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CONFIDENCE INTERVAL FOR QUANTILE RATIO OF THE DAGUM DISTRIBUTION

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Abstract:

- Inequality measures based on ratios of quantiles are frequently applied in economic research, especially to the analysis of income distributions. In the paper, we construct a confidence interval for such measures under the Dagum distribution which has widely been assumed as a model for income and wage distributions in empirical analysis and theoretical considerations. Its properties are investigated on the basis of computer simulations. The constructed confidence interval is further applied to the analysis of income inequality in Poland in 2015.

Keywords:

- *ratio of quantiles; confidence interval; Dagum distribution; quintile share ratio.*

AMS Subject Classification:

- 62F25, 62P20.

1. INTRODUCTION

In the Eurostat regional yearbook (2016), one of the basic measures of income distribution inequality is defined as the income quintile share ratio or the $S80/S20$ ratio. It is calculated as the ratio of total income received by the 20% of the population with the highest income (the top quintile) to that received by the 20% of the population with the lowest income (the bottom quintile), i.e. income quintile share ratio is defined as

$$r_{0.2,0.8} = \frac{F^{-1}(0.8)}{F^{-1}(0.2)},$$

where F denotes the distribution of the population income. The natural estimator of $r_{0.2,0.8}$ is the ratio of appropriate sample quintiles. However, the problem is in interval estimation. According to the best knowledge of the Authors such a problem has never been considered in the literature. In the paper a confidence interval for the population ratio of quintiles is constructed. The proposed confidence interval is based on the asymptotic distribution of the ratio of sample quintiles.

We confine ourselves to the Dagum ([1]) distribution as a probabilistic model of income. The Dagum distribution is widely used for income modeling in many countries all over the world (see for example Domański and Jędrzejczak [5], Jędrzejczak [10]). The Dagum distribution has many good mathematical as well as statistical properties. Basic properties of this distribution are presented in Appendix A; for more see Kleiber ([11]), Dey *et al.* ([4]). See also Encyclopedia ([6]) (pp. 3363–3378, also 3236–3248) and the references therein.

The paper is organized as follows. In the second section confidence interval for a ratio of quantiles is constructed. It is based on the ratio of sample quantiles of the Dagum distribution. It appears that the ends of the proposed confidence interval depend on a shape parameter which should be estimated from a sample. In the third section a short simulation study is provided. In this study two estimators of the shape parameter were applied. Namely, the estimator obtained by the method of moments and the one obtained by the method of probability-weighted moments. Results of the simulations are very similar for these two estimators. In the fourth section an application to income inequality analysis based on the data coming from the Polish Household Budget Survey is presented. In the last section some conclusions are presented as well as some remarks on further research on the subject.

We consider a more general set-up, namely a confidence interval for a ratio of α and β quantiles is constructed. To obtain a confidence interval for the quintile ratio it is enough to put $\alpha = 0.2$ and $\beta = 0.8$. The results of the paper may easily be generalized to other distributions applied in personal income modeling, such as Pareto, Burr Type XII, Beta, etc.

2. CONFIDENCE INTERVAL

Let $0 < \alpha < \beta < 1$ be given numbers and let

$$r_{\alpha,\beta} = \frac{F^{-1}(\beta)}{F^{-1}(\alpha)}$$

be the quantile ratio of interest, where $F(\cdot)$ is the cumulative distribution function (CDF) of income distribution. Let $X_1, ..., X_n$ be a sample of incomes of randomly drawn n individuals. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the ordered sample. As an estimator of $r_{\alpha,\beta}$ it is taken

$$r^*_{\alpha,\beta} = \frac{X_{\lfloor n\beta \rfloor + 1:n}}{X_{\lfloor n\alpha \rfloor + 1:n}},$$

where $\lfloor x \rfloor$ denotes the greatest integer not greater than x.

In our considerations we confine ourselves to the Dagum distribution, i.e. throughout the paper it will be assumed that the distribution of the population income is the Dagum one. As it was mentioned above, the Dagum distribution fits population income quite well for many countries around the world.

Consider the Dagum distribution with parameters $a, v > 0$ and $\lambda > 0$. Its cumulative distribution function (CDF) and probability density function (PDF) are as follows

$$F_{a,v,\lambda}(x) = \left(1 + \left(\frac{x}{\lambda}\right)^{-v}\right)^{-a} \quad \text{for } x > 0$$

and

$$f_{a,v,\lambda}(x) = \frac{av}{\lambda}\left(\frac{x}{\lambda}\right)^{av-1}\left(1 + \left(\frac{x}{\lambda}\right)^{v}\right)^{-a-1} \quad \text{for } x > 0.$$

Its quantile function equals

$$Q_{a,v,\lambda}(q) = \lambda\left(q^{-1/a} - 1\right)^{-1/v} \quad \text{for } 0 < q < 1.$$

For other interesting properties of the Dagum distribution see Appendix A.

The problem is in constructing a confidence interval at the confidence level δ for a ratio of quantiles of the Dagum distribution

$$r_{\alpha,\beta} = \frac{Q_{a,v,\lambda}(\beta)}{Q_{a,v,\lambda}(\alpha)} = \left(\frac{\beta^{-1/a} - 1}{\alpha^{-1/a} - 1}\right)^{-1/v}$$

on the basis of a random sample $X_1, ..., X_n$.

In what follows "large" sample sizes are considered, i.e. it is assumed that $n \to \infty$. There are two reasons for such an approach. The first one is that real sample sizes usually comprise many thousands of observations. The second one is rather technical — the finite sample size distribution of the ratio of sample quantiles of the Dagum distribution is analytically untractable (for exact distribution see Maswadah 2013).

Theorem 2.1. *For $0 < \alpha < \beta < 1$ the random variable $r^*_{\alpha,\beta}$ is strongly consistent estimator of $r_{\alpha,\beta}$, for all a, v, λ.*

Proof: The proof follows form the fact (David and Nagaraja [2]; Serfling [15]) that $X_{\lfloor n\alpha \rfloor + 1:n}$ is strongly consistent estimator of the α's quantile of the underlying distribution. Application of Slutsky theorem gives the thesis. $\square$

Theorem 2.2. *For $0 < \alpha < \beta < 1$ the estimator $r^*_{\alpha,\beta}$ is asymptotically normally distributed random variable.*

Proof: Let $Y_i = \ln X_i$. Of course $Y_{i:n} = \ln X_{i:n}$. Let γ^Y_α and γ^Y_β denote the quantiles of Y. For $\alpha < \beta$ we have (Serfling [15], th. 2.3.3; David and Nagaraja [2], th. 10.3):

$$\sqrt{n}\begin{bmatrix} Y_{\lfloor n\alpha\rfloor+1:n} - \gamma^Y_\alpha \\ Y_{\lfloor n\beta\rfloor+1:n} - \gamma^Y_\beta \end{bmatrix} \to N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\alpha(1-\alpha)}{(f_Y(\gamma^Y_\alpha))^2} & \frac{\alpha(1-\beta)}{\left(f_Y(\gamma^Y_\alpha)f_Y(\gamma^Y_\beta)\right)} \\ \frac{\alpha(1-\beta)}{\left(f_Y(\gamma^Y_\alpha)f_Y(\gamma^Y_\beta)\right)} & \frac{\beta(1-\beta)}{\left(f_Y(\gamma^Y_\beta)\right)^2} \end{bmatrix}\right),$$

where $f_Y(\cdot)$ is the PDF of Y.

Hence

$$\sqrt{n}\left[\left(Y_{\lfloor n\beta\rfloor+1:n} - Y_{\lfloor n\alpha\rfloor+1:n}\right) - \left(\gamma^Y_\beta - \gamma^Y_\alpha\right)\right] \to N\left(0, \sigma^2\right),$$

where

$$\sigma^2 = \frac{\beta(1-\beta)}{\left(f_Y(\gamma^Y_\beta)\right)^2} + \frac{\alpha(1-\alpha)}{\left(f_Y(\gamma^Y_\alpha)\right)^2} - 2\frac{\alpha(1-\beta)}{\left(f_Y(\gamma^Y_\beta)f_Y(\gamma^Y_\alpha)\right)}.$$

So we have

$$\sqrt{n}\left(\ln\frac{X_{\lfloor n\beta\rfloor+1:n}}{X_{\lfloor n\alpha\rfloor+1:n}} - \left(\gamma^Y_\beta - \gamma^Y_\alpha\right)\right) \to N\left(0, \sigma^2\right).$$

Applying Delta method (Greene [8], p. 913) with $g(t) = e^t$:

$$\sqrt{n}\left(\frac{X_{\lfloor n\beta\rfloor+1:n}}{X_{\lfloor n\alpha\rfloor+1:n}} - e^{\gamma^Y_\beta - \gamma^Y_\alpha}\right) \to e^{(\gamma^Y_\beta - \gamma^Y_\alpha)} N\left(0, \sigma^2\right).$$

Since in the Dagum distribution $\gamma^Y_\alpha = \ln\gamma_\alpha$ we have

$$\sqrt{n}\left(\frac{X_{\lfloor n\beta\rfloor+1:n}}{X_{\lfloor n\alpha\rfloor+1:n}} - \frac{\gamma_\beta}{\gamma_\alpha}\right) \to \left(\frac{\gamma_\beta}{\gamma_\alpha}\right) N\left(0, \sigma^2\right),$$

i.e.

$$(*) \qquad \sqrt{n}\left(r^*_{\alpha,\beta} - r_{\alpha,\beta}\right) \to r_{\alpha,\beta} N\left(0, \sigma^2\right). \qquad \square$$

Simple calculations show that

$$\sigma^2 = \frac{1}{(av)^2}\left(\frac{1-\beta}{\beta}\frac{1}{(1-\beta^{\frac{1}{a}})^2} + \frac{1-\alpha}{\alpha}\frac{1}{(1-\alpha^{\frac{1}{a}})^2} - 2\frac{1-\beta}{\beta}\frac{1}{(1-\alpha^{\frac{1}{a}})(1-\beta^{\frac{1}{a}})}\right).$$

Since we are interested in the estimation of the ratio $r_{\alpha,\beta}$ of quantiles, we reparametrize the considered model. It can be seen that

$$v = \frac{\log\left(\frac{\alpha^{-1/a}-1}{\beta^{-1/a}-1}\right)}{\log r_{\alpha,\beta}}.$$

The CDF of the Dagum distribution may be written in the following form

$$F_{a,r_{\alpha,\beta},\lambda}(x) = \left(1 + \left(\frac{x}{\lambda}\right)^{-\frac{\log\left(\frac{\alpha^{-1/a}-1}{\beta^{-1/a}-1}\right)}{\log r_{\alpha,\beta}}}\right)^{-a}$$

for $x > 0$ and $a > 0$, $r_{\alpha,\beta} > 0$ and $\lambda > 0$.

We have $\sigma^2 = (\log r_{\alpha,\beta})^2 w^2(a)$, where

$$w^2(a) = \left(\frac{1}{a \log\left(\frac{\alpha^{-1/a}-1}{\beta^{-1/a}-1}\right)}\right)^2 \left(\frac{1-\beta}{\beta}\frac{1}{\left(1-\beta^{\frac{1}{a}}\right)^2} + \frac{1-\alpha}{\alpha}\frac{1}{\left(1-\alpha^{\frac{1}{a}}\right)^2} - 2\frac{1-\beta}{\beta}\frac{1}{\left(1-\alpha^{\frac{1}{a}}\right)\left(1-\beta^{\frac{1}{a}}\right)}\right).$$

Let δ be a given confidence level. From ($*$) we have (the scale parameter λ is omitted)

$$P_{a,r_{\alpha,\beta}}\left\{\sqrt{n}\left|\frac{r^*_{\alpha,\beta} - r_{\alpha,\beta}}{w(a) r_{\alpha,\beta}\log r_{\alpha,\beta}}\right| \le u_{(1+\delta)/2}\right\} = \delta,$$

where $u_{(1+\delta)/2}$ is the quantile of $N(0,1)$ distribution.

Solving the above inequality with respect to $r_{\alpha,\beta}$ we obtain confidence interval with the ends

$$\frac{r^*_{\alpha,\beta} z_\pm(a)}{W\left(r^*_{\alpha,\beta} z_\pm(a)\exp\left(z_\pm(a)\right)\right)},$$

where $z_\pm(a) = \frac{\sqrt{n}}{u_{(1\pm\delta)/2} w(a)}$ and $W(\cdot)$ is the Lambert W function (see Appendix B).

Note that the ends of the confidence interval depend on an unknown shape parameter a. This parameter is a nuisance parameter and must be eliminated. There are at least two methods of eliminating such nuisance parameters: estimating or appropriate averaging. In our considerations the shape parameter a is to be estimated. Therefore, a problem arises what estimation method should be chosen. Because theoretical considerations seem to be impossible, a simulation study was carried out.

3. SIMULATION STUDY

The simulation study was performed for different values of quantile ratios $r_{\alpha,\beta}$ and shape parameter a (since scale parameter λ is not important in the problem of ratio of quantiles estimation, it has been set to 1). We take $\alpha = 0.2$, $\beta = 0.8$ and the nominal confidence level equal to 0.95.

From among various methods of parameter estimation for the Dagum distribution (Dey *et al.* [4]) two methods were chosen. The first one is the classical method of moments (MM). In this method theoretical moments of the distribution are compared with the empirical ones. Estimators obtained by this method are solutions of the following system of equations

$$\lambda^m \frac{\Gamma\left(a + \frac{m}{v}\right)\Gamma\left(1 - \frac{m}{v}\right)}{\Gamma(a)} = \frac{1}{n}\sum_{i=1}^n x_i^m, \quad \text{for } m = 1,2,3.$$

The left-hand side is the m^{th} moment of the Dagum distribution (see Appendix A).

The second method applied in the study was the probability-weighted moments (PWM) (see eg. Hosking *et al.* [9]; Małecka and Pekasiewicz [12]; Pekasiewicz [14]). Probability-weighted moments of the Dagum distribution are equal to (see Appendix A)

$$E_{a,v,\lambda}\left[X F^m_{a,v,\lambda}(X)\right] = \lambda \frac{\Gamma\left((m+1)a + \frac{1}{v}\right)\Gamma\left(1 - \frac{1}{v}\right)}{(m+1)\Gamma\left((m+1)a\right)}, \quad \text{for } m \ge 0.$$

Estimators obtained by this method are the solutions of the following system of equations (for $m = 0, 1, 2$)

$$\begin{cases} \dfrac{\lambda\Gamma\left(a+\frac{1}{v}\right)\Gamma\left(1-\frac{1}{v}\right)}{\Gamma\left(a\right)} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_{i:n}, \\ \dfrac{\lambda\Gamma\left(2a+\frac{1}{v}\right)\Gamma\left(1-\frac{1}{v}\right)}{2\Gamma\left(2a\right)} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \dfrac{(i-1)}{(n-1)} x_{i:n}, \\ \dfrac{\lambda\Gamma\left(3a+\frac{1}{v}\right)\Gamma\left(1-\frac{1}{v}\right)}{3\Gamma\left(3a\right)} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \dfrac{(i-1)(i-2)}{(n-1)(n-2)} x_{i:n}. \end{cases}$$

Estimated coverage probabilities based on 10000 repetitions of samples of size $n = 1000$ are given in Table 1 (MM) and Table 3 (PWM). In Table 2 (MM) and in Table 4 (PWM) average lengths of confidence intervals are presented.

Table 1: Coverage probability.

a	$r_{\alpha,\beta}$		
	1.2	1.6	2
0.1	0.9493	0.9492	0.9494
0.5	0.9497	0.9500	0.9530
1.0	0.9501	0.9518	0.9558
1.5	0.9491	0.9496	0.9549
2.0	0.9475	0.9477	0.9492

Table 2: Average length.

a	$r_{\alpha,\beta}$		
	1.2	1.6	2
0.1	0.03793	0.13175	0.24484
0.5	0.03205	0.11137	0.20909
1.0	0.03025	0.10541	0.20010
1.5	0.03014	0.10457	0.19901
2.0	0.03023	0.10442	0.19688

Table 3: Coverage probability.

a	$r_{\alpha,\beta}$		
	1.2	1.6	2
0.1	0.9496	0.9494	0.9494
0.5	0.9496	0.9491	0.9490
1.0	0.9495	0.9497	0.9492
1.5	0.9484	0.9481	0.9486
2.0	0.9479	0.9477	0.9483

Table 4: Average length.

a	$r_{\alpha,\beta}$		
	1.2	1.6	2
0.1	0.03803	0.13182	0.24483
0.5	0.03204	0.11077	0.20529
1.0	0.03020	0.10433	0.19326
1.5	0.03009	0.10393	0.19249
2.0	0.03021	0.10435	0.19325

Since in practical applications the samples usually comprise many thousands of observations (cf. Section 4), in our simulations samples of size 1000 have been used. It appears that such a size may be treated as large enough to do asymptotics: the simulated coverage probability is very close to the nominal confidence level. Of course, for larger sample sizes the coverage probability should be almost equal to the assumed confidence level.

It can also be noticed that whatever method of estimation (the method of moments or of probability-weighted moments) is applied, probability of covering the true value of the quintile share ratio is near the nominal confidence level. It is also seen that the lengths of obtained confidence intervals are similar; it may be concluded that the length does not depend on the applied method of estimation.

It is worth noting that the method of probability-weighted moments has an advantage over the classical method of moments. Namely, the method of moments is applicable for the distributions which have at least three moments, while the method of probability-weighted moments can be applied for the distributions which have at least the expected value (and thus present heavier tails). In the light of the presented results of the simulations, the method of probability-weighted moments may be recommended to the estimation of the shape parameter a of the Dagum distribution in the construction of the confidence interval for quintile share ratio.

4. EXAMPLE OF APPLICATION

In this section we present the application of the inequality measures based on the first and the fourth quintile, i.e. $r_{0.2,0.8}$, to income inequality analysis in Poland. Calculations are based on the sample coming form the Household Budget Survey (HBS) 2015 provided by the Statistics Poland and being the main source of information on income and expenditure of the population of households.

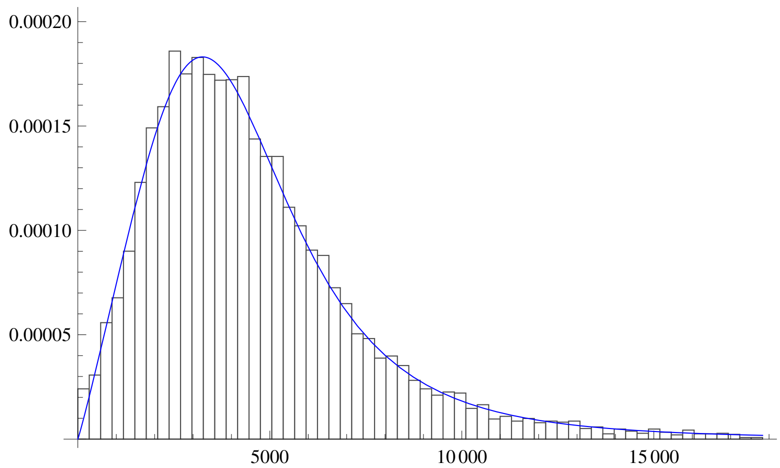

Figure 1: Income distribution in Poland and fitted Dagum distribution ($a = 0.6396$, $v = 3.2403$, $\lambda = 4961.36$).

Within the survey, the sample of size $n = 13420$ was drawn. Firstly, it was checked whether the Dagum distribution fits the data. In Figure 1 the histogram of collected data is shown along with the fitted Dagum distribution (the probability-weighted moments method was applied). The p-value of the standard Kolmogorov-Smirnov test equals 0.8983. Hence it may be concluded that the income distribution in Poland follows the Dagum model.

The sample quintile share ratio $r^*_{0.2,0.8}$ is 2.7600. Application of the formula ($*$) gives the confidence interval $(2.7081, 2.8160)$ for the population quintile share ratio $r_{0.2,0.8}$ (confidence level equals 0.95). It may be concluded that the income distribution in Poland is quite homogeneous, i.e. the poorest among the richest is about 2.76 times (at least 2.71 but at most 2.82) reacher then the richest among the poorest.

5. CONCLUSIONS

The main goal of the paper was to construct a confidence interval for the ratio of quantiles of the Dagum distribution. According to the best knowledge of the Authors, such a confidence interval has never been constructed. The confidence interval we propose is asymptotic. The first reason for such an approach is lack of finite sample results on the distribution of the ratio of sample quantiles for the Dagum model. Unfortunately, the distribution of the ratio of sample quantiles derived by Maswadah ([13]) was found to be analytically untractable. The second reason for considering asymptotics was that in practise the samples of income are really of large sizes. In a short simulation study it has been shown that sample size of 1000 may be treated as large enough to do asymptotics.

The ends of the obtained asymptotic confidence interval depend on shape parameter a of the Dagum distribution. This parameter should be estimated from a sample. In a simulation study two estimators of this parameter were applied. Both estimators gave similar results.

It will be interesting to check whether the length of the confidence interval depends on the choice of the estimation method (Maximum Likelihood, Method of L-Moments, Method of Maximum Product of Spacings and others) of the shape parameter a. Theoretical solutions seem unavailable, so relevant simulation studies are needed. Such studies are in preparation and will be published separately.

The confidence interval constructed above is symmetrical in the following sense: the risks of underestimation and overestimation are the same. It may also be interesting to consider the problem of constructing the shortest confidence interval. The idea of building such intervals is explained in detail in Zieliński ([16], [17]).

A. APPENDIX

Random variable X follows the Dagum distribution with parameters a, v, λ if its probability density function is given by the formula:

$$f_{a,v,\lambda}(x) = \frac{av}{\lambda}\left(\frac{x}{\lambda}\right)^{av-1}\left(1+\left(\frac{x}{\lambda}\right)^{v}\right)^{-a-1} \quad \text{for } x > 0.$$

Parameters a, v, λ are positive reals. Parameters a and v are shape parameters and λ is a scale parameter.

The distribution is unimodal if $av > 1$. Otherwise it is non-modal. If $av > 1$ the mode value is equal to

$$\lambda\left(\frac{av-1}{v+1}\right)^{\frac{1}{v}}.$$

Moments of the random variable X equal

$$E_{a,v,\lambda}X^m = \lambda^m \frac{\Gamma\left(1-\frac{m}{v}\right)\Gamma\left(a+\frac{m}{v}\right)}{\Gamma(a)}, \quad \text{for } m < v.$$

Empirical moment from a sample $X_1, ..., X_n$, i.e.

$$\frac{1}{n}\sum_{i=1}^{n} X_i^m$$

is the unbiased estimator of m^{th} moment of the random variable X.

Coefficient of skewness is equal to (for $v > 3$)

$$\frac{\Gamma^2(a)\,\Gamma\big(a+\frac{3}{v}\big)\,\Gamma\big(1-\frac{3}{v}\big) - 3\,\Gamma(a)\,\Gamma\big(a+\frac{1}{v}\big)\,\Gamma\big(a+\frac{2}{v}\big)\,\Gamma\big(1-\frac{2}{v}\big)\,\Gamma\big(1-\frac{1}{v}\big) + 2\,\Gamma^3\big(a+\frac{1}{v}\big)\,\Gamma^3\big(1-\frac{1}{v}\big)}{\Big(\Gamma(a)\,\Gamma\big(a+\frac{2}{v}\big)\,\Gamma\big(1-\frac{2}{v}\big) - \Gamma^2\big(a+\frac{1}{v}\big)\,\Gamma^2\big(1-\frac{1}{v}\big)\Big)^{3/2}}$$

and its kurtosis (for $v > 4$) is

$$\frac{\Gamma^2(a)\,\Big(\Gamma(a)\,\Gamma\big(a+\frac{4}{v}\big)\,\Gamma\big(1-\frac{4}{v}\big) + 3\,\Gamma^2\big(a+\frac{2}{v}\big)\,\Gamma^2\big(1-\frac{2}{v}\big) - 4\,\Gamma\big(a+\frac{1}{v}\big)\,\Gamma\big(a+\frac{3}{v}\big)\,\Gamma\big(1-\frac{3}{v}\big)\,\Gamma\big(1-\frac{1}{v}\big)\Big)}{\Big(\Gamma(a)\,\Gamma\big(a+\frac{2}{v}\big)\,\Gamma\big(1-\frac{2}{v}\big) - \Gamma^2\big(a+\frac{1}{v}\big)\,\Gamma^2\big(1-\frac{1}{v}\big)\Big)^{2}}.$$

The probability-weighted moments are equal to (for $m \geq 0$ and $v > 1$)

$$E_{a,v,\lambda}\left[XF^m_{a,v,\lambda}(X)\right] = \lambda\frac{\Gamma\left((m+1)a+\frac{1}{v}\right)\Gamma\left(1-\frac{1}{v}\right)}{(m+1)\Gamma\left((m+1)a\right)}.$$

Unbiased estimators (from a sample $X_1, ..., X_n$) of probability-weighted moments are

$$\frac{1}{n}\sum_{i=1}^{n} X_{i:n} \text{ (for } m = 0) \qquad \text{and} \qquad \frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)\cdots(i-m)}{(n-1)\cdots(n-m)}X_{i:n} \text{ (for } m \geq 1),$$

where $X_{1:n} \leq \cdots \leq X_{n:n}$ are ordered statistics (Hosking *et al.* [9]).

B. APPENDIX

Lambert function $W(\cdot)$ is defined as a solution with the respect to t of the equation

$$te^t = z \quad \Rightarrow \quad t = W(z).$$

It is seen that

$$W(z)e^{W(z)} = z \quad \Rightarrow \quad W(z) = \ln\left(\frac{z}{W(z)}\right) \quad \Rightarrow \quad z = \frac{z}{W(z)} \ln\left(\frac{z}{W(z)}\right).$$

Since the solution with respect to r of the equation $r \ln r = z$ is $r = \frac{z}{W(z)}$, hence

$$A\frac{x-r}{r\ln r} = 1 \quad \Rightarrow \quad Ax = r(\ln r + A) \quad \Rightarrow \quad e^A Ax = \left(re^A\right)\ln\left(re^A\right) \quad \Rightarrow \quad r = \frac{Ax}{W(Axe^A)}.$$

Application of the above to the equation

$$\sqrt{n}\frac{r^*_{\alpha,\beta} - r_{\alpha,\beta}}{w(a)r_{\alpha,\beta}\log r_{\alpha,\beta}} = u_{(1+\delta)/2}$$

gives the confidence interval for the ratio $r_{\alpha,\beta}$.

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