not obtained by Tahmasebi and Jafari [38] (2015) for FGM family, are given such as recurrence relations for the single, as well as the product, moments of bivariate concomitants of os's, the concomitant rank-os's, and the asymptotic behavior of the concomitants of os's. It is worth mentioning that, the same problem tackled by Barakat *et al.* [7, 6] (2019, 2018) for the Huang–Kotz FGM and Bairamov–Kotz–Becki FGM with GE marginals, respectively. Moreover, the FGM$(\lambda, \gamma\colon \theta_1, \alpha_1;\, \theta_2, \alpha_2)$ is not a special case of any of the latter models. Nowadays, we can find several recent relevant works on this subject. Among these works are Tahmasebi and Behboodian [36] (2012), Tahmasebi and Jafari [37] (2014) and Tahmasebi *et al.* [39, 40] (2015, 2016).

2. THE FGM$(\lambda, \gamma\colon \theta_1, \alpha_1;\, \theta_2, \alpha_2)$ FAMILY AND SOME OF ITS PROPERTIES

The joint df and pdf of (X, Y) are defined by (1.1) and (1.2), respectively, where $X \sim \mathrm{GE}(\theta_1; \alpha_1)$ and $Y \sim \mathrm{GE}(\theta_2; \alpha_2)$. Thus, it is easy to show that the (n, m)-th joint moments the of FGM$(\lambda, \gamma\colon \theta_1, \alpha_1;\, \theta_2, \alpha_2)$ family is given by

$$\begin{aligned}\mathrm{E}(X^nY^m) &= \mathrm{E}(X^n)\mathrm{E}(Y^m) + \lambda\big(\mathrm{E}(X^n) - \mathrm{E}(U_1^n)\big)\big(\mathrm{E}(Y^m) - \mathrm{E}(V_1^m)\big)\\ &\quad + \gamma\big(\mathrm{E}(U_1^n) - \mathrm{E}(U_2^n)\big)\big(\mathrm{E}(V_1^m) - \mathrm{E}(V_2^m)\big), \qquad n, m = 1, 2, \ldots,\end{aligned} \tag{2.1}$$

where $U_1 \sim \mathrm{GE}(\theta_1; 2\alpha_1)$, $U_2 \sim \mathrm{GE}(\theta_1; 3\alpha_1)$, $V_1 \sim \mathrm{GE}(\theta_2; 2\alpha_2)$ and $V_2 \sim \mathrm{GE}(\theta_2;\, 3\alpha_2)$. Thus, by combining (2.1) and (1.3), we get

$$\mathrm{E}(XY) = \frac{B(\alpha_1)B(\alpha_2) + \lambda D(2\alpha_1)D(2\alpha_2) + \gamma D(3\alpha_1)D(3\alpha_2)}{\theta_1\theta_2},$$

where $D((k+1)\alpha) = B((k+1)\alpha) - B(k\alpha)$, $k = 1, 2$. Therefore, the coefficient of correlation between X and Y is

$$\rho_{X,Y} = \frac{\lambda D(2\alpha_1)D(2\alpha_2) + \gamma D(3\alpha_1)D(3\alpha_2)}{\sqrt{C(\alpha_1)C(\alpha_2)}} = \lambda g_1(\alpha_1, \alpha_2) + \gamma g_2(\alpha_1, \alpha_2).$$

Clearly, the function $g_1(\alpha_1, \alpha_2)$ and $g_2(\alpha_1, \alpha_2,)$ is increasing and positive function with respect to each of $\alpha_i, i = 1, 2$. Therefore, if $\lambda, \gamma > 0$, then $\rho_{X,Y}$ is increasing and positive function and if $\lambda, \gamma < 0$, then $\rho_{X,Y}$ is decreasing and negative function with respect to each of α_1 and α_2. Moreover, we can show that $\lim_{\substack{\alpha_1\to\infty\\ \alpha_2\to\infty}} g_1(\alpha_1, \alpha_2) = \frac{6(\log(2))^2}{\pi^2}$, $\lim_{\substack{\alpha_1\to\infty\\ \alpha_2\to\infty}} g_2(\alpha_1, \alpha_2,) = \frac{6(\log(\frac{3}{2}))^2}{\pi^2}$, $\lim_{\substack{\alpha_1\to 0^+\\ \alpha_2\to 0^+}} g_1(\alpha_1,\, \alpha_2) = 0$ and $\lim_{\substack{\alpha_1\to 0^+\\ \alpha_2\to 0^+}} g_2(\alpha_1, \alpha_2) = 0$. Therefore, $\max \rho_{X,Y} = 0.392$ at corner point $(\lambda, \gamma) = (1, 1)$ and $\min \rho_{X,Y} = -0.292$ at corner point $(\lambda, \gamma) = (-1, 0)$.

The conditional df of Y given $X = x$ is given by

$$\begin{aligned}F_{Y|X}(y|x) = F_Y(y)\Big[&1 + \lambda\big(1 - F_Y(y)\big)\big(1 - 2F_X(x)\big)\\ &- \gamma F_X(x)F_Y(y)\big(1 - F_Y(y)\big)\big(2 - 3F_X(x)\big)\Big].\end{aligned} \tag{2.2}$$

Therefore, the regression curve of Y given $X=x$ for $\mathrm{FGM}(\lambda,\gamma\colon \theta_1,\alpha_1;\, \theta_2,\alpha_2)$ is

$$
\begin{aligned}
\mathrm{E}(Y|X=x) &= \mathrm{E}(Y) + \lambda\big(1-2F_X(x)\big)\big(\mathrm{E}(Y)-\mathrm{E}(V_1)\big)\\
&\qquad + \gamma F_X(x)\big(2-3F_X(x)\big)\big(\mathrm{E}(V_1)-\mathrm{E}(V_2)\big)\\
&= \frac{1}{\theta_2}\Big[B(\alpha_2) + \lambda D(2\alpha_2)\big(2F_X(x)-1\big) + \gamma F_X(x) D(3\alpha_2)\big(3F_X(x)-2\big)\Big],
\end{aligned} \tag{2.3}
$$

where $V_1 \sim \mathrm{GE}(\theta_2; 2\alpha_2)$ and $V_2 \sim \mathrm{GE}(\theta_2; 3\alpha_2)$ and the conditional expectation is non-linear with respect to x.

3. CONCOMITANTS OF OS'S BASED ON $\mathrm{FGM}(\lambda,\gamma\colon \theta_1,\alpha_1;\, \theta_2,\alpha_2)$

Suppose (X_i,Y_i), $i=1,2,\dots,n$, is a random sample from a bivariate df $F_{X,Y}(x,y)$. If we order the sample by the X-variate, and obtain the os's, $X_{1:n}\le X_{1:n}\le\cdots\le X_{n:n}$, for the X sample, then the Y-variate associated with the r-th order statistic $X_{r:n}$ is called the concomitant of the r-th order statistic, and is denoted by $Y_{[r:n]}$.

Let $X\sim \mathrm{GE}(\theta_1;\alpha_1)$ and $Y\sim\mathrm{GE}(\theta_2;\alpha_2)$. Since the conditional pdf of $Y_{[r:n]}$ given $X_{[r:n]}=x$ is $f_{Y_{[r:n]}|X_{r:n}}(y|x) = f_{Y|X}(y|x)$ (cf. Galambos [19], 1987, see also Tahmasebi and Jafari [38], 2015), then the pdf of $Y_{[r:n]}$ is given by

$$
\begin{aligned}
f_{[r:n]}(y) &= f_Y(y) + \Big[\lambda\big(f_Y(y)-f_{V_1}(y)\big) + \gamma\big(f_{V_2}(y)-f_{V_1}(y)\big)\Big]\Delta^{(1)}_{r,n}\\
&\qquad + \Big[\gamma\big(f_{V_1}(y)-f_{V_2}(y)\big)\Big]\Delta^{(2)}_{r,n},
\end{aligned} \tag{3.1}
$$

where

$$
\Delta^{(i)}_{r,n} = \frac{\beta(r,n-r+1)-(i+1)\beta(r+i,n-r+1)}{\beta(r,n-r+1)}, \qquad i=1,2.
$$

Therefore, the moment generating function of $Y_{[r:n]}$ is given by

$$
\begin{aligned}
M_{[r:n]}(t) = \alpha_2\Bigg[&\beta\left(\alpha_2, 1-\frac{t}{\theta_2}\right) + \lambda\Delta^{(1)}_{r,n}\left(\beta\left(\alpha_2,1-\frac{t}{\theta_2}\right) - \beta\left(2\alpha_2,1-\frac{t}{\theta_2}\right)\right)\\
&+ \gamma\Delta^{(2)}_{r,n}\left(\beta\left(2\alpha_2,1-\frac{t}{\theta_2}\right)-\beta\left(3\alpha_2,1-\frac{t}{\theta_2}\right)\right)\Bigg].
\end{aligned}
$$

Consequently, the k-th moment of $Y_{[r:n]}$ is given by

$$
\begin{aligned}
\mu^{(k)}_{[r:n]} = \mathrm{E}[Y^k_{[r:n]}] &= \mathrm{E}[Y^k] + \Delta^{(1)}_{r,n}\Big(\gamma\big(\mathrm{E}[V_2^k]-\mathrm{E}[V_1^k]\big) - \lambda\big(\mathrm{E}[V_1^k]-\mathrm{E}[Y^k]\big)\Big)\\
&\qquad - \gamma\Delta^{(2)}_{r,n}\big(\mathrm{E}[V_2^k]-\mathrm{E}[V_1^k]\big).
\end{aligned}
$$

Moreover, the mean of $Y_{[r:n]}$:

$$
\mu_{[r:n]} = \mu^{(1)}_{[r:n]} = \frac{1}{\theta_2}\left[B(\alpha_2) + \Delta^{(1)}_{r,n}\big(\gamma D(3\alpha_2)-\lambda D(2\alpha_2)\big) - \gamma\Delta^{(2)}_{r,n}D(3\alpha_2)\right]. \tag{3.2}
$$

Theorem 3.1. *For any $1 \le r \le n-3$, we get*

$$\Big[(n+2)A(\lambda,\gamma) - 3(r+1)\gamma D(3\alpha_2)\Big]\mu_{[r+2:n]} =$$
$$= \Big[2(n+2)A(\lambda,\gamma) - 3(2r+3)\gamma D(3\alpha_2)\Big]\mu_{[r+1:n]} - \Big[(n+2)A(\lambda,\gamma) - 3(r-2)\gamma D(3\alpha_2)\Big]\mu_{[r:n]}.$$

Moreover, for all $n > 2$, we get

$$\Big[A(\lambda,\gamma)\big(2 - n(n+1)\big) - 3(r+1)(n-1)\gamma D(3\alpha_2)\Big]\mu_{[r:n]} =$$
$$= (n+2)\Big[A(\lambda,\gamma)(n+1) + 3(r+1) + 3(r+1)\gamma D(3\alpha_2)\Big]\mu_{[r:n-2]}$$
$$-\Big[2A(\lambda,\gamma)(n+2) + 3(r+1)(2n+1)\gamma D(3\alpha_2)\Big]\mu_{[r:n-1]},$$

where $A(\lambda,\gamma) = \gamma D(3\alpha_2) - \lambda D(2\alpha_2)$.

Proof: It is easy, after some algebra, to show that the mean $\mu_{[r:n]}$, defined by (3.2), satisfies the following relation:

$$\frac{\mu_{[r+2:n]} - \mu_{[r:n]}}{\mu_{[r+1:n]} - \mu_{[r:n]}} = \frac{A(\lambda,\gamma)\left(\Delta^{(1)}_{r+2,n} - \Delta^{(1)}_{r,n}\right) + \gamma D(3\alpha_2)\left(\Delta^{(2)}_{r+2,n} - \Delta^{(2)}_{r,n}\right)}{A(\lambda,\gamma)\left(\Delta^{(1)}_{r+1,n} - \Delta^{(1)}_{r,n}\right) + \gamma D(3\alpha_2)\left(\Delta^{(2)}_{r+1,n} - \Delta^{(2)}_{r,n}\right)}. \tag{3.3}$$

On the other hand, we can check that $\Delta^{(1)}_{r+2,n} - \Delta^{(1)}_{r,n} = \frac{-4}{n+1}$, $\Delta^{(1)}_{r+1,n} - \Delta^{(1)}_{r,n} = \frac{-2}{n+1}$, $\Delta^{(2)}_{r+2,n} - \Delta^{(2)}_{r,n} = \frac{-12r-18}{(n+1)(n+2)}$ and $\Delta^{(2)}_{r+1,n} - \Delta^{(2)}_{r,n} = \frac{-6r-6}{(n+1)(n+2)}$. Thus, by combining the last four relations with (3.3), we get first recurrence relation in Theorem 3.1. Also, we can easily check that $\Delta^{(1)}_{r,n} - \Delta^{(1)}_{r,n-2} = \frac{4r}{(n-1)(n+1)}$, $\Delta^{(1)}_{r,n-1} - \Delta^{(1)}_{r,n-2} = \frac{2r}{n(n-1)}$, $\Delta^{(2)}_{r,n} - \Delta^{(2)}_{r,n-2} = \frac{6r(r+1)(2n+1)}{n(n-1)(n+1)(n+2)}$ and $\Delta^{(2)}_{r,n-1} - \Delta^{(1)}_{r,n-2} = \frac{6r(r+1)}{n(n-1)(n+1)}$. The last four relation and the relation (3.2) imply the second recurrence relation of the theorem. This completes the proof. □

Remark 3.1. By putting $\gamma = 0$ in the two recurrence relations defined in Theorem 3.1 (note that $A(\lambda, 0) = 0$), we get the two corresponding recurrence relations defined in Theorem 3.1 of Barakat *et al.* [7] (2019), at $p = 1$.

By multiplying the both sides of (3.1) by $(y - \mu_{[r:n]})^2$ and integrating, we obtain the variance of $Y_{[r:n]}$ as

$$\begin{aligned}\sigma^2_{[r:n]} = \frac{1}{\theta_2^2}\Big[&(1+\pi_1)\big(C(\alpha_2) - \pi_1 B^2(2\alpha_2)\big) + (\pi_2 - \pi_1)\big(C(2\alpha_2) + B^2(2\alpha_2)\big)\\ &- B^2(2\alpha_2)(\pi_1+\pi_2)^2 + \pi_2\big(C(3\alpha_2) - B^2(3\alpha_2)(1+\pi_2)\big) - 2B(\alpha_2)B(2\alpha_2)\pi_3\\ &- 2B(\alpha_2)B(3\alpha_2)\pi_2(1+\pi_1) - 2B(2\alpha_2)B(3\alpha_2)\pi_2(\pi_1+\pi_2)\Big],\end{aligned} \tag{3.4}$$

where $\pi_1 = \lambda\Delta^{(1)}_{r,n}$, $\pi_2 = \gamma\Delta^{(2)}_{r,n}$ and $\pi_3 = \pi_1(1+\pi_1) + \pi_2(1+\pi_1)$.

3.1. Joint df of concomitants of os's based on FGM($\lambda, \gamma\colon \theta_1, \alpha_1;\, \theta_2, \alpha_2$)

The joint pdf of concomitants $Y_{[r:n]}$ and $Y_{[s:n]}, r < s$, is (cf. Tahmasebi and Jafari [38], 2015)

$$f_{[r,s:n]}(y_1, y_2) = \int_0^{\infty}\int_0^{x_2} f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) f_{r,s:n}(x_1, x_2)\, dx_1 dx_2,$$

where $\beta(a,b,c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$ and

$$\begin{aligned} f_{r,s:n}(x_1,x_2) \;&=\; \frac{1}{\beta(r, s-r, n-s+1)} F_X^{r-1}(x_1) \\ &\times\; \big(F_X(x_2) - F_X(x_1)\big)^{s-r-1}\big(1 - F_X(x_2)\big)^{n-s} f_X(x_1) f_X(x_2), \qquad x_1 < x_2. \end{aligned}$$

Therefore,

$$\begin{aligned} f_{[r,s:n]}(y_1,y_2) = &\int_0^{\infty}\int_0^{x_2} f_Y(y_1)\Big[1 + \lambda\big(1 - 2F_X(x_1)\big)\big(1 - 2F_Y(y_1)\big) \\ &+ \gamma F_X(x_1) F_Y(y_1)\big(2 - 3F_X(x_1)\big)\big(2 - 3F_Y(y_1)\big)\Big]\Big[f_Y(y_2)\Big[1 + \lambda\big(1 - 2F_X(x_2)\big) \\ &\times \big(1 - 2F_Y(y_2)\big) + \gamma F_X(x_2) F_Y(y_2)\big(2 - 3F_X(x_2)\big)\big(2 - 3F_Y(y_2)\big)\Big]\Big] \\ &\times \left[\frac{F_X^{r-1}(x_1)\big(F_X(x_2) - F_X(x_1)\big)^{s-r-1}\big(1 - F_X(x_2)\big)^{n-s}}{\beta(r, s-r, n-s+1)} f_X(x_1) f_X(x_2)\right] dx_1 dx_2. \end{aligned} \tag{3.5}$$

On the other hand, after some algebra we can write the joint pdf $f_{[r,s:n]}(y_1, y_2)$, defined by (3.5), in the following compact form:

$$\begin{aligned} f_{[r,s:n]}(y_1,y_2) \;=\; &f_Y(y_1) f_Y(y_2)\Big[1 + \lambda\big(1 - 2F_Y(y_1)\big) I_1 + \lambda\big(1 - 2F_Y(y_2)\big) I_2 \\ &+ \lambda^2\big(1 - 2F_Y(y_1)\big)\big(1 - 2F_Y(y_2)\big) I_3 \;+\; \gamma F_Y(y_1)\big(2 - 3F_Y(y_1)\big) I_4 \\ &+ \gamma F_Y(y_2)\big(2 - 3F_Y(y_2)\big) I_5 \;+\; \gamma^2 F_Y(y_1) F_Y(y_2)\big(2 - 3F_Y(y_1)\big)\big(2 - 3F_Y(y_2)\big) I_6 \\ &+ \lambda\gamma F_Y(y_2)\big(1 - 2F_Y(y_1)\big)\big(2 - 3F_Y(y_2)\big) I_7 \\ &+ \lambda\gamma F_Y(y_1)\big(1 - 2F_Y(y_2)\big)\big(2 - 3F_Y(y_1)\big) I_8\Big], \end{aligned}$$

where $I_1 = \Delta^{(1)}_{r,s,n}$, $I_2 = \Delta^{(2)}_{r,s,n}$, $I_3 = \Delta^{(1)}_{r,s,n} + \Delta^{(2)}_{r,s,n} - \Delta^{(3)}_{r,s,n}$, $I_4 = \Delta^{(4)}_{r,s,n} - \Delta^{(1)}_{r,s,n}$, $I_5 = \Delta^{(5)}_{r,s,n} - \Delta^{(2)}_{r,s,n}$, $I_6 = \big(\Delta^{(6)}_{r,s,n} + \Delta^{(7)}_{r,s,n}\big) - \big(\Delta^{(3)}_{r,s,n} - \Delta^{(8)}_{r,s,n}\big)$, $I_7 = \Delta^{(5)}_{r,s,n} - \Delta^{(2)}_{r,s,n} + \Delta^{(3)}_{r,s,n} - \Delta^{(7)}_{r,s,n}$ and $I_8 = \Delta^{(4)}_{r,s,n} - \Delta^{(1)}_{r,s,n} + \Delta^{(3)}_{r,s,n} - \Delta^{(6)}_{r,s,n}$. Moreover,

$$\begin{aligned} \Delta^{(i)}_{r,s,n} \;&=\; \frac{\beta(r, s-r, n-s+1) - (p_i+1)\beta(r+p_i, s-r, n-s+1)}{\beta(r, s-r, n-s+1)}, \qquad i = 1, 4, \\ \Delta^{(i)}_{r,s,n} \;&=\; \frac{\beta(r, s-r, n-s+1) - (p_i+1)\beta(s+p_i, n-s+1)\beta(r, s-r)}{\beta(r, s-r, n-s+1)}, \qquad i = 2, 5, \\ \Delta^{(i)}_{r,s,n} \;&=\; \frac{\beta(r, s-r, n-s+1) - (p_i+1)^2\beta(s+2p_i, n-s+1)\beta(r+p_i, s-r)}{\beta(r, s-r, n-s+1)}, \qquad i = 3, 8, \\ \Delta^{(i)}_{r,s,n} \;&=\; \frac{\beta(r, s-r, n-s+1) - 6\beta(s+3, n-s+1)\beta(r+p_i, s-r)}{\beta(r, s-r, n-s+1)}, \qquad i = 6, 7, \end{aligned}$$

where $p_1 = p_2 = p_3 = p_7 = 1$ and $p_4 = p_5 = p_6 = p_8 = 2$. Therefore, the product moment $\mathrm{E}\big[Y_{[r:n]}Y_{[s:n]}\big]$ is obtained directly as

$$\begin{aligned}\mu_{[r,s:n]} = \frac{1}{\theta_2^2}\Big[&B^2(\alpha_2)\xi_1(\lambda,r,s,n) - B(\alpha_2)B(2\alpha_2)\xi_2(\gamma,\lambda,r,s,n)\\ &+ B^2(2\alpha_2)\xi_3(\gamma,\lambda,r,s,n) - B(\alpha_2)B(2\alpha_2)\xi_4(\gamma,\lambda,r,s,n) + \gamma^2B^2(3\alpha_2)I_6\Big],\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\xi_1(\lambda,r,s,n) &= 1+\lambda(I_1+I_2+I_3),\\ \xi_2(\gamma,\lambda,r,s,n) &= \lambda(I_1+I_2+2\lambda I_3) - \gamma(I_4+I_5) - \lambda\gamma(I_7+I_8),\\ \xi_3(\gamma,\lambda,r,s,n) &= \lambda^2 I_3 + \gamma^2 I_6 - \lambda\gamma(I_7+I_8)\end{aligned}$$

and

$$\xi_4(\gamma,\lambda,r,s,n) = \gamma(I_4+I_5) + \lambda\gamma(I_7+I_8).$$

Therefore, by using (3.2) and (3.6) we can after some algebra calculate the covariance between $Y_{[r:n]}$ and $Y_{[s:n]}$ as

$$\begin{aligned}\sigma_{[r,s:n]} = \frac{1}{\theta_2^2}\Big[&B^2(\alpha_2)\delta^{(1)}_{r,s,n} - B(\alpha_2)B(2\alpha_2)\delta^{(2)}_{r,s,n}\\ &+ B^2(2\alpha_2)\delta^{(3)}_{r,s,n} - B(\alpha_2)B(3\alpha_2)\delta^{(4)}_{r,s,n} + B^2(3\alpha_2)\delta^{(5)}_{r,s,n}\Big].\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\delta^{(1)}_{r,s,n} &= 1+\lambda\Big(I_1+I_2+\lambda I_3 - \Delta^{(1)}_{r,n} - \Delta^{(1)}_{s,n}\Big),\\ \delta^{(2)}_{r,s,n} &= \lambda\Big(I_1+I_2+2\lambda I_3 - \Delta^{(1)}_{r,n} - \Delta^{(1)}_{s,n}\Big) - \gamma\Big(I_4+I_5-\Delta^{(2)}_{r,n}-\Delta^{(2)}_{s,n}\Big) - \lambda\gamma(I_7+I_8),\\ \delta^{(3)}_{r,s,n} &= \lambda^2\Big(I_3+\Delta^{(1)}_{r,n}\Delta^{(1)}_{s,n}\Big) + \gamma^2\Big(I_6+\Delta^{(2)}_{r,n}\Delta^{(2)}_{s,n}\Big) - \lambda\gamma(I_7+I_8),\\ \delta^{(4)}_{r,s,n} &= \gamma\Big(I_4+I_5-\Delta^{(2)}_{r,n}\Delta^{(2)}_{s,n}\Big) + \lambda\gamma(I_7+I_8)\end{aligned}$$

and

$$\delta^{(5)}_{r,s,n} = \gamma^2\Big(I_6+\Delta^{(2)}_{r,n}\Delta^{(2)}_{s,n}\Big).$$

We can now use (3.7) and (3.4) to obtain the coefficient of correlation between $Y_{[r:n]}$ and $Y_{[s:n]}$ as $\rho_{[r,s:n]} = \frac{\sigma_{[r,s:n]}}{\sigma_{[r:n]}\sigma_{[s:n]}}$. By putting $\gamma = 0$ in (3.4) and (3.7), we can easily check that the $\rho_{[r,s:n]}$ is exactly the coefficient of correlation between $Y_{[r:n]}$ and $Y_{[s:n]}$ calculated by Barakat *et al.* [7] (2019), at $p = 1$.

Theorem 3.2. *For any $1 \le r \le n-3$, we get*

$$\mu_{[r+2,s:n]} = 2\mu_{[r+1,s:n]} - \mu_{[r,s:n]} - \tau_n(s;\lambda,\gamma;\alpha_2), \tag{3.8}$$

where

$$\tau_n(s;\lambda,\gamma;\alpha_2) = \frac{6A_1(n+3)(n+4) + 12A_2(s+2)(n+4) + 18A_3(s+2)(s+3)}{(n+1)(n+2)(n+3)(n+4)}.$$

Moreover, for any $1 \le s \le n-3$, we get

$$\mu_{[r,s+2:n]} = 2\mu_{[r,s+1:n]} - \mu_{[r,s:n]} - \omega_n(r;\lambda,\gamma;\alpha_2), \tag{3.9}$$

where

$$\omega_n(r;\lambda,\gamma;\alpha_2) = \frac{6A_4(n+3)(n+4) + 12rA_5(n+4) + 18A_3r(r+1)}{(n+1)(n+2)(n+3)(n+4)}.$$

Finally, for all $n > 2$, we get

$$(n+1)\mu_{[r,s:n]} = 2n\mu_{[r,s:n-1]} - (n-1)\mu_{[r,s:n-2]} + \zeta_n(r,s;\lambda,\gamma;\alpha_2), \tag{3.10}$$

where

$$\begin{aligned}\zeta_n(r,s;\lambda,\gamma;\alpha_2) \;=\; & \frac{3A_4 s(s+1)(n+3)(n+4) + 36rA_2 r(r+1)(s+2)(n+4)}{n(n+1)(n+2)(n+3)(n+4)} \\ & + \frac{36A_5(s+1)(s+2)(n+4) + 108A_3(s+2)(s+3)r(r+1)}{n(n+1)(n+2)(n+3)(n+4)} \\ & + \frac{8A_6 r(s+1)(n+3)(n+4) + 6A_1 r(r+1)(n+3)(n+4)}{n(n+1)(n+2)(n+3)(n+4)},\end{aligned}$$

$$A_1 = \frac{1}{\theta_2^2}\Big[\lambda\gamma\big(B^2(2\alpha_2) + B(\alpha_2)B(2\alpha_2) - B(\alpha_2)B(3\alpha_2)\big) - \gamma B(\alpha_2)B(2\alpha_2)\Big],$$

$$A_2 = \frac{1}{\theta_2^2}\Big[\gamma^2\big(B^2(2\alpha_2) + B^2(3\alpha_2)\big) + \lambda\gamma\big(B(\alpha_2)B(3\alpha_2) - B(\alpha_2)B(2\alpha_2)\big)\Big],$$

$$A_3 = \frac{-\gamma^2}{\theta_2^2}\big(B^2(2\alpha_2) + B^2(3\alpha_2)\big),$$

$$A_4 = \frac{1}{\theta_2^2}\Big[\gamma\big(B(\alpha_2)B(2\alpha_2) - B(\alpha_2)B(3\alpha_2)\big) + \lambda\gamma\big(B(\alpha_2)B(2\alpha_2) + B(\alpha_2)B(3\alpha_2) - B^2(\alpha_2)\big)\Big],$$

$$A_5 = \frac{1}{\theta_2^2}\Big[\gamma^2\big(B^2(2\alpha_2) + B^2(3\alpha_2)\big) + \lambda\gamma\big(B(\alpha_2)B(3\alpha_2) - B(\alpha_2)B(2\alpha_2) + B^2(\alpha_2)\big)\Big]$$

and

$$A_6 = \frac{1}{\theta_2^2}\Big[\lambda^2\big(B(2\alpha_2) - B(\alpha_2)\big)^2 + 2\lambda\gamma B(\alpha_2)\big(B(2\alpha_2) - B(3\alpha_2)\big) - \gamma^2 B^2(3\alpha_2)\Big].$$

Proof: It is easy to check that

$$\Delta^{(i)}_{r+2,s,n} - \Delta^{(i)}_{r,s,n} = 2\big(\Delta^{(i)}_{r+1,s,n} - \Delta^{(i)}_{r,s,n}\big), \qquad i = 1,3,6, \tag{3.11}$$

$$\Delta^{(i)}_{r+2,s,n} - \Delta^{(i)}_{r,s,n} = \big(\Delta^{(i)}_{r+1,s,n} - \Delta^{(i)}_{r,s,n}\big)\,\frac{2r+3}{r+1}, \qquad i = 4,7,8, \tag{3.12}$$

and

$$\Delta^{(2)}_{r,s,n} = \Delta^{(2)}_{r+1,s,n} = \Delta^{(2)}_{r+2,s,n}, \qquad \Delta^{(5)}_{r,s,n} = \Delta^{(5)}_{r+1,s,n} = \Delta^{(5)}_{r+2,s,n}. \tag{3.13}$$

The recurrence relation (3.8) is now followed by combining (3.11) and (3.12) with (3.13). Now, we turn to prove (3.9). First, we notice that

$$\Delta^{(1)}_{r,s,n} = \Delta^{(1)}_{r,s+1,n} = \Delta^{(1)}_{r,s+2,n} \tag{3.14}$$

and

$$\Delta^{(4)}_{r,s,n} = \Delta^{(4)}_{r,s+1,n} = \Delta^{(4)}_{r,s+2,n}. \tag{3.15}$$

Moreover, it is easy to check that

$$\Delta^{(i)}_{r,s+2,n} - \Delta^{(i)}_{r,s,n} = 2\big(\Delta^{(i)}_{r,s+1,n} - \Delta^{(i)}_{r,s,n}\big), \qquad i = 2,3,6, \tag{3.16}$$

and

$$\Delta^{(i)}_{r,s+2,n} - \Delta^{(i)}_{r,s,n} = \big(\Delta^{(i)}_{r,s+1,n} - \Delta^{(i)}_{r,s,n}\big)\,\frac{2s+2p_i+1}{s+p_i}, \tag{3.17}$$

where $i = 5, 7, 8$ and $p_5 = 1$, $p_7 = 2$, $p_8 = 3$. Therefore, the recurrence relation (3.9) is followed by combining (3.14), (3.15), (3.16) and (3.17). In order to prove the recurrence relation (3.10), we first notice that

$$\Delta^{(i)}_{r,s,n-2:p} - \Delta^{(i)}_{r,s,n:p_i} = \left(\Delta^{(i)}_{r,s,n-1:p_i} - \Delta^{(i)}_{r,s,n:p_i}\right) \frac{2n + p_i - 1}{n - 1}, \tag{3.18}$$

where $i = 1, 2, ..., 8$ and $p_1 = p_2 = 1, p_3 = p_4 = p_5 = 2, p_6 = p_7 = 3, p_8 = 4$. The recurrence relation (3.10) is now followed by using (3.18). The proof is completed. □

Remark 3.2. By putting $\gamma = 0$ in (3.8), (3.9) and (3.10), we get (3.24), (3.25) and (3.26) in Theorem 3.3 of Barakat *et al.* [7] (2019), at $p = 1$.

4. CONCOMITANTS OF RECORD VALUES BASED ON FGM(λ,γ: θ_1,α_1; θ_2,α_2)

Let (X_i, Y_i), $i = 1, 2, ...$, be a random sample from FGM$(\lambda, \gamma; \theta_1, \alpha_1; \theta_2, \alpha_2)$. When the experimenter interests in studying just the sequence of records of the first component X_i's, the second component associated with the record value of the first one is termed as the concomitant of that record value. The concomitants of record values arise in a wide variety of practical experiments, e.g., see Bdair and Raqab [8] (2014) and Arnold *et al.* [5] (1998). Let $\{R_n, n \geq 1\}$ be the sequence of record values in the sequence of X's, while $R_{[n]}$ be the corresponding concomitant. Houchens [24] (1984) has obtained the pdf of concomitant of n-th record value for $n \geq 1$, as $h_{[n]}(y) = \int_0^\infty f_Y(y|x) g_n(x) dx$, where $g_n(x) = \frac{1}{\Gamma(n)}\big(-\log(1 - F_X(x))\big)^{n-1} f_X(x)$ is the pdf of R_n. Therefore, after some algebra, we get

$$h_{[n]}(y) = (1 + \lambda \Upsilon_{n:1}) f_Y(y) + (\gamma \Upsilon_{n:2} - \lambda \Upsilon_{n:1}) f_{V_1}(y) - \gamma \Upsilon_{n:2} f_{V_2}(y), \tag{4.1}$$

where $V_1 \sim \mathrm{GE}(\theta_2; 2\alpha_2)$, $V_2 \sim \mathrm{GE}(\theta_2; 3\alpha_2)$ and

$$\Upsilon_{n:p} = \left[1 - (1 + p) \sum_{i=0}^{\aleph(p)} \frac{(-1)^i \binom{p}{i}}{(i + 1)^n}\right]$$

(clearly, $\Upsilon_{n:1} = (2^{-(n-1)} - 1)$). The representation (4.1) enables us to derive the mean and the variance of $R_{[n]}$ as

$$\mu_{[R_n]} = \frac{1}{\theta_2}\Big[B(\alpha_2) - \lambda \Upsilon_{n:1} D(2\alpha_2) - \gamma \Upsilon_{n:2} D(3\alpha_2)\Big]$$

and

$$\begin{aligned} \sigma^2_{[R_n]} = \frac{1}{\theta_2^2}\Big[&C(\alpha_2) + \lambda \Upsilon_{n:1}\big(C(\alpha_2) - C(2\alpha_2)\big) + \gamma \Upsilon_{n:2}\big(C(2\alpha_2) - C(3\alpha_2)\big) \\ &- (1 + \lambda \Upsilon_{n:1}) \lambda \Upsilon_{n:1} D^2(2\alpha_2) - (1 + \gamma \Upsilon_{n:2}) \gamma \Upsilon_{n:2} D^2(3\alpha_2) \\ &- \lambda \gamma \Upsilon_{n:1} \Upsilon_{n:2} D(2\alpha_2) D(3\alpha_2)\Big]. \end{aligned} \tag{4.2}$$

Again, by putting $\gamma = 0$, we get the mean and the variance of $R_{[n]}$ for the Huang–Kotz FGM family based on the GE marginals at $p = 1$ (cf. Barakat *et al.* [7], 2019).

The joint pdf of the concomitants $R_{[n]}$ and $R_{[m]}$, $n<m$, is given by

$$h_{[n,m]}(y_1,y_2)=\int_0^\infty\int_{x_1}^\infty f_{Y|X}(y_1|x_1)f_{Y|X}(y_2|x_2)g_{m,n}(x_1,x_2)\,dx_2dx_1,$$

where

$$g_{m,n}(x)=\frac{1}{\Gamma(n)\Gamma(m-n)}\Big(-\log\big(1-F_X(x_1)\big)\Big)^{n-1}\Big(-\log\frac{1-F_X(x_2)}{1-F_X(x_1)}\Big)^{m-n-1}\frac{f_X(x_1)f_X(x_1)}{1-F_X(x_1)}$$

is the joint pdf of R_n and R_m. Therefore, after some algebra, we get

$$\begin{aligned}h_{[n,m]}(y_1,y_2)=f_Y(y_1)f_Y(y_2)\Big[&1+\lambda\big(1-2F_Y(y_1)\big)J_1+\lambda\big(1-2F_Y(y_2)\big)J_2\\&+\lambda^2\big(1-2F_Y(y_1)\big)\big(1-2F_Y(y_2)\big)J_3+\gamma F_Y(y_1)\big(2-3F_Y(y_1)\big)J_4\\&+\gamma F_Y(y_2)\big(2-3F_Y(y_2)\big)J_5+\gamma^2F_Y(y_1)F_Y(y_2)\big(2-3F_Y(y_1)\big)\big(2-3F_Y(y_2)\big)J_6\\&+\lambda\gamma F_Y(y_2)\big(1-2F_Y(y_1)\big)\big(2-3F_Y(y_2)\big)J_7\\&+\lambda\gamma F_Y(y_1)\big(1-2F_Y(y_2)\big)\big(2-3F_Y(y_1)\big)J_8\Big],\end{aligned}\tag{4.3}$$

where $J_1=\Upsilon_{n:1}$, $J_2=\Upsilon_{m:1}$, $J_3=4\Upsilon_{n:1}+\Upsilon_{m:1}-\Upsilon_{n,m:1,1}$, $J_4=\Upsilon_{n:2}-\Upsilon_{n:1}$, $J_5=\Upsilon_{m:2}-\Upsilon_{m:1}$, $J_6=\Upsilon_{n,m:2,1}+\Upsilon_{n,m:1,2}-\Upsilon_{n,m:1,1}-\Upsilon_{n,m:2,2}$, $J_7=\Upsilon_{m:2}+\Upsilon_{n,m:1,1}-\Upsilon_{m:1}-\Upsilon_{n,m:1,2}$, $J_8=\Upsilon_{n:2}+\Upsilon_{n,m:1,1}-\Upsilon_{n:1}-\Upsilon_{n,m:2,1}$ and

$$\Upsilon_{n,m:p,q}=\left[1-(1+p)(1+q)\sum_{i=0}^{\aleph(p)}\sum_{j=0}^{\aleph(q)}\frac{(-1)^{i+j}\binom{p}{i}\binom{q}{j}}{(i+j+1)^n(j+1)^{m-n}}\right].$$

The representation (4.3) enables us to derive the product moment and the covariance of $R_{[n]}$ and $R_{[m]}$, respectively, as

$$\begin{aligned}\mu_{[R_n,R_m]:p}&=\frac{1}{\theta_2^2}\Big[B^2(\alpha_2)\xi_1(\lambda,n,m)-B(\alpha_2)B(2\alpha_2)\xi_2(\gamma,\lambda,n,m)\\&\quad+B^2(2\alpha_2)\xi_3(\gamma,\lambda,n,m)-B(\alpha_2)B(2\alpha_2)\xi_4(\gamma,\lambda,n,m)+B^2(3\alpha_2)\gamma^2J_6\Big],\end{aligned}$$

where $\xi_1(\lambda,n,m)=1+\lambda(J_1+J_2+J_3)$, $\xi_2(\gamma,\lambda,n,m)=\lambda(J_1+J_2+2\lambda J_3)-\gamma(J_4+J_5)-\lambda\gamma(J_7+J_8)$, $\xi_3(\gamma,\lambda,n,m)=\lambda^2J_3+\gamma^2J_6-\lambda\gamma(J_7+J_8)$ and $\xi_4(\gamma,\lambda,n,m)=\gamma(J_4+J_5)+\lambda\gamma(J_7+J_8)$ and

$$\begin{aligned}\sigma_{[R_n,R_m]}=\frac{1}{\theta_2^2}\Big[&B^2(\alpha_2)\eta^{(1)}_{n,m}-B(\alpha_2)B(2\alpha_2)\eta^{(2)}_{n,m}\\&+B^2(2\alpha_2)\eta^{(3)}_{n,m}-B(\alpha_2)B(3\alpha_2)\eta^{(4)}_{n,m}+B^2(3\alpha_2)\eta^{(5)}_{n,m}\Big],\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}\eta^{(1)}_{n,m}&=1+\lambda(J_1+J_2+\lambda J_3-\Upsilon_{n:1}-\Upsilon_{m:1}),\\\eta^{(2)}_{n,m}&=\lambda(J_1+J_2+2\lambda J_3-\Upsilon_{n:1}-\Upsilon_{m:1})-\gamma(J_4+J_5-\Upsilon_{n:2}-\Upsilon_{m:2})-\lambda\gamma(J_7+J_8),\\\eta^{(3)}_{n,m}&=\lambda^2(J_3+\Upsilon_{n:1}\Upsilon_{m:1})+\gamma^2(J_6+\Upsilon_{n:2}\Upsilon_{m:2})-\lambda\gamma(J_7+J_8),\\\eta^{(4)}_{n,m}&=\gamma(J_4+J_5-\Upsilon_{n:2}\Upsilon_{m:2})+\lambda\gamma(J_7+J_8)\end{aligned}$$

and

$$\eta^{(5)}_{n,m}=\gamma^2(J_6+\Upsilon_{n:2}\Upsilon_{m:2}).$$

Finally, by combining (4.2) and (4.4), we get the correlation coefficient of the concomitants $R_{[n]}$ and $R_{[m]}$ as

$$\rho_{[R_n,R_m]} = \frac{\sigma_{[R_n,R_m]}}{\sqrt{\sigma^2_{[R_n]}\sigma^2_{[R_m]}}}.$$

Clearly, by putting $\gamma = 0$ in (4.2) and (4.4), we can easily check that the $\rho_{[R_n,R_m]}$ is exactly the coefficient of correlation between R_n and R_m calculated by Barakat *et al.* [7] (2019), at $p = 1$.

5. APPLICATIONS

Concomitants of os's and record values have received a continued remarkable attention in recent years due to their applicability in many problems. The most striking application of concomitants of os's arises in biological selection problems. For example, in choosing the top k out of n rams as judged by their genetic make up is selected for breeding, then $Y_{[n-k+1:n]}, ..., Y_{[n:n]}$, might represent the quality of the wool of one of their female offspring. In such type of experiments a geneticist is more likely to choose the best set of offsprings with less number of trials than one in which all trials are undertaken which is much expensive and time consuming. Examples of such application can be found in Scaria and Thomas [34] (2014).

Estimation of the parameters associated with the df of the rv Y of primary interest using concomitants of os's or record values on the auxiliary rv X is an another important application, where extensive works are seen carried out. For example see, Begum and Khan [9] (2000), Scaria [33] (2003), Philip and Thomas [31] (2015), Veena and Thomas [42] (2015) and Domma and Giordano [16] (2016).

Another important application of concomitants of os's and record values is a method of sampling known as ranked set sampling. Namely, when we have an auxiliary rv X, which is easily measurable while the measurement of the rv Y of primary interest is hard and expensive. In order to achieve observational economy, we choose n^2 units randomly from the population and arrange them in n groups of n units each for measurement of the observed rv X. Therefore, based on the observations on X, units in each group are ranked among themselves and from the j-th group the unit ranked j is chosen for measurement of the variable Y of primary interest for $j = 1, 2, ..., n$. Clearly the observations finally measured on Y are concomitants of os's. For some references in this area one may refer, Chen *et al.* [12] (2004), Chacko and Thomas [10, 11] (2008, 2009), Lesitha and Thomas [28] (2013), Paul and Thomas [30] (2017) and Philip and Thomas [32] (2017).

Moreover, some results on characterization of bivariate distributions by properties of concomitants of os's are available in Thomas and Veena [41] (2011). Besides the preceding applications, there are important other recent applications, For example, Jung *et al.* [27] (2008) presented an application of generalized FGM copula function in exchange markets using directional dependence concept. Hlubinka and Kotz [23] (2010) used the generalized FGM distribution and related copulas as bivariate models for the distribution of spheroidal characteristics. Sheikhi and Tata [35] (2013) modeled the joint distribution of a linear combination of concomitants of os's and linear combinations of their os's as a unified skew-normal family assuming a multivariate normal distribution. Eryilmaz [17] (2016) has shown that the concomitants are potentially useful in reliability modeling.

Eryilmaz [17] (2016) has analysed the FGM with exponential marginals from a reliability point of view. We extend some of these results to the FGM$(\lambda, \gamma\colon \theta_1, \alpha_1;\, \theta_2, \alpha_2)$.

Let $X_i \sim \mathrm{GE}(\theta_1; \alpha_1)$ and $Y_i \sim \mathrm{GE}(\theta_2; \alpha_2)$ denote respectively the lifetime of the i-th component, and the utility of the i-th component during its lifetime, $i = 1, ..., n$. Total utility of n components is defined by the rv $\sum_{i=1}^n Y_i$. Moreover, the residual performance after the first failure in the system is given by $\sum_{i=1}^n Y_i - Y_{[1:n]}$. Although the components are identical, they may have different contribution/utility to the performance of the whole system since the components may be located in different positions or they may be used by different operators. The utility of the component is positively correlated with its lifetime. Such a dependence can be modeled by FGM$(\lambda, \gamma\colon \theta_1, \alpha_1;\, \theta_2, \alpha_2)$.

The residual performance after time t is defined by the process (cf. Eryilmaz [17], 2016)

$$S(t) = \sum_{i=N(t)+1}^{n} Y_{[i:n]}, \qquad t > 0,$$

where the process $N(t)$ denotes the number of failures up to time t, i.e., $P(N(t) = r) = \binom{n}{r} F_X^r(t)\,(1 - F_X(t))^{n-r}$, $r = 0, 1, ..., n$, with $P(N(t) = 0) = 1$. Clearly, knowing the mean value of $S(t)$ may help to an engineer at various stages such as design, and preventive maintenance. By using Proposition 1 of Eryilmaz [17] (2016) and after some algebra, we can show that

$$\mathrm{E}\big(S(t)\big) \;=\; \frac{n}{\theta_2}\Big[B(\alpha_2)\big(1 - F_X(t)\big) + \lambda D(2\alpha_2)\big(F_X(t) - F_{U_1}(t)\big) + \gamma D(3\alpha_2)\big(F_{U_1}(t) - F_{U_2}(t)\big)\Big],$$

where $U_1 \sim \mathrm{GE}(\theta_1; 2\alpha_1)$ and $U_2 \sim \mathrm{GE}(\theta_1; 3\alpha_1)$.

On the other hand, it is useful to know about the mean residual performance of the system when at a specific time there are exactly m working components. For this purpose, we consider the conditional mean residual performance defined by $\psi_m(t) = \mathrm{E}(S(t) = j | M(t) = n - N(t) = m)$, where $M(t)$ is the number of working components at time t. Now, using Theorem 1 of Eryilmaz [17] (2016), we get after some algebra

$$\begin{aligned}\psi_m(t) \;&=\; \frac{m}{n}\frac{\mathrm{E}\big(S(t)\big)}{1 - F_X(t)}\\ &=\; \frac{m}{\theta_2}\left[B(\alpha_2) + \lambda D(2\alpha_2)\frac{F_X(t) - F_{U_1}(t)}{1 - F_X(t)} + \gamma D(3\alpha_2)\frac{F_{U_1}(t) - F_{U_2}(t)}{1 - F_X(t)}\right].\end{aligned}$$

By using applying L'Hospital's rule, we get

$$\lim_{t\to\infty} \psi_m(t) \;=\; \frac{m}{\theta_2}\Big[B(\alpha_2) + \lambda D(2\alpha_2) + \gamma D(3\alpha_2)\Big] \;=\; \lim_{t\to\infty} \mathrm{E}(Y | X = t)$$

($\mathrm{E}(Y|X = t)$ is given by (2.3)).

Furthermore, we can consider the random time until the total output of the system first falls below the critical level k. Clearly, the waiting time until the total output first falls below k is of special importance in the analysis. The corresponding time is defined by the rv $T(k) = \inf\{t\colon S(t) < k\}$. Since this waiting time corresponds to one of the failure time of the components, the two events $\{T(k) = X_{r:n}\}$ and $\{S(X_{r-1:n}) \ge k$ and $S(X_{r:n}) < k\}$ are equivalent, where the rv $S(X_{r:n}) = \sum_{i=r+1}^{n} Y_{[i:n]}$ defines the residual performance after the r-th failure in the system (cf. Eryilmaz [17], 2016). For a system consisting of $n = 3$ components, using Proposition 3 of Eryilmaz [17], 2016), we get

$$\begin{aligned}
P\big(T(k) = X_{2:3}\big) &= \int_0^\infty P\big(Y_1^\star + Y_2^\star \ge k\big)\, dF_{X_{1:3}}(x) - \int_0^\infty P\big(Y_1^\star \ge k\big)\, dF_{X_{2:3}}(x),\\
P\big(T(k) = X_{3:3}\big) &= \int_0^\infty P\big(Y_1^\star \ge k\big)\, dF_{X_{2:3}}(x),
\end{aligned}$$

and $P(T(k) = X_{1:3}) = 1 - P(T(k) = X_{2:3}) - P(T(k) = X_{3:3})$, where $P(Y^\star < y) = F_{Y|X}(y|x)$ is defined by (2.2). Thus,

$$\begin{aligned}
P\big(Y_1^\star + Y_2^\star > k\big) &= \int_0^\infty P\big(Y_1^\star + Y_2^\star > k \,|\, Y_1^\star = y\big)\, dy\\
&= \int_0^k P\big(Y_1^\star + Y_2^\star > k \,|\, Y_1^\star = y\big)\, dy + \int_k^\infty P\big(Y_1^\star + Y_2^\star > k \,|\, Y_1^\star = y\big)\, dy\\
&= \int_0^k \Big(1 - P\big(Y_2^\star \le k - y\big)\Big) f_{Y_1^\star}(y)\, dy + 1\\
&\quad - F_Y(k)\Big[1 - \lambda F_X(x)\bar{F}_Y(k) - \gamma F_X^2(x) F_Y(k)\bar{F}_Y(k)\Big].
\end{aligned}$$

By using the binomial theorem, the above integration can be easily explicitly evaluated. However, Eryilmaz [17] (2016) presented a simple Monte-Carlo simulation algorithm to compute the probability $P(T(k) = X_{r:n})$ for general bivariate df $F_{X,Y}$.

6. CONCLUDING REMARKS

While introducing the iterated FGM distribution by Huang and Kotz [25] (1984), and thereby showed that the maximum correlation is higher than was previously known. Moreover, Huang and Kotz [25] (1984) showed that just one single iteration can result in tripling the covariance for certain marginals. Other than this a systematic study (by Huang and Kotz [25], 1984) of the properties of this promising distribution and its application does not appear to have been discussed in literature. The present paper is an attempt in this direction. Some new distributional properties of concomitants of os's of the iterated FGM based on the GE df were presented in Section 2. Moreover, several new useful recurrence relations between single and product moments of concomitants were established. Finally, by relying of the results of Section 2, we gave an application of this model in reliability theory. Besides this application we reviewed some various applications for concomitants and the FGM distribution. Most probably, the utilization of the iterated FGM distribution instead FGM distribution for studying these applications will give more accurate results.

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REFERENCES

[1] Ahsanullah, M. (2009). Records and concomitants, *Bull. Malays. Math.*, **32**(2), 101–117.

[2] Ahsanullah, M. and Shakil, M. (2013). Characterizations of Rayleigh distribution based on order statistics and record values, *Bull. Malays. Math. Sci. Soc.*, **36**(3), 625–635.

[3] Ahsanullah, M.; Shakil, M. and Kibria, G.M. (2013). A note on a characterization of Gompeertz–Verhulst distribution, *J. of Statist. Theory and Appl.*, **13**(1), 17–26.

[4] AL-Hussaini, E.K. and Ahsanullah, M. (2015). *Exponentiated Distributions*, Atlantis Studies in Probability and Statistics, Atlantis Press, Paris.

[5] Arnold, B.C.; Balakrishnan, N. and Nagaraja, H.N. (1998). *Records*, Wiley Series in Probability and Statistics, Wiley, New York.

[6] Barakat, H.M.; Nigm, E.M. and Syam, A.H. (2018). Concomitants of order statistics and record values from Bairamov–Kotz–Bekci FGM bivariate-generalized exponential distribution, *Filomat Journal*, **32**(9), 3313–3324.

[7] Barakat, H.M.; Nigm, E.M. and Syam, A.H. (2019). Concomitants of ordered variables from Huang–Kotz FGM type bivariate generalized exponential distribution, *Bull. Malays. Math. Sci. Soc.*, **42**, 337–353.

[8] Bdair, O.M. and Raqab, M.Z. (2014). Mean residual life of k-th records under double monitoring, *Bull. Malays. Math. Sci. Soc.*, **37**(2), 457–464.

[9] Begum, A.A. and Khan, A.H. (2000). Concomitants of order statistics from Marshall and Olkin's bivariate Weibull distribution, *Calcutta Statist. Assoc. Bul.*, **50**, 65–70.

[10] Chacko, M. and Thomas, P.Y. (2008). Estimation of a parameter of Morgenstern type bivariate exponential distribution by ranked set sampling, *Ann. Ins. Statist. Math.*, **60**, 301–318.

[11] Chacko, M. and Thomas, P.Y. (2009). Estimation of parameters of Morgenstern type bivariate logistic distribution by ranked set sampling, *J. of Indian Soc. of Agricultural Statist.*, **63**, 77–83.

[12] Chen, Z.; Bai, Z. and Sinha, B.K. (2004). *Ranked Set Sampling, Theory and Applications*, Lecture Notes in Statistics, Springer, New York.

[13] David, H.A. (1973). Concomitants of order statistics, *Bull. of Inter. Statist. Inst.*, **45**(2), 295–300.

[14] David, H.A. and Nagaraja, H.N. (1998). *Concomitants of order statistics.* In "Order Statistics: Theory & Methods" (N. Balakrishnan and C.R. Rao, Eds.), Elsevier, Amsterdam, pp. 487–513.

[15] David, H.A. and Nagaraja, H.N. (2003). *Order Statistics*, John Wiley Sons, Inc.

[16] Domma, F. and Giordano, S. (2016). Concomitants of m-generalized order statistics from generalized Farlie–Gumbel–Morgenstern distribution family, *J. of Comp. Appl. Math.*, **294**, 413–435.

[17] Eryilmaz, S. (2016). On an application of concomitants of order statistics, *Comm. in Statist. – Theory and Meth.*, **45**(19), 5628–5636.

[18] Farlie, D.J.G. (1960). The performance of some correlation coefficients for a general bivariate distribution, *Biometrica*, **47**, 307–323.

[19] Galambos, J. (1987). *The Asymptotic Theory of Extreme Order Statistics*, Krieger, FL (2nd Edn.).

[20] Gumbel, E.J. (1960). Bivariate exponential distributions, *J. Amer. Statist. Assoc.*, **55**, 698–707.

[21] Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions, *Austral. & New Zealand J. Statist.*, **41**, 173–188.

[22] Hanif, S. (2007). *Concomitants of order random variables*, Ph.D. Thesis, National College of Business Administration & Economics, Labore.

[23] Hlubinka, D. and Kotz, S. (2010). The generalized FGM distribution and its application to stereology of extremes, *Appl. of Math.*, **55**, 495–512.

[24] Houchens, R.L. (1984). *Record Value Theory and Inference*, ProQuest LLC, Ann Arbor.

[25] Huang, J.S. and Kotz, S. (1984). Correlation structure in iterated Farlie–Gumbel–Morgenstern distributions, *Biometrika*, **71**(3), 633–636.

[26] Huang, J.S. and Kotz, S. (1999). Modifications of the Farlie–Gumbel–Morgenstern distributions. A tough hill to climb, *Metrika*, **49**, 135–145.

[27] Jung, Y.S.; Kim, J.M. and Kim, J. (2008). New approach of directional dependence in exchange markets using generalized FGM copula function, *Comm. in Statist. – Sim. and Comp.*, **37**, 772–788.

[28] Lesitha, G. and Thomas, P.Y. (2013). Estimation of the scale parameter of a log-logistic distribution, *Metrika*, **76**, 427–448.

[29] Morgenstern, D. (1956). Einfache Beispiele zweidimensionaler Verteilungen, *Mitt. Math. Stat*, **8**, 234–235.

[30] Paul, J. and Thomas, P.Y. (2017). Concomitant record ranked set sampling, *Comm. in Statist. – Theory and Meth.*, **46**(19), 9518–9540.

[31] Philip, A. and Thomas, P.Y. (2015). On concomitants of order statistics arising from the extended Farlie–Gumbel–Morgenstern bivariate logistic distribution and its application in estimation, *Statistical Method.*, **25**, 59–73.

[32] Philip, A. and Thomas, P.Y. (2017). On concomitants of order statistics and its application in defining ranked set sampling from Farlie–Gumbel–Morgenstern bivariate Lomax distribution, *JIRSS*, **16**(2), 67–95.

[33] Scaria, J. (2003). *Concomitants of order statistics from Morgenstern family*, Ph.D. Thesis, Cochin University of Science and Technology, Cochin − 682 022, India.

[34] Scaria, J. and Thomas, B. (2014). Second order concomitants from the Morgenstern family of distributions, *J. App. Statist. Science*, **21**, 63–76.

[35] Sheikhi, A. and Tata, M. (2013). The exact joint distribution of concomitants of order statistics and their order statistics under normality, *REVSTAT*, **11**(2), 121–134.

[36] Tahmasebi, S. and Behboodian, J. (2012). Shannon information for concomitants of generalized order statistics in Farlie–Gumbel–Morgenstern (FGM) family, *Bull. Malays. Math. Sci. Soc.*, **35**(4), 975–981.

[37] Tahmasebi, S. and Jafari, A.A. (2014). Estimators for the parameter mean of Morgenstern type bivariate generalized exponential distribution using ranked set sampling, *SORT*, **38**(2), 161–180.

[38] Tahmasebi, S. and Jafari, A.A. (2015). Concomitants of order statistics and record values from Morgenstern type bivariate-generalized exponential distribution, *Bull. Malays. Math. Sci. Soc.*, **38**, 1411–1423.

[39] Tahmasebi, S.; Jafari, A.A. and Afshari, M. (2015). Concomitants of dual generalized order statistics from Morgenstern type bivariate generalized exponential distribution, *J. Statist. Theory and Appl.*, **14**(1), 1–12.

[40] Tahmasebi, S.; Jafari, A.A. and Ahsanullah, M. (2016). Properties on concomitants of generalized order statistics from a bivariate Rayleigh distribution, *Bull. Malays. Math. Sci. Soc.*.

[41] Thomas, P.Y. and Veena, T.G. (2011). On an application of concomitants of order statistics in characterizing a family of bivariate distributions, *Comm. in Statist. – Theory and Meth.*, **40**(1), 1445–1452.

[42] Veena, T.G. and Thomas, P.Y. (2015). Application of concomitants of order statistics of independent non-identically distributed random variables in estimation, *Comm. in Statist. – Theory and Meth.*, **44**, 2530–2545.

[43] Veena, T.G. and Thomas, P.Y. (2017). Role of concomitants of order statistics in determining of parent bivariate distributions, *Comm. in Statist. – Theory and Meth.*, **46**(16), 7976–7997.