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STATISTICAL INFERENCE FOR A GENERAL CLASS OF NONCENTRAL ELLIPTICAL DISTRIBUTIONS

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Abstract:

- In this paper we introduce a new family of noncentral elliptical distributions. This family is generated as the quotient of two independent random variables, one with noncentral standard elliptical distribution and the other the power of a $\mathrm{U}(0,1)$ random variable. For this family of distributions, we derive general properties, including the moments and discuss some special cases based on the family of scale mixtures of normal distributions, where the main advantage is easy simulation and nice hierarchical representation facilitating the implementation of an EM algorithm for maximum likelihood estimation. This new family of distributions provides a robust alternative for parameter estimation in asymmetric distributions. The results and methods are applied to three real datasets, showing that this new distribution fits better than other models reported in the recent statistical literature.

Keywords:

- *noncentral slash-elliptical distribution; elliptical distribution; moments; kurtosis; EM-algorithm.*

AMS Subject Classification:

- 62E10, 62F10.

1. INTRODUCTION

Many univariate or multivariate distributions have been generalized to noncentral versions. These include numerous continuous univariate (Student-t, chi-squared, gamma, beta) distributions. The noncentral Student-t (NCt) distribution is a skewed distribution that has received attention in the statistical inference context. When the mean of a normal distribution is tested, the noncentral distribution describes how a test statistic t is distributed when the null hypothesis is false. That is

$$t_\nu(\lambda) = \frac{Z+\lambda}{\sqrt{U/\nu}},$$

where $Z \sim \mathrm{N}(0,1)$ and $U \sim \chi^2_\nu$ are independent random variables. Lahiri and Teigland ([18]), and Dasgupta and Lahiri ([7]) found the NCt distribution is useful in analyzing survey data and forecasting record data. Tsionas ([29]) used the NCt distribution in linear regression models and applied it to stock market data. Applications of the NCt distribution have been limited by the fact that the probability density function is not expressible in closed form, making the maximum likelihood (ML) estimation difficult. On the other hand, the symmetric Student-t (t) distribution has a long history in statistics to model data with outliers as does as the elliptical (EL) distribution; see for example, Lange *et al.* ([19]), Fang *et al.* ([10]), and Cambanis *et al.* ([6]). A random variable X is said to have an EL distribution with location μ and scale parameter σ, denoted as $X \sim \mathrm{EL}(\mu, \sigma^2; g)$ if its probability density function (pdf) is given by

$$f_X(x) = \frac{1}{\sigma}\, g\left(\left(\frac{x-\mu}{\sigma}\right)^2\right), \tag{1.1}$$

for some nonnegative function $g(u)$, $u \geq 0$, referred to as the density generator which satisfies $\int_0^\infty u^{-\frac{1}{2}}\, g(u)\, du = 1$. Based on this family of EL distributions, Gómez *et al.* ([13]) and Gómez and Venegas ([15]) introduced the slash-elliptical (SEL) family of distributions. These distributions originate from the ratio between two independent random variables, one the standard EL distribution and the other a uniform $(0,1)$ distribution,

$$Y = \frac{Z}{U^{\frac{1}{q}}}, \tag{1.2}$$

where $Z \sim \mathrm{EL}(0,1;g)$ and $U \sim \mathrm{U}(0,1)$ are independent random variables with $q > 0$. The resulting distribution is denoted by $Y \sim \mathrm{SEL}(0,1,q)$, and has heavier tails than the standard normal distribution. On the other hand,when q tends to ∞, the resulting distribution is the standard EL distribution. For example, if $Z \sim \mathrm{N}(0,1)$ and $q = 1$, one obtains the canonic slash distribution,

$$f(y) = \begin{cases} \dfrac{\phi(0)-\phi(y)}{y}, & \text{if } y \neq 0, \\[2ex] \dfrac{\phi(0)}{2}, & \text{if } y = 0, \end{cases} \tag{1.3}$$

where $\phi(\cdot)$ is the pdf of the standard normal distribution. This distribution has heavier tails than the normal distribution, that is, it has higher kurtosis. Properties of this family are discussed in Rogers and Tukey ([28]), Mosteller and Tukey ([24]) and Johnson *et al.* ([16]).

ML estimators for location and scale parameters are discussed in Kafadar ([17]). Wang and Genton ([31]) described multivariate symmetrical and skew-multivariate extensions of the slash (S) distribution. Arslan and Genc ([3]) discussed a symmetric extension of the multivariate slash distribution and Genc ([12]) discussed a symmetric generalization of the slash distribution.

The aim of this paper is to provide an extension of the family of SEL distributions to a family of noncentral (NC) distributions. We derive its properties and method of estimating the model parameters. Also, we present a multivariate extension.

The paper is organized as follows: In Section 2, we present the pdf of the noncentral slash-elliptical (NCSEL) distribution, and some of its properties. Also, moments of order r are obtained, including the asymmetry and kurtosis coefficients. In Section 3, we discuss derivation of moment method and maximum likelihood estimation and report results of using the proposed model in three real applications. Section 4 reports examples using both simulated and real data to illustrate the performance of the proposed method. Section 5 presents a discussion of the multivariate case. Finally, some concluding remarks are given in Section 6.

2. NONCENTRAL SLASH-ELLIPTICAL DISTRIBUTIONS

In this section, we introduce a family of NCSEL distributions, which is defined through the following stochastic representation. A random variable Y represented as

$$Y = \frac{W+\lambda}{U^{\frac{1}{q}}}, \qquad \lambda \in \mathbb{R}, \quad q > 0, \tag{2.1}$$

where $W \sim \mathrm{EL}(0,1;g)$ and $U \sim \mathrm{U}(0,1)$ are independent random variables, is said to have a NCSEL distribution, with λ being the non-centrality parameter and q the kurtosis parameter. This distribution will be denoted by $Y \sim \mathrm{NCSEL}(1,q,\lambda;g)$. Before presenting some of its important properties, we present two special cases. If $W \sim \mathrm{N}(0,1)$, then Y follows a noncentral slash (NCS) distribution, denoted by $Y \sim \mathrm{NCS}(1,q,\lambda)$, while if W follows a t distribution, $t(0,1;\nu)$, then the resulting distribution is a noncentral slash-Student-t (NCSt) distribution, denoted by $Y \sim \mathrm{NCSt}(1,q,\lambda;\nu)$. For the special case of $q=1$, this distribution is called the canonical NCSEL distribution.

2.1. Density function

The stochastic representation in (2.1) is useful to obtain the pdf of Y, as shown in the following result.

Proposition 2.1. *Let $Y \sim \mathrm{NCSEL}(1,q,\lambda;g)$. Then, the pdf of Y is given by*

$$f_Y(y;1,q,\lambda) = \begin{cases} \dfrac{q}{y^{q+1}} \displaystyle\int_{-\lambda}^{y-\lambda} (u+\lambda)^q \, g(u^2)\, du, & \text{if } y \neq 0, \\[2ex] \dfrac{q}{q+1}\, g(\lambda^2), & \text{if } y = 0. \end{cases}$$

Proof: From (2.1), using the fact that U and W are independent and standard calculations (based on the Jacobian of the appropriate transformation), we obtain

$$f_{Y,U}(y,u) = u^{\frac{1}{q}}\, g\Big(\big(y\,u^{\frac{1}{q}} - \lambda\big)^2\Big), \qquad y \in \mathbb{R}, \quad 0 < u < 1.$$

Hence, the marginal pdf of Y is given by

$$f_Y(y;1,q,\lambda) = \int_0^1 u^{\frac{1}{q}}\, g\Big(\big(y\,u^{\frac{1}{q}} - \lambda\big)^2\Big)\, du.$$

Now, by substituting u for $u = y\,t^{\frac{1}{q}} - \lambda$, we have the required results for $y \neq 0$. For $y = 0$, the result is immediate. □

Corollary 2.1. *For the special case $q = 1$, the pdf reduces to the form*

$$f_Y(y;1,1,\lambda) = \begin{cases} \dfrac{1}{y^2}\displaystyle\int_{-\lambda}^{y-\lambda} (u+\lambda)\, g(u^2)\, du, & \text{if } y \neq 0,\\[2ex] \dfrac{1}{2}\, g(\lambda^2), & \text{if } y = 0. \end{cases}$$

Corollary 2.2. *If $W \sim \mathrm{N}(0,1)$, then*

i) *The pdf of Y is*

$$f_Y(y;1,q,\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^1 u^{\frac{1}{q}}\, e^{-\frac{1}{2}\left(y\,u^{\frac{1}{q}} - \lambda\right)^2} du;$$

ii) *For $q = 1$, the pdf of Y can be expressed as*

$$f(y;1,1,\lambda) = \begin{cases} \dfrac{1}{2}\Big\{\phi(\lambda) - \phi(y-\lambda) + \lambda\Big(\Phi(y-\lambda) + \Phi(\lambda) - 1\Big)\Big\}, & \text{if } y \neq 0,\\[2ex] \dfrac{\phi(\lambda)}{2}, & \text{if } y = 0, \end{cases}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and the cumulative distribution function (cdf) of the standard normal distribution, respectively.

Proof: Both parts are direct consequences of Proposition 2.1. In Part i) consider $g(u) = (1/\sqrt{2\pi})\exp(-u/2)$, and in Part ii), for $y \neq 0$, we have

$$f_Y(y;\lambda) = \int_0^1 \frac{u}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y\,u-\lambda)^2} du.$$

Letting $w = y\,u - \lambda$, $f_Y(y;\lambda)$ can be expressed as

$$\begin{aligned} f_Y(y;\lambda) &= \frac{1}{\sqrt{2\pi}\,y^2} \int_0^1 (w+\lambda)\, e^{-\frac{w^2}{2}}\, dw \\ &= \frac{1}{y^2}\left[\frac{1}{\sqrt{2\pi}}\Big(e^{-\frac{\lambda^2}{2}} + e^{-\frac{(y-\lambda)^2}{2}}\Big) + \int_{-\lambda}^{y-\lambda} \phi(w)\, dw\right] \\ &= \frac{1}{y^2}\Big\{\phi(\lambda) - \phi(y-\lambda) + \lambda\Big(\Phi(y-\lambda) + \Phi(\lambda) - 1\Big)\Big\}. \end{aligned}$$

Finally, for $y = 0$, the result is direct. □

Figure 1 illustrates some possible shapes of the pdf of Y for some parameter values of λ. It can be seen that the parameter λ controls the skewness of the distribution. It is also possible to observe that, as $|\lambda|$ increases, the density becomes more skewed. Figure 2 displays some possible shapes of the pdf of Y for some parameter values of q and $\sigma = 1$. From this figure, we note that the parameter q controls the kurtosis of the distribution. Moreover, for smaller values of q we have a heavy-tailed distribution.

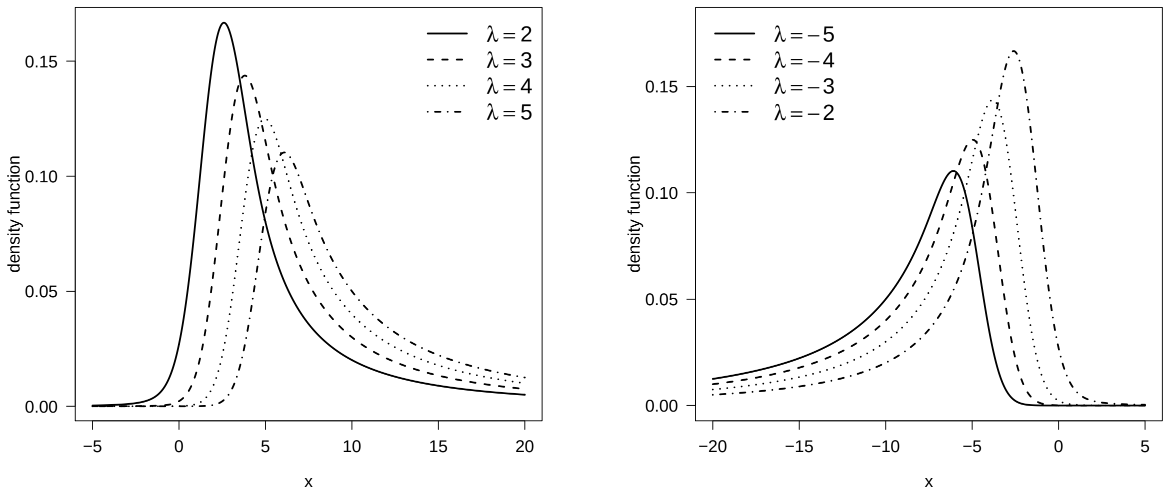

Figure 1: NCS pdf plots for $q = 1$ and different values of λ.

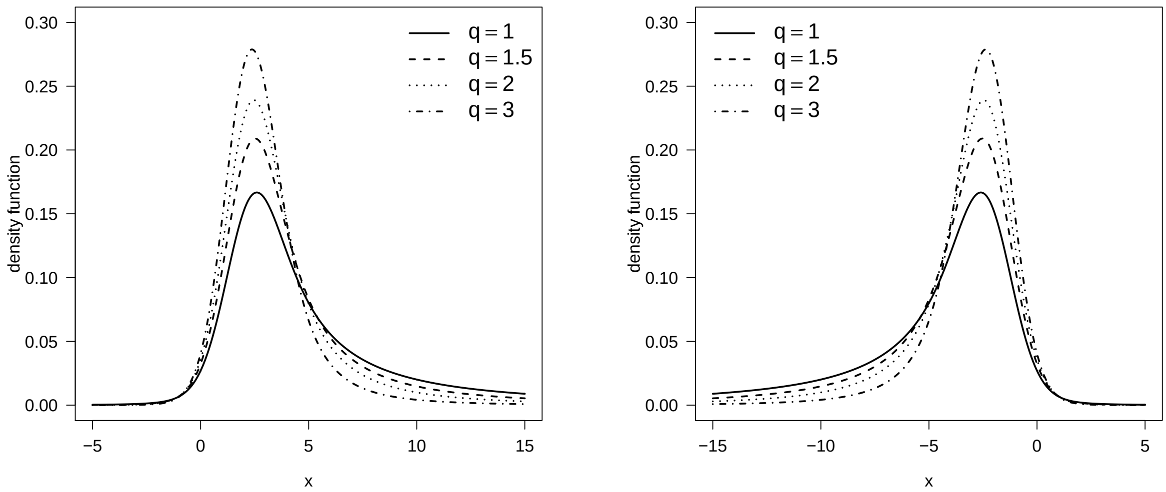

Figure 2: NCS pdf plots for $\lambda = 2$ (left panel) and $\lambda = -2$ (right panel) and different values of q.

A slight extension of the NCSEL distribution is obtained by introducing a scale parameter through the representation

$$Y = \frac{\sigma W + \lambda}{U^{\frac{1}{q}}} = \sigma \frac{W + \delta}{U^{\frac{1}{q}}}, \tag{2.2}$$

where $\delta = \lambda/\sigma$, $W \sim \mathrm{EL}(0, 1; g)$ and $U \sim \mathrm{U}(0, 1)$ are independent, and σ is a scale parameter.

This distribution is denoted by NCSEL$(\sigma, q, \lambda; g)$, and its pdf is given by

$$f_Y(y;\sigma,q,\lambda) \;=\; \frac{1}{\sigma}\int_0^1 u^{\frac{1}{q}}\, g\left(\left(\frac{y\,u^{\frac{1}{q}}-\lambda}{\sigma}\right)^2\right) du.$$

An important class of symmetric distributions is the family of normal/independent (NI) (or scale mixture of normal) distributions, which contains many important unimodal distributions such as the contaminated normal (CN), S, t and Laplace (L) distributions, among others, all possessing heavier tails than the normal. For more information on this family of distributions, see for example, Andrews and Mallows ([2]) and Lange and Sinsheimer ([20]). A random variable W is said to have a standard NI distribution, if it can be related to the normal distribution through the stochastic representation $W = V^{-1/2} Z_0$, where $Z_0 \sim \mathrm{N}(0,1)$ is independent of the positive random variable V. The pdf of W can be expressed as

$$\phi_{\mathrm{NI}}(w) \;=\; \int_0^\infty \frac{v^{1/2}}{\sqrt{2\pi}}\, \exp\left\{-\frac{v}{2}\,w^2\right\} dH_V(v;\boldsymbol{\nu}), \tag{2.3}$$

where $H_V(\,\cdot\,;\boldsymbol{\nu})$ is the cdf of V, indexed by a scalar or vector of parameters $\boldsymbol{\nu}$. The distribution of W is denoted by $W \sim \mathrm{NI}(0,1;H_V)$. In the EL distribution context, the generator function $g(\cdot)$ for an NI distribution is

$$g(u) \;=\; \int_0^\infty \frac{v^{1/2}}{\sqrt{2\pi}}\, \exp\left\{-\frac{v}{2}\,u\right\} dH_V(v), \qquad v>0. \tag{2.4}$$

Some special cases of the family of NI distributions are for example:

1) *The* CN *distribution:* Here V has pdf given by $h_V(v) = \nu\,\mathbb{I}_{\{\gamma\}}(v) + (1-\nu)\,\mathbb{I}_{\{1\}}(v)$, $0<\nu<1$, $0<\gamma<1$, where $\mathbb{I}_A(\cdot)$ denotes the indicator function of the set A and $\boldsymbol{\nu} = (\nu,\gamma)^\top$. Then, the pdf of W is

$$\phi_{\mathrm{NI}}(w) \;=\; \Big[\nu\,\sqrt{\gamma}\;\phi\big(\sqrt{\gamma}\,w\big) + (1-\nu)\,\phi\big(w\big)\Big], \qquad y\in\mathbb{R}.$$

2) *The* S *distribution:* Here $V \sim \mathrm{Beta}(\nu,1)$ and the pdf of W is

$$\phi_{\mathrm{NI}}(w) \;=\; \nu\int_0^1 v^{\nu-1}\,\phi(w;0,v^{-1})\,dv, \qquad w\in\mathbb{R}.$$

3) *The t distribution:* Here $V \sim \mathrm{Gamma}(\nu/2,\nu/2)$, so the t distribution has as special cases the Cauchy model for $\nu=1$ and the normal model as $\nu\to\infty$, and the pdf of W is

$$\phi_{\mathrm{NI}}(w) = k(\nu)\,\nu^{\nu/2}\,\big(\nu+w^2\big)^{-\left(\frac{\nu+1}{2}\right)}, \qquad w\in\mathbb{R},$$

where $k(\nu) = \Gamma\big(\frac{\nu+1}{2}\big)/\big[\sqrt{\pi}\;\Gamma\big(\frac{\nu}{2}\big)\big]$.

Remark 2.1. The special case $Y \sim \mathrm{NCS}(\sigma,1,\lambda)$, i.e. $q=1$, will be called as the canonical NCS and its pdf is

$$f(y,\sigma,\lambda) \;=\; \begin{cases} \dfrac{\sigma^2}{y^2}\left[\phi\left(\dfrac{\lambda}{\sigma}\right) - \phi\left(\dfrac{y-\lambda}{\sigma}\right) + \dfrac{\lambda}{\sigma}\left(\Phi\left(\dfrac{y-\lambda}{\sigma}\right) + \Phi\left(\dfrac{\lambda}{\sigma}\right) - 1\right)\right], & \text{if } \; y\neq 0,\\[2ex] \dfrac{\phi(\frac{\lambda}{\sigma})}{2}, & \text{if } \; y=0, \end{cases}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively.

2.2. Properties

In this section, we present some properties of the NCSEL distribution.

Proposition 2.2. *Let* $Y \sim \mathrm{NCSEL}(\sigma, q, \lambda; g)$. *Then,*

i) *If* $\lambda = 0$ *and* $q \to \infty$, *then* $Y \sim \mathrm{EL}(0, \sigma^2; g)$;

ii) *If* $\lambda = 0$, *then* $Y \sim \mathrm{SEL}(0, \sigma^2, q; g)$;

iii) *If* $U_1 = U^{1/q}$ *in* (2.2), *then the conditional pdf of* $U_1 = u$, *given* $Y = y$, *is*

$$f_{U_1|Y}(u|y) \;=\; \frac{q\,u^{q-1}}{f_Y(y)}\; f_{Y|U_1}(y|u)\, I_{(0,1)}(u),$$

where $f_{Y|U_1}(\cdot\,|u)$ *is the pdf of* $\mathrm{EL}\big(\frac{\lambda}{u}, \frac{\sigma^2}{u^2}; g\big)$ *distribution;*

iv) *If* $W = V^{-1/2} Z_0 \sim \mathrm{NI}(0, 1; H_V)$ *in* (2.2), *then the conditional mean of* $U^r V^s$ *for* $r \geq 0$, $s \geq 0$, *given* $Y = y$, *is*

$$\mathrm{E}\big[U^r V^s\,|y\big] \;=\; \frac{q}{\sigma\, f_Y(y)} \int_0^1 u^{r+q} \left[\int_0^\infty \frac{v^{s+1/2}}{\sqrt{2\pi\,\sigma^2}}\, \exp\!\left\{-\frac{v}{2\sigma^2}\,(u y - \lambda)^2\right\} f_V(v)\, dv\right] du.$$

For the special case $V \sim \mathrm{Gamma}(\nu/2, \nu/2)$,

$$\mathrm{E}\big[U^r V^s\,|y\big] \;=\; \frac{q\, d(\nu, s)}{\sigma\, f_Y(y)} \int_0^1 u^{r+q} \left[\nu + \left(\frac{u y - \lambda}{\sigma}\right)^2\right]^{-\frac{2s+\nu+1}{2}} du,$$

where

$$d(\nu, s) \;=\; 2^s\, \nu^{\nu/2}\, \Gamma\!\left(\frac{2s+\nu+1}{2}\right) \Big/ \left(\sqrt{\pi}\; \Gamma\!\left(\frac{\nu}{2}\right)\right).$$

Remark 2.2. We now present some comments on the usefulness of the results proposed in Proposition 2.2:

1) Parts i) and ii) state that the NCSEL distribution contains the elliptical distribution as a special case as $q \to \infty$ and the noncentral parameter is zero ($\lambda = 0$). Moreover, the NCSEL distribution contains as special case the SEL distribution when $\lambda = 0$.

2) Letting $U_2 = U^{-1/q}$ in the representation in (2.2), we can get the following model

$$Y = \mu + \lambda U_2 + U_2 W, \tag{2.5}$$

where $W \sim \mathrm{EL}(0, \sigma^2; g)$ and U_2 are independent and $\mu \in \mathbb{R}$. We note that the conditional distribution of Y, given $U_2 = u$ follows a $Y\,|\,(U_2 = u) \sim \mathrm{EL}(\mu + \lambda u, u\sigma^2; g)$ for some density generator $g(\cdot)$.

3) The distribution in (2.5) is like a variance-mean mixture of the EL distribution proposed by Barndorff-Nielsen ([5]), in which W follows a normal distribution, which has been used in financial empirical studies.

4) Finally, Part iv) is useful to implement the EM-algorithm in ML estimation.

2.3. Moments

In this section, we discuss distributional moments of the NCSEL distribution, an important need in any statistical analysis. Some of the important characteristics of a distribution can be studied through moments, which are used to derive moment estimators, and skewness and kurtosis coefficients.

Proposition 2.3. *Let* $Y \sim \mathrm{NCSEL}(\sigma, q, \lambda; g)$ *such that*

$$Y = \frac{\sigma W + \lambda}{U^{\frac{1}{q}}} = \sigma \frac{W + \delta}{U^{\frac{1}{q}}},$$

where $\delta = \frac{\lambda}{\sigma}$*. Then, for* $r = 1, 2, 3, \ldots$ *and* $q > r$, $\mathrm{E}[Y^r] = \sigma^r \mu_r$, *where*

$$\mu_r = \mathrm{E}[X^r] = \frac{q}{q-r} \sum_{k=0}^{r} \binom{r}{k} \delta^{r-k} a_{k/2},$$

with $X \sim \mathrm{NCSEL}(1, q, \delta; g)$ *and* $a_{k/2} = \int_{-\infty}^{\infty} x^k g(x^2)\, dx$.

Proof: Using the stochastic representation of X and Y, and the independence of W and U, we have

$$\mu_r = \mathrm{E}[X^r] = \mathrm{E}\left[\left(\frac{W+\delta}{U^{\frac{1}{q}}}\right)^r\right] = \mathrm{E}\big[(W+\delta)^r\big]\, \mathrm{E}\big[U^{-\frac{r}{q}}\big].$$

Using the binomial theorem for $(W + \delta)^r$ and applying expectation, we have

$$\mathrm{E}\big[(W+\delta)^r\big] = \sum_{k=0}^{r} \binom{r}{k} \delta^{r-k}\, \mathrm{E}\big[W^k\big],$$

where $\mathrm{E}[W^k] = a_{k/2} = \int_{-\infty}^{\infty} x^k g(x^2)\, dx$. Since $\mathrm{E}[U^{-\frac{r}{q}}] = \frac{q}{q-r}$, $q > r$, we obtain the required result. □

Corollary 2.3. *Let* $Y \sim \mathrm{NCSEL}(\sigma, q, \lambda; g)$*. Then, the mean and variance of* Y *are given by*

$$\mathrm{E}[Y] = \frac{\lambda q}{q-1}, \quad q > 1, \qquad \text{and} \qquad \mathrm{Var}(Y) = \frac{\sigma^2 q}{q-2}\left(\left(\frac{\lambda}{\sigma(q-1)}\right)^2 + a_1\right), \quad q > 2.$$

Proposition 2.4. *Let* $Y \sim \mathrm{NCSEL}(\sigma, q, \lambda; g)$*. Then, the asymmetry and kurtosis coefficients of* Y *are respectively*

$$\gamma_1 = \frac{\frac{q}{q-3}(\delta^3 + 3a_1) - \frac{3\delta q}{(q-1)(q-2)}(\delta^2 + a_1) + \frac{2\delta^3 q^3}{(q-1)^3}}{\left[\frac{q}{q-2}\left(\frac{\delta^2}{(q-1)^2} + a_1\right)\right]^{\frac{3}{2}}}, \qquad q > 3,$$

$$\beta_2 = \frac{\frac{q}{q-4}(\delta^4 + 6\delta a_1 + a_2) - \frac{4\delta q^2}{(q-1)(q-3)}(\delta^3 + 3a_1) + \frac{6\delta^2 q^3}{(q-1)^2(q-2)}(\delta^2 + a_1) - \frac{3\delta^4 q^4}{(q-1)^4}}{\left[\frac{q}{q-2}\left(\frac{\delta^2}{(q-1)^2} + a_1\right)\right]^2}, \qquad q > 4.$$

Proof: The proof follows by using the formulas of asymmetry and kurtosis coefficients given respectively by

$$\gamma_1 = \frac{\mu_3 - 3\mu_1\mu_2 + 2\,\mu_1^3}{(\mu_2 - \mu_1^2)^{\frac{3}{2}}} \qquad \text{and} \qquad \beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\,\mu_1^2\mu_2 - 3\,\mu_1^4}{(\mu_2 - \mu_1^2)^2},$$

where μ_k, $k = 1, ..., 4$, as given in Proposition 2.3. □

Figure 3 displays graphs for the asymmetry coefficient and kurtosis coefficient of the NCS distribution.

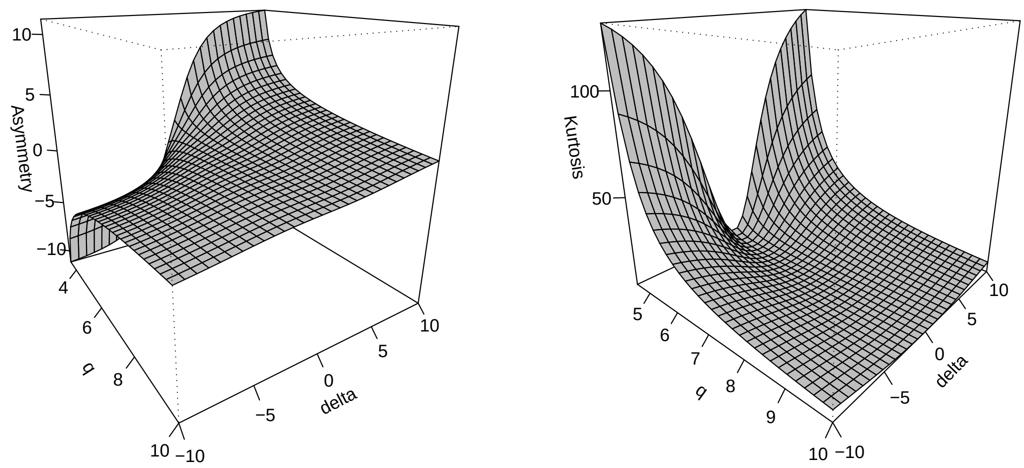

Figure 3: Graphs for the asymmetry coefficient (left) and the kurtosis coefficient (right) of the NCS distribution.

3. INFERENCE

Here, we discuss the moment method (MM) and ML estimation for parameters λ, σ and q of the NCSEL distribution based on a random sample $Y_1, ..., Y_n$ of $Y \sim \text{NCSEL}(\sigma, q, \lambda; g)$. We present the MM estimation and then the ML estimation.

3.1. Method of moment estimation

We discuss an MM estimation based on the distributional moments which are presented in the following result.

Proposition 3.1. *The moment estimators of λ and σ are*

$$\widehat{\lambda}_M(\widehat{q}_M) = \frac{\overline{Y}(\widehat{q}_M - 1)}{\widehat{q}_M} \qquad \text{and} \qquad \widehat{\sigma}_M(\widehat{q}_M) = \sqrt{\frac{1}{a_1}\left(\frac{S^2(\widehat{q}_M - 2)}{\widehat{q}_M} - \frac{\widehat{\lambda}^2_M}{(\widehat{q}_M - 1)^2}\right)},$$

where $a_1 = \mathrm{E}[W^2]$, whereas the moment estimator of q is the solution in the interval $(3, \infty)$ for the nonlinear equation

$$(q-3)\,\overline{Y^3} - q\big(\sigma_M(q)\big)^3 \left(\left(\frac{\lambda_M(q)}{\sigma_M(q)}\right)^3 + 3\,a_1\right) = 0.$$

Proof: These equations follow from Proposition 2.3 and Corollary 2.3. □

3.2. Maximum likelihood estimation

We now discuss the ML estimation for a sample of size n. The log-likelihood function for the parameters σ, q and λ can be written as

$$\ell(\sigma, q, \lambda) = -n\log(\sigma) + \sum_{i=1}^{n} \log G(y_i), \tag{3.1}$$

where $G(y_i) = G(y_i; \sigma, q, \lambda) = \int_0^1 v^{\frac{1}{q}}\, g\left(\left(\frac{y_i v^{\frac{1}{q}} - \lambda}{\sigma}\right)^2\right) dv$ and hence the likelihood equations are given by

$$\sum_{i=1}^{n} \frac{G_\sigma(y_i)}{G(y_i)} = \frac{n}{\sigma}, \qquad \sum_{i=1}^{n} \frac{G_q(y_i)}{G(y_i)} = 0, \qquad \sum_{i=1}^{n} \frac{G_\lambda(y_i)}{G(y_i)} = 0, \tag{3.2}$$

where $G_\sigma(y_i) = \frac{\partial}{\partial\sigma}G(y_i)$, $G_q(y_i) = \frac{\partial}{\partial q}G(y_i)$, $G_\lambda(y_i) = \frac{\partial}{\partial\lambda}G(y_i)$, which can be expressed as

$$\begin{aligned}
G_\sigma(y_i) &= -\frac{2}{\sigma}\int_0^1 u^{\frac{1}{q}}\, g'(t_i^2)\, t_i^2\, du, \\
G_q(y_i) &= -\frac{1}{\sigma q^2}\int_0^1 u^{\frac{1}{q}} \log(u)\left(\sigma\, g(t_i^2) + 2\, t_i\, y_i\, g'(t_i^2)\right) du, \\
G_\lambda(y_i) &= -\frac{2}{\sigma}\int_0^1 u^{\frac{1}{q}}\, g'(t_i^2)\, t_i\, du,
\end{aligned}$$

where $t_i = (y_i\, u^{\frac{1}{q}} - \lambda)/\sigma$. Solutions for equations in (3.2) can be obtained using numerical procedures such as the Newton–Raphson procedure. This procedure requires the maximization of the log-likelihood function which involves integrals that make the maximization difficult, especially when the NCSEL model is based on a bimodal elliptical distribution. But when the NCSEL model is based on the family of the NI distributions, an EM algorithm can be implemented to obtain the ML estimates of the model parameters, as we show next.

3.3. EM algorithm

The EM-algorithm is a well known technique for the ML estimation when unobserved (or missing) data or latent variables are present while modeling. This estimation algorithm enables computationally efficient determination of the ML estimates when iterative methods are required. For a random sample of size n of the NCSEL$(\sigma, q, \lambda; \boldsymbol{\nu})$ model, let $\mathbf{y} = (y_1, ..., y_n)^\top$ be observed data, and let $\mathbf{u} = (u_1, ..., u_n)^\top$ and $\mathbf{v} = (v_1, ..., v_n)$ be unobserved data, so the complete dataset is $\mathbf{y}_\mathrm{c} = (\mathbf{y}^\top, \mathbf{v}^\top, \mathbf{u}^\top)^\top$. In what follows, we describe the implementation of the EM-algorithm for the ML estimation of the parameters of the NCSEL model. For this purpose, we first present the NCSEL model in an incomplete-data framework, where the model can be written hierarchically as

$$\begin{aligned} \mathbf{Y} \mid U_i = u_i, V_i = v_i &\sim \mathrm{N}\big(u_i^{-1}\lambda, \sigma^2 u_i^{-2} v_i^{-1}\big), \\ U_i \mid V_i = v_i &\sim \mathrm{Beta}(q, 1), \\ V_i &\sim h(\cdot). \end{aligned} \tag{3.3}$$

The complete-data log-likelihood function for $\boldsymbol{\theta} = (\sigma, q, \lambda)^\top$ given $\mathbf{y}_\mathrm{c}$ (without the additive constant) is given by

$$\ell_\mathrm{c}(\boldsymbol{\theta}|\mathbf{y}_\mathrm{c}) = -\frac{n}{2}\log\sigma^2 + \frac{1}{2}\sum_{i=1}^{n}\log(u_i^2 v_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\Big(u_i^2 v_i y_i^2 - 2 u_i v_i \lambda y_i + \lambda^2 v_i\Big) + \ell_\mathrm{c}(q|\mathbf{y}_\mathrm{c}),$$

where $\ell_\mathrm{c}(q|\mathbf{y}_\mathrm{c}) = \sum_{i=1}^{n} \ell_{\mathrm{c}i}(q|\mathbf{y}_\mathrm{c})$, with $\ell_{\mathrm{c}i}(q|\mathbf{y}_\mathrm{c}) = \log q + (q-1)\log u_i$. Letting $\widehat{u_i v_i} = \mathrm{E}(U_i V_i | \mathbf{y}_i, \boldsymbol{\theta} = \widehat{\boldsymbol{\theta}})$, $\widehat{u_i^2 v_i} = \mathrm{E}(U_i^2 V_i | \mathbf{y}_i, \boldsymbol{\theta} = \widehat{\boldsymbol{\theta}})$ and $\widehat{v_i} = \mathrm{E}(V_i | \mathbf{y}_i, \boldsymbol{\theta} = \widehat{\boldsymbol{\theta}})$. The conditional expectation of the complete-data log-likelihood function (without the additive constant) is given by $Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}) = \mathrm{E}\big[\ell_\mathrm{c}(\boldsymbol{\theta}|\mathbf{y}_\mathrm{c}) \,|\, \mathbf{y}, \widehat{\boldsymbol{\theta}}\big] = \sum_{i=1}^{n} Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})$, where $Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})$ has the form

$$Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}) = -\frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Big(\widehat{u_i^2 v_i}\, y_i^2 - 2\lambda\, \widehat{u_i v_i}\, y_i + \lambda^2 \widehat{v_i}\Big) + Q_{\mathrm{c}i}(q|\widehat{\boldsymbol{\theta}}),$$

where $Q_{\mathrm{c}i}(q|\widehat{\boldsymbol{\theta}}) = \log q + (q-1)S_i$, with $S_i = \mathrm{E}\big[\log U_i \,|\, \mathbf{y}_i\big]$, $i = 1, ..., n$. Since the quantity S_i does not have closed form, it must be computed numerically. We follow the idea from Lee and Xu ([21]) and Reyes *et al.* ([27]) to compute $Q_{\mathrm{c}i}(q|\widehat{\boldsymbol{\theta}})$. Specifically, let $\{u_r; r = 1, ..., R\}$ be a sample randomly drawn from the conditional distribution $U_i \,|\, (Y_i = y_i, \boldsymbol{\theta} = \widehat{\boldsymbol{\theta}})$, so the quantity $Q_{\mathrm{c}i}(q|\widehat{\boldsymbol{\theta}})$ can be approximated as follows:

$$Q_{\mathrm{c}i}(q|\widehat{\boldsymbol{\theta}}) \approx \frac{1}{R}\sum_{r=1}^{R} \ell_{\mathrm{c}i}(q|u_r).$$

We then have the EM-algorithm for the ML estimation of the parameters of the NCSEL model as follows:

E-Step: Given $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}^{(k)} = \big(\widehat{\sigma}^{(k)}, \widehat{q}^{(k)}, \widehat{\lambda}^{(k)}\big)^\top$, compute $\widehat{u_i v_i}^{(k)}$, $\widehat{u_i^2 v_i}^{(k)}$ and $\widehat{v_i}^{(k)}$, for $i = 1, ..., n$;

CM-step I: Update $\widehat{\lambda}^{(k)}$ and $\widehat{\sigma}^{(k)}$ and maximize $Q\big(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}^{(k)}\big)$ over λ and σ, which leads to the expressions:

$$\begin{aligned}\widehat{\lambda}^{(k+1)} &= \frac{\sum_{i=1}^{n} \widehat{u_i v_i}^{(k)} y_i}{\sum_{i=1}^{n} \widehat{v_i}^{(k)}},\\ \widehat{\sigma}^{2(k+1)} &= \frac{1}{n}\sum_{i=1}^{n}\Big(\widehat{u_i^2 v_i}^{(k)} y_i^2 - 2\,\widehat{\lambda}^{(k+1)}\,\widehat{u_i v_i}^{(k)} y_i + \widehat{\lambda}^{2(k+1)}\,\widehat{v_i}^{(k)}\Big);\end{aligned}$$

CM-step II: Fix $\lambda = \widehat{\lambda}^{(k)}$ and $\sigma^2 = \widehat{\sigma^2}^{(k)}$, update $q^{(k)}$ by

$$\widehat{q}^{(k+1)} = \arg\max_{q} Q\Big(\widehat{\lambda}^{(k)}, \widehat{\sigma^2}^{(k)}, q|\widehat{\boldsymbol{\theta}}^{(k)}\Big).$$

The iterations are repeated until a suitable convergence rule is satisfied, say $\big\|\boldsymbol{\theta}^{(l+1)} - \boldsymbol{\theta}^{(l)}\big\|$ sufficiently small. Useful starting values are required to implement this algorithm, and the moment estimates can be used effectively as initial values in the iterative procedure for computing the ML estimates.

3.4. Estimation of standard errors

To compute the standard errors of the ML estimates, we follow the information-based method exploited by Louis ([22]) and Meilijson ([23]), who proposed using of empirical information matrix, which is computed as

$$I_{\mathrm{c}}(\boldsymbol{\theta}|\mathbf{y}) = \sum_{i=1}^{n} s(y_i|\boldsymbol{\theta})\, s(y_i|\boldsymbol{\theta})^{\top} - \frac{1}{n}\, S(\mathbf{y}|\boldsymbol{\theta})\, S(\mathbf{y}|\boldsymbol{\theta})^{\top},$$

where $S(\mathbf{y}|\boldsymbol{\theta}) = \sum_{i=1}^{n} s(y_i|\boldsymbol{\theta})$, with $s(y_i|\boldsymbol{\theta}) = \mathrm{E}\big[\big(\partial\ell(\boldsymbol{\theta}|\mathbf{y}_{\mathrm{c}i})/\partial\boldsymbol{\theta}\big)\,\big|\,y_i, \boldsymbol{\theta}\big]$ being the empirical score function for the i-th individual, which can be written as

$$s(y_i|\boldsymbol{\theta}) = \Big(\partial Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})/\partial\sigma,\ \partial Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})/\partial q,\ \partial Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})/\partial\lambda\Big)^{\top},$$

whose elements are given by

$$\begin{aligned}\partial Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})/\partial\sigma &= -\frac{1}{\sigma} + \frac{1}{\sigma^3}\Big(\widehat{u_i^2 v_i}\, y_i^2 - 2\,\lambda\,\widehat{u_i v_i}\, y_i + \lambda^2\,\widehat{v_i}\Big),\\ \partial Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})/\partial q &= \frac{1}{q} + \mathrm{E}\big[\log U_i\,|\,y_i\big],\\ \partial Q_i(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}})/\partial\lambda &= \frac{1}{\sigma^2}\Big(\widehat{u_i v_i}\, y_i - \lambda\,\widehat{v_i}\Big).\end{aligned}$$

Now, replacing $\boldsymbol{\theta}$ by its ML estimates $\widehat{\boldsymbol{\theta}}$ in $I_{\mathrm{c}}(\boldsymbol{\theta}|\mathbf{y})$, we obtain

$$I_{\mathrm{c}}(\widehat{\boldsymbol{\theta}}|\mathbf{y}) = \sum_{i=1}^{n} s(y_i|\widehat{\boldsymbol{\theta}})\, s(y_i|\widehat{\boldsymbol{\theta}})^{\top} - \frac{1}{n}\, S(\mathbf{y}|\widehat{\boldsymbol{\theta}})\, S(\mathbf{y}|\widehat{\boldsymbol{\theta}})^{\top},$$

which is used to compute the standard errors of the ML estimates.

4. ILLUSTRATIVE EXAMPLES

4.1. Simulation study

For each scenario, we simulate data based on the stochastic representation of the model presented in (2.1). The objective of this simulation study is to evaluate if the estimation algorithm developed in Section 3.3 can recover the parameters with which the simulation is performed. We consider two special cases of NCSEL models based on the NCS distribution (Table 1) and the NCSt distribution with $\nu = 5$ (Table 2), while for $\nu = 10$ the result is reported in the Appendix (see Table 9). We consider three cases for λ: -0.5, 0.5 and 1.0;

Table 1: Simulation for the NCS distribution.

true values			$\widehat{\theta}$	$n = 50$			$n = 100$			$n = 200$		
λ	σ	q		mean	s.e.	$\sqrt{\text{MSE}}$	mean	s.e.	$\sqrt{\text{MSE}}$	mean	s.e.	$\sqrt{\text{MSE}}$
−0.5	0.5	1	$\widehat{\lambda}$	−0.5223	0.1201	0.1238	−0.5137	0.0837	0.0899	−0.5128	0.0584	0.0678
			$\widehat{\sigma}$	0.5250	0.1203	0.1333	0.5127	0.0826	0.1046	0.5129	0.0579	0.0730
			$\widehat{q}$	1.1230	0.2760	0.4144	1.0610	0.1693	0.1810	1.0391	0.1143	0.1229
		3	$\widehat{\lambda}$	−0.5237	0.1067	0.1125	−0.5160	0.0739	0.0762	−0.5054	0.0502	0.0518
			$\widehat{\sigma}$	0.5150	0.1054	0.1062	0.5118	0.0733	0.0794	0.5032	0.0494	0.0494
			$\widehat{q}$	4.4343	3.7121	3.1171	3.7622	1.7486	2.0391	3.2533	0.7639	0.9762
	1.0	1	$\widehat{\lambda}$	−0.5244	0.1922	0.1874	−0.5143	0.1328	0.1406	−0.5105	0.0926	0.0965
			$\widehat{\sigma}$	1.0459	0.2533	0.2627	1.0377	0.1737	0.2160	1.0269	0.1207	0.1564
			$\widehat{q}$	1.1265	0.3081	0.4225	1.0654	0.1807	0.1958	1.0447	0.1222	0.1346
		3	$\widehat{\lambda}$	−0.5162	0.1782	0.1732	−0.5184	0.1230	0.1256	−0.5065	0.0843	0.0843
			$\widehat{\sigma}$	1.0549	0.2290	0.2290	1.0349	0.1585	0.1624	1.0150	0.1068	0.1120
			$\widehat{q}$	4.9179	5.0027	3.6413	3.9935	2.3857	2.4501	3.3853	1.0160	1.3326
0.5	0.5	1	$\widehat{\lambda}$	0.5257	0.1206	0.1239	0.5151	0.0838	0.0896	0.5104	0.0588	0.0669
			$\widehat{\sigma}$	0.5261	0.1207	0.1270	0.5149	0.0831	0.0982	0.5139	0.0583	0.0812
			$\widehat{q}$	1.1091	0.2599	0.2906	1.0582	0.1691	0.1826	1.0365	0.1147	0.1271
		3	$\widehat{\lambda}$	0.5201	0.1061	0.1006	0.5133	0.0741	0.0755	0.5085	0.0504	0.0504
			$\widehat{\sigma}$	0.5185	0.1053	0.1077	0.5140	0.0741	0.0804	0.5071	0.0497	0.0508
			$\widehat{q}$	4.4930	3.6785	3.1798	3.8143	1.8557	2.1013	3.2819	0.7595	0.8955
	1.0	1	$\widehat{\lambda}$	0.5206	0.1930	0.2006	0.5186	0.1325	0.1336	0.5085	0.0924	0.0941
			$\widehat{\sigma}$	1.0612	0.2536	0.2761	1.0364	0.1736	0.1792	1.0254	0.1204	0.1304
			$\widehat{q}$	1.1279	0.3017	0.3844	1.0627	0.1793	0.1973	1.0433	0.1217	0.1272
		3	$\widehat{\lambda}$	0.5269	0.1779	0.1758	0.5128	0.1230	0.1304	0.5086	0.0848	0.0842
			$\widehat{\sigma}$	1.0445	0.2312	0.2270	1.0398	0.1593	0.1701	1.0184	0.1082	0.1115
			$\widehat{q}$	4.8398	5.0577	3.5896	4.0944	2.5002	2.5839	3.3558	0.9935	1.2306
1.0	0.5	1	$\widehat{\lambda}$	1.0273	0.1600	0.1719	1.0230	0.1120	0.1271	1.0211	0.0785	0.1279
			$\widehat{\sigma}$	0.5068	0.1157	0.1313	0.5051	0.0797	0.1007	0.5123	0.0554	0.1152
			$\widehat{q}$	1.0760	0.2167	0.2387	1.0495	0.1457	0.1567	1.0396	0.1010	0.1253
		3	$\widehat{\lambda}$	1.0303	0.1318	0.1380	1.0150	0.0899	0.0939	1.0087	0.0620	0.0610
			$\widehat{\sigma}$	0.5076	0.0980	0.1059	0.5045	0.0670	0.0690	0.5015	0.0460	0.0435
			$\widehat{q}$	3.8927	2.1092	2.2969	3.3113	0.8944	1.1456	3.1370	0.5151	0.5628
	1.0	1	$\widehat{\lambda}$	1.0566	0.2452	0.2488	1.0311	0.1686	0.1752	1.0307	0.1186	0.1524
			$\widehat{\sigma}$	1.0597	0.2465	0.2595	1.0353	0.1674	0.1953	1.0326	0.1172	0.1834
			$\widehat{q}$	1.1176	0.2673	0.3278	1.0721	0.1720	0.1955	1.0404	0.1156	0.1274
		3	$\widehat{\lambda}$	1.0501	0.2142	0.2096	1.0324	0.1478	0.1531	1.0138	0.1011	0.1031
			$\widehat{\sigma}$	1.0414	0.2118	0.2181	1.0255	0.1468	0.1573	1.0130	0.0999	0.1032
			$\widehat{q}$	4.5624	3.8165	3.2397	3.7511	1.7692	2.0399	3.2569	0.7691	0.9233

two for σ: 0.5 and 1.0; two for q: 1 and 3; and three for the sample size: $n = 50$, $n = 100$ and $n = 200$. Each combination of parameters and sample size was replicated 1000 times. We present the mean of the obtained estimators, the mean of the standard deviations calculated based on the observed information matrix and the root mean square error. Note that the bias of the estimators is acceptable and decreases as the sample size increases. Additionally, when the sample size increases, the mean of the estimated deviations approximates the term $\sqrt{\text{MSE}}$, suggesting consistent estimators.

Table 2: Simulation for the NCSt distribution with $\nu = 5$ degrees of freedom.

true values			$\widehat{\theta}$	$n=50$			$n=100$			$n=200$		
λ	σ	q		mean	s.e.	$\sqrt{\text{MSE}}$	mean	s.e.	$\sqrt{\text{MSE}}$	mean	s.e.	$\sqrt{\text{MSE}}$
−0.5	0.5	1	$\widehat{\lambda}$	−0.5299	0.1315	0.1346	−0.5174	0.0906	0.0894	−0.5121	0.0632	0.0601
			$\widehat{\sigma}$	0.5361	0.1395	0.1441	0.5207	0.0953	0.0946	0.5191	0.0665	0.0620
			$\widehat{q}$	1.1435	0.2982	0.3739	1.0772	0.1830	0.2004	1.0560	0.1239	0.1254
		3	$\widehat{\lambda}$	−0.5174	0.1245	0.1151	−0.5128	0.0872	0.0849	−0.5091	0.0599	0.0603
			$\widehat{\sigma}$	0.5196	0.1336	0.1200	0.5149	0.0946	0.0902	0.5086	0.0651	0.0656
			$\widehat{q}$	4.7202	5.7034	3.4817	4.1062	3.3006	2.6697	3.4869	1.4701	1.5523
	1.0	1	$\widehat{\lambda}$	−0.5331	0.2151	0.2202	−0.5146	0.1443	0.1431	−0.5173	0.1017	0.1055
			$\widehat{\sigma}$	1.0902	0.2921	0.3318	1.0422	0.1947	0.1942	1.0312	0.1363	0.1294
			$\widehat{q}$	1.1882	0.4338	0.6419	1.0734	0.1946	0.2034	1.0517	0.1320	0.1383
		3	$\widehat{\lambda}$	−0.5181	0.2047	0.1882	−0.5214	0.1453	0.1395	−0.5163	0.1008	0.1009
			$\widehat{\sigma}$	1.0402	0.2914	0.2394	1.0381	0.2103	0.1863	1.0252	0.1457	0.1435
			$\widehat{q}$	5.0573	8.1198	3.8670	4.2968	4.4232	2.9118	3.7668	2.3982	2.1330
0.5	0.5	1	$\widehat{\lambda}$	0.5302	0.1315	0.1326	0.5180	0.0898	0.0904	0.5146	0.0632	0.0608
			$\widehat{\sigma}$	0.5426	0.1403	0.1523	0.5208	0.0945	0.0893	0.5174	0.0660	0.0619
			$\widehat{q}$	1.1508	0.3073	0.3942	1.0856	0.1845	0.2021	1.0604	0.1246	0.1299
		3	$\widehat{\lambda}$	0.5195	0.1260	0.1164	0.5142	0.0882	0.0832	0.5108	0.0608	0.0609
			$\widehat{\sigma}$	0.5219	0.1358	0.1246	0.5165	0.0962	0.0910	0.5113	0.0661	0.0672
			$\widehat{q}$	4.8069	6.2872	3.5639	4.0846	3.3858	2.5787	3.5687	1.7137	1.7158
	1.0	1	$\widehat{\lambda}$	0.5380	0.2131	0.2281	0.5261	0.1464	0.1445	0.5123	0.1011	0.0956
			$\widehat{\sigma}$	1.0749	0.2865	0.3011	1.0518	0.1959	0.2048	1.0345	0.1359	0.1298
			$\widehat{q}$	1.1655	0.3513	0.4978	1.0810	0.1974	0.2185	1.0543	0.1324	0.1372
		3	$\widehat{\lambda}$	0.5269	0.2062	0.1976	0.5206	0.1462	0.1400	0.5160	0.1004	0.0992
			$\widehat{\sigma}$	1.0444	0.2898	0.2434	1.0442	0.2135	0.1956	1.0208	0.1455	0.1376
			$\widehat{q}$	4.9896	7.5749	3.8015	4.3185	4.5664	2.9438	3.7512	2.3581	2.0826
1.0	0.5	1	$\widehat{\lambda}$	1.0530	0.1734	0.1826	1.0363	0.1197	0.1211	1.0298	0.0855	0.0804
			$\widehat{\sigma}$	0.5341	0.1403	0.1475	0.5235	0.0952	0.0956	0.5187	0.0676	0.0653
			$\widehat{q}$	1.1284	0.2458	0.3075	1.0859	0.1590	0.1768	1.0677	0.1109	0.1215
		3	$\widehat{\lambda}$	1.0383	0.1514	0.1523	1.0149	0.1026	0.1044	1.0084	0.0698	0.0718
			$\widehat{\sigma}$	0.5177	0.1227	0.1225	0.5076	0.0838	0.0851	0.5061	0.0574	0.0583
			$\widehat{q}$	4.1493	3.1966	2.6803	3.4830	1.4792	1.5935	3.2059	0.6775	0.7802
	1.0	1	$\widehat{\lambda}$	1.0635	0.2631	0.2795	1.0414	0.1807	0.1830	1.0265	0.1257	0.1239
			$\widehat{\sigma}$	1.0739	0.2784	0.3209	1.0503	0.1903	0.1866	1.0335	0.1317	0.1249
			$\widehat{q}$	1.1522	0.3307	0.5153	1.0766	0.1817	0.1832	1.0615	0.1248	0.1327
		3	$\widehat{\lambda}$	1.0436	0.2494	0.2362	1.0352	0.1751	0.1659	1.0115	0.1193	0.1195
			$\widehat{\sigma}$	1.0400	0.2683	0.2404	1.0316	0.1897	0.1813	1.0202	0.1299	0.1308
			$\widehat{q}$	4.7107	5.8531	3.4608	4.0555	3.2062	2.5689	3.5171	1.5251	1.6419

4.2. Applications to real data

In this section, we use two real datasets to show the flexibility and applicability of the proposed NCSEL model. In these applications, we present analyses of the two real datasets to show the flexibility and applicability of the proposed NCSEL model by illustrating the fit of the proposed model and the use of the proposed EM-algorithm. We compare the results of these fits with other models that have been used. All the computations were done using the R package.

4.2.1. Nickel dataset

In this application, we consider a dataset consisting details regarding of Nickel (Ni) concentrations in 86 soil samples analyzed at the Mining Department of the University of Atacama, Chile. We report the ML estimates obtained under other models such as the Epsilon Skew-Normal (ESN) distribution (Mudholkar and Hutson ([25])) and Skew-Normal (SN) distribution (Azzalini ([4])), and compare them with our NCS model. A descriptive summary of this dataset is displayed in Table 3 where b_1 and b_2 are sample asymmetry and kurtosis coefficients, respectively.

Table 3: Nickel data: Descriptive summary of the mineral data.

n	$\overline{X}$	S	b_1	b_2
86	21.3372	16.6392	2.4483	12.0429

We observe that the data have positive asymmetry and high kurtosis. For this dataset, the NCS model moment estimators are given by $\widehat{\lambda}_M = 15.340$, $\widehat{\sigma}_M = 9.234$ and $\widehat{q}_M = 3.558$, which were used as initial values to start the EM algorithm. The ML estimates of the parameters of the ESN, SN and NCS models are found in Table 4. The AIC values Akaike ([1]) are given in Table 4. The model that provides the best fit for these data is the NCS model, which is supported by results in Figure 4 and the Q-Q plot in Figure 5.

Table 4: Nickel data: ML estimates and corresponding standard error (SE) for ESN, SN and NCS models.

Parameter	ESN	SN	NCS
μ	4.006 (1.249)	2.626 (2.066)	
σ	13.398 (1.022)	24.975 (2.454)	5.329 (0.735)
q			2.190 (0.398)
λ		10.259 (9.603)	12.030 (1.044)
ϵ	−0.856 (0.057)		
AIC	696.419	695.523	680.363

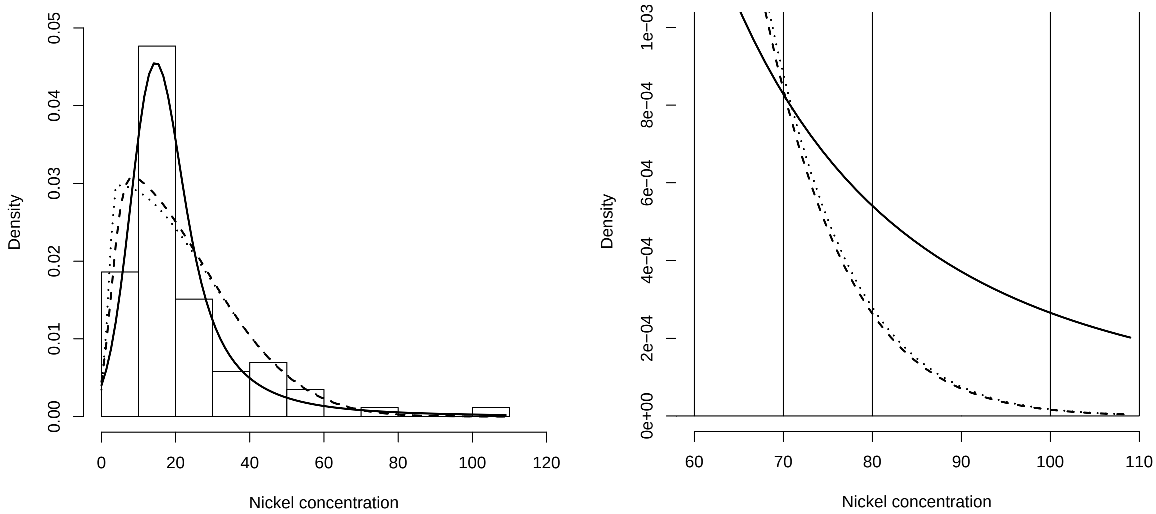

Figure 4: Nickel data: Fitted models, ESN (dotted line), estimated SN (dashed line) and estimated NCS (solid line) (Left panel). Upper tail of histogram with estimated ESN (dotted line), estimated SN (dashed line) and estimated NCS (solid line) (Right panel).

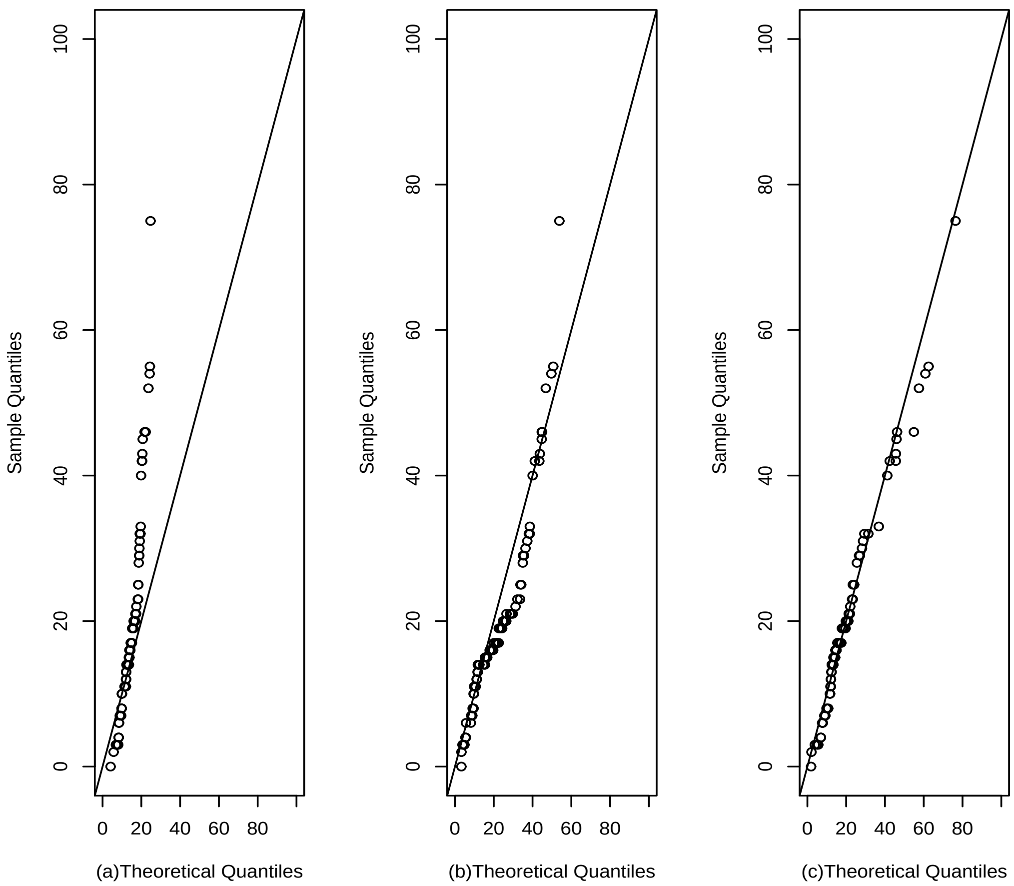

Figure 5: Nickel data: Q-Q plots; ESN model (a), SN model (b) and NCS model (c).

4.2.2. Copper data

This dataset refers to the soluble copper concentration of 1933 samples (Fuentes ([11])), for which the ML estimates are obtained for the Epsilon Skew-t (ESt) model (Gómez *et al.* ([14])), the Skew-t (St) model and for our NCSt model. A descriptive summary of this dataset is reported in Table 5. For this dataset, we observe positive asymmetry and kurtosis coefficients.

Table 5: Copper data: Descriptive statistics of the dataset.

n	$\overline{y}$	S	b_1	b_2
1933	0.591	0.302	1.196	4.633

Moreover, the moment estimates under the NCS model are given by $\widehat{\lambda}_M = 0.441$, $\widehat{\sigma}_M = 0.150$ and $\widehat{q}_M = 3.950$, which were used as initial values to start the EM algorithm. Table 6 reports the estimates of the degrees of freedom, ν, for each model based on the Student-t distribution, which are obtained by maximizing the profile log-likelihood function, as in Vilca *et al.* ([30]). The estimates of ν is obtained for the ESt, St and NCSt models, as reported in Table 7. This table also includes the AIC values, revealing that the NCSt model fits the data well.

Table 6: Copper data: Estimation of ν for the St, ESt and NCSt models by maximizing the log-likelihood function.

ν	Log-likelihood St	Log-likelihood ESt	Log-likelihood NCSt
1	−359.599	−416.062	−709.297
2	−327.326	−266.335	−328.671
3	−209.663	−227.596	−223.877
4	−197.641	−213.163	−192.580
5	−191.688	−206.886	−188.485
6	−188.635	−203.988	**−185.151**
7	−187.067	−202.669	−189.139
8	−186.303	−202.151	−189.365
9	−185.992	**−202.068**	−190.621
10	**−185.941**	−202.195	−192.370
11	−186.042	−202.455	−193.973

Table 7: Copper data: ML estimates and the corresponding SE (in parentheses) for the St ($\nu = 10$), ESt ($\nu = 9$) and NCSt ($\nu = 6$) models.

Parameter	St	ESt	NCSt
μ	0.253 (0.008)	0.351 (0.013)	—
σ	0.404 (0.010)	0.2396 (0.004)	0.1355 (0.022)
q			5.991 (0.168)
λ	4.262 (0.360)		3.162 (0.475)
ϵ		−0.563 (0.029)	
AIC	377.881	410.115	376.301

Moreover, we present other results to show the performance of our approach. Figure 6 depicts plots of the fitted St, ESt and NCSt models using the ML estimates. We note that the fitted NCSt model presents heavier tails than the other models. Figure 7 shows the Q-Q plots for these fitted models. From all these summaries and plots, we can conclude that the NCSt model provides the best fit to the data.

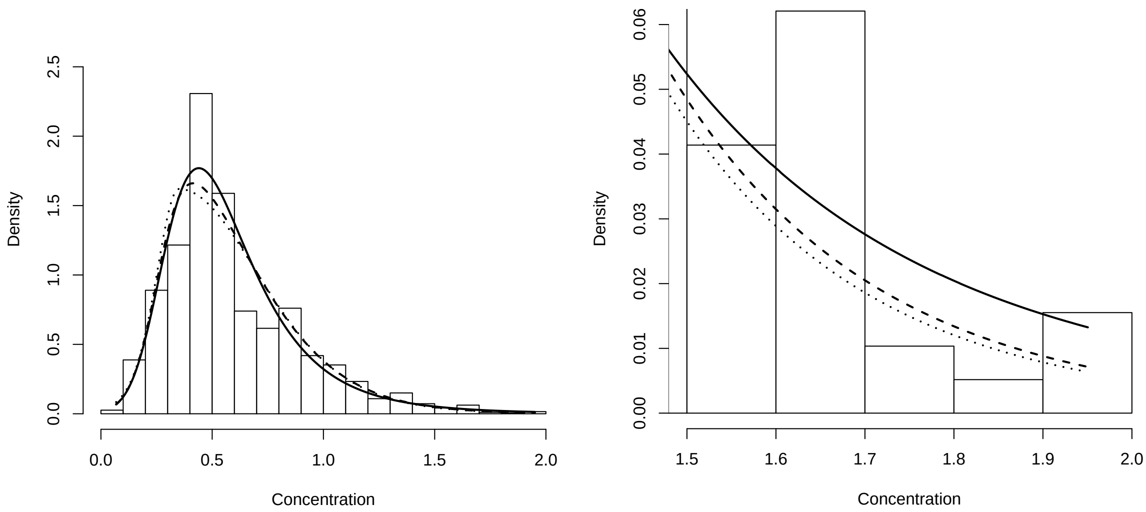

Figure 6: Copper data: Fitted models, NCSt (solid line), St (dashed line) and ESt (dotted line) (Left panel). Plots of the tails for the models (Right panel).

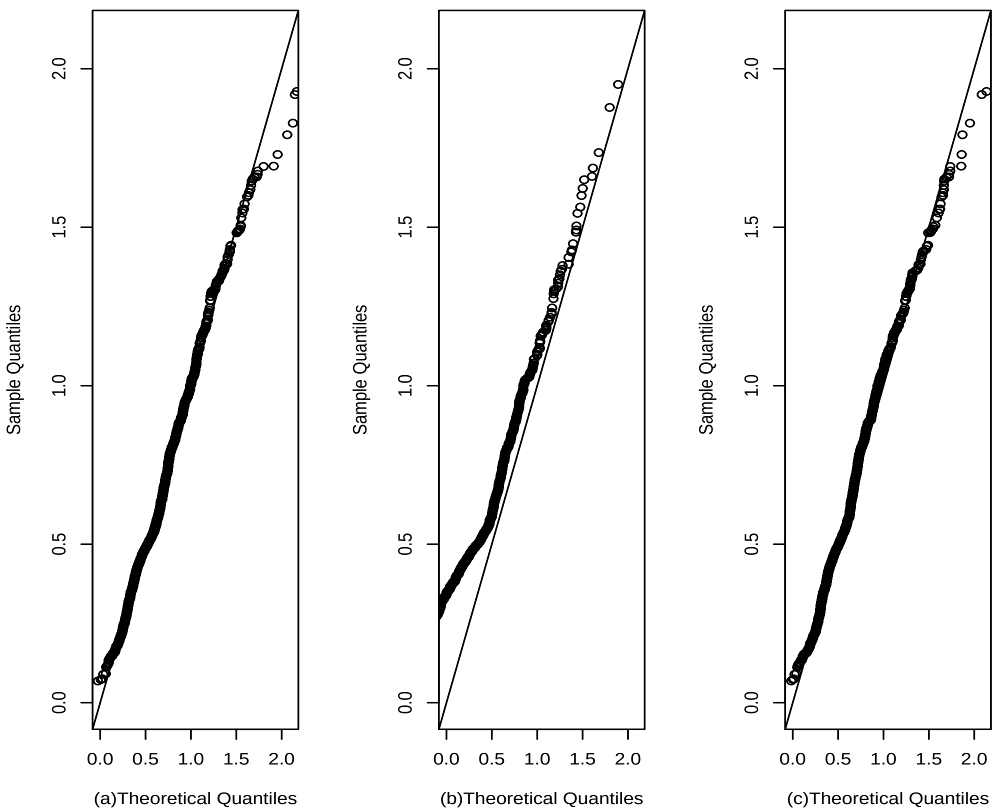

Figure 7: Copper data: Q-Q plots; St model (a), ESt model (b) and NCSt model (c).

4.2.3. Snack data

We consider in this application part of the data of an experiment performed in the Department of Nutrition of the Faculty of Public Health of the University of São Paulo, in which 5 different forms of a new type of snack, with low content of saturated fat and fatty acids, were compared over the course of 20 weeks. In this new product the hydrogenated vegetable fat has been replaced, in whole or in part, by canola oil. The forms are as follow: A (22% of fat 0% of canola oil), B (0% fat, 22% canola oil), C (17% fat, 5% canola oil), D (11% fat, 11% canola oil) and E (5% fat, 17% canola oil). The experiment was conducted so that in the even weeks 15 packs of each of the products A, B, C, D and E were analyzed in the laboratory and several variables were observed. In particular, we study the texture behavior of the products through the force necessary for shear (y). For more details on the study, see Paula ([26]), Section 2.8.1. The equation is

$$y_i = \beta_0 + \beta_1 x_{i\mathrm{B}} + \beta_2 x_{i\mathrm{C}} + \beta_3 x_{i\mathrm{D}} + \beta_4 x_{i\mathrm{D}} + \beta_5 x_{i\mathrm{E}} + \beta_6\, weeks_i + \varepsilon_i, \qquad i = 1, ..., n,$$

where $x_{iT} = 1$ if measurement i corresponds to a snack of type T, for $T = \mathrm{B}, \mathrm{D}, \mathrm{C}, \mathrm{E}$, and $weeks_i$ is the number of weeks that passed until measurement i was made.

We assume that $\varepsilon_i \sim \mathrm{NCS}(\sigma, q, \lambda)$, where $\lambda = -\beta_0(q-1)/q$, with $q > 1$. This condition is to obtain that $\mathrm{E}(\varepsilon_i) = 0$, $i = 1, ..., n$, with the purpose of comparing the fit under $\varepsilon_i \sim \mathrm{ESN}(\sigma, \epsilon, \mu_1)$ and $\mathrm{SN}(\sigma, \lambda, \mu_2)$ distributions. We also consider appropriate restrictions such as $\mu_1 = g(\sigma, \epsilon, \beta_0)$ and $\mu_2 = g(\sigma, \lambda, \beta_0)$, in order to obtain that $\mathrm{E}(\varepsilon_i) = 0$, $i = 1, ..., n$.

Results of the fit of the models are reported in Table 8. Note that, according to the AIC criterion, the best fit is provided by the NCS regression model. This is confirmed by the randomized quantile residuals, see Dunn and Smyth ([8]). If the model is correctly specified for the data, such residuals should be a random sample from the standard normal distribution. Figure 8 confirms that the NCS regression model provides a better fit than the ESN and SN regression models.

Table 8: Snack data: ML estimates and corresponding standard errors (SE) for ESN, SN and NCS regression models.

Parameter	ESN		SN		NCS	
	Estimate	SE	Estimate	SE	Estimate	SE
β_0	58.044	2.095	57.958	46.910	56.483	1.512
β_1	−10.907	1.755	−10.907	1.680	−8.167	1.626
β_2	−4.569	1.68	−4.569	1.680	−4.762	1.634
β_3	−15.174	1.84	−15.174	1.680	−11.708	1.632
β_4	−15.945	1.858	−15.944	1.680	−12.624	1.627
β_5	0.742	0.094	0.742	0.092	0.713	0.082
σ	14.550	0.399	14.551	0.379	9.233	0.527
ϵ	0.000	0.054	—		—	
λ	—		0.007	4.038	—	
q	—		—		6.444	0.614
AIC	6160.839		6160.839		6083.816	

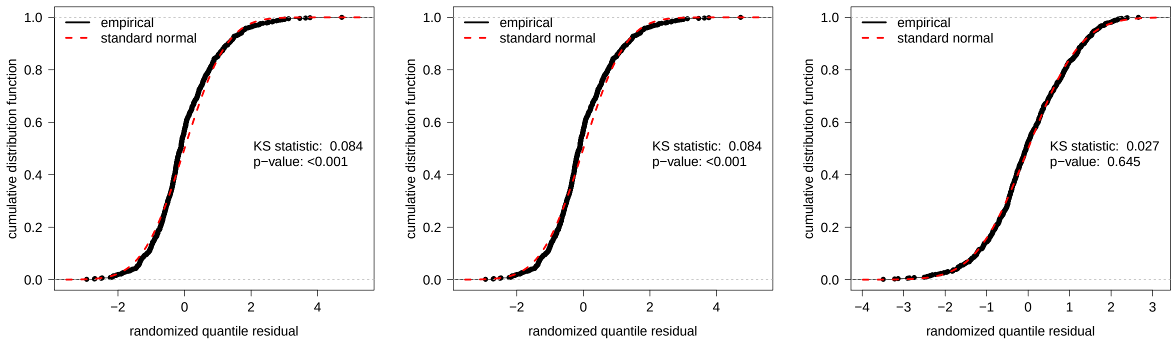

Figure 8: Snack data: Empirical cdf for randomized quantile residual versus cdf of standard normal distribution for ESN, SN and NCS regression models. Also provided are the statistics and p-values for the Kolmogorov–Smirnov (KS) test to compare both curves.

5. MULTIVARIATE NCSEL DISTRIBUTIONS

In this section the multivariate NCSEL distribution is introduced, its pdf is derived and some additional properties are studied.

In the multivariate setup, a k-dimensional random vector $\mathbf{Y} = (Y_1, ..., Y_k)^\top$ follows an EL distribution with location parameter vector $\boldsymbol{\mu}$ and scale parameter matrix $\boldsymbol{\Sigma}$, which is positive definite, if its pdf is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = |\boldsymbol{\Sigma}|^{-1/2} g\Big((\mathbf{y}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\Big), \qquad \mathbf{y} \in \mathbb{R}^k,$$

where g is the density generator function satisfying

$$\int_0^\infty u^{k-1} g(u^2)\, du < \infty.$$

We use the notation $\mathbf{Y} \sim \mathrm{EL}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$. If the moments of each element of the random vector $\mathbf{Y}$ are finite, then it follows that $\mathrm{E}(\mathbf{Y}) = \boldsymbol{\mu}$ and $\mathrm{Var}(\mathbf{Y}) = \alpha_g \boldsymbol{\Sigma}$, where α_g is a positive constant, as seen for example, in Fang *et al.* ([10]). Now a multivariate NCSEL distribution is proposed, where a k-variate vector $\mathbf{Y}$ is said to have a multivariate noncentral slash-elliptical(MNCSEL) distribution with scale matrix $\boldsymbol{\Sigma}$ positive definite, $\boldsymbol{\lambda}$ being the non-centrality parameter and q the kurtosis parameter

$$\mathbf{Y} = \frac{\boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{X} + \boldsymbol{\lambda}}{U^{\frac{1}{q}}}, \tag{5.1}$$

where $\boldsymbol{X} \sim \mathrm{EL}_k(\mathbf{0}, \mathbf{I}_k; g)$ is independent of $U \sim \mathrm{U}(0,1)$. The resulting distribution is denoted by $\mathbf{Y} \sim \mathrm{MNCSEL}_k(\boldsymbol{\Sigma}, q, \boldsymbol{\lambda}; g)$. The pdf of $\mathbf{Y}$ is presented in the following result.

Proposition 5.1. *Let $\mathbf{Y} \sim \mathrm{MNCSEL}_k(\boldsymbol{\Sigma}, q, \boldsymbol{\lambda}; g)$. Then, the pdf of $\mathbf{Y}$ is given by*

$$f(\boldsymbol{y}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \int_0^1 z^{\frac{k}{q}} g\Big[\big(\boldsymbol{y} z^{\frac{1}{q}} - \boldsymbol{\lambda}\big)^\top \boldsymbol{\Sigma}^{-1} \big(\boldsymbol{y} z^{\frac{1}{q}} - \boldsymbol{\lambda}\big)\Big]\, dz. \tag{5.2}$$

Proof: Using the fact that $\boldsymbol{X}$ and U are independent and standard calculations of the Jacobian transformation of

$$\mathbf{Y} = \frac{\boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{X} + \boldsymbol{\lambda}}{U^{\frac{1}{q}}} \qquad \text{and} \qquad Z = U,$$

we obtain the join pdf of $\mathbf{Y}$ and Z given by

$$f_{\mathbf{Y},Z}(\boldsymbol{y}, z) \;=\; |\boldsymbol{\Sigma}|^{-\frac{1}{2}}\, z^{\frac{k}{q}}\, g\Big[\big(\boldsymbol{y}\, z^{\frac{1}{q}} - \boldsymbol{\lambda}\big)^{\top}\, \boldsymbol{\Sigma}^{-1}\big(\boldsymbol{y}\, z^{\frac{1}{q}} - \boldsymbol{\lambda}\big)\Big].$$

The required result is obtained by integrating the above joint pdf with respect to z. □

Remark 5.1. If $\boldsymbol{\lambda} = 0$, we obtain the family of distributions discussed by Gómez *et al.* ([13]) and Gómez and Venegas ([15]). On the other hand for $\boldsymbol{\lambda} = 0$ and under normality of $\boldsymbol{X}$, we obtain the slash multivariate introduced by Wang and Genton ([31]).

Another important property is that the MNCSEL distribution can be written as a scale mixture of an elliptical distribution and a uniform distribution in the unit interval.

Proposition 5.2. *Let* $\mathbf{Y}\,|\,(Z\!=\!z) \sim \mathrm{EL}_k\big(z^{-\frac{1}{q}}\boldsymbol{\lambda}, z^{-\frac{2}{q}}\boldsymbol{\Sigma}; g\big)$ *and* $Z \sim \mathrm{U}(0,1)$. *Then* $\mathbf{Y} \sim \mathrm{MNCSEL}_k(\boldsymbol{\Sigma}, q, \boldsymbol{\lambda}; g)$.

Proof: We can write

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \int_0^1 f_{\mathbf{Y}|Z}(\mathbf{y})\, f_Z(z)\, dz \\ &= \int_0^1 \big|z^{-\frac{2}{q}}\boldsymbol{\Sigma}\big|^{-1/2} g\Big[\big(\mathbf{y} - z^{-\frac{1}{q}}\boldsymbol{\lambda}\big)^{\top}\big(z^{-\frac{2}{q}}\boldsymbol{\Sigma}\big)^{-1}\big(\mathbf{y} - z^{-\frac{1}{q}}\boldsymbol{\lambda}\big)\Big]\, dz. \end{aligned}$$

The result follows using properties of determinants. □

Proposition 5.3. *Let* $\mathbf{Y} \sim \mathrm{MNCSEL}_k(\boldsymbol{\Sigma}, q, \boldsymbol{\lambda}; g)$. *Then,*

$$\mathrm{E}[\mathbf{Y}] = \frac{q\boldsymbol{\lambda}}{q-1}, \quad q > 1, \qquad \text{and} \qquad \mathrm{Var}(\mathbf{Y}) = \frac{q}{q-2}\left(\frac{\boldsymbol{\lambda}\boldsymbol{\lambda}^{\top}}{(q-1)^2} + \alpha_g\boldsymbol{\Sigma}\right), \quad q > 2.$$

Proof: Following the procedure in Proposition 5.2, we have $\mathbf{Y}\,|\,(Z\!=\!z) \sim \mathrm{EL}_k\big(z^{-\frac{1}{q}}\boldsymbol{\lambda}, z^{-\frac{2}{q}}\boldsymbol{\Sigma}\big)$. So, using the fact that $\mathrm{E}\big[Z^{-\frac{r}{q}}\big] = \frac{q}{q-r}$, $q > r$ and the conditional expectation properties:

$$\mathrm{E}[\mathbf{Y}] = \mathrm{E}\big[\mathrm{E}(\mathbf{Y}|Z)\big] = \mathrm{E}\big[Z^{-\frac{1}{q}}\boldsymbol{\lambda}\big] = \frac{q\boldsymbol{\lambda}}{q-1}, \qquad q > 1.$$

Moreover, following the same idea we obtain the variance of $\mathbf{Y}$ as follows:

$$\begin{aligned} \mathrm{Var}(\mathbf{Y}) &= \mathrm{Var}\big[\mathrm{E}(\mathbf{Y}|Z)\big] + \mathrm{E}\big[\mathrm{Var}(\mathbf{Y}|Z)\big] \\ &= \mathrm{Var}\big[Z^{-\frac{1}{q}}\boldsymbol{\lambda}\big] + \mathrm{E}\big[Z^{-\frac{2}{q}}\alpha_g\boldsymbol{\Sigma}\big] \\ &= \boldsymbol{\lambda}\,\mathrm{Var}\big[Z^{-\frac{1}{q}}\big]\,\boldsymbol{\lambda}^{\top} + \alpha_g\mathrm{E}\big[Z^{-\frac{2}{q}}\big]\,\boldsymbol{\Sigma} \\ &= \boldsymbol{\lambda}\boldsymbol{\lambda}^{\top}\frac{q}{(q-2)\,(q-1)^2} + \alpha_g\boldsymbol{\Sigma}\,\frac{q}{q-2}, \qquad q > 2 \\ &= \frac{q}{q-2}\left(\frac{\boldsymbol{\lambda}\boldsymbol{\lambda}^{\top}}{(q-1)^2} + \alpha_g\boldsymbol{\Sigma}\right), \qquad q > 2. \end{aligned}$$

□

6. CONCLUSION

Here we have introduced a new distribution called the NCSEL distribution. The main idea is to incorporate a non-centrality parameter in the usual SEL distribution. The resulting distribution is an asymmetric distribution that contains as special cases the EL and SEL distributions. For this family of distributions we point out some important characteristics and properties that allow us to obtain qualitatively robust ML estimates and efficiently compute them by using the EM-algorithm for a special class based on the family of NI distributions. We illustrate our results by using three numerical examples. They show the flexibility and inherent robustness of the estimation procedure in the NCSEL model.

Finally, the NCSEL can be used along the same lines as the skew distributions in the context of regression. This issue is currently under investigation, and we hope to report these findings in a future paper.

A. APPENDIX — Simulation study with $\nu = 10$ degrees of freedom

Table 9: Simulation for the NCSt distribution with $\nu = 10$ degrees of freedom.

true values			$\widehat{\theta}$	$n = 50$			$n = 100$			$n = 200$		
λ	σ	q		mean	s.e.	$\sqrt{\text{MSE}}$	mean	s.e.	$\sqrt{\text{MSE}}$	mean	s.e.	$\sqrt{\text{MSE}}$
−0.5	0.5	1	$\widehat{\lambda}$	−0.5237	0.1261	0.1303	−0.5150	0.0874	0.0862	−0.5085	0.0606	0.0594
			$\widehat{\sigma}$	0.5286	0.1306	0.1324	0.5209	0.0901	0.0889	0.5125	0.0631	0.0603
			$\widehat{q}$	1.0905	0.2628	0.2901	1.0666	0.1759	0.1848	1.0399	0.1178	0.1219
		3	$\widehat{\lambda}$	−0.5195	0.1147	0.1113	−0.5132	0.0796	0.0803	−0.5078	0.0549	0.0556
			$\widehat{\sigma}$	0.5171	0.1188	0.1110	0.5122	0.0825	0.0826	0.5100	0.0568	0.0611
			$\widehat{q}$	4.4553	4.3511	3.1883	3.8335	2.2236	2.2079	3.4520	1.1176	1.3898
	1.0	1	$\widehat{\lambda}$	−0.5300	0.2026	0.2120	−0.5131	0.1377	0.1423	−0.5142	0.0974	0.0966
			$\widehat{\sigma}$	1.0666	0.2710	0.2854	1.0309	0.1827	0.1846	1.0346	0.129	0.1309
			$\widehat{q}$	1.1467	0.3292	0.4879	1.0635	0.1852	0.1984	1.0458	0.126	0.1296
		3	$\widehat{\lambda}$	−0.5410	0.1955	0.1905	−0.5144	0.1340	0.1342	−0.5102	0.0924	0.0944
			$\widehat{\sigma}$	1.0590	0.2610	0.2388	1.0393	0.1832	0.1805	1.0285	0.1242	0.1291
			$\widehat{q}$	5.1102	6.5417	3.8243	4.2180	3.4874	2.8081	3.6203	1.5291	1.7002
0.5	0.5	1	$\widehat{\lambda}$	0.5294	0.1266	0.1269	0.5181	0.0871	0.0873	0.5099	0.0612	0.0581
			$\widehat{\sigma}$	0.5319	0.1303	0.1420	0.5224	0.0895	0.0897	0.5146	0.0628	0.0603
			$\widehat{q}$	1.1341	0.3178	0.4400	1.0661	0.1738	0.1850	1.0458	0.1194	0.1248
		3	$\widehat{\lambda}$	0.5292	0.1171	0.1142	0.5147	0.0800	0.0802	0.5058	0.0543	0.053
			$\widehat{\sigma}$	0.5273	0.1206	0.1170	0.5170	0.0828	0.0811	0.5061	0.056	0.0577
			$\widehat{q}$	4.8062	4.9793	3.4829	3.8846	2.3178	2.2996	3.3449	0.9999	1.213
	1.0	1	$\widehat{\lambda}$	0.5249	0.2025	0.2049	0.5109	0.1383	0.1383	0.5146	0.0978	0.0995
			$\widehat{\sigma}$	1.0680	0.2701	0.2850	1.0363	0.1849	0.1837	1.0337	0.1295	0.1285
			$\widehat{q}$	1.1345	0.3277	0.4665	1.0620	0.1851	0.1953	1.0371	0.1244	0.1284
		3	$\widehat{\lambda}$	0.5214	0.1928	0.1852	0.5266	0.1347	0.1325	0.5096	0.0917	0.094
			$\widehat{\sigma}$	1.0617	0.2578	0.2414	1.0418	0.1835	0.1784	1.0192	0.1231	0.1261
			$\widehat{q}$	5.1212	6.3880	3.8663	4.2130	3.3634	2.7218	3.5264	1.4173	1.5681
1.0	0.5	1	$\widehat{\lambda}$	1.0358	0.1656	0.1707	1.0212	0.1147	0.1091	1.0168	0.0801	0.0757
			$\widehat{\sigma}$	0.5075	0.1261	0.1363	0.5079	0.0870	0.0862	0.5106	0.0607	0.0601
			$\widehat{q}$	1.0947	0.2264	0.2533	1.0569	0.1496	0.1536	1.0458	0.1031	0.1122
		3	$\widehat{\lambda}$	1.0376	0.1399	0.1438	1.0137	0.0952	0.0996	1.0064	0.0653	0.0656
			$\widehat{\sigma}$	0.5153	0.1090	0.1133	0.5087	0.0747	0.0791	0.5028	0.051	0.0507
			$\widehat{q}$	4.0107	2.4780	2.4694	3.4097	1.0933	1.3396	3.1600	0.5736	0.6723
	1.0	1	$\widehat{\lambda}$	1.0500	0.2525	0.2634	1.0242	0.1724	0.1668	1.0138	0.1207	0.1197
			$\widehat{\sigma}$	1.0579	0.2605	0.2797	1.0320	0.1775	0.1751	1.0235	0.1243	0.118
			$\widehat{q}$	1.1121	0.2715	0.3156	1.0594	0.1729	0.1839	1.0375	0.1172	0.1208
		3	$\widehat{\lambda}$	1.0392	0.2312	0.2183	1.0212	0.1581	0.1585	1.0148	0.1085	0.1091
			$\widehat{\sigma}$	1.0405	0.2388	0.2193	1.0229	0.1638	0.1628	1.0136	0.1116	0.1131
			$\widehat{q}$	4.5632	4.5408	3.2450	3.8152	2.1794	2.1591	3.3253	0.9405	1.0617

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