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COMPARISON OF WEIBULL TAIL-COEFFICIENT ESTIMATORS

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Abstract:

- We address the problem of estimating the Weibull tail-coefficient which is the regular variation exponent of the inverse failure rate function. We propose a family of estimators of this coefficient and an associate extreme quantile estimator. Their asymptotic normality are established and their asymptotic mean-square errors are compared. The results are illustrated on some finite sample situations.

Key-Words:

- *Weibull tail-coefficient; extreme quantile; extreme value theory; asymptotic normality.*

AMS Subject Classification:

- 62G32, 62F12, 62G30.

1. INTRODUCTION

Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed random variables with cumulative distribution function F. We denote by $X_{1,n} \le ... \le X_{n,n}$ their associated order statistics. We address the problem of estimating the Weibull tail-coefficient $\theta > 0$ defined when the distribution tail satisfies

$$\textbf{(A.1)} \qquad 1 - F(x) = \exp\big(-H(x)\big) , \qquad H^{\leftarrow}(t) = \inf\big\{x,\ H(x) \ge t\big\} = t^\theta \ell(t) ,$$

where ℓ is a slowly varying function, *i.e.*,

$$\ell(\lambda x)/\ell(x) \to 1 \qquad \text{as} \quad x \to \infty \quad \text{for all} \ \lambda > 0 .$$

The inverse cumulative hazard function $H^{\leftarrow}$ is said to be regularly varying at infinity with index θ and this property is denoted by $H^{\leftarrow} \in \mathcal{R}_\theta$, see [7] for more details on this topic. As a comparison, Pareto type distributions satisfy $(1/(1-F))^{\leftarrow} \in \mathcal{R}_\gamma$, and $\gamma > 0$ is the so-called extreme value index. Weibull tail-distributions include for instance Gamma, Gaussian and, of course, Weibull distributions.

Let (k_n) be a sequence of integers such that $1 \le k_n < n$ and (T_n) be a positive sequence. We examine the asymptotic behavior of the following family of estimators of θ:

$$\hat\theta_n = \frac{1}{T_n}\,\frac{1}{k_n}\sum_{i=1}^{k_n}\Big(\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})\Big) . \tag{1.1}$$

Following the ideas of [10], an estimator of the extreme quantile x_{p_n} can be deduced from (1.1) by:

$$\hat x_{p_n} = X_{n-k_n+1,n}\left(\frac{\log(1/p_n)}{\log(n/k_n)}\right)^{\hat\theta_n} =:\ X_{n-k_n+1,n}\,\tau_n^{\hat\theta_n} . \tag{1.2}$$

Recall that an extreme quantile x_{p_n} of order p_n is defined by the equation

$$1 - F(x_{p_n}) = p_n , \qquad \text{with} \quad 0 < p_n < 1/n .$$

The condition $p_n < 1/n$ is very important in this context. It usually implies that x_{p_n} is larger than the maximum observation of the sample. This necessity to extrapolate sample results to areas where no data are observed occurs in reliability [8], hydrology [21], finance [9], ... We establish in Section 2 the asymptotic normality of $\hat\theta_n$ and $\hat x_{p_n}$. The asymptotic mean-square error of some particular members of (1.1) are compared in Section 3. In particular, it is shown that family (1.1) encompasses the estimator introduced in [12] and denoted by $\hat\theta_n^{(2)}$ in the sequel. In this paper, the asymptotic normality of $\hat\theta_n^{(2)}$ is obtained under weaker conditions. Furthermore, we show that other members of family (1.1) should be preferred in some typical situations. We also quote some other estimators of θ which do not belong to family (1.1): [4, 3, 6, 19]. We refer to [12] for a comparison with $\hat\theta_n^{(2)}$. The asymptotic results are illustrated in Section 4 on finite sample situations. Proofs are postponed to Section 5.

2. ASYMPTOTIC NORMALITY

To establish the asymptotic normality of $\hat{\theta}_n$, we need a second-order condition on ℓ:

(A.2) There exist $\rho \le 0$ and $b(x) \to 0$ such that uniformly locally on $\lambda \ge 1$

$$\log\left(\frac{\ell(\lambda x)}{\ell(x)}\right) \sim b(x)\, K_\rho(\lambda)\,, \qquad \text{when} \quad x \to \infty\,,$$

with $K_\rho(\lambda) = \int_1^\lambda u^{\rho-1} du$.

It can be shown [11] that necessarily $|b| \in \mathcal{R}_\rho$. The second order parameter $\rho \le 0$ tunes the rate of convergence of $\ell(\lambda x)/\ell(x)$ to 1. The closer ρ is to 0, the slower is the convergence. Condition (A.2) is the cornerstone in all proofs of asymptotic normality for extreme value estimators. It is used in [18, 17, 5] to prove the asymptotic normality of estimators of the extreme value index γ. In regular case, as noted in [13], one can choose $b(x) = x\,\ell'(x)/\ell(x)$ leading to

$$b(x) \;=\; \frac{x\,e^{-x}}{F^{-1}(1-e^{-x})\, f\big(F^{-1}(1-e^{-x})\big)} - \theta\,, \tag{2.1}$$

where f is the density function associated to F. Let us introduce the following functions: for $t > 0$ and $\rho \le 0$,

$$\mu_\rho(t) \;=\; \int_0^\infty K_\rho\Big(1+\frac{x}{t}\Big)\, \mathrm{e}^{-x}\, dx\,,$$

$$\sigma_\rho^2(t) \;=\; \int_0^\infty K_\rho^2\Big(1+\frac{x}{t}\Big)\, \mathrm{e}^{-x}\, dx \;-\; \mu_\rho^2(t)\,,$$

and let $a_n = \mu_0(\log(n/k_n))/T_n - 1$. As a preliminary result, we propose an asymptotic expansion of $(\hat{\theta}_n - \theta)$:

Proposition 2.1. *Suppose (A.1) and (A.2) hold. If $k_n \to \infty$, $k_n/n \to 0$, $T_n \log(n/k_n) \to 1$ and $k_n^{1/2} b(\log(n/k_n)) \to \lambda \in \mathbb{R}$ then,*

$$k_n^{1/2}(\hat{\theta}_n - \theta) \;=\; \\ =\; \theta\xi_{n,1} + \theta\mu_0\big(\log(n/k_n)\big)\xi_{n,2} + k_n^{1/2}\,\theta a_n + k_n^{1/2}\, b\big(\log(n/k_n)\big)\big(1 + o_{\mathrm{P}}(1)\big)\,,$$

where $\xi_{n,1}$ and $\xi_{n,2}$ converge in distribution to a standard normal distribution.

Similar distributional representations exist for various estimators of the extreme value index γ. They are used in [16] to compare the asymptotic properties of several tail index estimators. In [15], a bootstrap selection of k_n is derived from such a representation. It is also possible to derive bias reduction method as in [14]. The asymptotic normality of $\hat{\theta}_n$ is a straightforward consequence of Proposition 2.1.

Theorem 2.1. *Suppose (A.1) and (A.2) hold. If* $k_n \to \infty$, $k_n/n \to 0$, $T_n \log(n/k_n) \to 1$ *and* $k_n^{1/2}\, b(\log(n/k_n)) \to \lambda \in \mathbb{R}$ *then,*

$$k_n^{1/2}\Big(\hat{\theta}_n - \theta - b\big(\log(n/k_n)\big) - \theta a_n\Big) \xrightarrow{d} \mathcal{N}(0, \theta^2)\ . \qquad \square$$

Theorem 2.1 implies that the Asymptotic Mean Square Error (AMSE) of $\hat{\theta}_n$ is given by:

$$AMSE(\hat{\theta}_n) \;=\; \Big(\theta a_n + b\big(\log(n/k_n)\big)\Big)^2 + \frac{\theta^2}{k_n}\ . \tag{2.2}$$

It appears that all estimators of family (1.1) share the same variance. The bias depends on two terms $b(\log(n/k_n))$ and θa_n. A good choice of T_n (depending on the function b) could lead to a sequence a_n cancelling the bias. Of course, in the general case, the function b is unknown making difficult the choice of a "universal" sequence T_n. This is discussed in the next section.

Clearly, the best rate of convergence in Theorem 2.1 is obtained by choosing $\lambda \neq 0$. In this case, the expression of the intermediate sequence (k_n) is known.

Proposition 2.2. *If* $k_n \to \infty$, $k_n/n \to 0$ *and* $k_n^{1/2}\, b(\log(n/k_n)) \to \lambda \neq 0$,

$$k_n \sim \left(\frac{\lambda}{b\big(\log(n)\big)}\right)^2 = \lambda^2 \big(\log(n)\big)^{-2\rho}\, L\big(\log(n)\big)\ ,$$

where L is a slowly varying function.

The "optimal" rate of convergence is thus of order $(\log(n))^{-\rho}$, which is entirely determined by the second order parameter ρ: small values of $|\rho|$ yield slow convergence. The asymptotic normality of the extreme quantile estimator (1.2) can be deduced from Theorem 2.1:

Theorem 2.2. *Suppose (A.1) and (A.2) hold. If moreover,* $k_n \to \infty$, $k_n/n \to 0$, $T_n \log(n/k_n) \to 1$, $k_n^{1/2} b(\log(n/k_n)) \to 0$ *and*

$$1 \;\le\; \liminf \tau_n \;\le\; \limsup \tau_n \;<\; \infty \tag{2.3}$$

then,

$$\frac{k_n^{1/2}}{\log \tau_n}\left(\frac{\hat{x}_{p_n}}{x_{p_n}} - \tau_n^{\theta a_n}\right) \xrightarrow{d} \mathcal{N}(0, \theta^2)\ .$$

3. COMPARISON OF SOME ESTIMATORS

First, we propose some choices of the sequence (T_n) leading to different estimators of the Weibull tail-coefficient. Their asymptotic distributions are provided, and their AMSE are compared.

3.1. Some examples of estimators

– The natural choice is clearly to take

$$T_n = T_n^{(1)} =: \mu_0\big(\log(n/k_n)\big) ,$$

in order to cancel the bias term a_n. This choice leads to a new estimator of θ defined by:

$$\hat{\theta}_n^{(1)} = \frac{1}{\mu_0\big(\log(n/k_n)\big)} \frac{1}{k_n} \sum_{i=1}^{k_n} \Big(\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})\Big) .$$

Remarking that

$$\mu_\rho(t) = \mathrm{e}^t \int_1^\infty \mathrm{e}^{-tu}\, u^{\rho-1}\, du$$

provides a simple computation method for $\mu_0(\log(n/k_n))$ using the Exponential Integral (EI), see for instance [1], Chapter 5, pages 225–233.

– Girard [12] proposes the following estimator of the Weibull tail-coefficient:

$$\hat{\theta}_n^{(2)} = \sum_{i=1}^{k_n} \Big(\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})\Big) \Big/ \sum_{i=1}^{k_n} \Big(\log_2(n/i) - \log_2(n/k_n)\Big) ,$$

where $\log_2(x) = \log(\log(x))$, $x > 1$. Here, we have

$$T_n = T_n^{(2)} =: \frac{1}{k_n} \sum_{i=1}^{k_n} \log\left(1 - \frac{\log(i/k_n)}{\log(n/k_n)}\right) .$$

It is interesting to remark that $T_n^{(2)}$ is a Riemann's sum approximation of $\mu_0(\log(n/k_n))$ since an integration by parts yields:

$$\mu_0(t) = \int_0^1 \log\left(1 - \frac{\log(x)}{t}\right) dx .$$

– Finally, choosing T_n as the asymptotic equivalent of $\mu_0(\log(n/k_n))$,

$$T_n = T_n^{(3)} =: 1/\log(n/k_n)$$

leads to the estimator:

$$\hat{\theta}_n^{(3)} = \frac{\log(n/k_n)}{k_n} \sum_{i=1}^{k_n} \Big(\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})\Big) .$$

For $i = 1, 2, 3$, let us denote by $\hat{x}_{p_n}^{(i)}$ the extreme quantile estimator built on $\hat{\theta}_n^{(i)}$ by (1.2). Asymptotic normality of these estimators is derived from Theorem 2.1 and Theorem 2.2. To this end, we introduce the following conditions:

(C.1) $k_n/n \to 0$,

(C.2) $\log(k_n)/\log(n) \to 0$,

(C.3) $k_n/n \to 0$ and $k_n^{1/2}/\log(n/k_n) \to 0$.

Our result is the following:

Corollary 3.1. *Suppose (A.1) and (A.2) hold together with* $k_n \to \infty$ *and* $k_n^{1/2}\, b(\log(n/k_n)) \to 0$. *For* $i = 1, 2, 3$*:*

i) *If (C.i) hold then*

$$k_n^{1/2}\big(\hat\theta_n^{(i)} - \theta\big) \xrightarrow{d} \mathcal{N}(0, \theta^2)\ .$$

ii) *If (C.i) and (2.3) hold, then*

$$\frac{k_n^{1/2}}{\log \tau_n}\left(\frac{\hat x_{p_n}^{(i)}}{x_{p_n}} - 1\right) \xrightarrow{d} \mathcal{N}(0, \theta^2)\ .$$

In view of this corollary, the asymptotic normality of $\hat\theta_n^{(1)}$ is obtained under weaker conditions than $\hat\theta_n^{(2)}$ and $\hat\theta_n^{(3)}$, since (C.2) implies (C.1). Let us also highlight that the asymptotic distribution of $\hat\theta_n^{(2)}$ is obtained under less assumptions than in [12], Theorem 2, the condition $k_n^{1/2}/\log(n/k_n) \to 0$ being not necessary here. Finally, note that, if b is not ultimately zero, condition $k_n^{1/2} b(\log(n/k_n)) \to 0$ implies (C.2) (see Lemma 5.1).

3.2. Comparison of the AMSE of the estimators

We use the expression of the AMSE given in (2.2) to compare the estimators proposed previously.

Theorem 3.1. *Suppose (A.1) and (A.2) hold together with* $k_n \to \infty$, $\log(k_n)/\log(n) \to 0$ *and* $k_n^{1/2}\, b(\log(n/k_n)) \to \lambda \in \mathbb{R}$. *Several situations are possible:*

i) *b is ultimately non-positive. Introduce* $\alpha = -4 \lim_{n\to\infty} b(\log n) \dfrac{k_n}{\log k_n} \in [0, +\infty]$.
If $\alpha > \theta$, *then, for n large enough,*

$$AMSE\big(\hat\theta_n^{(2)}\big) \;<\; AMSE\big(\hat\theta_n^{(1)}\big) \;<\; AMSE\big(\hat\theta_n^{(3)}\big)\ .$$

If $\alpha < \theta$, *then, for n large enough,*

$$AMSE\big(\hat\theta_n^{(1)}\big) \;<\; \min\Big(AMSE\big(\hat\theta_n^{(2)}\big), AMSE\big(\hat\theta_n^{(3)}\big)\Big)\ .$$

ii) *b is ultimately non-negative. Let us introduce $\beta = 2 \lim_{x\to\infty} xb(x) \in [0, +\infty]$. If $\beta > \theta$ then, for n large enough,*

$$AMSE\big(\hat\theta_n^{(3)}\big) \;<\; AMSE\big(\hat\theta_n^{(1)}\big) \;<\; AMSE\big(\hat\theta_n^{(2)}\big) \;.$$

If $\beta < \theta$ then, for n large enough,

$$AMSE\big(\hat\theta_n^{(1)}\big) \;<\; \min\Big(AMSE\big(\hat\theta_n^{(2)}\big), AMSE\big(\hat\theta_n^{(3)}\big)\Big) \;.$$

It appears that, when b is ultimately non-negative (case ii)), the conclusion does not depend on the sequence (k_n). The relative performances of the estimators is entirely determined by the nature of the distribution: $\hat\theta_n^{(1)}$ has the best behavior, in terms of AMSE, for distributions close to the Weibull distribution (small b and thus, small β). At the opposite, $\hat\theta_n^{(3)}$ should be preferred for distributions far from the Weibull distribution.

The case when b is ultimately non-positive (case i)) is different. The value of α depends on k_n, and thus, for any distribution, one can obtain $\alpha = 0$ by choosing small values of k_n(for instance $k_n = -1/b(\log n)$) as well as $\alpha = +\infty$ by choosing large values of k_n (for instance $k_n = (1/b(\log n))^2$ as in Proposition 2.2).

4. NUMERICAL EXPERIMENTS

4.1. Examples of Weibull tail-distributions

Let us give some examples of distributions satisfying (A.1) and (A.2).

Absolute Gaussian distribution: $|\mathcal{N}(\mu, \sigma^2)|$, $\sigma > 0$.

From [9], Table 3.4.4, we have $H^{\leftarrow}(x) = x^\theta \ell(x)$, where $\theta = 1/2$ and an asymptotic expansion of the slowly varying function is given by:

$$\ell(x) \;=\; 2^{1/2}\sigma - \frac{\sigma}{2^{3/2}}\,\frac{\log x}{x} + O(1/x) \;.$$

Therefore $\rho = -1$ and $b(x) = \log(x)/(4x) + O(1/x)$. b is ultimately positive, which corresponds to case ii) of Theorem 3.1 with $\beta = +\infty$. Therefore, one always has, for n large enough:

$$AMSE\big(\hat\theta_n^{(3)}\big) \;<\; AMSE\big(\hat\theta_n^{(1)}\big) \;<\; AMSE\big(\hat\theta_n^{(2)}\big) \;. \tag{4.1}$$

Gamma distribution: $\Gamma(a, \lambda),\ a, \lambda > 0$.

We use the following parameterization of the density

$$f(x) = \frac{\lambda^a}{\Gamma(a)}\, x^{a-1} \exp(-\lambda x) .$$

From [9], Table 3.4.4, we obtain $H^{\leftarrow}(x) = x^\theta \ell(x)$ with $\theta = 1$ and

$$\ell(x) = \frac{1}{\lambda} + \frac{a-1}{\lambda} \frac{\log x}{x} + O(1/x) .$$

We thus have $\rho = -1$ and $b(x) = (1-a)\log(x)/x + O(1/x)$. If $a > 1$, b is ultimately negative, corresponding to case i) of Theorem 3.1. The conclusion depends on the value of k_n as explained in the preceding section. If $a < 1$, b is ultimately positive, corresponding to case ii) of Theorem 3.1 with $\beta = +\infty$. Therefore, we are in situation (4.1).

Weibull distribution: $\mathcal{W}(a, \lambda),\ a, \lambda > 0$.

The inverse failure rate function is $H^{\leftarrow}(x) = \lambda x^{1/a}$, and then $\theta = 1/a$, $\ell(x) = \lambda$ for all $x > 0$. Therefore $b(x) = 0$ and we use the usual convention $\rho = -\infty$. One may apply either i) or ii) of Theorem 3.1 with $\alpha = \beta = 0$ to get for n large enough,

$$AMSE\big(\hat\theta_n^{(1)}\big) < \min\Big(AMSE\big(\hat\theta_n^{(2)}\big), AMSE\big(\hat\theta_n^{(3)}\big)\Big) . \tag{4.2}$$

4.2. Numerical results

The finite sample performance of the estimators $\hat\theta_n^{(1)}$, $\hat\theta_n^{(2)}$ and $\hat\theta_n^{(3)}$ are investigated on 5 different distributions: $\Gamma(0.5, 1)$, $\Gamma(1.5, 1)$, $|\mathcal{N}(0, 1)|$, $\mathcal{W}(2.5, 2.5)$ and $\mathcal{W}(0.4, 0.4)$. In each case, $N = 200$ samples $(\mathcal{X}_{n,i})_{i=1,...,N}$ of size $n = 500$ were simulated. On each sample $(\mathcal{X}_{n,i})$, the estimates $\hat\theta_{n,i}^{(1)}(k)$, $\hat\theta_{n,i}^{(2)}(k)$ and $\hat\theta_{n,i}^{(3)}(k)$ are computed for $k = 2, ..., 150$. Finally, the associated Mean Square Error (MSE) plots are built by plotting the points

$$\left(k,\ \frac{1}{N}\sum_{i=1}^{N}\Big(\hat\theta_{n,i}^{(j)}(k) - \theta\Big)^2\right), \qquad j = 1, 2, 3 .$$

They are compared to the AMSE plots (see (2.2) for the definition of the AMSE):

$$\left(k,\ \Big(\theta a_n^{(j)} + b\big(\log(n/k)\big)\Big)^2 + \frac{\theta^2}{k}\right), \qquad j = 1, 2, 3 ,$$

and where b is given by (2.1). It appears on Figure 1 – Figure 5 that, for all the above mentioned distributions, the MSE and AMSE have a similar qualitative behavior. Figure 1 and Figure 2 illustrate situation (4.1) corresponding to ultimately positive bias functions. The case of an ultimately negative bias function is presented on Figure 3 with the $\Gamma(1.5, 1)$ distribution. It clearly appears that the MSE associated to $\hat\theta_n^{(3)}$ is the largest. For small values of k, one has $MSE(\hat\theta_n^{(1)}) < MSE(\hat\theta_n^{(2)})$ and $MSE(\hat\theta_n^{(1)}) > MSE(\hat\theta_n^{(2)})$ for large value of k. This phenomenon is the illustration of the asymptotic result presented in Theorem 3.1i). Finally, Figure 4 and Figure 5 illustrate situation (4.2) of asymptotically null bias functions. Note that, the MSE of $\hat\theta_n^{(1)}$ and $\hat\theta_n^{(2)}$ are very similar. As a conclusion, it appears that, in all situations, $\hat\theta_n^{(1)}$ and $\hat\theta_n^{(2)}$ share a similar behavior, with a small advantage to $\hat\theta_n^{(1)}$. They provide good results for null and negative bias functions. At the opposite, $\hat\theta_n^{(3)}$ should be preferred for positive bias functions.

5. PROOFS

For the sake of simplicity, in the following, we note k for k_n. We first give some preliminary lemmas. Their proofs are postponed to the appendix.

5.1. Preliminary lemmas

We first quote a technical lemma.

Lemma 5.1. *Suppose that b is ultimately non-zero. If $k \to \infty$, $k/n \to 0$ and $k^{1/2}\, b(\log(n/k)) \to \lambda \in \mathbb{R}$, then $\log(k)/\log(n) \to 0$.*

The following two lemmas are of analytical nature. They provide first-order expansions which will reveal useful in the sequel.

Lemma 5.2. *For all $\rho \le 0$ and $q \in \mathbb{N}^*$, we have*

$$\int_0^\infty K_\rho^q\Big(1+\frac{x}{t}\Big)\mathrm{e}^{-x}\,dx \sim \frac{q!}{t^q} \qquad \text{as} \quad t \to \infty\ .$$

Let $a_n^{(i)} = \mu_0(\log(n/k_n))/T_n^{(i)} - 1$, for $i = 1, 2, 3$.

Lemma 5.3. *Suppose $k \to \infty$ and $k/n \to 0$.*

i) $T_n^{(1)} \log(n/k) \to 1$ *and* $a_n^{(1)} = 0$.

ii) $T_n^{(2)} \log(n/k) \to 1$. *If moreover* $\log(k)/\log(n) \to 0$ *then* $a_n^{(2)} \sim \log(k)/(2k)$.

iii) $T_n^{(3)} \log(n/k) = 1$ *and* $a_n^{(3)} \sim -1/\log(n/k)$.

The next lemma presents an expansion of $\hat{\theta}_n$.

Lemma 5.4. *Suppose $k \to \infty$ and $k/n \to 0$. Under (A.1) and (A.2), the following expansions hold:*

$$\hat{\theta}_n = \frac{1}{T_n}\Big(\theta U_n^{(0)} + b\big(\log(n/k)\big)\, U_n^{(\rho)}\big(1+o_{\mathrm{P}}(1)\big)\Big) ,$$

where

$$U_n^{(\rho)} = \frac{1}{k}\sum_{i=1}^{k-1} K_\rho\bigg(1 + \frac{F_i}{E_{n-k+1,n}}\bigg) , \qquad \rho \le 0$$

and where $E_{n-k+1,n}$ is the $(n-k+1)$-th order statistics associated to n independent standard exponential variables and $\{F_1, ..., F_{k-1}\}$ are independent standard exponential variables and independent from $E_{n-k+1,n}$.

The next two lemmas provide the key results for establishing the asymptotic distribution of $\hat{\theta}_n$. Their describe they asymptotic behavior of the random terms appearing in Lemma 5.4.

Lemma 5.5. *Suppose $k \to \infty$ and $k/n \to 0$. Then, for all $\rho \le 0$,*

$$\mu_\rho(E_{n-k+1,n}) \overset{P}{\sim} \sigma_\rho(E_{n-k+1,n}) \overset{P}{\sim} \frac{1}{\log(n/k)} .$$

Lemma 5.6. *Suppose $k \to \infty$ and $k/n \to 0$. Then, for all $\rho \le 0$,*

$$\frac{k^{1/2}}{\sigma_\rho(E_{n-k+1,n})}\Big(U_n^{(\rho)} - \mu_\rho(E_{n-k+1,n})\Big) \xrightarrow{d} \mathcal{N}(0,1) .$$

5.2. Proofs of the main results

Proof of Proposition 2.1: Lemma 5.6 states that for $\rho \le 0$,

$$\frac{k^{1/2}}{\sigma_\rho(E_{n-k+1,n})}\Big(U_n^{(\rho)} - \mu_\rho(E_{n-k+1,n})\Big) = \xi_n(\rho) ,$$

where $\xi_n(\rho) \xrightarrow{d} \mathcal{N}(0,1)$ for $\rho \le 0$. Then, by Lemma 5.4

$$\begin{aligned} k^{1/2}(\hat{\theta}_n - \theta) &= \\ &= \theta\,\frac{\sigma_0(E_{n-k+1,n})}{T_n}\,\xi_n(0) \,+\, k^{1/2}\theta\bigg(\frac{\mu_0(E_{n-k+1,n})}{T_n} - 1\bigg) \\ &\quad + k^{1/2}\, b\big(\log(n/k)\big)\bigg(\frac{\sigma_\rho(E_{n-k+1,n})}{T_n}\,\frac{\xi_n(\rho)}{k^{1/2}} + \frac{\mu_\rho(E_{n-k+1,n})}{T_n}\bigg)\big(1+o_{\mathrm{P}}(1)\big) . \end{aligned}$$

Since $T_n \sim 1/\log(n/k)$ and from Lemma 5.5, we have

$$
\begin{aligned}
& k^{1/2}(\hat{\theta}_n - \theta) \;= \\
(5.1) \qquad & \;= \theta\xi_{n,1} + k^{1/2}\,\theta\left(\frac{\mu_0(E_{n-k+1,n})}{T_n} - 1\right) + k^{1/2}\, b\big(\log(n/k)\big)\big(1 + o_{\mathrm{P}}(1)\big) \;,
\end{aligned}
$$

where $\xi_{n,1} \xrightarrow{d} \mathcal{N}(0,1)$. Moreover, a first-order expansion of μ_0 yields

$$
\frac{\mu_0(E_{n-k+1,n})}{\mu_0\big(\log(n/k)\big)} \;=\; 1 + \Big(E_{n-k+1,n} - \log(n/k)\Big)\,\frac{\mu_0^{(1)}(\eta_n)}{\mu_0\big(\log(n/k)\big)} \;,
$$

where $\eta_n \in \,]\min(E_{n-k+1,n}, \log(n/k)),\, \max(E_{n-k+1,n}, \log(n/k))[$ and

$$
\mu_0^{(1)}(t) \;=\; \frac{d}{dt}\int_0^\infty \log\Big(1 + \frac{x}{t}\Big) e^{-x}\, dx \;=:\; \frac{d}{dt}\int_0^\infty f(x,t)\, dx \;.
$$

Since for $t \geq T > 0$, $f(.,t)$ is integrable, continuous and

$$
\left|\frac{\partial f(x,t)}{\partial t}\right| \;=\; \frac{x}{t^2}\Big(1 + \frac{x}{t}\Big)^{-1} e^{-x} \;\leq\; x\,\frac{e^{-x}}{T^2} \;,
$$

we have that

$$
\mu_0^{(1)}(t) \;=\; -\int_0^\infty \frac{x}{t^2}\Big(1 + \frac{x}{t}\Big)^{-1} e^{-x}\, dx \;.
$$

Then, Lebesgue Theorem implies that $\mu_0^{(1)}(t) \sim -1/t^2$ as $t \to \infty$. Therefore, $\mu_0^{(1)}$ is regularly varying at infinity and thus

$$
\frac{\mu_0^{(1)}(\eta_n)}{\mu_0\big(\log(n/k)\big)} \;\overset{P}{\sim}\; \frac{\mu_0^{(1)}\big(\log(n/k)\big)}{\mu_0\big(\log(n/k)\big)} \;\sim\; -\frac{1}{\log(n/k)} \;.
$$

Since $k^{1/2}(E_{n-k+1,n} - \log(n/k)) \xrightarrow{d} \mathcal{N}(0,1)$ (see [12], Lemma 1), we have

$$
(5.2) \qquad \frac{\mu_0(E_{n-k+1,n})}{\mu_0\big(\log(n/k)\big)} \;=\; 1 - \frac{k^{-1/2}}{\log(n/k)}\,\xi_{n,2} \;,
$$

where $\xi_{n,2} \xrightarrow{d} \mathcal{N}(0,1)$. Collecting (5.1), (5.2) and taking into account that $T_n \log(n/k) \to 1$ concludes the proof. □

Proof of Proposition 2.2: Lemma 5.1 entails $\log(n/k) \sim \log(n)$. Since $|b|$ is a regularly varying function, $b(\log(n/k)) \sim b(\log(n))$ and thus, $k^{1/2} \sim \lambda/b(\log(n))$. □

Proof of Theorem 2.2: The asymptotic normality of $\hat{x}_{p_n}$ can be deduced from the asymptotic normality of $\hat{\theta}_n$ using Theorem 2.3 of [10]. We are in the situation, denoted by (S.2) in the above mentioned paper, where the limit distribution of $\hat{x}_{p_n}/x_{p_n}$ is driven by $\hat{\theta}_n$. Following, the notations of [10], we denote by $\alpha_n = k_n^{1/2}$ the asymptotic rate of convergence of $\hat{\theta}_n$, by $\beta_n = \theta a_n$ its asymptotic bias, and by $\mathcal{L} = \mathcal{N}(0, \theta^2)$ its asymptotic distribution. It suffices to verify that

$$\log(\tau_n)\log(n/k) \to \infty \; . \tag{5.3}$$

To this end, note that conditions (2.3) and $p_n < 1/n$ imply that there exists $0 < c < 1$ such that

$$\log(\tau_n) \;>\; c(\tau_n - 1) \;>\; c\left(\frac{\log(n)}{\log(n/k)} - 1\right) \;=\; c\,\frac{\log(k)}{\log(n/k)} \;,$$

which proves (5.3). We thus have

$$\frac{k^{1/2}}{\log \tau_n}\, \tau_n^{-\theta a_n}\left(\frac{\hat{x}_{p_n}}{x_{p_n}} - \tau_n^{\theta a_n}\right) \xrightarrow{d} \mathcal{N}(0, \theta^2) \; .$$

Now, remarking that, from Lemma 5.2, $\mu_0(\log(n/k)) \sim 1/\log(n/k) \sim T_n$, and thus $a_n \to 0$ gives the result. □

Proof of Corollary 3.1: Lemma 5.3 shows that the assumptions of Theorem 2.1 and Theorem 2.2 are verified and that, for $i = 1, 2, 3$, $k^{1/2} a_n^{(i)} \to 0$. □

Proof of Theorem 3.1:

i) First, from (2.2) and Lemma 5.3 iii), since b is ultimately non-positive,

$$AMSE\big(\hat{\theta}_n^{(1)}\big) - AMSE\big(\hat{\theta}_n^{(3)}\big) \;=\; -\theta\big(a_n^{(3)}\big)^2\left(\theta + 2\,\frac{b\big(\log(n/k)\big)}{a_n^{(3)}}\right) \;<\; 0 \; . \tag{5.4}$$

Second, from (2.2),

$$AMSE\big(\hat{\theta}_n^{(2)}\big) - AMSE\big(\hat{\theta}_n^{(1)}\big) \;=\; \theta\big(a_n^{(2)}\big)^2\left(\theta + 2\,\frac{b\big(\log(n/k)\big)}{a_n^{(2)}}\right) \; . \tag{5.5}$$

If b is ultimately non-zero, Lemma 5.1 entails that $\log(n/k) \sim \log(n)$ and consequently, since $|b|$ is regularly varying, $b(\log(n/k)) \sim b(\log(n))$. Thus, from Lemma 5.3 ii),

$$2\,\frac{b\big(\log(n/k)\big)}{a_n^{(2)}} \;\sim\; 4\,b(\log n)\,\frac{k}{\log(k)} \;\to\; -\alpha \; . \tag{5.6}$$

Collecting (5.4)–(5.6) concludes the proof of i).

ii) First, (5.5) and Lemma 5.3 ii) yields

$$AMSE\big(\hat{\theta}_n^{(2)}\big) - AMSE\big(\hat{\theta}_n^{(1)}\big) \;>\; 0 \; , \tag{5.7}$$

since b is ultimately non-negative. Second, if b is ultimately non-zero, Lemma 5.1 entails that $\log(n/k) \sim \log(n)$ and consequently, since $|b|$ is regularly varying, $b(\log(n/k)) \sim b(\log(n))$. Thus, observe that in (5.4),

$$2\,\frac{b\big(\log(n/k)\big)}{a_n^{(3)}} \;\sim\; -2\,b(\log n)(\log n) \;\to\; -\beta\;. \tag{5.8}$$

Collecting (5.4), (5.7) and (5.8) concludes the proof of ii). The case when b is ultimately zero is obtained either by considering $\alpha = 0$ in (5.6), or $\beta = 0$ in (5.8). □

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APPENDIX: PROOF OF LEMMAS

Proof of Lemma 5.1: Remark that, for n large enough,

$$\left|k^{1/2}\, b\big(\log(n/k)\big)\right| \;\le\; \left|k^{1/2}\, b\big(\log(n/k)\big) - \lambda\right| + |\lambda| \;\le\; 1 + |\lambda| \;,$$

and thus, if b is ultimately non-zero,

$$0 \le \frac{1}{2}\,\frac{\log(k)}{\log(n/k)} \le \frac{\log\,(1+|\lambda|)}{\log(n/k)} - \frac{\log\big|b\big(\log(n/k)\big)\big|}{\log(n/k)} \;. \tag{5.9}$$

Since $|b|$ is a regularly varying function, we have that (see [7], Proposition 1.3.6.)

$$\frac{\log\big|b\big(\log(x)\big)\big|}{\log(x)} \to 0 \qquad \text{as} \quad x \to \infty \;.$$

Then, (5.9) implies $\log(k)/\log(n/k) \to 0$ which entails $\log(k)/\log(n) \to 0$. □

Proof of Lemma 5.2: Since for all $x, t > 0$, $tK_\rho(1{+}x/t) < x$, Lebesgue Theorem implies that

$$\begin{aligned}\lim_{t\to\infty} \int_0^\infty \Big(tK_\rho\Big(1+\frac{x}{t}\Big)\Big)^q \mathrm{e}^{-x}\,dx \;&=\; \int_0^\infty \lim_{t\to\infty} \Big(tK_\rho\Big(1+\frac{x}{t}\Big)\Big)^q \mathrm{e}^{-x}\,dx \\ &=\; \int_0^\infty x^q\,\mathrm{e}^{-x}\,dx \;=\; q! \;,\end{aligned}$$

which concludes the proof. □

Proof of Lemma 5.3:

i) Lemma 5.2 shows that $\mu_0(t) \sim 1/t$ and thus $T_n^{(1)} \log(n/k) \to 1$. By definition, $a_n^{(1)} = 0$.

ii) The well-known inequality $-x^2/2 \le \log(1+x) - x \le 0$, $x > 0$ yields

$$(5.10) \qquad -\frac{1}{2}\,\frac{1}{\log(n/k)}\,\frac{1}{k}\sum_{i=1}^{k} \log^2(k/i) \;\le\; \log(n/k)\,T_n^{(2)} - \frac{1}{k}\sum_{i=1}^{k}\log(k/i) \;\le\; 0\;.$$

Now, since when $k \to \infty$,

$$\frac{1}{k}\sum_{i=1}^{k}\log^2(k/i) \to \int_0^1 \log^2(x)\,dx = 2 \quad \text{and} \quad \frac{1}{k}\sum_{i=1}^{k}\log(k/i) \to -\int_0^1 \log(x)\,dx = 1\,,$$

it follows that $T_n^{(2)} \log(n/k) \to 1$. Let us now introduce the function defined on $(0,1]$ by:

$$f_n(x) \;=\; \log\left(1 - \frac{\log(x)}{\log(n/k)}\right)\,.$$

We have:

$$\begin{aligned}
a_n^{(2)} \;&=\; -\frac{1}{T_n^{(2)}}\Big(T_n^{(2)} - \mu_0\big(\log(n/k)\big)\Big)\\
&=\; -\frac{1}{T_n^{(2)}}\left(\frac{1}{k}\sum_{i=1}^{k-1} f_n(i/k) \;-\; \int_0^1 f_n(t)\,dt\right)\\
&=\; -\frac{1}{T_n^{(2)}}\sum_{i=1}^{k-1}\int_{i/k}^{(i+1)/k}\big(f_n(i/k) - f_n(t)\big)\,dt \;+\; \frac{1}{T_n^{(2)}}\int_0^{1/k} f_n(t)\,dt\,.
\end{aligned}$$

Since

$$f_n(t) \;=\; f_n(i/k) + (t - i/k)\,f_n^{(1)}(i/k) + \int_{i/k}^{t}(t-x)\,f_n^{(2)}(x)\,dx\,,$$

where $f_n^{(p)}$ is the pth derivative of f_n, we have:

$$\begin{aligned}
a_n^{(2)} \;&=\; \frac{1}{T_n^{(2)}}\sum_{i=1}^{k-1}\int_{i/k}^{(i+1)/k}(t-i/k)\,f_n^{(1)}(i/k)\,dt\\
&\quad+\; \frac{1}{T_n^{(2)}}\sum_{i=1}^{k-1}\int_{i/k}^{(i+1)/k}\int_{i/k}^{t}(t-x)\,f_n^{(2)}(x)\,dx\,dt \;+\; \frac{1}{T_n^{(2)}}\int_0^{1/k} f_n(t)\,dt\\
&=:\; \Psi_1 + \Psi_2 + \Psi_3\,.
\end{aligned}$$

Let us focus first on the term Ψ_1:

$$\begin{aligned}
\Psi_1 &= \frac{1}{T_n^{(2)}}\,\frac{1}{2k^2}\sum_{i=1}^{k-1} f_n^{(1)}(i/k)\\
&= \frac{1}{2k\,T_n^{(2)}}\int_{1/k}^{1} f_n^{(1)}(x)\,dx \;+\; \frac{1}{2k\,T_n^{(2)}}\left(\frac{1}{k}\sum_{i=1}^{k-1} f_n^{(1)}(i/k) - \int_{1/k}^{1} f_n^{(1)}(x)\,dx\right)\\
&= \frac{1}{2k\,T_n^{(2)}}\Big(f_n(1)-f_n(1/k)\Big) - \frac{1}{2k\,T_n^{(2)}}\sum_{i=1}^{k-1}\int_{i/k}^{(i+1)/k}\Big(f_n^{(1)}(x)-f_n^{(1)}(i/k)\Big)\,dx\\
&=: \Psi_{1,1}-\Psi_{1,2}\;.
\end{aligned}$$

Since $T_n^{(2)} \sim 1/\log(n/k)$ and $\log(k)/\log(n)\to 0$, we have:

$$\Psi_{1,1} \;=\; -\frac{1}{2k\,T_n^{(2)}}\,\log\!\left(1+\frac{\log(k)}{\log(n/k)}\right) \;=\; -\frac{\log(k)}{2k}\big(1+o(1)\big)\;.$$

Furthermore, since, for n large enough, $f_n^{(2)}(x)>0$ for $x\in[0,1]$,

$$\begin{aligned}
O \;\le\; \Psi_{1,2} &\le \frac{1}{2k\,T_n^{(2)}}\sum_{i=1}^{k-1}\int_{i/k}^{(i+1)/k}\Big(f_n^{(1)}\big((i+1)/k\big)-f_n^{(1)}(i/k)\Big)\,dx\\
&= \frac{1}{2k^2\,T_n^{(2)}}\Big(f_n^{(1)}(1)-f_n^{(1)}(1/k)\Big)\\
&= \frac{1}{2k^2\,T_n^{(2)}}\left(-\frac{1}{\log(n/k)}+\frac{k}{\log(n/k)}\left(1+\frac{\log(k)}{\log(n/k)}\right)^{-1}\right)\\
&\sim \frac{1}{2k} \;=\; o\!\left(\frac{\log(k)}{k}\right)\;.
\end{aligned}$$

Thus,

$$\Psi_1 \;=\; -\frac{\log(k)}{2k}\big(1+o(1)\big)\;. \tag{5.11}$$

Second, let us focus on the term Ψ_2. Since, for n large enough, $f_n^{(2)}(x)>0$ for $x\in[0,1]$,

$$\begin{aligned}
0 \;\le\; \Psi_2 &\le \frac{1}{T_n^{(2)}}\sum_{i=1}^{k-1}\int_{i/k}^{(i+1)/k}\int_{i/k}^{(i+1)/k}(t-i/k)\,f_n^{(2)}(x)\,dx\,dt\\
&= \frac{1}{2k^2\,T_n^{(2)}}\Big(f_n^{(1)}(1)-f_n^{(1)}(1/k)\Big) \;=\; o\!\left(\frac{\log(k)}{k}\right)\;.
\end{aligned}\tag{5.12}$$

Finally,

$$\Psi_3 \;=\; \frac{1}{T_n^{(2)}}\int_0^{1/k} -\frac{\log(t)}{\log(n/k)}\,dt + \frac{1}{T_n^{(2)}}\int_0^{1/k}\left(f_n(t)+\frac{\log(t)}{\log(n/k)}\right)dt \;=:\; \Psi_{3,1}+\Psi_{3,2}\;,$$

and we have:

$$\Psi_{3,1} \;=\; \frac{1}{\log(n/k)\,T_n^{(2)}}\,\frac{1}{k}\big(\log(k)+1\big) \;=\; \frac{\log(k)}{k}\big(1+o(1)\big)\;.$$

Furthermore, using the well known inequality: $|\log(1+x)-x| \le x^2/2,\; x>0$, we have:

$$\begin{aligned}
|\Psi_{3,2}| \;&\le\; \frac{1}{2\,T_n^{(2)}}\int_0^{1/k}\bigg(\frac{\log(t)}{\log(n/k)}\bigg)^2 dt\\
&=\; \frac{1}{2\,T_n^{(2)}}\,\frac{1}{k\big(\log(n/k)\big)^2}\Big(\big(\log(k)\big)^2+2\log(k)+2\Big)\\
&\sim\; \frac{\big(\log(k)\big)^2}{2k\log(n/k)} \;=\; o\bigg(\frac{\log(k)}{k}\bigg)\;,
\end{aligned}$$

since $\log(k)/\log(n)\to 0$. Thus,

$$\Psi_3 \;=\; \frac{\log(k)}{k}\,\big(1+o(1)\big)\;. \tag{5.13}$$

We conclude the proof of i) by collecting (5.11)–(5.13).

iii) First, $T_n^{(3)}\log(n/k)=1$ by definition. Besides, we have

$$\begin{aligned}
a_n^{(3)} \;&=\; \frac{\mu_0\big(\log(n/k)\big)}{T_n^{(3)}}-1\\
&=\; \log(n/k)\,\mu_0\big(\log(n/k)\big)-1\\
&=\; \int_0^\infty \log(n/k)\log\bigg(1+\frac{x}{\log(n/k)}\bigg)\mathrm{e}^{-x}\,dx \;-\; 1\\
&=\; \int_0^\infty x\,\mathrm{e}^{-x}\,dx \;-\; \frac{1}{2}\int_0^\infty \frac{x^2}{\log(n/k)}\,\mathrm{e}^{-x}\,dx \;-\; 1+R_n \;=\; -\frac{1}{\log(n/k)}+R_n\;,
\end{aligned}$$

where

$$R_n \;=\; \int_0^\infty \log(n/k)\Bigg(\log\bigg(1+\frac{x}{\log(n/k)}\bigg)-\frac{x}{\log(n/k)}+\frac{x^2}{2\big(\log(n/k)\big)^2}\Bigg)\mathrm{e}^{-x}\,dx\;.$$

Using the well known inequality: $|\log(1+x)-x+x^2/2| \le x^3/3,\; x>0$, we have,

$$|R_n| \;\le\; \frac{1}{3}\int_0^\infty \frac{x^3}{\big(\log(n/k)\big)^2}\,\mathrm{e}^{-x}\,dx \;=\; o\bigg(\frac{1}{\log(n/k)}\bigg)\;,$$

which finally yields $a_n^{(3)} \sim -1/\log(n/k)$. □

Proof of Lemma 5.4: Recall that

$$\hat\theta_n \;=:\; \frac{1}{T_n}\,\frac{1}{k}\sum_{i=1}^{k-1}\Big(\log(X_{n-i+1,n})-\log(X_{n-k+1,n})\Big)\;,$$

and let $E_{1,n},...,E_{n,n}$ be ordered statistics generated by n independent standard exponential random variables. Under (A.1), we have

$$\begin{aligned}\hat\theta_n &\stackrel{d}{=} \frac{1}{T_n}\frac{1}{k}\sum_{i=1}^{k-1}\Big(\log H^{\leftarrow}(E_{n-i+1,n})-\log H^{\leftarrow}(E_{n-k+1,n})\Big)\\ &\stackrel{d}{=} \frac{1}{T_n}\left(\theta\,\frac{1}{k}\sum_{i=1}^{k-1}\log\left(\frac{E_{n-i+1,n}}{E_{n-k+1,n}}\right)+\frac{1}{k}\sum_{i=1}^{k-1}\log\left(\frac{\ell(E_{n-i+1,n})}{\ell(E_{n-k+1,n})}\right)\right).\end{aligned}$$

Define $x_n = E_{n-k+1,n}$ and $\lambda_{i,n} = E_{n-i+1,n}/E_{n-k+1,n}$. It is clear, in view of [12], Lemma 1 that $x_n \xrightarrow{P} \infty$ and $\lambda_{i,n} \xrightarrow{P} 1$. Thus, (A.2) yields that uniformly in $i = 1,...,k-1$:

$$\hat\theta_n \stackrel{d}{=} \frac{1}{T_n}\left(\theta\frac{1}{k}\sum_{i=1}^{k-1}\log\left(\frac{E_{n-i+1,n}}{E_{n-k+1,n}}\right)+\big(1+o_p(1)\big)\,b(E_{n-k+1,n})\,\frac{1}{k}\sum_{i=1}^{k-1}K_\rho\left(\frac{E_{n-i+1,n}}{E_{n-k+1,n}}\right)\right).$$

The Rényi representation of the Exp(1) ordered statistics (see [2], p. 72) yields

$$\left\{\frac{E_{n-i+1,n}}{E_{n-k+1,n}}\right\}_{i=1,...,k-1} \stackrel{d}{=} \left\{1+\frac{F_{k-i,k-1}}{E_{n-k+1,n}}\right\}_{i=1,...,k-1}, \tag{5.14}$$

where $\{F_{1,k-1},...,F_{k-1,k-1}\}$ are ordered statistics independent from $E_{n-k+1,n}$ and generated by $k-1$ independent standard exponential variables $\{F_1,...,F_{k-1}\}$. Therefore,

$$\begin{aligned}\hat\theta_n \stackrel{d}{=} \frac{1}{T_n}\Bigg(&\theta\,\frac{1}{k}\sum_{i=1}^{k-1}\log\left(1+\frac{F_i}{E_{n-k+1,n}}\right)\\ &+\big(1+o_p(1)\big)\,b(E_{n-k+1,n})\,\frac{1}{k}\sum_{i=1}^{k-1}K_\rho\left(1+\frac{F_i}{E_{n-k+1,n}}\right)\Bigg).\end{aligned}$$

Remarking that $K_0(x) = \log(x)$ concludes the proof. □

Proof of Lemma 5.5: Lemma 5.2 implies that,

$$\mu_\rho(E_{n-k+1,n}) \stackrel{P}{\sim} \frac{1}{E_{n-k+1,n}} \stackrel{P}{\sim} \frac{1}{\log(n/k)},$$

since $E_{n-k+1,n}/\log(n/k) \xrightarrow{P} 1$ (see [12], Lemma 1). Next, from Lemma 5.2,

$$\begin{aligned}\sigma^2_\rho(E_{n-k+1,n}) &= \frac{2}{E^2_{n-k+1,n}}\big(1+o_{\mathrm P}(1)\big)-\frac{1}{E^2_{n-k+1,n}}\big(1+o_{\mathrm P}(1)\big)\\ &= \frac{1}{E^2_{n-k+1,n}}\big(1+o_{\mathrm P}(1)\big) = \frac{1}{\big(\log(n/k)\big)^2}\big(1+o_{\mathrm P}(1)\big),\end{aligned}$$

which concludes the proof. □

Proof of Lemma 5.6: Remark that

$$\frac{k^{1/2}}{\sigma_\rho(E_{n-k+1,n})}\Big(U_n^{(\rho)} - \mu_\rho(E_{n-k+1,n})\Big) =$$
$$= \frac{k^{-1/2}}{\sigma_\rho(E_{n-k+1,n})}\sum_{i=1}^{k-1}\left(K_\rho\left(1+\frac{F_i}{E_{n-k+1,n}}\right) - \mu_\rho(E_{n-k+1,n})\right) - k^{-1/2}\,\frac{\mu_\rho(E_{n-k+1,n})}{\sigma_\rho(E_{n-k+1,n})} .$$

Let us introduce the following notation:

$$S_n(t) = \frac{(k-1)^{-1/2}}{\sigma_\rho(t)}\sum_{i=1}^{k-1}\left(K_\rho\left(1+\frac{F_i}{t}\right) - \mu_\rho(t)\right) .$$

Thus,

$$\frac{k^{1/2}}{\sigma_\rho(E_{n-k+1,n})}\Big(U_n^{(\rho)} - \mu_\rho(E_{n-k+1,n})\Big) = S_n(E_{n-k+1,n})\big(1+o(1)\big) + o_{\mathrm{P}}(1) ,$$

from Lemma 5.5. It remains to prove that for $x \in \mathbb{R}$,

$$\mathrm{P}\Big(S_n(E_{n-k+1,n}) \le x\Big) - \Phi(x) \to 0 \qquad \text{as} \quad n\to\infty ,$$

where Φ is the cumulative distribution function of the standard Gaussian distribution. Lemma 5.2 implies that for all $\varepsilon \in \,]0,1[$, there exists T_ε such that for all $t \ge T_\varepsilon$,

$$\frac{q!}{t^q}(1-\varepsilon) \le \mathbb{E}\left(\left(K_\rho\left(1+\frac{F_1}{t}\right)\right)^q\right) \le \frac{q!}{t^q}(1+\varepsilon) . \tag{5.15}$$

Furthermore, for $x \in \mathbb{R}$,

$$\mathrm{P}\Big(S_n(E_{n-k+1,n}) \le x\Big) - \Phi(x) =$$
$$= \int_0^{T_\varepsilon}\Big(\mathrm{P}\big(S_n(t)\le x\big) - \Phi(x)\Big)\,h_n(t)\,dt + \int_{T_\varepsilon}^{\infty}\Big(\mathrm{P}\big(S_n(t)\le x\big) - \Phi(x)\Big)\,h_n(t)\,dt$$
$$=: A_n + B_n ,$$

where h_n is the density of the random variable $E_{n-k+1,n}$. First, let us focus on the term A_n. We have,

$$|A_n| \le 2\,\mathrm{P}\big(E_{n-k+1,n} \le T_\varepsilon\big) .$$

Since $E_{n-k+1,n}/\log(n/k) \xrightarrow{P} 1$ (see [12], Lemma 1), it is easy to show that $A_n \to 0$. Now, let us consider the term B_n. For the sake of simplicity, let us denote:

$$\left\{Y_i = K_\rho\left(1+\frac{F_i}{t}\right) - \mu_\rho(t), \quad i=1,...,k-1\right\} .$$

Clearly, $Y_1, ..., Y_{k-1}$ are independent, identically distributed and centered random variables. Furthermore, for $t \geq T_\varepsilon$,

$$\begin{aligned}
\mathbb{E}\big(|Y_1|^3\big) &\leq \mathbb{E}\left(\left(K_\rho\left(1+\frac{F_1}{t}\right)+\mu_\rho(t)\right)^3\right) \\
&= \mathbb{E}\left(\left(K_\rho\left(1+\frac{F_1}{t}\right)\right)^3\right) + \big(\mu_\rho(t)\big)^3 + 3\,\mathbb{E}\left(\left(K_\rho\left(1+\frac{F_1}{t}\right)\right)^2\right)\mu_\rho(t) \\
&\quad + 3\,\mathbb{E}\left(K_\rho\left(1+\frac{F_1}{t}\right)\right)\big(\mu_\rho(t)\big)^2 \\
&\leq \frac{1}{t^3}\,C_1(q,\varepsilon) \;<\; \infty\;,
\end{aligned}$$

from (5.15) where $C_1(q,\varepsilon)$ is a constant independent of t. Thus, from Esseen's inequality (see [20], Theorem 3), we have:

$$\sup_x \Big|\mathrm{P}\big(S_n(t) \leq x\big) - \Phi(x)\Big| \;\leq\; C_2\,L_n\;,$$

where C_2 is a positive constant and

$$L_n \;=\; \frac{(k-1)^{-1/2}}{\big(\sigma_\rho(t)\big)^3}\;\mathbb{E}\big(|Y_1|^3\big)\;.$$

From (5.15), since $t \geq T_\varepsilon$,

$$\big(\sigma_\rho(t)\big)^2 \;=\; \mathbb{E}\left(\left(K_\rho\left(1+\frac{F_1}{t}\right)\right)^2\right) - \left(\mathbb{E}\left(K_\rho\left(1+\frac{F_1}{t}\right)\right)\right)^2 \;\geq\; \frac{1}{t^2}\,C_3(\varepsilon)\;,$$

where $C_3(\varepsilon)$ is a constant independent of t. Thus, $L_n \leq (k-1)^{-1/2}\,C_4(q,\varepsilon)$ where $C_4(q,\varepsilon)$ is a constant independent of t, and therefore

$$|B_n| \;\leq\; C_4(q,\varepsilon)\,(k-1)^{-1/2}\,\mathrm{P}\big(E_{n-k+1,n} \geq T_\varepsilon\big) \;\leq\; C_4(q,\varepsilon)\,(k-1)^{-1/2} \;\to\; 0\;,$$

which concludes the proof. □

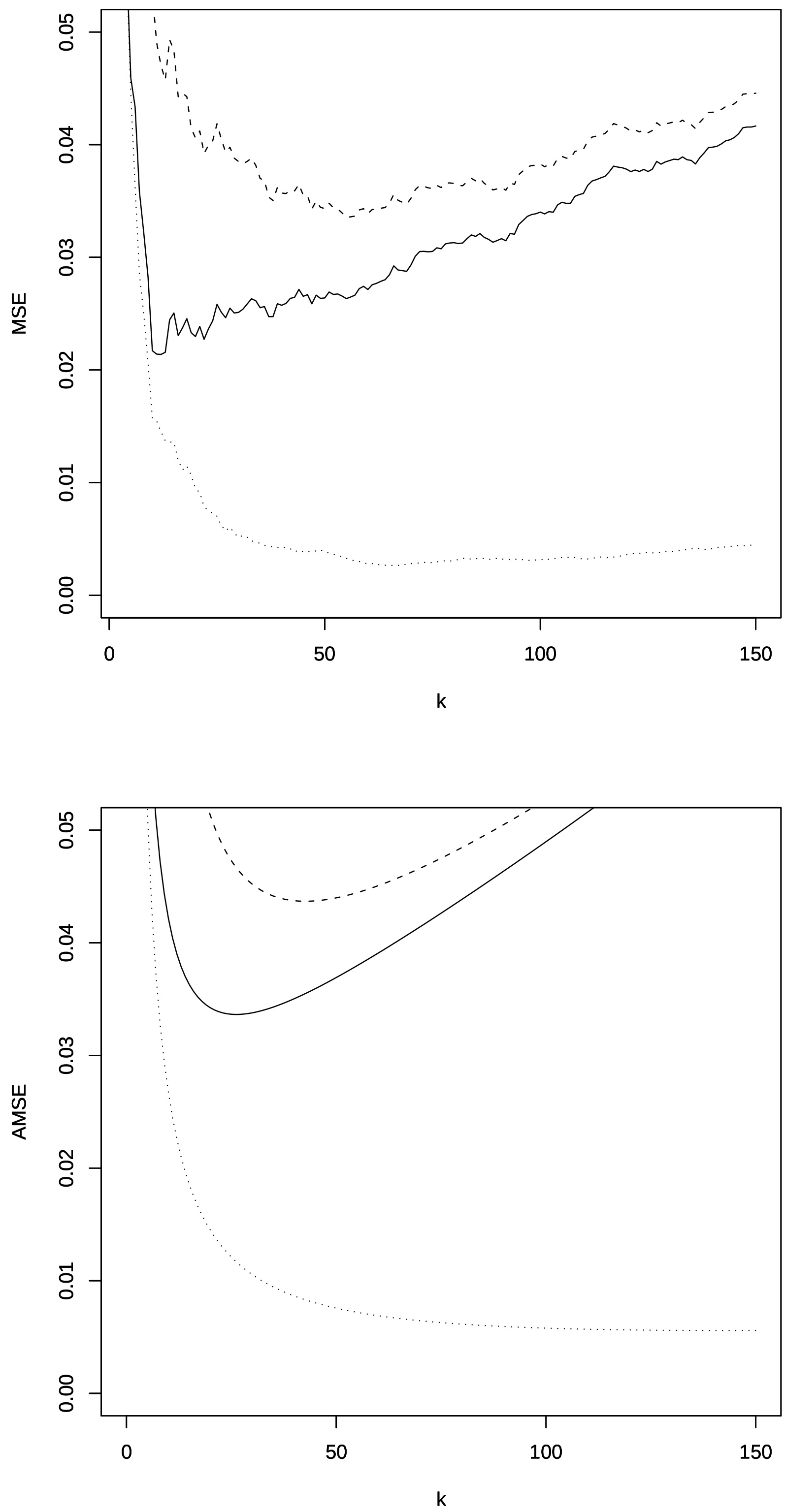


Figure 1: Comparison of estimates $\hat{\theta}_n^{(1)}$ (solid line), $\hat{\theta}_n^{(2)}$ (dashed line) and $\hat{\theta}_n^{(3)}$ (dotted line) for the $|\mathcal{N}(0,1)|$ distribution. Up: MSE, down: AMSE.

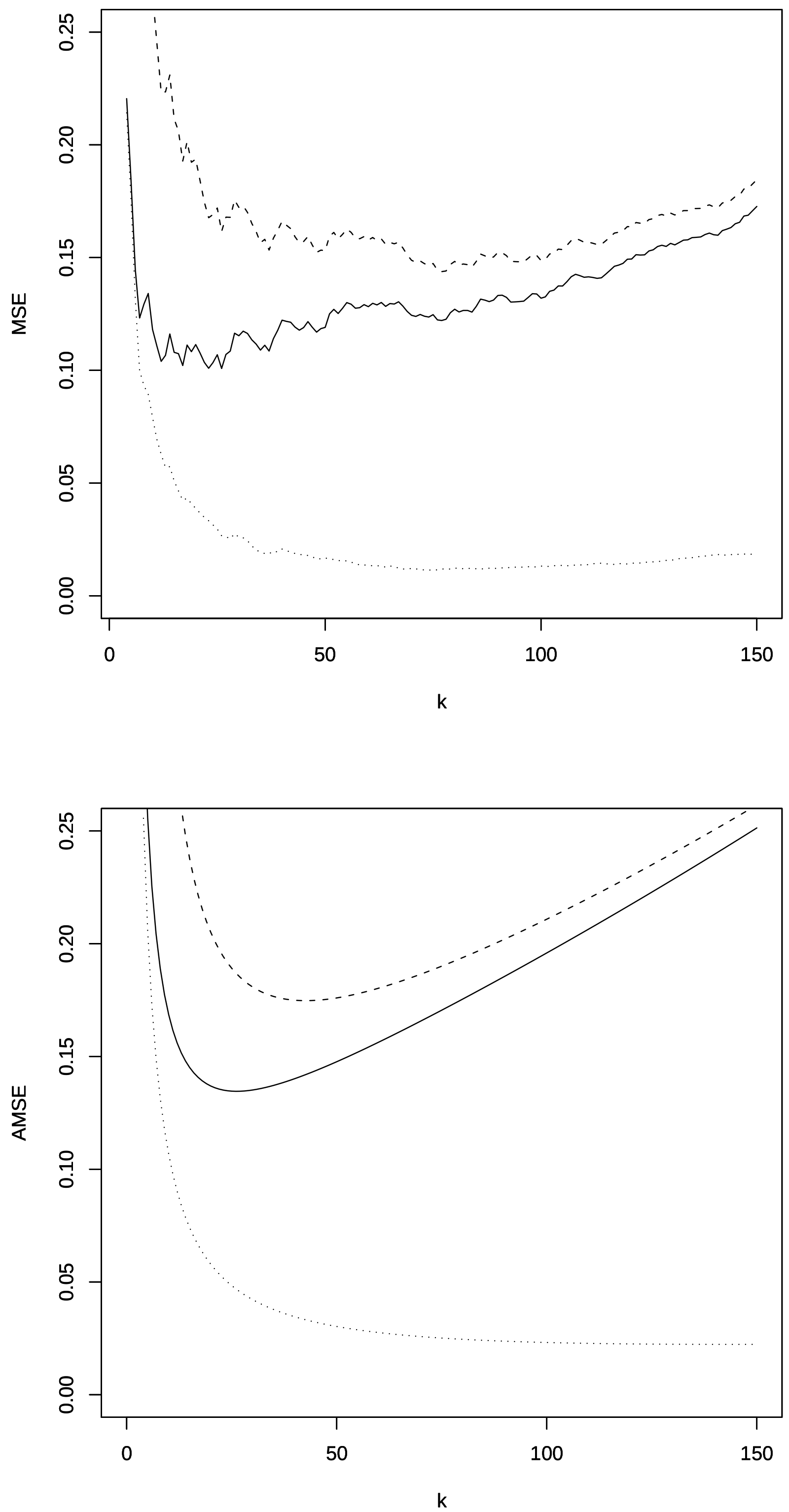


Figure 2: Comparison of estimates $\hat{\theta}_n^{(1)}$ (solid line), $\hat{\theta}_n^{(2)}$ (dashed line) and $\hat{\theta}_n^{(3)}$ (dotted line) for the $\Gamma(0.5, 1)$ distribution. Up: MSE, down: AMSE.

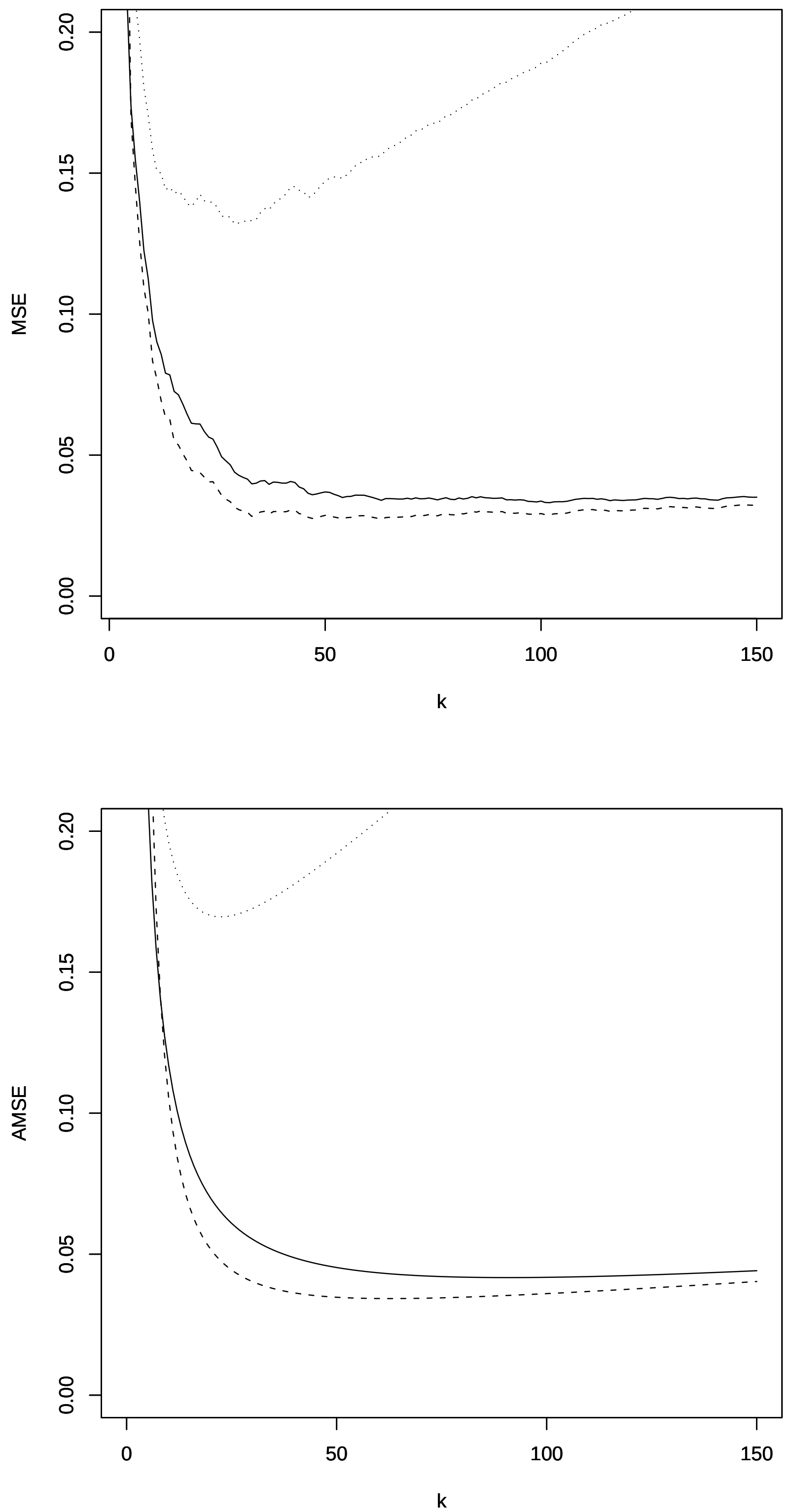
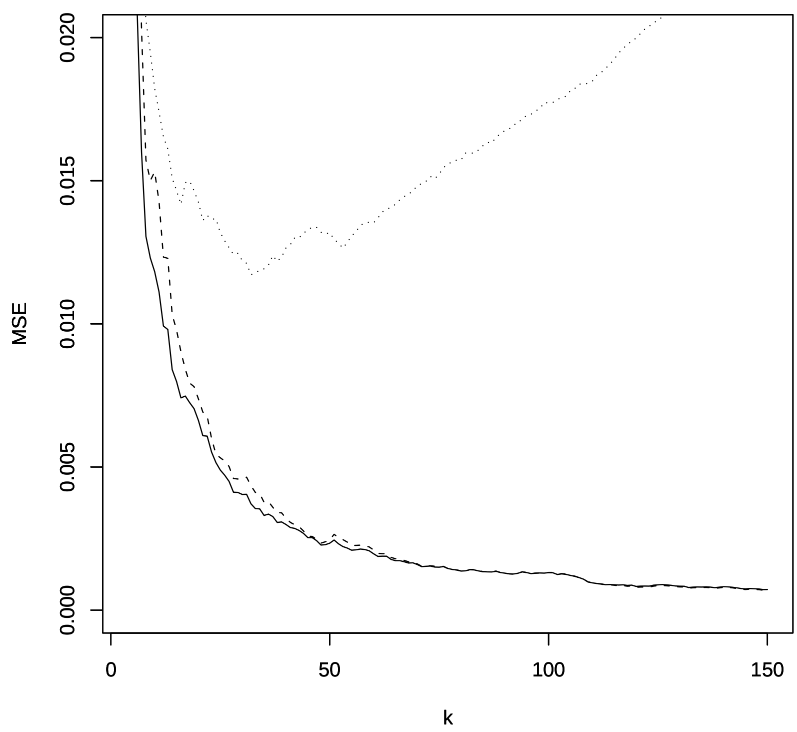


Figure 3: Comparison of estimates $\hat{\theta}_n^{(1)}$ (solid line), $\hat{\theta}_n^{(2)}$ (dashed line) and $\hat{\theta}_n^{(3)}$ (dotted line) for the $\Gamma(1.5, 1)$ distribution. Up: MSE, down: AMSE.



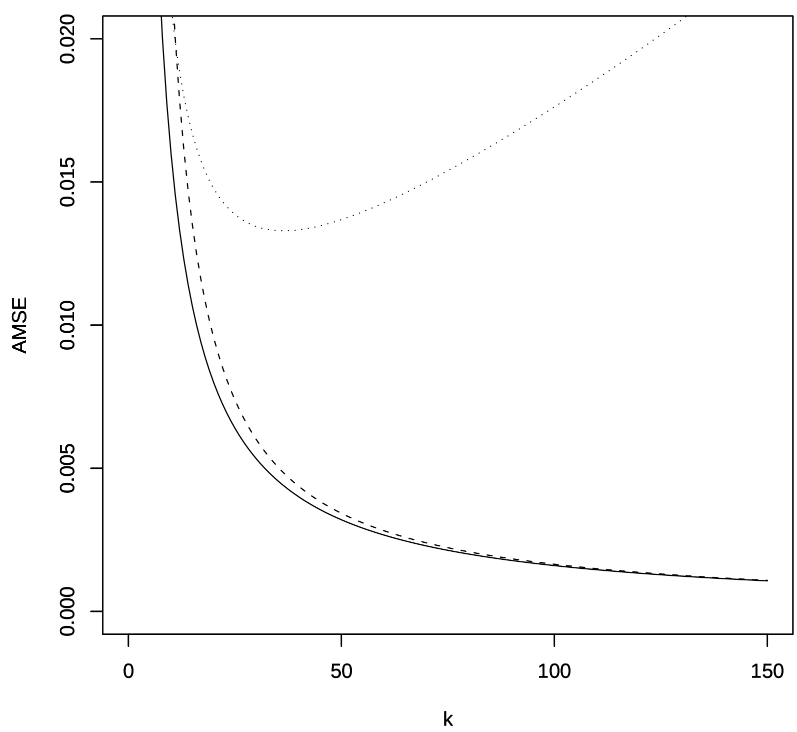


Figure 4: Comparison of estimates $\hat{\theta}_n^{(1)}$ (solid line), $\hat{\theta}_n^{(2)}$ (dashed line) and $\hat{\theta}_n^{(3)}$ (dotted line) for the $\mathcal{W}(2.5, 2.5)$ distribution. Up: MSE, down: AMSE.

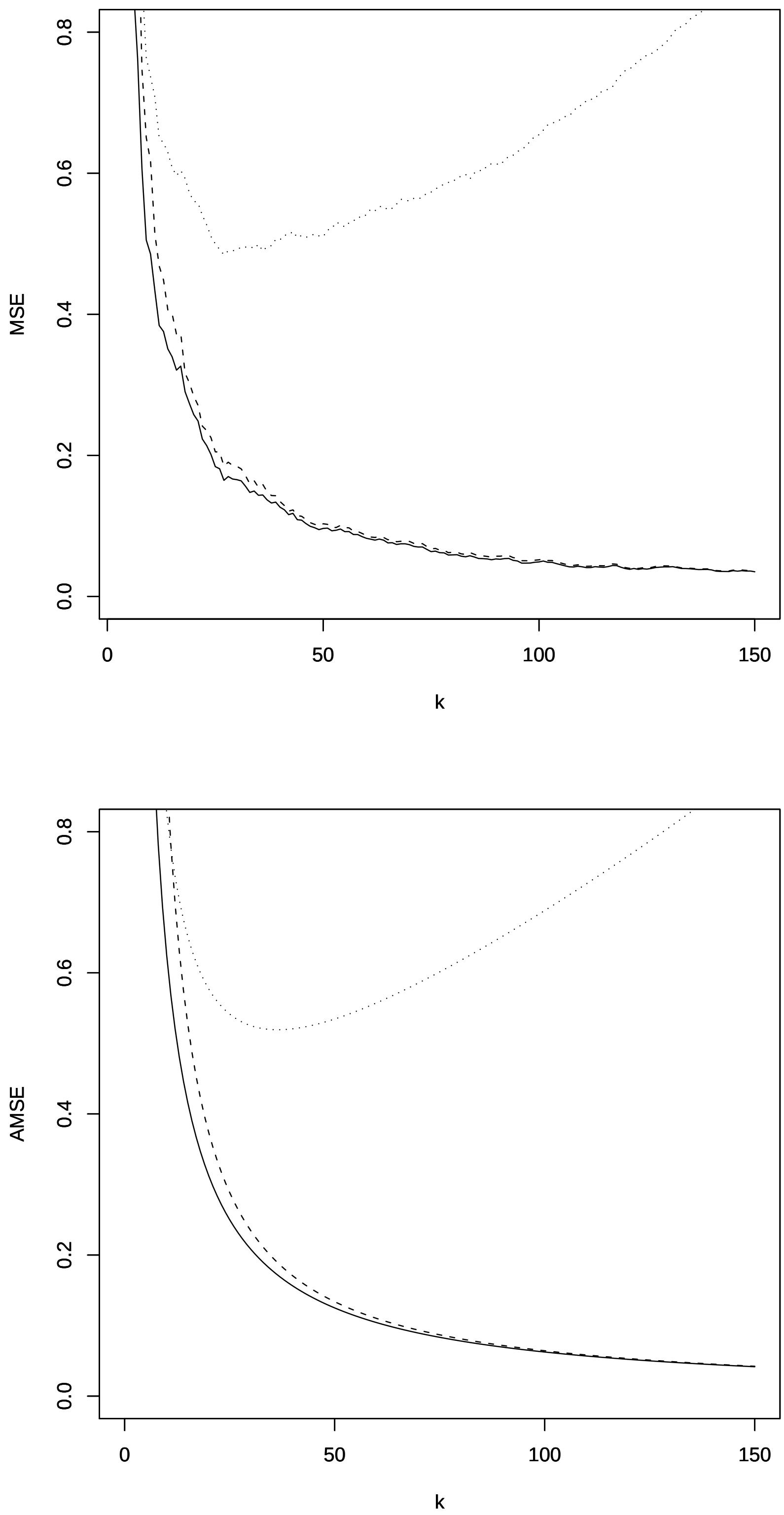


Figure 5: Comparison of estimates $\hat{\theta}_n^{(1)}$ (solid line), $\hat{\theta}_n^{(2)}$ (dashed line) and $\hat{\theta}_n^{(3)}$ (dotted line) for the $\mathcal{W}(0.4, 0.4)$ distribution. Up: MSE, down: AMSE.