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ON THE ESTIMATION FOR COMPOUND POISSON INARCH PROCESSES

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Abstract:

- Considering the wide class of discrete Compound Poisson INARCH models, introduced in [6], the main goal of this paper is to develop and compare parametric estimation procedures for first-order models, applicable without specifying the conditional distribution of the process. Therefore, two-step estimation procedures, combining either the conditional least squares (CLS) or the Poisson quasi-maximum likelihood (PQML) methods with that of the moment's estimation, are introduced and discussed. Specifying the process conditional distribution, we develop also within this class of models the conditional maximum likelihood (CML) methodology. A simulation study illustrates, particularly, the competitive performance of the two-step approaches regarding the more classical CML one which requires the conditional distribution knowledge. A final real-data example shows the relevance of this wide class of models, as it will be clear the better performance in the data fitting of some new models emerging in such class.

Keywords:

- *integer-valued time series; CP-INGARCH processes; estimation.*

AMS Subject Classification:

- 62M10, 60G12, 62F10.

1. INTRODUCTION

The family of discrete compound Poisson distributions, which includes as particular cases the Poisson, the Neyman type-A or the geometric Poisson laws, was recently used to define a new class of integer-valued GARCH models, the compound Poisson INGARCH ones [6], specified through the characteristic function of the conditional law of the process given its past. Namely, $X = (X_t, t \in \mathbb{Z})$ follows a CP-INGARCH process if the characteristic function of X_t conditioned on $\underline{X}_{t-1}$ is such that

$$\begin{cases} \Phi_{X_t|\underline{X}_{t-1}}(u) = \exp\left\{ i \dfrac{\lambda_t}{\varphi_t'(0)} \left[\varphi_t(u) - 1\right] \right\}, \quad u \in \mathbb{R}, \\ E(X_t|\underline{X}_{t-1}) = \lambda_t = \alpha_0 + \displaystyle\sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \end{cases}$$

where $\alpha_0 > 0$, $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q \geq 0$, $\underline{X}_{t-1}$ represents the σ-field generated by $\{X_{t-s}, s \geq 1\}$ and $(\varphi_t, t \in \mathbb{Z})$ is a family of characteristic functions on $\mathbb{R}$, $\underline{X}_{t-1}$-measurables, associated to a family of discrete laws with support in $\mathbb{N}_0$ and finite mean. If $\beta_k = 0$, $k = 1, ..., q$, the CP-INGARCH(p, q) model is simply denoted CP-INARCH(p). The functional form of the conditional characteristic function $\Phi_{X_t|\underline{X}_{t-1}}$ allows a wide flexibility of the class of CP-INGARCH models. In fact, as it is assumed that the family of discrete characteristic functions $(\varphi_t, t \in \mathbb{Z})$ is $\underline{X}_{t-1}$-measurable it means that its elements may be random functions or deterministic ones. Thus, this general model unifies and enlarges substantially the family of conditionally heteroscedastic integer-valued processes. In fact, it is possible to present new specific models with conditional distributions with interest in practical applications as, for instance, the geometric Poisson INGARCH ([6]) or the Neyman type-A INGARCH ([5]) ones, and also recover recent contributions such as the Poisson INGARCH ([4]), the generalized Poisson INGARCH ([15]), the negative binomial INGARCH ([14]) and the negative binomial DINARCH ([13]) processes (corresponding to random or deterministic functions φ_t, respectively). In addition to having the ability to describe different distributional behaviors and consequently different kinds of conditional heteroscedasticity, the CP-INGARCH model is able to incorporate simultaneously the overdispersion characteristic that has been recorded in real count data.

In this paper, we focus on the case where φ_t is deterministic and constant in time which still includes many of the particular cases referred above. For that reason, from now on we will refer these functions simply as φ. In this subclass of models, there exists a strictly stationary and ergodic solution with finite first and second order moments under $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$ ([6]). For $p = q = 1$, Gonçalves, Mendes-Lopes and Silva [7] stated that this simple coefficient condition is also necessary and sufficient to establish the existence of all the moments of X_t.

In this class of models we have, additionally to the usual estimation of the parameters of the conditional mean, the estimation of φ. We observe that a related problem with the knowledge of φ has been discussed in [12] in which a testing methodology was proposed to distinguish between a simple Poisson INARCH model ($\varphi(u) = \exp(iu)$) and a true CP-INARCH one ($\varphi(u) \neq \exp(iu)$). In order to analyse φ, in this paper we propose a two-step estimation procedure that lead us to its consistent estimation after estimating the conditional mean parameters.

The remainder of the paper proceeds as follows. In Section 2 we consider the subclass of CP-INARCH models of order one, with $\varphi_t = \varphi$ deterministic, and deduce its moments, central moments and cumulants up to the order 4. These results are particularly important in Section 3, devoted to estimation procedures, to deduce explicit expressions for the asymptotic distribution of the Conditional Least Squares (CLS) estimators of the conditional mean parameters, α_0 and α_1. In a second step, the method of moments is used to estimate the additional parameter associated to the function φ. Another two-step estimation procedure, combining the Poisson Quasi Maximum Likelihood (PQML) and the moment methods, is also proposed in this section, followed by the Conditional Maximum Likelihood (CML) estimation for the NTA-INARCH(1) and GEOMP2-INARCH(1) models. Section 4 presents some simulation studies that illustrate and compare the performance of these three methodologies of estimation. In Section 5 an integer-valued time series related to the prices of electricity in Portugal and Spain between July 2016 and June 2017 is considered. The data is fitted by several CP-INARCH(1) models estimated by the three estimation approaches considered and the quality of the fitting is discussed using for the CML method, in particular, the values of the log likelihood function, Akaike and Bayesian information criteria. Detailed calculations are included in the Appendices.

2. THE CP-INARCH(1) PROCESS

Let us consider now the subclass of CP-INARCH(1) models. Supposing $\varphi_t = \varphi$ constant in time and deterministic we recall that $\alpha_1 < 1$ is a necessary and sufficient condition to assure the existence of a strictly stationary and ergodic solution of the model. Moreover the process has moments of all the orders.

Setting $X = (X_t, t \in \mathbb{Z})$ a CP-INARCH(1) process we derive in the following closed-form expressions for the joint (central) moments and cumulants of the CP-INARCH(1) up to order 4. In fact, setting the notations below (used, for instance, by Weiß in [10]),

$$f_k = \frac{\alpha_0}{\prod_{j=1}^{k}(1-\alpha_1^j)}, \qquad k \in \mathbb{N},$$

$$\begin{aligned} \mu(s_1,...,s_{r-1}) &= E\big(X_t X_{t+s_1} \cdots X_{t+s_{r-1}}\big), \\ \widetilde{\mu}(s_1,...,s_{r-1}) &= E\Big((X_t-\mu)(X_{t+s_1}-\mu)\cdots(X_{t+s_{r-1}}-\mu)\Big), \\ \kappa(s_1,...,s_{r-1}) &= \mathrm{Cum}\big[X_t, X_{t+s_1},...,X_{t+s_{r-1}}\big], \end{aligned} \tag{2.1}$$

with $r = 2,3,4$ and $0 \le s_1 \le \cdots \le s_{r-1}$, and

$$v_0 = -i\,\frac{\varphi''(0)}{\varphi'(0)}, \qquad d_0 = -\frac{\varphi'''(0)}{\varphi'(0)}, \qquad c_0 = i\,\frac{\varphi^{(\mathrm{IV})}(0)}{\varphi'(0)},$$

we establish the following results whose proofs may be found in Appendices A and B, respectively.

Theorem 2.1 (Moments of a CP-INARCH(**1**) process). *We have:*

(**a**) *For any* $k \geq 0$, $\mu(k) = f_2\big(v_0\,\alpha_1^k + \alpha_0(1+\alpha_1)\big)$.

(**b**) *For any* $l \geq k \geq 0$,

$$\mu(k,l) \;=\; \Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\,\alpha_1^2)\Big] f_3\,\alpha_1^{l+k} + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1}\,f_2\,\alpha_1^l + v_0\,f_1\,f_2\,\alpha_1^{l-k} + f_1\,\mu(k).$$

(**c**) *For any* $m \geq l \geq k \geq 0$,

$$\begin{aligned}\mu(k,l,m) \;=\; \alpha_1^{m-l}\bigg[&\Big\{(c_0-4v_0d_0+3v_0^3)+3v_0(v_0^2-d_0)\alpha_1+(3v_0d_0-c_0)\alpha_1^2\\ &+(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\Big\}f_4\alpha_1^{2l+k}\\ &+\frac{2v_0+\alpha_0}{1-\alpha_1}f_3\Big[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\Big]\alpha_1^{2l}\\ &+\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\Big[2v_0\alpha_0+d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\Big]\alpha_1^{2l-k}\\ &+\frac{\alpha_0 f_3}{1-\alpha_1}\Big\{d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\Big\}\alpha_1^{2(l-k)}+\frac{v_0+\alpha_0}{1-\alpha_1}\mu(k,l)\\ &-f_2\mu(k)\Big[\alpha_0+(v_0+\alpha_0)\alpha_1\Big]\bigg]+f_1\mu(k,l).\end{aligned}$$

Corollary 2.1 (Central Moments and Cumulants of a CP-INARCH(**1**) process). *We have:*

(**a**) *For any* $s \geq 0$, $\widetilde{\mu}(s) = \kappa(s) = v_0\alpha_1^s f_2$.

(**b**) *For any* $l \geq s \geq 0$, *we have*

$$\widetilde{\mu}(s,l) \;=\; \kappa(s,l) \;=\; f_3\alpha_1^l\bigg[v_0^2(1+\alpha_1+\alpha_1^2)-\Big\{v_0^2(1+\alpha_1-2\alpha_1^2)-d_0(1-\alpha_1^2)\Big\}\alpha_1^s\bigg].$$

(**c**) *For any* $m \geq l \geq s \geq 0$,

$$\begin{aligned}\kappa(s,l,m) \;=\; \alpha_1^m f_4\bigg[&\Big\{c_0+3v_0^3-4v_0d_0+3v_0(v_0^2-d_0)\alpha_1+(3\alpha_0d_0-c_0)\alpha_1^2\\ &+(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\Big\}\alpha_1^{l+s}\\ &+v_0(1+\alpha_1+\alpha_1^2+\alpha_1^3)\Big[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\Big](2\alpha_1^l+\alpha_1^s)\\ &+v_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)\Big[(1+\alpha_1)v_0^2+\big(d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\big)\alpha_1^{l-s}\Big]\bigg],\end{aligned}$$

$$\widetilde{\mu}(s,l,m) \;=\; \kappa(s,l,m)+v_0^2f_2^2(\alpha_1^{m-l+s}+2\alpha_1^{m+l-s}).$$

From Theorem 2.1 we deduce, for instance,

$$E(X_t^2) \;=\; \mu(0) \;=\; \frac{\alpha_0\big(v_0+\alpha_0(1+\alpha_1)\big)}{(1-\alpha_1)(1-\alpha_1^2)}, \tag{2.2}$$

$$E(X_t^3) \;=\; \mu(0,0) \;=\; \frac{\alpha_0}{(1-\alpha_1)^3}\left[\frac{d_0+(3v_0^2-d_0)\alpha_1^2}{(1+\alpha_1)(1+\alpha_1+\alpha_1^2)}+\frac{3v_0\alpha_0}{1+\alpha_1}+\alpha_0^2\right].$$

These results generalize those of Weiß [10] for the INARCH(1) model and the two last equalities are important to deduce explicit expressions for the asymptotic distribution of the CLS estimators of the parameters α_0 and α_1 provided in the next section. As we will take in our study some important particular cases concerning the process conditional law, we conclude this section recalling such cases and deducing the corresponding values of v_0, d_0 and c_0, previously introduced.

a) The INARCH(1) model ([4]) corresponds to a CP-INARCH model considering φ the characteristic function of the Dirac's law concentrated in $\{1\}$, that is, with a Poisson conditional distribution; we denote it by Poisson-INARCH(1) model. So, we deduce that $v_0 = d_0 = c_0 = 1$.

b) When φ is the characteristic function of the Poisson distribution with mean $\phi > 0$, $X_t|\underline{X}_{t-1}$ follows a Neyman type-A law with parameter $(\lambda_t/\phi, \phi)$, and we have the NTA-INARCH(1) model introduced in [5]. For this case, $v_0 = 1+\phi$, $d_0 = 1+3\phi+\phi^2$ and $c_0 = 1+7\phi+6\phi^2+\phi^3$.

c) Considering in the above expressions $v_0 = (2-p^*)/p^*$, $d_0 = \big(6-6p^*+(p^*)^2\big)/(p^*)^2$ and $c_0 = \big((2-p^*)\left(12-12p^*+(p^*)^2\right)\big)/(p^*)^3$, we obtain the expressions for the GEOMP2-INARCH(1) model ([6]). In fact, this process is defined considering φ the characteristic function of the geometric distribution with parameter $p^* \in]0,1[$ and $X_t|\underline{X}_{t-1}$ following a geometric Poisson $(p^*\lambda_t, p^*)$ law.

d) Another particular case of the CP-INARCH model is the NB2-INARCH (that is identical to the NB-DINARCH model proposed by Xu *et al.*, [13]), where $X_t|\underline{X}_{t-1}$ follows a negative binomial distribution with parameter $\big(\lambda_t/(\beta-1), 1/\beta\big)$ and $\beta>0$. This process is stated when φ is the characteristic function of the logarithmic distribution with parameter $(\beta-1)/\beta$ and then we deduce $v_0=\beta$, $d_0 = 2\beta^2-\beta$ and $c_0 = 6\beta^2(\beta-1)+\beta$.

e) When φ is the characteristic function of the Borel law with parameter $\kappa \in]0,1[$, $X_t|\underline{X}_{t-1}$ follows a generalized Poisson distribution with parameter $\big((1-\kappa)\lambda_t, \kappa\big)$ and we recover the GP-INARCH model ([15]). So, $v_0 = (1-\kappa)^{-2}$, $d_0 = (2\kappa+1)(1-\kappa)^{-4}$ and $c_0 = (6\kappa^2+8\kappa+1)(1-\kappa)^{-6}$.

3. ESTIMATION PROCEDURES

In this section, we focus on the estimation of the vector $\theta = (\alpha_0, \alpha_1, v_0)^\top$, where v_0 includes the additional parameter associated to the conditional distribution of the CP-INARCH(1) model (for example, $v_0 = 1+\phi$ in the NTA-INARCH(1) model and $v_0 = (2-p^*)/p^*$ in the GEOMP2-INARCH(1)). To estimate the true value of θ, we start by discussing a two-step approach using the conditional least squares and moment estimation methods; after we consider the combination of the Poisson Quasi-Maximum Likelihood and moments estimation methods and finally develop the conditional maximum likelihood estimation. For this purpose, let $(x_1, ..., x_n)$ be n particular values, arbitrarily fixed, of the process X.

3.1. Two-step estimation procedures

3.1.1. Conditional Least Squares and Moments estimation methods

In the first step, we discuss the conditional least squares (CLS) approach for the estimation of the conditional mean parameters α_0 and α_1 and, for parameter v_0 associated to the CP-INARCH(1) conditional distribution, an approach based on the moment estimation method is developed.

The CLS estimator of $\alpha = (\alpha_0, \alpha_1)$ is obtained by minimizing the sum of squares

$$Q_n(\alpha) = \sum_{t=2}^{n} \Big[x_t - E\big(X_t \,|\, X_{t-1} = x_{t-1}\big)\Big]^2 = \sum_{t=2}^{n} \Big[x_t - \alpha_0 - \alpha_1 x_{t-1}\Big]^2,$$

with respect to α. Solving the least squares equations

$$\begin{cases} \dfrac{\partial Q_n(\alpha)}{\partial \alpha_0} = -2 \displaystyle\sum_{t=2}^{n} (x_t - \alpha_0 - \alpha_1 x_{t-1}) = 0, \\ \dfrac{\partial Q_n(\alpha)}{\partial \alpha_1} = -2 \displaystyle\sum_{t=2}^{n} x_{t-1} (x_t - \alpha_0 - \alpha_1 x_{t-1}) = 0, \end{cases}$$

we obtain the following explicit expressions for the CLS estimator $\widehat{\alpha}_n = (\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$:

$$\widehat{\alpha}_{1,n} = \frac{\sum_{t=2}^{n} X_t X_{t-1} - \frac{1}{n-1} \cdot \sum_{t=2}^{n} X_t \cdot \sum_{s=2}^{n} X_{s-1}}{\sum_{t=2}^{n} X_{t-1}^2 - \frac{1}{n-1} \big(\sum_{t=2}^{n} X_{t-1}\big)^2},$$

$$\widehat{\alpha}_{0,n} = \frac{\sum_{t=2}^{n} X_t - \widehat{\alpha}_{1,n} \sum_{t=2}^{n} X_{t-1}}{n-1}. \tag{3.1}$$

The consistency and the asymptotic distribution of these estimators are stated in the next theorem. This theorem generalizes the results obtained in [11], Section 4.2, where the CLS estimators of α_0 and α_1 are obtained and studied in the particular case of a Poisson-INARCH model.

Theorem 3.1. *Let $\widehat{\alpha}_n = (\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$ be the CLS estimator of $\alpha = (\alpha_0, \alpha_1)$ given in (3.1). Then $\widehat{\alpha}_n$ converges almost surely to α and*

$$\sqrt{n}\,(\widehat{\alpha}_n - \alpha) \xrightarrow{d} N\big(\mathbf{0}_{2\times 1}, \mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}\big),$$

as $n \to \infty$, where the entries of the matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1} = (b_{ij})$, $i, j = 1, 2$, are given by

$$b_{11} = \frac{\alpha_0}{1-\alpha_1} \left(\alpha_0(1+\alpha_1) + \frac{v_0^2 + (d_0 - v_0^2)\,\alpha_1(1+\alpha_1-\alpha_1^2) + (3v_0^2 - d_0)\,\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)} \right),$$

$$b_{12} = b_{21} = v_0\alpha_1 - \alpha_0(1+\alpha_1) - \frac{\alpha_1(1+\alpha_1)\,\big(d_0 + (3v_0^2 - d_0)\,\alpha_1^2\big)}{v_0(1+\alpha_1+\alpha_1^2)},$$

$$b_{22} = (1-\alpha_1^2) \left(1 + \frac{\alpha_1\big(d_0 + (3v_0^2 - d_0)\,\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)} \right),$$

and $\xrightarrow{d}$ means convergence in distribution.

Proof: The results announced are proved using those of Klimko and Nelson [9, Section 3]. In fact, it is easily checked that the regularity conditions (i) to (iii) defined on [9, p. 634] are satisfied taking into account that $g(\alpha; X_{t-1}) = E\left(X_t | \underline{X}_{t-1}\right) = \alpha_0 + \alpha_1 X_{t-1}$, and thus, by their Theorem 3.1, it follows that the CLS estimators are strongly consistent. Furthermore, the matrix $\mathbf{V}$ is invertible as it is given by

$$\mathbf{V} = \begin{bmatrix} E\left(\frac{\partial g}{\partial \alpha_0}\frac{\partial g}{\partial \alpha_0}\right) & E\left(\frac{\partial g}{\partial \alpha_0}\frac{\partial g}{\partial \alpha_1}\right) \\ E\left(\frac{\partial g}{\partial \alpha_1}\frac{\partial g}{\partial \alpha_0}\right) & E\left(\frac{\partial g}{\partial \alpha_1}\frac{\partial g}{\partial \alpha_1}\right) \end{bmatrix} = \begin{bmatrix} E(1) & E(X_{t-1}) \\ E(X_{t-1}) & E(X_{t-1}^2) \end{bmatrix} = \begin{bmatrix} 1 & \frac{\alpha_0}{1-\alpha_1} \\ \frac{\alpha_0}{1-\alpha_1} & \frac{\alpha_0\left(v_0 + \alpha_0(1+\alpha_1)\right)}{(1-\alpha_1)\,(1-\alpha_1^2)} \end{bmatrix},$$

considering the expressions stated in Theorem 2.1. Thus, Theorem 3.2 of [9] is satisfied implying the asymptotic normality of the CLS estimators. The entries of the covariance matrix of the asymptotic distribution $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$ are derived in Appendix C. □

To estimate the parameter v_0 we propose to use the moments estimation method. Taking into consideration the expression (2.2) of the second order moment of the CP-INARCH(1) model, an estimator for v_0, whose strong consistence is a consequence from the strict stationarity and ergodicity of the process X, is given by solving the equation

$$\frac{\widehat{\alpha}_{0,n}\left(v_0 + \widehat{\alpha}_{0,n}(1+\widehat{\alpha}_{1,n})\right)}{(1-\widehat{\alpha}_{1,n})\,(1-\widehat{\alpha}_{1,n}^2)} = \frac{1}{n}\sum_{t=1}^{n} X_t^2$$

in order to v_0. In this way we get the two-step CLS+M estimator for $(\alpha_0, \alpha_1, v_0)$.

We note that the estimation of v_0 doesn't involve the knowledge of the conditional law, as it is totally determined by the estimators of α_0 and α_1 and the empirical second order moment.

3.1.2. Poisson Quasi-Maximum Likelihood and Moments estimation methods

One of the advantages of using the above CLS+M approach is the fact that we do not need to specify entirely the conditional distribution of the CP-INARCH(1) model to estimate its parameters. We refer now another two-step approach where it is used the Poisson quasi-conditional maximum likelihood estimator (PQMLE) to estimate the conditional mean parameters α_0 and α_1 and, as previously, the moment estimation method for parameter v_0. The resulting estimator is denoted PQML+M.

The PQMLE provides a general approach for estimating the conditional mean parameters of the CP-INARCH(1) models by maximizing a pseudo-likelihood function considering the conditional distribution the Poisson one, that is, the function

$$\widetilde{L}_n(\theta|\mathbf{x}) = \sum_{t=2}^{n}\Big(x_t\log(\lambda_t) - \lambda_t - \log(x_t!)\Big).$$

Ahmad and Francq [1] found some regularity conditions to establish the consistency and asymptotic normality of the Poisson quasi-maximum likelihood estimator of the conditional mean parameters of a count time series. These regularity conditions are easily satisfied by a CP-INARCH(1) process with $\alpha_1 < 1$, and so the PQML estimator of (α_0, α_1) is consistent and asymptotically Gaussian. The almost sure convergence of the v_0 estimator follows as previously.

3.2. Conditional Maximum Likelihood Estimation

When the distribution of $X_t|\underline{X}_{t-1}$ is known, we can estimate its parameters using the conditional maximum likelihood estimation (CMLE) method. In this section, we discuss this procedure by considering NTA-INARCH(1) and GEOMP2-INARCH(1) models, as developed in [11], Section 4.1, for a Poisson-INARCH(1) model .

Starting by a NTA-INARCH(1) process, we have the conditional probability mass function of X_t ([8]) given by

$$P\big[X_t = x_t \,|\, \underline{X}_{t-1}\big] \;=\; \frac{e^{-\frac{\lambda_t}{\phi}}\,\phi^{x_t}}{x_t!}\, Z(\lambda_t, x_t, \phi), \qquad Z(\lambda_t, X_t, \phi) \;=\; \sum_{j=0}^{\infty} \frac{\left(\lambda_t\, e^{-\phi}\right)^j j^{X_t}}{\phi^j\, j!},$$

for $x_t = 0, 1, ...$. The conditional likelihood function is then

$$L_n(\theta|\mathbf{x}) \;=\; \prod_{t=2}^{n} \frac{e^{-\frac{\lambda_t}{\phi}}\,\phi^{x_t}}{x_t!}\, Z(\lambda_t, x_t, \phi),$$

where for convenience $\theta = (\alpha_0, \alpha_1, \phi)$ as $v_0 = 1+\phi$. So the log-likelihood function has the form

$$\log L_n(\theta|\mathbf{x}) \;=\; \sum_{t=2}^{n} l_t(\theta) \;=\; \sum_{t=2}^{n} \left\{ -\frac{\lambda_t}{\phi} + x_t \log(\phi) - \log(x_t!) + \log\big(Z(\lambda_t, x_t, \phi)\big) \right\}.$$

The first derivatives of l_t are given as

$$\frac{\partial l_t(\theta)}{\partial \phi} \;=\; \frac{\lambda_t}{\phi^2} + \frac{x_t}{\phi} - \left(\frac{\phi+1}{\phi}\right) \frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)},$$

$$\frac{\partial l_t(\theta)}{\partial \alpha_j} \;=\; \left[-\frac{1}{\phi} + \frac{1}{\lambda_t}\,\frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)}\right] \frac{\partial \lambda_t}{\partial \alpha_j}, \qquad j = 0, 1,$$

and the second derivatives of l_t are

$$\frac{\partial^2 l_t(\theta)}{\partial \phi^2} \;=\; -\frac{2\lambda_t}{\phi^3} - \frac{x_t}{\phi^2} + \frac{Z(\lambda_t, x_t+1, \phi)}{\phi^2 Z(\lambda_t, x_t, \phi)} + \left(\frac{\phi+1}{\phi}\right)^2 \left[\frac{Z(\lambda_t, x_t+2, \phi)}{Z(\lambda_t, x_t, \phi)} - \frac{Z^2(\lambda_t, x_t+1, \phi)}{Z^2(\lambda_t, x_t, \phi)}\right],$$

$$\frac{\partial^2 l_t(\theta)}{\partial \phi \partial \alpha_j} \;=\; \left[\frac{1}{\phi^2} - \frac{\phi+1}{\phi\lambda_t}\left\{\frac{Z(\lambda_t, x_t+2, \phi)}{Z(\lambda_t, x_t, \phi)} - \frac{Z^2(\lambda_t, x_t+1, \phi)}{Z^2(\lambda_t, x_t, \phi)}\right\}\right] \frac{\partial \lambda_t}{\partial \alpha_j},$$

$$\frac{\partial^2 l_t(\theta)}{\partial \alpha_j \partial \alpha_k} \;=\; \frac{1}{\lambda_t^2}\left[-\frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)} + \frac{Z(\lambda_t, x_t+2, \phi)}{Z(\lambda_t, x_t, \phi)} - \frac{Z^2(\lambda_t, x_t+1, \phi)}{Z^2(\lambda_t, x_t, \phi)}\right] \frac{\partial \lambda_t}{\partial \alpha_j}\,\frac{\partial \lambda_t}{\partial \alpha_k},$$

for $0 \le j, k \le 1$, where the expressions for $\partial\lambda_t/\partial\alpha_j$ and $\partial^2\lambda_t/\partial\alpha_j\partial\alpha_k$ are easily deduced.

Analogously, for the GEOMP2-INARCH(1) process we obtain the following expression:

$$\begin{aligned}\log L_n(\theta|\mathbf{x}) &= \sum_{t=2}^{n} l_t(\theta)\\ &= \sum_{t=2}^{n}\left\{-\lambda_t+\log\left(1_{x_t=0}+\left[\sum_{n=1}^{x_t}\frac{\lambda_t^n}{n!}\binom{x_t-1}{n-1}(p^*)^n(1-p^*)^{x_t-n}\right]1_{x_t\neq 0}\right)\right\},\end{aligned}$$

where $\theta=(\alpha_0,\alpha_1,p^*)$, as $v_0=(2-p^*)/p^*$ and taking into consideration that the conditional probability mass function of X_t is given by

$$\begin{aligned}P\big[X_t=0\,|\,\underline{X}_{t-1}\big] &= e^{-\lambda_t},\\ P\big[X_t=x_t\,|\,\underline{X}_{t-1}\big] &= \sum_{n=1}^{x_t} e^{-\lambda_t}\frac{\lambda_t^n}{n!}\binom{x_t-1}{n-1}(p^*)^n(1-p^*)^{x_t-n}, \qquad x_t=1,2,...\end{aligned}$$

Similarly to the previous case, the first and second derivatives of l_t in order to α_0, α_1 and p^* are deduced.

4. A SIMULATION STUDY

Some simulation studies are now developed to examine and compare the performance of the different estimators considered in Section 3 for the model parameters. We begin by illustrating the two-step approach based on CLS and moments estimation methods by computing the estimates and analyzing its performance. In the sequel, the several estimation procedures are discussed and compared. The study is developed considering the NTA-INARCH(1) and the GEOMP2-INARCH(1) models. So, after estimating α_0, α_1 and v_0, we deduce the estimator of ϕ, in the first case, given by

$$\widehat{\phi}_n = -1-\widehat{\alpha}_{0,n}(1+\widehat{\alpha}_{1,n})+\frac{(1-\widehat{\alpha}_{1,n})\,(1-\widehat{\alpha}_{1,n}^2)}{n\,\widehat{\alpha}_{0,n}}\sum_{t=1}^{n}X_t^2,$$

and, in the second one, that of p^* namely

$$\widehat{p^*_n} = 2\left[1-\widehat{\alpha}_{0,n}(1+\widehat{\alpha}_{1,n})+\frac{(1-\widehat{\alpha}_{1,n})\,(1-\widehat{\alpha}_{1,n}^2)}{n\,\widehat{\alpha}_{0,n}}\sum_{t=1}^{n}X_t^2\right]^{-1}.$$

4.1. CLS estimators performance

4.1.1. NTA-INARCH(1) model

To illustrate the CLS method, we focus on the NTA-INARCH(1) model with true parameters $\alpha_0=2$, $\alpha_1=0.2$ and $\phi=2$ and, for different sample sizes $n=100$, 250, 500, 750, 1 000, we present in Table 1 the expected values, variances and covariance of $\widehat{\alpha}_{0,n}$, $\widehat{\alpha}_{1,n}$ and $\widehat{\phi}_n$, considering 10 000 replications. In the last column of this table we present the true values of α_0, α_1 and ϕ, as well as the entries of the asymptotic matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$, respectively b_{11}, b_{22} and b_{12}, given in Theorem 3.1. We verify that the asymptotic and the sample values are quite similar for large values of n.

Table 1: Means, variances and covariances for the CLS+M estimates of the NTA-INARCH(1) model with coefficients $\alpha_0 = 2$, $\alpha_1 = 0.2$, $\phi = 2$ and for different sample sizes n.

n	100	250	500	750	1 000	
$E_{\text{est}}(\widehat{\alpha}_0)$	2.0444	2.0161	2.0090	2.0090	2.0041	2
$E_{\text{est}}(\widehat{\alpha}_1)$	0.1797	0.1918	0.1956	0.1973	0.1981	0.2
$E_{\text{est}}(\widehat{\phi})$	1.9238	1.9670	1.9842	1.9899	1.9929	2
$n \cdot V_{\text{est}}(\widehat{\alpha}_0)$	12.2393	12.3125	12.3782	12.3133	12.3133	12.3774
$n \cdot V_{\text{est}}(\widehat{\alpha}_1)$	1.1793	1.1957	1.2227	1.2594	1.2776	1.2604
$n \cdot V_{\text{est}}(\widehat{\phi})$	21.9663	21.7000	21.3637	22.2183	22.1552	
$n \cdot \text{Cov}_{\text{est}}(\widehat{\alpha}_0, \widehat{\alpha}_1)$	−2.3311	−2.4081	−2.4814	−2.5270	−2.5911	−2.5510

Figure 1 displays a multiple boxplot for samples of length $n = 250$, 750 and 2 000 of the CLS estimator of α_0 and α_1 based on 10 000 model replications as well as the histogram of the corresponding standardized values, for $n = 2\,000$, of a NTA-INARCH(1) model with $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$. These multiple boxplots show a significant stability and allow to infer a high rate of convergence to the limit distribution. In agreement with Theorem 3.1, the plots indicate the adequacy of the normal for the empirical marginal distributions of the estimators $\widehat{\alpha_0}$, $\widehat{\alpha_1}$. Let us observe that the Kolmogorov–Smirnov test for the sampling laws of the standardized CLS estimation gives large p-values for testing the standard normal distribution as, for instance, when we consider $n = 2\,000$ and 1 000 replications we obtain 0.9454 and 0.4051.

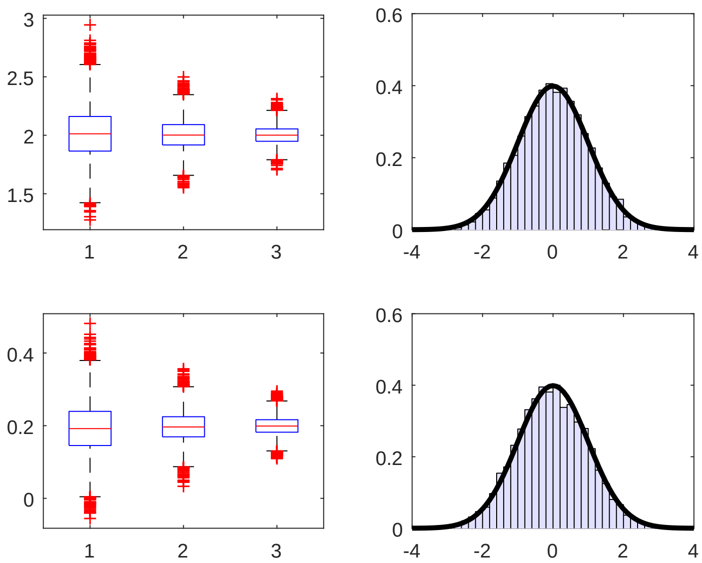

Figure 1: Boxplots for $n = 250$, 750, 2 000 (from left to right) and histogram for $n = 2\,000$ of the empirical law of $\widehat{\alpha}_0$ (on top) and $\widehat{\alpha}_1$ (below) for a NTA-INARCH(1) process with $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$. Superimposed is the standard normal density function. The results are based on 10 000 replications.

In Figure 2 we present now a multiple boxplot and the histogram of the distribution of $\sqrt{n}\,(\widehat{\phi}_n - \phi)$. Figure 3 shows the similarity between the empirical cumulative distribution function of $\sqrt{n}\,(\widehat{\phi}_n - \phi)$ (represented in solid line) and the cumulative distribution function of the normal(0, 4.7) law (in dashed line), whose parameters are the sample mean and variance of $\sqrt{n}\,(\widehat{\phi}_n - \phi)$. The stability previously observed appears also here and, once again, the p-value of the Kolmogorov–Smirnov test, namely 0.8231 when $n = 2\,000$ and for 1 000 replications, indicates the adequacy of the normal for the empirical distribution of $\sqrt{n}\,(\widehat{\phi}_n - \phi)$.

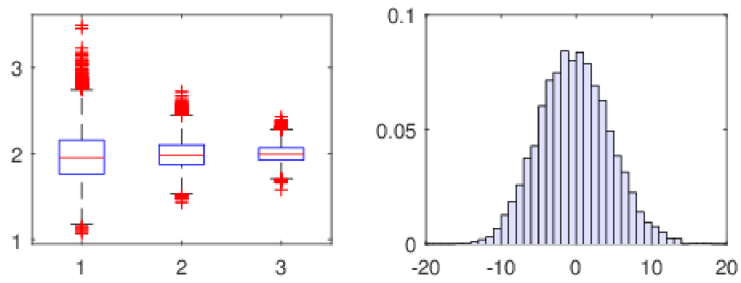

Figure 2: Boxplots for $n = 250$, 750, 2 000 (from left to right) and histogram for $n = 2\,000$ of the empirical law of $\sqrt{n}\,(\widehat{\phi}_n - \phi)$ when $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$ for a NTA-INARCH(1).

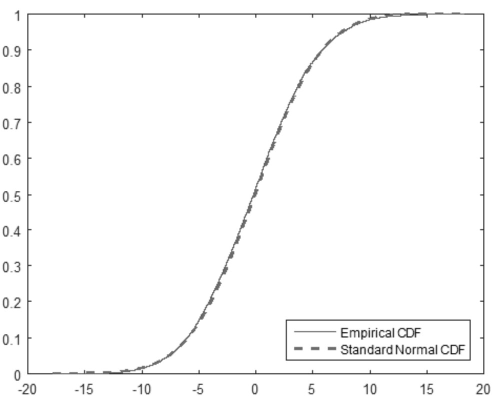

Figure 3: Empirical CDF of the law of $\sqrt{n}\,(\widehat{\phi}_n - \phi)$ when $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$ for a NTA-INARCH(1) (in solid line) and the CDF of the normal(0, 4.7) law (in dashed line), for $n = 2\,000$.

From the empirical results presented in the two last lines of Table 2, we can presume that the estimators of α_0 (resp., α_1) and ϕ are asymptotically uncorrelated. In fact, for the NTA-INARCH(1) model in study, the empirical correlations $\rho_{\text{est}}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$ and $\rho_{\text{est}}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$ are significantly low. To support this statement we use the Monte Carlo method to determine confidence intervals for the mean of $\rho_{\text{est}}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$ and for the mean of $\rho_{\text{est}}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$ which we denote by $m_{0,n,\widetilde{n}}$ and $m_{1,n,\widetilde{n}}$, respectively. The confidence intervals are obtained considering $\widetilde{n} = 35$ and $\widetilde{n} = 50$ replications of n-dimensional samples ($n = 500$ and $n = 1\,000$) of a NTA-INARCH(1) model with $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$.

Table 2: Empirical correlations for the CLS+M estimates of the NTA-INARCH(1) model with coefficients $\alpha_0 = 2$, $\alpha_1 = 0.2$, $\phi = 2$ and for different sample sizes n.

n	250	750	1 000	5 000	10 000
$\rho_{\text{est}}(\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$	−0.6276	−0.6417	−0.6385	−0.6482	−0.6402
$\rho_{\text{est}}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$	0.0883	0.0962	0.1139	0.1059	0.0911
$\rho_{\text{est}}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$	0.0272	0.0192	0.0078	0.0246	0.0438

Such intervals with confidence level 0.99 are presented in Table 3, where we stress the lower values when n or $\widetilde{n}$ increase. So we have estimated (α_0, α_1) and ϕ separately likely without loss of efficiency.

Table 3: Confidence intervals for the mean of $\rho_{\text{est}}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$ and for the mean of $\rho_{\text{est}}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$, with confidence level $\gamma = 0.99$ and for different sample sizes n and $\widetilde{n}$.

	$\widetilde{n} = 35$		$\widetilde{n} = 50$	
	$n = 500$	$n = 1\,000$	$n = 500$	$n = 1\,000$
$m_{0,n,\widetilde{n}}$	$[0.0917, 0.1180]$	$[0.0883, 0.1162]$	$[0.0940, 0.1160]$	$[0.0814, 0.1064]$
$m_{1,n,\widetilde{n}}$	$[0.0113, 0.0412]$	$[0.0165, 0.0412]$	$[0.0137, 0.0354]$	$[0.0132, 0.0397]$

4.1.2. GEOMP2-INARCH(1) model

Let us consider now the GEOMP2-INARCH(1) model with true parameters $\alpha_0 = 2$, $\alpha_1 = 0.4$ and $p^* = 0.1$. As in the previous section, for different sample sizes n, we compute the expected values, variances and covariances of $\widehat{\alpha}_{0,n}$, $\widehat{\alpha}_{1,n}$ and $\widehat{p^*_n}$ (see Table 4, where in the last column we present the true values of α_0, α_1 and p^* as well as the entries b_{11}, b_{22} and b_{12} of the asymptotic matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$) and for samples of length $n = 250$, 750 and 2 000 we plot a multiple boxplot and for $n = 2\,000$ the histograms for 10 000 values of the CLS+M estimators (in Figure 4) and similar conclusions to the previous case may be deduced.

Table 4: Expected values, variances and covariances for the CLS+M estimates of the GEOMP2-INARCH(1) model with $\alpha_0 = 2$, $\alpha_1 = 0.4$, $p^* = 0.1$ and different sample sizes n.

n	100	250	500	750	1 000	
$E_{\text{est}}(\widehat{\alpha}_0)$	2.1401	2.0705	2.0381	2.0265	2.0241	2
$E_{\text{est}}(\widehat{\alpha}_1)$	0.3267	0.3655	0.3803	0.3875	0.3900	0.4
$E_{\text{est}}(\widehat{p^*})$	0.1171	0.1068	0.1038	0.1025	0.1019	0.1
$n \cdot V_{\text{est}}(\widehat{\alpha}_0)$	54.8720	54.9255	57.1036	57.6511	58.7167	61.5325
$n \cdot V_{\text{est}}(\widehat{\alpha}_1)$	2.7975	3.2809	3.6923	3.8768	3.9021	4.3979
$n \cdot V_{\text{est}}(\widehat{p^*})$	0.2011	0.0879	0.0867	0.0884	0.0886	
$n \cdot \text{Cov}_{\text{est}}(\widehat{\alpha}_0, \widehat{\alpha}_1)$	−1.4509	−3.4576	−4.8393	−5.3491	−5.5056	−7.0598

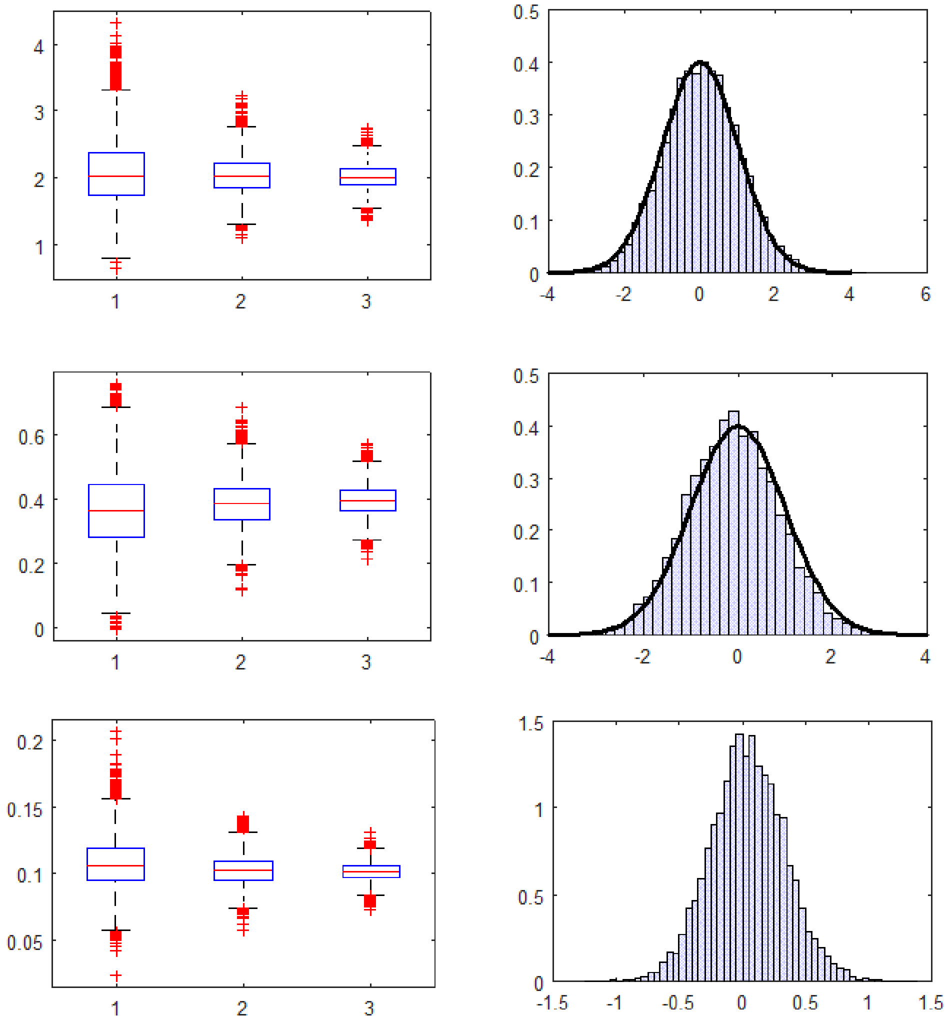

Figure 4: Boxplots for $n = 250,\ 750,\ 2\,000$ (from left to right) and histogram for $n = 2\,000$ of the empirical law of $\widehat{\alpha}_0$ (on top), $\widehat{\alpha}_1$ (in the middle) and $\widehat{p^*}$ (below) when $\alpha_0 = 2$, $\alpha_1 = 0.4$ and $p^* = 0.1$ for a GEOMP2-INARCH(1) process. Superimposed is the standard normal density function. The results are based on 10 000 replications.

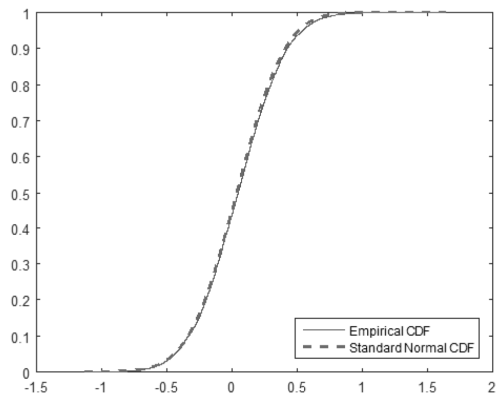

Figure 5: Empirical CDF of the law of $\sqrt{n}\left(\widehat{p^*_n} - p^*\right)$ when $\alpha_0 = 2$, $\alpha_1 = 0.4$ and $p^* = 0.1$ for a GEOMP2-INARCH(1) model (in solid line) and the CDF of the normal(0, 0.3) law (in dashed line).

To show the adequacy of the normal for the empirical distribution of $\sqrt{n}\,(\widehat{p^*}_n - p^*)$, in Figure 5 we present the empirical cumulative distribution function of $\sqrt{n}\,(\widehat{p^*}_n - p^*)$ (represented in solid line) and the cumulative distribution function of the normal(0, 0.3) law (in dashed line). Analogously to the previous study, we can also presume that the estimators of α_0, (resp., α_1) and p^* are asymptotically uncorrelated.

4.2. Comparative analysis of the estimation procedures

To examine and compare the finite sample performances of the CLS+M, PQML+M and CML methods, we consider two different NTA-INARCH(1) models with parameter values $\alpha_0 = 2$, $\alpha_1 = 0.2$, $\phi = 2$ and $\alpha_0 = 5$, $\alpha_1 = 0.3$, $\phi = 1$, and two different GEOMP2-INARCH(1) models with parameter values $\alpha_0 = 2$, $\alpha_1 = 0.2$, $p^* = 0.1$ and $\alpha_0 = 5$, $\alpha_1 = 0.3$, $p^* = 0.6$. The sample sizes considered are $n = 500$ and 1 000 and the number of replications $m = 10\,000$.

For the maximization of the log-likelihood functions, we use the MATLAB function `fmincon` where the estimates obtained using the CLS+M method were used as the initial values and the constrained conditions are $\alpha_0 > 0$, $0 < \alpha_1 < 1$, $\phi > 0$ (for the NTA) and $0 < p^* < 1$ (for the GEOMP2). The performance of the estimators is evaluated by the mean square error, i.e.,

$$\frac{1}{m}\sum_{k=1}^{m}\left(\widehat{\theta}_{j,k} - \theta_j\right)^2, \qquad j = 1, 2, 3.$$

The results of the simulation experiments are presented in Tables 5 and 6 where the smallest values of the mean square errors are highlighted in italics.

Table 5: Mean estimates (in bold) and mean square errors (within parentheses) for the NTA-INARCH(1) model with different sample sizes n.

n	Method	$\alpha_0 = 2$	$\alpha_1 = 0.2$	$\phi = 2$	$\alpha_0 = 5$	$\alpha_1 = 0.3$	$\phi = 1$
500	CLS+M	**2.0071** (0.0248)	**0.1967** (0.0025)	**1.9832** (0.0458)	**5.0288** (0.1169)	**0.2956** (0.0021)	**0.9915** (0.0180)
	PQML+M	**2.0061** (0.0239)	**0.1971** (0.0023)	**1.9831** (0.0459)	**5.0259** (0.1123)	**0.2960** *(0.0020)*	**0.9912** (0.0181)
	CML	**2.0047** *(0.0233)*	**0.1977** *(0.0022)*	**1.9937** *(0.0174)*	**5.0249** *(0.1115)*	**0.2961** *(0.0020)*	**0.9928** *(0.0141)*
1 000	CLS+M	**2.0023** (0.0124)	**0.1982** (0.0013)	**1.9906** (0.0219)	**5.0117** (0.0582)	**0.2979** *(0.0010)*	**0.9946** (0.0089)
	PQML+M	**2.0020** (0.0120)	**0.1983** (0.0012)	**1.9907** (0.0221)	**5.0103** (0.0558)	**0.2981** *(0.0010)*	**0.9945** (0.0090)
	CML	**2.0017** *(0.0116)*	**0.1985** *(0.0011)*	**1.9960** *(0.0085)*	**5.0105** *(0.0552)*	**0.2981** *(0.0010)*	**0.9948** *(0.0072)*

From this study we may conclude that the three methods seem to perform quite well, although the CML gives slightly smaller mean square errors in most cases.

Table 6: Mean estimates (in bold) and mean square errors (within parentheses) for the GEOMP2-INARCH(1) model with different sample sizes n.

n	Method	$\alpha_0 = 2$	$\alpha_1 = 0.2$	$p^* = 0.1$	$\alpha_0 = 5$	$\alpha_1 = 0.3$	$p^* = 0.6$
500	CLS+M	**2.0142** (0.0964)	**0.1898** (0.0058)	**0.1035** (0.0002)	**5.0269** (0.1219)	**0.2963** (0.0021)	**0.6033** (0.0009)
	PQML+M	**2.0070** (0.0913)	**0.1926** (0.0052)	**0.1036** (0.0002)	**5.0250** (0.1173)	**0.2966** *(0.0020)*	**0.6033** (0.0009)
	CML	**1.9967** *(0.0807)*	**0.1968** *(0.0036)*	**0.1013** *(0.0001)*	**5.0240** *(0.1141)*	**0.2967** *(0.0020)*	**0.6027** *(0.0007)*
1 000	CLS+M	**2.0072** (0.0481)	**0.1959** (0.0030)	**0.1017** (0.0001)	**5.0100** (0.0600)	**0.2985** (0.0011)	**0.6020** (0.0004)
	PQML+M	**2.0032** (0.0450)	**0.1975** (0.0026)	**0.1018** (0.0001)	**5.0084** (0.0578)	**0.2988** *(0.0010)*	**0.6020** (0.0004)
	CML	**1.9995** *(0.0397)*	**0.1989** *(0.0018)*	**0.1006** *(0.0000)*	**5.0080** *(0.0566)*	**0.2988** *(0.0010)*	**0.6016** *(0.0003)*

5. REAL DATA EXAMPLE — COUNTS OF DIFFERENCES IN THE PRICES OF ELECTRICITY IN PORTUGAL AND SPAIN

OMIE (http://www.omie.es) is the company that manages the wholesale electricity market on the Iberian Peninsula. Electricity prices in Europe are set on a daily basis (every day of the year) at 12 noon, for the twenty-four hours of the following day, known as daily market. The market splitting is the mechanism used for setting the price of electricity on the daily market. When the price of electricity is the same in Portugal and Spain, which corresponds to the desired situation, it means that the integration of the Iberian market is working properly.

In the following, we consider the time series that represents the number of hours in a day in which the prices of electricity for Portugal and Spain are different. The data presented in Figure 6 consists of 365 observations, starting from July 2016 and ending in June 2017.

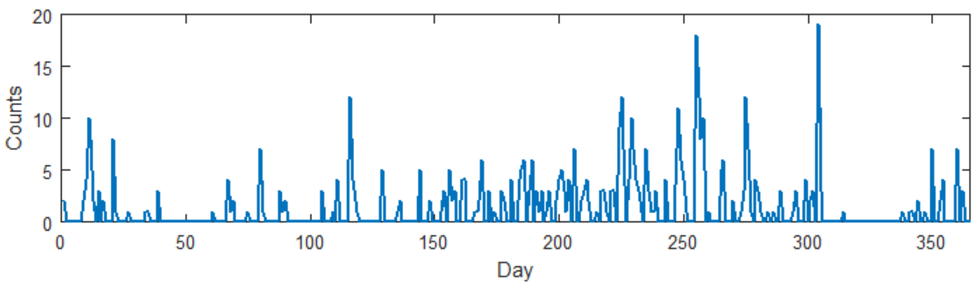

Figure 6: Daily number of hours in which the price of electricity of Portugal and Spain are different, starting from July 2016 and ending in June 2017.

Empirical mean and variance of the data are 1.4082 and 7.3027, respectively, indicating that the true marginal distribution is overdispersed. Let us observe that this time series exhibits also volatility clusters suggesting characteristics of conditional heteroscedasticity.

The partial autocorrelation function presented in Figure 7, suggests an order 1 dependence and so a CP-INARCH(1) model may be a reasonable choice to fit the data within the CP-INGARCH class. Despite the support bounding of this variable, the empirical analysis of the data set observed allows us to infer that its distributional characteristics (see histogram in Figure 7) are compatible with some compound Poisson laws.

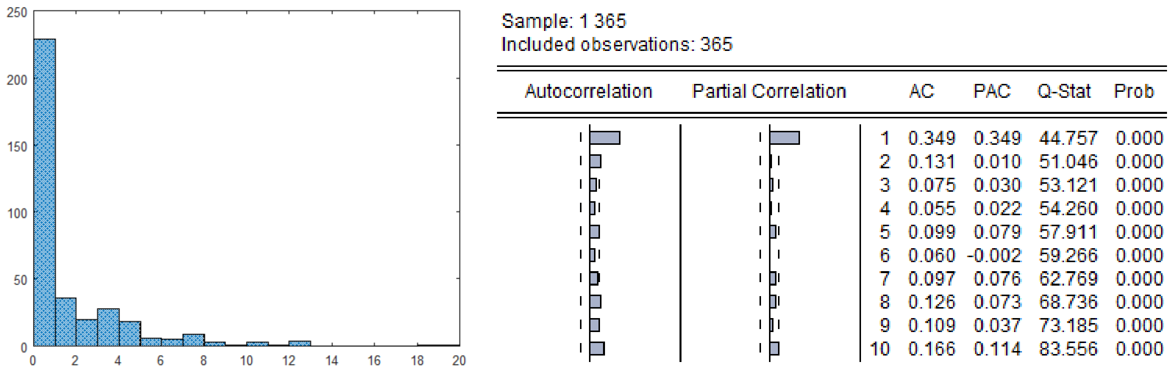

Figure 7: Sample histogram, autocorrelations and partial autocorrelations.

Trying to obtain a suitable model for this count time series, we present a comparative study between five CP-INARCH(1) processes, namely those associated to the Poisson ([4]), the generalized Poisson ([15]), the Neyman type-A, the geometric Poisson and the negative binomial ([13]) laws. Considering the slightly better performance observed in Section 3 for the CML estimator, we begin by using this methodology to estimate the models parameters and take a decision on the model fitting. The results, obtained with the help of MATLAB software, are displayed in Table 7. So, based on the values of the log likelihood function, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), we conclude that the GEOMP2-INARCH(1) model gives better fit than the other CP-INARCH(1) models considered. The NB2 model follows closely and the Poisson model shows the worst adequacy.

Table 7: CML parameters estimates for several CP-INARCH(1) models. Standard errors are shown in parentheses. The best values of the criteria −Log L, AIC and BIC are emphasised in italics.

	Model	$\widehat{\alpha}_{0,365}$	$\widehat{\alpha}_{1,365}$	Additional parameter	$-$Log L	AIC	BIC
INARCH(1)	Poisson	0.9751 (0.0008)	0.3055 (0.0018)		786.3	1576.5	1584.3
	GP	0.8971 (0.0012)	0.3608 (0.0018)	$\widehat{\kappa}_{365} = 0.3736$ (0.0073)	524.6	1055.2	1066.9
	NTA	0.9558 (0.0051)	0.3192 (0.0125)	$\widehat{\phi}_{365} = 2.4368$ (0.0502)	524.7	1055.4	1067.1
	GEOMP2	0.9338 (0.0060)	0.3349 (0.0022)	$\widehat{p^*_{365}} = 0.3599$ (0.0024)	*516.2*	*1038.4*	*1050.1*
	NB2	0.9129 (0.0078)	0.3496 (0.0031)	$\widehat{\beta}_{365} = 5.5659$ (0.0968)	519.8	1045.6	1057.3

The mean, variance and the first-order autocorrelation coefficient (FOAC) for the fitted CP-INARCH(1) models are summarized in Table 8. The results are in accordance with the previous conclusion as, although the similarity of the mean values, the variance and FOAC values point to a GEOMP2 or NB2-INARCH(1) choice. The two other methodologies are also considered to estimate the previous models and it should be noted in Table 9 the close proximity between each of the three parameters and those obtained by the CML method in the case of the GEOMP2 and also NB2 models. This conclusion is validated by the values referred in Table 10 for the sample and estimated means, variances and FOAC values under the two methods, particularly for the CLS+M one. Thus these methodologies seem to capture the same models as the powerful but distribution-demanding CML approach, which is in line with the previous conclusions of the simulation study.

Table 8: Sample and estimated means, variances and FOACs under CP-INARCH(1) models.

Method	Model	Sample	Poisson	GP	NTA	GEOMP2	NB2
CML	Mean	1.4082	1.4040	1.4034	1.4039	1.4040	1.4036
	Variance	7.3027	1.5485	4.1125	5.3723	7.2064	8.9001
	FOAC	0.349	0.3055	0.3608	0.3192	0.3349	0.3496

Table 9: Estimated parameters of several CP-INARCH(1) models based on CLS+M and PQML+M approaches. Standard errors are shown in parentheses. (a) means all models.

Method	Model			Additional parameter
CLS+M	(a)	$\widehat{\alpha}_{0,365} = 0.9138$	$\widehat{\alpha}_{1,365} = 0.3490$	
	Poisson	(0.0862)	(0.0564)	—
	GP	(0.1493)	(0.0928)	0.5319
	NTA	(0.1337)	(0.0799)	3.5645
	GEOMP2	(0.1392)	(0.0845)	0.3594
	NB2	(0.1445)	(0.0889)	4.5645
PQML+M	(a)	$\widehat{\alpha}_{0,365} = 0.9751$	$\widehat{\alpha}_{1,365} = 0.3055$	
	GP	(0.0008)	(0.0018)	0.5392
	NTA	(0.0008)	(0.0018)	3.7096
	GEOMP2	(0.0008)	(0.0018)	0.3503
	NB2	(0.0008)	(0.0018)	4.7096

Table 10: Sample and estimated means, variances and FOACs under CP-INARCH(1) models.

Method	Model	Sample	GEOMP2	NB2
CLS+M	Mean	1.4082	1.4037	1.4037
	Variance	7.3027	7.2964	7.2958
	FOAC	0.349	0.3490	0.3490
PQML+M	Mean	1.4082	1.4040	1.4040
	Variance	7.3027	7.2926	7.2929
	FOAC	0.349	0.3055	0.3055

The statistical study that was developed in this Section was naturally circumscribed to the class of CP-INARCH(1) models considered here. However, this observed time series has characteristics that can also be taken into account if the adjustment is done in other classes of models, namely, in view of its histogram, the zero-inflated CP-INGARCH models ([7]).

6. CONCLUSION

The class of integer-valued GARCH models, specified through the characteristic function of the compound Poisson law and denoted CP-INGARCH ([6]) unifies and enlarges substantially the family of conditionally heteroscedastic integer-valued processes. With this new class, we may capture simultaneously different kinds of conditional volatility and the overdispersion characteristic often recorded in real count data. The probabilistic analysis of these models, concerning stationarity and ergodicity properties as well as moments studies, was the goal of previous works among which we may refer those established in [5] and [6]. The aim of this paper is to develop some statistical studies, regarding the parametric estimation of the CP-INARCH models, that allow the use of this general class with real data and show its true practical usefulness. We concentrate our study on the CP-INARCH models of order one, and a two-step estimation methodology, involving the conditional least squares or the Poisson quasi-maximum likelihood methods in a first step, and the moment's estimation method in the second one, has been introduced and developed. We point out the great advantage of this procedure regarding the more classical conditional maximum likelihood one, as its application is independent from the specific conditional distribution of the process. In fact, the simulation study presented allows concluding that the two-step methodology performance is strongly competitive with that of the conditional maximum likelihood estimation. We should also stress that the practical relevance of this wide class is clearly shown with the real-data example presented which illustrates the better quality of the fitting performed by new models emerged from that class.

Future developments of the present study should concern, particularly, the establishment of the conjectured Gaussian asymptotic distribution of the additional parameter estimator. The development of the parametric estimation of a more general CP-INGARCH model should also be considered.

A. APPENDIX — Proof of Theorem 2.1

To establish the results present in Theorem 2.1 let us begin by recalling the expression of the following conditional moments:

$$\text{(A.1)}\qquad \begin{aligned} E(X_t|\underline{X}_{t-1}) &= \lambda_t = \alpha_0+\alpha_1 X_{t-1},\\ E(X_t^2|\underline{X}_{t-1}) &= v_0\lambda_t+\lambda_t^2 = \alpha_1^2X_{t-1}^2+\alpha_1(2\alpha_0+v_0)X_{t-1}+\alpha_0(\alpha_0+v_0),\end{aligned}$$

$$\text{(A.2)}\qquad \begin{aligned} E(X_t^3|\underline{X}_{t-1}) &= i\,\Phi'''_{X_t|\underline{X}_{t-1}}(0)\\ &= d_0\lambda_t+3v_0\lambda_t^2+\lambda_t^3\\ &= \alpha_1^3X_{t-1}^3+3\alpha_1^2(v_0+\alpha_0)X_{t-1}^2+\alpha_1(3\alpha_0^2+6v_0\alpha_0+d_0)X_{t-1}\\ &\quad+\alpha_0(d_0+3v_0\alpha_0+\alpha_0^2).\end{aligned}$$

(**a**) Using the fact that for $k\geq 0$, $\Gamma(k)=\alpha_1^k f_2$, we get

$$\text{(A.3)}\qquad \mu(k) = E(X_tX_{t+k}) = \text{Cov}(X_t,X_{t+k})+E(X_t)^2 = f_2\Big(v_0\alpha_1^k+\alpha_0(1+\alpha_1)\Big).$$

(**b**) To derive $\mu(k,l)$, $0\leq k\leq l$, we distinguish the following three cases:

Case 1: $l>k$. We have

$$\begin{aligned}\mu(k,l) &= E(X_tX_{t+k}X_{t+l})\\ &= E\Big[X_tX_{t+k}E(X_{t+l}|\underline{X}_{t+l-1})\Big]\\ &= \alpha_0E(X_tX_{t+k})+\alpha_1E(X_tX_{t+k}X_{t+l-1})\\ &= \alpha_0\mu(k)+\alpha_1\mu(k,l-1)\\ &= \alpha_0\mu(k)+\alpha_1\Big[\alpha_0\mu(k)+\alpha_1\mu(k,l-2)\Big]\\ &= \cdots\\ &= \alpha_1^{l-k}\Big[\mu(k,k)-f_1\mu(k)\Big]+f_1\mu(k).\end{aligned}$$

Case 2: $l=k>0$. We have

$$\begin{aligned}\mu(k,k) &= E\Big[X_tE(X_{t+k}^2|\underline{X}_{t+k-1})\Big]\\ &= \alpha_1^2E(X_tX_{t+k-1}^2)+\alpha_1(2\alpha_0+v_0)\,E(X_tX_{t+k-1})+\alpha_0(\alpha_0+v_0)\,E(X_t)\\ &= \alpha_1^2\mu(k-1,k-1)+\alpha_1(2\alpha_0+v_0)\,\mu(k-1)+\alpha_0(\alpha_0+v_0)f_1\\ &= \cdots\\ &= \alpha_1^{2k}\left[\mu(0,0)-\frac{v_0(2\alpha_0+v_0)f_2}{1-\alpha_1}-f_1\mu(0)\right]+\frac{v_0(2\alpha_0+v_0)f_2\alpha_1^k}{1-\alpha_1}+f_1\mu(0).\end{aligned}$$

Case 3: $l=k=0$. According to the relations between the moments and the cumulants (e.g., formula (15.10.4) in [3, p. 186]) and Theorem 4.2 of [7], we have

$$\begin{aligned}\mu(0,0) &= E(X_t^3)\\ &= \kappa_3+3\kappa_2\mu+\mu^3\\ &= f_3\Big[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2\Big]+3v_0f_2f_1+f_1^3\\ &= \Big[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2\Big]f_3+\frac{2\alpha_0v_0}{1-\alpha_1}f_2+f_1\mu(0),\end{aligned}$$

since $f_1 = (1-\alpha_1^2) f_2$. So the above formula for $\mu(k,k)$ simplifies to

$$\begin{aligned}\mu(k,k) &= \alpha_1^{2k}\left[\Big[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2\Big]f_3 - \frac{v_0^2}{1-\alpha_1}f_2\right] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k + f_1\mu(0)\\ &= \alpha_1^{2k}f_3\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k + f_1\mu(0),\end{aligned}$$

which also holds for $k=0$. Replacing this expression in $\mu(k,l)$ above, it follows that

$$\begin{aligned}\mu(k,l) &= \alpha_1^{l-k}\left[\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]f_3\alpha_1^{2k} + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k\right.\\ &\qquad \left. + f_1\mu(0) - f_1\mu(k)\right] + f_1\mu(k).\end{aligned}$$

As

$$f_1\mu(0) - f_1\mu(k) = v_0 f_1 f_2 - \frac{v_0\alpha_0}{1-\alpha_1}f_2\alpha_1^k,$$

we finally obtain, for any $0 \le k \le l$,

$$\begin{aligned}\mu(k,l) &= \Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]f_3\alpha_1^{l+k} + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^l\\ &\quad + v_0 f_1 f_2 \alpha_1^{l-k} + f_1\mu(k).\end{aligned}$$

(c) In what concerns the fourth-order moments $\mu(k,l,m)$ with $0 \le k \le l \le m$, we proceed in a similar way as above and distinguish the following four cases:

Case 1: $m > l$. As above we have

$$\begin{aligned}\mu(k,l,m) &= E(X_t X_{t+k} X_{t+l} X_{t+m})\\ &= \alpha_1^{m-l}\Big[\mu(k,l,l) - f_1\mu(k,l)\Big] + f_1\mu(k,l).\end{aligned}$$

Case 2: $m = l > k$. For this case, using formula ($A.1$), we obtain

$$\begin{aligned}\mu(k,l,l) &= E\Big[X_t X_{t+k} E(X_{t+l}^2 | \underline{X}_{t+l-1})\Big]\\ &= \alpha_1^2\mu(k,l-1,l-1) + \alpha_1(v_0+2\alpha_0)\,\mu(k,l-1) + \alpha_0(v_0+\alpha_0)\,\mu(k).\end{aligned}$$

Replacing $\mu(k,l-1)$, using $\mu(0) = \big(v_0 + \alpha_0(1+\alpha_1)\big)f_2$ and replacing $\mu(k)$, we obtain

$$\begin{aligned}\mu(k,l,l) &= \alpha_1^{2(l-k)}\mu(k,k,k) + \mu(k)\,\mu(0)\\ &\quad - f_2 v_0\left[f_2\Big(v_0+\alpha_0(1+\alpha_1)\Big) + \frac{(v_0+2\alpha_0)\,(v_0+\alpha_0)}{(1-\alpha_1)^2}\right]\alpha_1^{2l-k}\\ &\quad - f_1\left[f_1\mu(0) + \frac{v_0(v_0+2\alpha_0)}{1-\alpha_1}f_2\right]\alpha_1^{2(l-k)} + \frac{v_0+2\alpha_0}{1-\alpha_1}\Big[\mu(k,l) - f_1\mu(k)\Big]\\ &\quad - \frac{v_0+2\alpha_0}{1-\alpha_1}\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]f_3\alpha_1^{2l}.\end{aligned}$$

So, replacing $\mu(0)$, recalling $\mu(0,0)$ and taking into account that $\frac{f_1}{1-\alpha_1} = (1+\alpha_1)f_2$, we get

$$\begin{aligned}\mu(k,l,l) \;=\; & \alpha_1^{2(l-k)}\mu(k,k,k) - \mu(k)\, f_2\Big[\alpha_0 + (v_0+\alpha_0)\alpha_1\Big] \\ & - \frac{f_2 v_0}{(1-\alpha_1)(1-\alpha_1^2)}\Big[v_0^2(1+\alpha_1) + v_0\alpha_0(4+3\alpha_1) + 3\alpha_0^2(1+\alpha_1)\Big]\alpha_1^{2l-k} \\ & - f_1\left\{\mu(0,0) - \Big[d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2\Big] f_3 + \frac{v_0^2 f_2}{1-\alpha_1}\right\}\alpha_1^{2(l-k)} \\ & + \frac{v_0+2\alpha_0}{1-\alpha_1}\,\mu(k,l) - \frac{v_0+2\alpha_0}{1-\alpha_1}\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big] f_3\alpha_1^{2l}. \end{aligned} \tag{A.4}$$

Case 3: $m = l = k > 0$. From formula ($A.2$) we have

$$\begin{aligned}\mu(k,k,k) \;=\; & E\Big[X_t E(X_{t+k}^3 | \underline{X}_{t+k-1})\Big] \\ =\; & \alpha_1^3\mu(k-1,k-1,k-1) + 3\alpha_1^2(v_0+\alpha_0)\,\mu(k-1,k-1) \\ & + \alpha_1(d_0 + 6v_0\alpha_0 + 3\alpha_0^2)\,\mu(k-1) + \alpha_0(d_0 + 3v_0\alpha_0 + \alpha_0^2)\,\mu.\end{aligned}$$

Replacing $\mu(k-1,k-1)$ and thereafter $\mu(k-1)$, we deduce

$$\begin{aligned}\mu(k,k,k) \;=\; & \alpha_1^3\mu(k-1,k-1,k-1) \\ & + 3(v_0+\alpha_0)\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big] f_3\alpha_1^{2k} \\ & + \frac{v_0 f_2}{1-\alpha_1}\Big[3\alpha_1(v_0+\alpha_0)^2 + 3\alpha_1(v_0+\alpha_0)\alpha_0 + (d_0 + 6v_0\alpha_0 + 3\alpha_0^2)(1-\alpha_1)\Big]\alpha_1^k \\ & + f_1 f_2\Big\{3\alpha_1^2(v_0+\alpha_0)\Big(v_0 + \alpha_0(1+\alpha_1)\Big) + (d_0 + 6v_0\alpha_0 + 3\alpha_0^2)\,\alpha_1(1-\alpha_1)\,(1+\alpha_1) \\ & + (d_0 + 3v_0\alpha_0 + \alpha_0^2)\,(1-\alpha_1)\,(1-\alpha_1^2)\Big\}.\end{aligned}$$

Making some calculations and then recalling the expression of $\mu(0,0)$, we obtain

$$\begin{aligned}\mu(k,k,k) \;=\; & \alpha_1^3\mu(k-1,k-1,k-1) \\ & + 3(v_0+\alpha_0)\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big] f_3\alpha_1^{2k} \\ & + \frac{v_0 f_2}{1-\alpha_1}\Big[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\Big]\alpha_1^k \\ & + f_1(1-\alpha_1^3)\,\mu(0,0).\end{aligned}$$

Replacing successively the expression of $\mu(k-j,k-j,k-j)$, $j = 1,...,k-1$, it remains

$$\begin{aligned}\mu(k,k,k) \;=\; & \alpha_1^{3k}\Big\{\mu(0,0,0) - 3(v_0+\alpha_0)\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]\frac{f_3}{1-\alpha_1} \\ & - \frac{v_0 f_2}{(1-\alpha_1)(1-\alpha_1^2)}\Big[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\Big] - f_1\mu(0,0)\Big\} \\ & + \frac{3(v_0+\alpha_0) f_3\alpha_1^{2k}}{1-\alpha_1}\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big] \\ & + \frac{v_0 f_2\alpha_1^k}{(1-\alpha_1)(1-\alpha_1^2)}\Big[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\Big] + f_1\mu(0,0).\end{aligned} \tag{A.5}$$

Replacing $\mu(0,0)$, highlighting $\frac{f_3}{1-\alpha_1^2}$, noting that $f_2 = (1-\alpha_1^3)f_3$ and $\frac{f_3}{1-\alpha_1^2} = f_4(1+\alpha_1^2)$ and developing the calculations, we finally get

$$\begin{aligned}\mu(k,k,k) &= \Big\{\mu(0,0,0) - f_4\Big[4v_0d_0 - 3v_0^3 + 3v_0(d_0-v_0^2)\alpha_1 + v_0(3v_0^2+d_0)\alpha_1^2 \\ &\quad + v_0(6v_0^2-d_0)\alpha_1^3 + 3v_0(2v_0^2-d_0)\alpha_1^4 + v_0(9v_0^2-4d_0)\alpha_1^5 \\ &\quad + \alpha_0(1+\alpha_1^2)\Big[3v_0^2+4d_0+(3v_0^2+4d_0)\alpha_1+(15v_0^2-4d_0)\alpha_1^2+(12v_0^2-4d_0)\alpha_1^3\Big] \\ &\quad + 6v_0\alpha_0^2(1+\alpha_1^2)(1+\alpha_1)(1+\alpha_1+\alpha_1^2) + \alpha_0^3(1+\alpha_1^2)(1+\alpha_1)^2(1+\alpha_1+\alpha_1^2)\Big]\Big\}\alpha_1^{3k} \\ &\quad + 3\frac{v_0+\alpha_0}{1-\alpha_1}f_3\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]\alpha_1^{2k} + f_1\mu(0,0) \\ &\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\Big[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\Big]\alpha_1^k. \qquad \text{(A.6)}\end{aligned}$$

Case 4: $m = l = k = 0$. Once again, according to the relations between the moments and the cumulants, we obtain

$$\begin{aligned}\mu(0,0,0) &= E(X_t^4) \\ &= \kappa_4 + 3\kappa_2^2 + 6\kappa_2\mu^2 + 4\kappa_3\mu + \mu^4 \\ &= f_4\Big\{c_0 + (3v_0^3+4v_0d_0-c_0)\alpha_1^2 + (6v_0d_0-c_0)\alpha_1^3 + (15v_0^3-10v_0d_0+c_0)\alpha_1^5 \\ &\quad + \alpha_0(1+\alpha_1^2)\Big[3v_0^2+4d_0+(3v_0^2+4d_0)\alpha_1+(15v_0^2-4d_0)\alpha_1^2+(12v_0^2-4d_0)\alpha_1^3\Big] \\ &\quad + \ + 6v_0\alpha_0^2(1+\alpha_1)(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2) + \alpha_0^3(1+\alpha_1)^2(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2)\Big\}.\end{aligned}$$

So the formula ($A.6$) for $\mu(k,k,k)$ studied in case 3 simplifies to

$$\begin{aligned}\mu(k,k,k) &= f_4\Big\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2-d_0)\alpha_1 + (3v_0d_0-c_0)\alpha_1^2 \\ &\quad + (7v_0d_0-6v_0^3-c_0)\alpha_1^3 + 3v_0(d_0-2v_0^2)\alpha_1^4 + (6v_0^3-6v_0d_0+c_0)\alpha_1^5\Big\}\alpha_1^{3k} \\ &\quad + 3\,\frac{v_0+\alpha_0}{1-\alpha_1}f_3\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]\alpha_1^{2k} + f_1\mu(0,0) \\ &\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\Big[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\Big]\alpha_1^k.\end{aligned}$$

Inserting into the formula ($A.4$) for $\mu(k,l,l)$ stated in case 2, we obtain

$$\begin{aligned}\mu(k,l,l) &= f_4\Big\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2-d_0)\alpha_1 + (3v_0d_0-c_0)\alpha_1^2 \\ &\quad + (7v_0d_0-6v_0^3-c_0)\alpha_1^3 + 3v_0(d_0-2v_0^2)\alpha_1^4 + (6v_0^3-6v_0d_0+c_0)\alpha_1^5\Big\}\alpha_1^{2l+k} \\ &\quad + \frac{2v_0+\alpha_0}{1-\alpha_1}f_3\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]\alpha_1^{2l} \\ &\quad + \Big\{\frac{\alpha_0 f_3}{1-\alpha_1}\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\Big]\Big\}\alpha_1^{2(l-k)} \\ &\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\Big[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1-1)\Big]\alpha_1^{2l-k} \\ &\quad + \frac{v_0+2\alpha_0}{1-\alpha_1}\mu(k,l) - f_2\,\mu(k)\Big[\alpha_0 + (v_0+\alpha_0)\alpha_1\Big].\end{aligned}$$

So it follows that we have

$$\begin{aligned}
\mu(k,l,m) \;=\; & \alpha_1^{m-l}\Big[\mu(k,l,l) - f_1\mu(k,l)\Big] + f_1\mu(k,l) \\
=\; & \alpha_1^{m-l}\Bigg[f_4\Big\{ c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3v_0d_0 - c_0)\alpha_1^2 \\
& + (7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\Big\}\, \alpha_1^{2l+k} \\
& + \frac{2v_0 + \alpha_0}{1-\alpha_1}\, f_3\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)\Big]\alpha_1^{2l} \\
& + \left\{ \frac{\alpha_0 f_3}{1-\alpha_1}\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)\Big]\right\} \alpha_1^{2(l-k)} \\
& + \frac{v_0}{(1-\alpha_1)\,(1-\alpha_1^2)}\, f_2\Big[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1)\Big]\, \alpha_1^{2l-k} \\
& + \frac{v_0 + \alpha_0}{1-\alpha_1}\,\mu(k,l) - f_2\,\mu(k)\Big[\alpha_0 + (v_0 + \alpha_0)\alpha_1\Big]\Bigg] + f_1\mu(k,l),
\end{aligned}$$

which holds for all $0 \le k \le l \le m$.

B. APPENDIX — Proof of Corollary 2.1

To establish the results present in Corollary 2.1 we use the general relations between joint moments and joint cumulants (see [2], p. 5):

(**a**) The second-order central moments and cumulants of X, for any $s \geq 0$, are given by

$$\widetilde{\mu}(s) \,=\, \kappa(s) \,=\, \mathrm{Cov}(X_t, X_{t+s}) \,=\, v_0 \alpha_1^s f_2.$$

(**b**) The third-order central moments and cumulants, for any $l \geq s \geq 0$, are given by

$$\begin{aligned}\widetilde{\mu}(s,l) &= \kappa(s,l)\\ &= f_3 \alpha_1^l \Big[v_0^2(1+\alpha_1+\alpha_1^2) - \Big\{ v_0^2(1+\alpha_1-2\alpha_1^2) - d_0(1-\alpha_1^2) \Big\}\, \alpha_1^s \Big].\end{aligned}$$

(**c**) In what concerns the fourth-order cumulants we have, for $m \geq l \geq s \geq 0$,

$$\begin{aligned}
\kappa(s,l,m) \,=\, &\boldsymbol{\alpha_1^{m-l}} \Bigg[\alpha_1^{2l+s} f_4 \Big\{ c_0 - 4v_0 d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3v_0 d_0 - c_0)\alpha_1^2 \\
&+ (7v_0 d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0 d_0 + c_0)\alpha_1^5 \Big\} \\
&+ \frac{2v_0 + \alpha_0}{1-\alpha_1}\, f_3 \Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2) \Big]\, \alpha_1^{2l} \\
&+ \boldsymbol{\left\{ \frac{\alpha_0 f_3}{1-\alpha_1} \Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2) \Big] \right\} \alpha_1^{2(l-s)}} \\
&+ \frac{v_0}{(1-\alpha_1)\,(1-\alpha_1^2)}\, f_2 \Big[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1) \Big]\, \alpha_1^{2l-s} \\
&+ \frac{v_0+\alpha_0}{1-\alpha_1}\, \mu(s,l) - f_2\, \mu(s) \Big[\alpha_0 + (v_0+\alpha_0)\alpha_1 \Big] \Bigg] \boldsymbol{+ f_1\mu(s,l) - f_1\mu(s,l)} \\
&\boldsymbol{- f_1} \Bigg(\boldsymbol{\Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2) \Big] f_3\, \alpha_1^{m+l-2s}} + \frac{v_0(v_0+\alpha_0)}{1-\alpha_1}\, f_2\, \alpha_1^{m-s} \\
&+ v_0 f_1 f_2\, \alpha_1^{m-l} \boldsymbol{+ f_1\mu(l-s) - f_1 f_2 (v_0 \alpha_1^{l-s} + \alpha_0(1+\alpha_1))} \\
&+ \Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2) \Big] f_3 \alpha_1^{m+l} \\
&+ \frac{v_0(v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^m + f_1 f_2 v_0 \alpha_1^{m-l} \boldsymbol{+ f_1\mu(l) - f_1\mu(l)} + \frac{v_0(v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^m \\
&+ \Big[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2) \Big] f_3 \alpha_1^{m+s} + v_0 f_1 f_2 \alpha_1^{m-s} \boldsymbol{+ f_1\mu(s) - f_1\mu(s)} \Bigg) \\
&- \Big(f_2 \Big[v_0\alpha_1^s + \alpha_0(1+\alpha_1) \Big] - f_1^2 \Big) \Big(f_2 \Big[v_0 \alpha_1^{m-l} + \alpha_0(1+\alpha_1) \Big] - f_1^2 \Big) \\
&- \Big(f_2 \Big[v_0\alpha_1^l + \alpha_0(1+\alpha_1) \Big] - f_1^2 \Big) \Big(f_2 \Big[v_0 \alpha_1^{m-s} + \alpha_0(1+\alpha_1) \Big] - f_1^2 \Big) \\
&- \Big(f_2 \Big[v_0\alpha_1^m + \alpha_0(1+\alpha_1) \Big] - f_1^2 \Big) \Big(f_2 \Big[v_0 \alpha_1^{l-s} + \alpha_0(1+\alpha_1) \Big] - f_1^2 \Big) \\
&+ f_1^2 \Big(f_2 \Big[v_0\alpha_1^m + \alpha_0(1+\alpha_1) \Big] + f_2 \Big[v_0\alpha_1^{m-s} + \alpha_0(1+\alpha_1) \Big] \\
&+ f_2 \Big[v_0\alpha_1^{m-l} + \alpha_0(1+\alpha_1) \Big] - 3f_1^2 \Big),
\end{aligned}$$

where we highlight, using bold, expressions whose sum equals zero.

So, taking into account that

$$-f_2\,\mu(s)\Big[\alpha_0+(v_0+\alpha_0)\alpha_1\Big]\,\alpha_1^{m-l} \;=\; \Big[-f_1\frac{\alpha_0+v_0}{1-\alpha_1}\,\mu(s)+v_0 f_2\,\mu(s)\Big]\alpha_1^{m-l}$$

and

$$\begin{aligned}
&-\;\Big(f_2\Big[v_0\alpha_1^s+\alpha_0(1+\alpha_1)\Big]-f_1^2\Big)\Big(f_2\Big[v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1)\Big]-f_1^2\Big)\\
&-\;\Big(f_2\Big[v_0\alpha_1^l+\alpha_0(1+\alpha_1)\Big]-f_1^2\Big)\Big(f_2\Big[v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1)\Big]-f_1^2\Big)\\
&-\;\Big(f_2\Big[v_0\alpha_1^m+\alpha_0(1+\alpha_1)\Big]-f_1^2\Big)\Big(f_2\Big[v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1)\Big]-f_1^2\Big)\\
&+\;f_1^2\Big(f_2\Big[v_0\alpha_1^m+\alpha_0(1+\alpha_1)\Big]+f_2\Big[v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1)\Big]\\
&+\;f_2\Big[v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1)\Big]-3f_1^2\Big)\;=\\
&\qquad\qquad=\;-v_0^2f_2^2\Big[\alpha_1^{m-l+s}+2\alpha_1^{m+l-s}\Big]+v_0f_1^2f_2\Big[\alpha_1^{m-l}+\alpha_1^{m-s}+\alpha_1^m\Big]
\end{aligned}$$

we obtain, by replacing $\mu(s,l)$,

$$\begin{aligned}
\kappa(s,l,m) \;=\; \alpha_1^m f_4\Bigg[&\Big\{c_0-4v_0d_0+3v_0^3+3v_0(v_0^2-d_0)\alpha_1+(3\alpha_0d_0-c_0)\alpha_1^2\\
&+\;(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\Big\}\,\alpha_1^{l+s}\\
&+\;v_0(1+\alpha_1+\alpha_1^2+\alpha_1^3)\Big[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\Big](2\alpha_1^l+\alpha_1^s)\\
&+\;v_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)\Big[(1+\alpha_1)v_0^2+\big(d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\big)\,\alpha_1^{l-s}\Big]\Bigg],
\end{aligned}$$

for any $m\geq l\geq s\geq 0$.

Finally, the fourth-order central moments of X are given by

$$\begin{aligned}
\widetilde{\mu}(s,l,m) \;&=\; \kappa(s,l,m)+v_0\alpha_1^s f_2v_0\alpha_1^{m-l}f_2+v_0\alpha_1^l f_2v_0\alpha_1^{m-s}f_2+v_0\alpha_1^{l-s}f_2v_0\alpha_1^m f_2\\
&=\; \kappa(s,l,m)+v_0^2f_2^2\alpha_1^{m-l+s}+2v_0^2f_2^2\alpha_1^{m+l-s}.
\end{aligned}$$

C. APPENDIX — Covariance matrix of the asymptotic distribution of CLS estimators in CP-INARCH model

To obtain the entries of the covariance matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$, let us begin by deducing the inverse of $\mathbf{V}$:

$$\mathbf{V}^{-1} = \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\begin{bmatrix} \dfrac{\alpha_0\big(v_0+\alpha_0(1+\alpha_1)\big)}{(1-\alpha_1)(1-\alpha_1^2)} & -\dfrac{\alpha_0}{1-\alpha_1} \\ -\dfrac{\alpha_0}{1-\alpha_1} & 1 \end{bmatrix} = \begin{bmatrix} 1+\dfrac{\alpha_0}{v_0}(1+\alpha_1) & -\dfrac{1}{v_0}(1-\alpha_1^2) \\ -\dfrac{1}{v_0}(1-\alpha_1^2) & \dfrac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0} \end{bmatrix}.$$

Furthermore, considering $u_t(\alpha) = X_t - g(\alpha, X_{t-1})$,

$$\begin{aligned} E\Big[f(X_{t-1})\cdot u_t^2(\alpha)\Big] &= E\Big[f(X_{t-1})\cdot E\Big[(X_t-\alpha_0-\alpha_1X_{t-1})^2\,|\,X_{t-1}\Big]\Big] \\ &= E\Big[f(X_{t-1})\cdot V\Big[X_t-\alpha_0-\alpha_1X_{t-1}\,|\,X_{t-1}\Big]+0\Big] \\ &= E\Big[f(X_{t-1})\cdot V\Big[X_t|X_{t-1}\Big]\Big] = E\Big[f(X_{t-1})\cdot v_0(\alpha_0+\alpha_1X_{t-1})\Big], \end{aligned}$$

because of the conditional compound Poisson distribution, and then

$$\begin{aligned} \mathbf{W} &= \begin{bmatrix} E\left(u_t^2\dfrac{\partial g}{\partial\alpha_0}\dfrac{\partial g}{\partial\alpha_0}\right) & E\left(u_t^2\dfrac{\partial g}{\partial\alpha_0}\dfrac{\partial g}{\partial\alpha_1}\right) \\ E\left(u_t^2\dfrac{\partial g}{\partial\alpha_1}\dfrac{\partial g}{\partial\alpha_0}\right) & E\left(u_t^2\dfrac{\partial g}{\partial\alpha_1}\dfrac{\partial g}{\partial\alpha_1}\right) \end{bmatrix} \\ &= \begin{bmatrix} E\Big[1\cdot v_0(\alpha_0+\alpha_1X_{t-1})\Big] & E\Big[X_{t-1}\cdot v_0(\alpha_0+\alpha_1X_{t-1})\Big] \\ E\Big[X_{t-1}\cdot v_0(\alpha_0+\alpha_1X_{t-1})\Big] & E\Big[X_{t-1}^2\cdot v_0(\alpha_0+\alpha_1X_{t-1})\Big] \end{bmatrix} \\ &= \frac{v_0\alpha_0}{1-\alpha_1}\begin{bmatrix} 1 & \dfrac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} \\ \dfrac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} & \dfrac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\dfrac{\alpha_0^2}{(1-\alpha_1)^2}+\dfrac{\alpha_1\big(d_0+(3v_0^2-d_0)\,\alpha_1^2\big)}{(1-\alpha_1^2)(1-\alpha_1^3)} \end{bmatrix}, \end{aligned}$$

since

$$E\Big[v_0(\alpha_0+\alpha_1X_{t-1})\Big] = v_0\left[\alpha_0+\alpha_1\frac{\alpha_0}{1-\alpha_1}\right] = \frac{v_0\alpha_0}{1-\alpha_1},$$

$$\begin{aligned} E\Big[X_{t-1}v_0(\alpha_0+\alpha_1X_{t-1})\Big] &= v_0\left[\frac{\alpha_0^2}{1-\alpha_1}+\frac{\alpha_1\alpha_0\big(v_0+\alpha_0(1+\alpha_1)\big)}{(1-\alpha_1)(1-\alpha_1^2)}\right] \\ &= \frac{v_0\alpha_0}{1-\alpha_1}\cdot\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2}, \end{aligned}$$

$$\begin{aligned}
&E\Big[X_{t-1}^2\cdot v_0(\alpha_0+\alpha_1X_{t-1})\Big] = \\
&= v_0\left[\frac{\alpha_0^2\big(v_0+\alpha_0(1+\alpha_1)\big)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_1\alpha_0}{(1-\alpha_1)^3}\left(\frac{d_0+(3v_0^2-d_0)\alpha_1^2}{(1+\alpha_1)(1+\alpha_1+\alpha_1^2)}+\frac{3v_0\alpha_0}{1+\alpha_1}+\alpha_0^2\right)\right]\\
&= \frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{v_0\alpha_0(1-\alpha_1)+3v_0\alpha_0\alpha_1}{(1-\alpha_1)^2(1+\alpha_1)}+\frac{\alpha_0^2(1-\alpha_1)+\alpha_0^2\alpha_1}{(1-\alpha_1)^2}+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right]\\
&= \frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right],
\end{aligned}$$

using again the expressions stated in Theorem 2.1.

Now, the product of $\mathbf{V}^{-1}\mathbf{W}$ is given by

$$\begin{bmatrix} 1+\frac{\alpha_0}{v_0}(1+\alpha_1) & -\frac{1}{v_0}(1-\alpha_1^2) \\ -\frac{1}{v_0}(1-\alpha_1^2) & \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\end{bmatrix}\cdot$$

$$\cdot\begin{bmatrix} 1 & \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} \\ \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} & \frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{(1-\alpha_1^2)(1-\alpha_1^3)}\end{bmatrix}=$$

$$=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix} 1-\alpha_1 & \frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0(1-\alpha_1^3)} \\ \frac{\alpha_1(1-\alpha_1)}{\alpha_0} & 1+\alpha_1+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\end{bmatrix},$$

since

$$\begin{aligned}
a_{11} &= 1+\frac{\alpha_0(1+\alpha_1)}{v_0}-\frac{1-\alpha_1^2}{v_0}\,\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} = 1-\alpha_1,\\
a_{12} &= \left(1+\frac{\alpha_0}{v_0}(1+\alpha_1)\right)\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2}\\
&\quad -\frac{1-\alpha_1^2}{v_0}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right]\\
&= \frac{v_0\alpha_1}{1-\alpha_1^2}+\frac{\alpha_0}{1-\alpha_1}+\frac{\alpha_0\alpha_1}{1-\alpha_1}+\frac{\alpha_0^2(1+\alpha_1)}{v_0(1-\alpha_1)}-\frac{\alpha_0(1+2\alpha_1)}{1-\alpha_1}-\frac{\alpha_0^2(1+\alpha_1)}{v_0(1-\alpha_1)}\\
&\quad -\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0(1-\alpha_1^3)}\\
&= \frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0(1-\alpha_1^3)},\\
a_{21} &= -\frac{(1-\alpha_1^2)}{v_0}+\frac{(1-\alpha_1)(1-\alpha_1^2)\big(v_0\alpha_1+\alpha_0(1+\alpha_1)\big)}{v_0\alpha_0(1-\alpha_1^2)}\\
&= -\frac{(1-\alpha_1^2)}{v_0}+\frac{\alpha_1(1-\alpha_1)}{\alpha_0}+\frac{(1-\alpha_1^2)}{v_0} = \frac{\alpha_1(1-\alpha_1)}{\alpha_0},
\end{aligned}$$

$$
\begin{aligned}
a_{22} &= -\frac{(1-\alpha_1^2)\big(v_0\alpha_1+\alpha_0(1+\alpha_1)\big)}{v_0(1-\alpha_1^2)} \\
&\quad + \frac{(1-\alpha_1)\,(1-\alpha_1^2)}{v_0\alpha_0}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)\,(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{(1-\alpha_1^2)\,(1-\alpha_1^3)}\right] \\
&= -\alpha_1-\frac{\alpha_0(1+\alpha_1)}{v_0}+1+2\alpha_1+\frac{\alpha_0(1+\alpha_1)}{v_0}+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)} \\
&= 1+\alpha_1+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}.
\end{aligned}
$$

So, the asymptotic covariance matrix is such that

$$
\begin{aligned}
\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1} &= \begin{bmatrix} b_{11} & b_{12} \\ \\ b_{21} & b_{22} \end{bmatrix} \\
&= \frac{v_0\alpha_0}{1-\alpha_1}\begin{bmatrix} 1-\alpha_1 & \dfrac{v_0\alpha_1}{1-\alpha_1^2}-\dfrac{\alpha_0\alpha_1}{1-\alpha_1}-\dfrac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0(1-\alpha_1^3)} \\ \dfrac{\alpha_1(1-\alpha_1)}{\alpha_0} & 1+\alpha_1+\dfrac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)} \end{bmatrix}. \\
&\quad \cdot \begin{bmatrix} 1+\dfrac{\alpha_0}{v_0}(1+\alpha_1) & -\dfrac{1}{v_0}(1-\alpha_1^2) \\ -\dfrac{1}{v_0}(1-\alpha_1^2) & \dfrac{(1-\alpha_1)\,(1-\alpha_1^2)}{v_0\alpha_0} \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
b_{11} &= \frac{\alpha_0}{1-\alpha_1}\left(\alpha_0(1+\alpha_1)+\frac{v_0^2+(d_0-v_0^2)\,\alpha_1(1+\alpha_1-\alpha_1^2)+(3v_0^2-d_0)\,\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)}\right), \\
b_{12} &= b_{21} = v_0\alpha_1-\alpha_0(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0(1+\alpha_1+\alpha_1^2)}, \\
b_{22} &= (1-\alpha_1^2)\left(1+\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right).
\end{aligned}
$$

In fact, we have

$$
\begin{aligned}
b_{11} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[(1-\alpha_1)\left(1+\frac{\alpha_0}{v_0}(1+\alpha_1)\right)\right. \\
&\quad \left. -\frac{1}{v_0}\,(1-\alpha_1^2)\left(\frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)}{v_0(1-\alpha_1^3)}\right)\right] \\
&= \frac{\alpha_0}{1-\alpha_1}\left[v_0(1-\alpha_1)+\alpha_0(1-\alpha_1^2)-v_0\alpha_1+\alpha_0\alpha_1(1+\alpha_1)\right. \\
&\quad \left. +\frac{\alpha_1\big(d_0+(3v_0^2-d_0)\alpha_1^2\big)\,(1+\alpha_1)}{v_0(1+\alpha_1+\alpha_1^2)}\right] =
\end{aligned}
$$

$$= \frac{\alpha_0}{1-\alpha_1}\left[\alpha_0(1+\alpha_1) + \frac{v_0^2(1-2\alpha_1)(1+\alpha_1+\alpha_1^2) + \alpha_1\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)(1+\alpha_1)}{v_0(1+\alpha_1+\alpha_1^2)}\right]$$

$$= \frac{\alpha_0}{1-\alpha_1}\left(\alpha_0(1+\alpha_1) + \frac{v_0^2 + (d_0 - v_0^2)\,\alpha_1(1+\alpha_1-\alpha_1^2) + (3v_0^2 - d_0)\,\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)}\right),$$

$$\begin{aligned}
b_{12} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[-\frac{(1-\alpha_1)\,(1-\alpha_1^2)}{v_0}\right.\\
&\quad \left. + \frac{(1-\alpha_1)\,(1-\alpha_1^2)}{v_0\alpha_0}\left(\frac{v_0\alpha_1}{1-\alpha_1^2} - \frac{\alpha_0\alpha_1}{1-\alpha_1} - \frac{\alpha_1\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0(1-\alpha_1^3)}\right)\right]\\
&= -\alpha_0(1-\alpha_1^2) + v_0\alpha_1 - \alpha_0\alpha_1(1+\alpha_1) - \frac{\alpha_1(1+\alpha_1)\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0(1+\alpha_1+\alpha_1^2)}\\
&= v_0\alpha_1 - \alpha_0(1+\alpha_1) - \frac{\alpha_1(1+\alpha_1)\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0(1+\alpha_1+\alpha_1^2)},
\end{aligned}$$

$$\begin{aligned}
b_{21} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{\alpha_1(1-\alpha_1)}{\alpha_0}\left(1 + \frac{\alpha_0(1+\alpha_1)}{v_0}\right) - \frac{1-\alpha_1^2}{v_0}\left(1+\alpha_1 + \frac{\alpha_1\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\right]\\
&= v_0\alpha_1 + \alpha_0\alpha_1(1+\alpha_1) - \alpha_0(1+\alpha_1) - \alpha_0\alpha_1(1+\alpha_1) - \frac{\alpha_1(1+\alpha_1)\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0(1+\alpha_1+\alpha_1^2)}\\
&= v_0\alpha_1 - \alpha_0(1+\alpha_1) - \frac{\alpha_1(1+\alpha_1)\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0(1+\alpha_1+\alpha_1^2)},
\end{aligned}$$

$$\begin{aligned}
b_{22} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[-\frac{\alpha_1(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0} + \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\left(1+\alpha_1 + \frac{\alpha_1\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\right]\\
&= -\alpha_1(1-\alpha_1^2) + \alpha_1(1-\alpha_1^2) + (1-\alpha_1^2)\left(1 + \frac{\alpha_1\big(d_0 + (3v_0^2 - d_0)\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\\
&= (1-\alpha_1^2)\left(1 + \frac{\alpha_1\big(d_0 + (3v_0^2 - d_0)\,\alpha_1^2\big)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right).
\end{aligned}$$

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