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ON A SUM AND DIFFERENCE OF TWO LINDLEY DISTRIBUTIONS: THEORY AND APPLICATIONS

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Abstract:

- This paper investigates theoretical and practical aspects of two basic random variables constructed from Lindley distribution. The first one is defined as the sum of two independent random variables following the Lindley distribution (with the same parameter) and the second one is defined as the difference of two independent random variables following the Lindley distribution (with the same parameter). Then, statistical inference is performed. In both the cases, we assess the performance of the maximum likelihood estimators using simulation studies. The usefulness of the corresponding models are proved using goodness-of-fit tests based on different real datasets.

Key-Words:

- *convolution; data analysis; Lindley distribution; maximum likelihood estimation; moment estimator.*

AMS Subject Classification:

- 60E05, 62E15, 62F10.

1. INTRODUCTION

Statistical distributions have been widely applied over the past decades for modeling data in several areas. In fact, the statistics literature is filled with hundreds or thousands of continuous univariate distributions. Among them, the exponential distribution is perhaps the most widely applied statistical distribution in various fields, mainly because of the simplicity of its mathematical quantities like moments, moment generating function, etc. However, under some comparison criteria, it was shown that the Lindley distribution is a reliable alternative to the exponential distribution in modeling lifetime data. The Lindley distribution has a cumulative density function (cdf) of the form

$$F_*(x) = 1 - \left(1 + \frac{\theta}{1+\theta}x\right) e^{-\theta x}, \quad x, \theta > 0.$$

The corresponding probability density function (pdf) is given by

$$f_*(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \quad x, \theta > 0. \tag{1.1}$$

As indicated by its name, this distribution was introduced by [14, 15] to illustrate a difference between fiducial distribution and posterior distribution. In the recent years, the Lindley distribution is mainly used for studying stress-strength reliability modeling. It finds applications in various areas such as engineering, demography, reliability, medicine and biology. Its detailed properties can be found in [6], [10], [3], [1], [24], [27], etc.

In the last decades, its different generalizations have been emerged in distribution theory and applications. In particular, the reader is refereed to the three parameters-Lindley distribution [31], generalized Poisson-Lindley distribution [16], generalized Lindley distribution [22], Marshall-Olkin Lindley distribution [32], power Lindley distribution [5], two-parameter Lindley distribution [25], quasi Lindley distribution [26], transmuted Lindley distribution [18], transmuted Lindley-geometric distribution [19], beta-Lindley distribution [20] and discrete Harris extended Lindley distribution [29], among others. Moreover, a latest version of the Lindley distribution, called modified Lindley distribution, is given by [2]. Further, Lindley distribution and its generalizations have been studied extensively by [30].

In this note, we consider two independent random variables following the Lindley distribution with appropriate parameter and study the convolutions (sum and difference) of their distributions. In addition, we investigate applications and structural properties of the new models. In fact, this is a pioneering work in investigating comprehensively the applications and properties of exact distributions of the sum and difference of Lindley random variables. The article is outlined as follows: Section 2 deals with a detailed study of sum of two independent Lindley distributions. Section 3 presents difference of two independent Lindley distributions. Finally, Section 4 offers some concluding remarks.

2. ON THE SUM OF TWO INDEPENDENT LINDLEY DISTRIBUTION

This section is devoted to the sum of two independent random variables following the Lindley distribution with pdf given by equation (1.1), including its main theoretical properties and modeling.

2.1. Definition

We consider the pdf given by

$$f(x) = \frac{\theta^4}{(1+\theta)^2} x \left(\frac{x^2}{6} + x + 1 \right) e^{-\theta x}, \quad x, \theta > 0. \tag{2.1}$$

The feature of this distribution is the following: let X and Y be two independent random variables following the Lindley distribution with parameter θ. Then, the random variable $Z = X + Y$ has the pdf given by (2.1). This result is a particular case of [8, Theorem 2]. A crystal clear proof is given below. Since X and Y are independent, the pdf of Z is given by the following convolution product: for $x > 0$,

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} f_*(x-t) f_*(t) dt = \int_0^x \frac{\theta^2}{1+\theta} (1+x-t) e^{-\theta(x-t)} \frac{\theta^2}{1+\theta} (1+t) e^{-\theta t} dt \\ &= \frac{\theta^4}{(1+\theta)^2} e^{-\theta x} \int_0^x (1+x-t)(1+t) dt = \frac{\theta^4}{(1+\theta)^2} x \left(\frac{x^2}{6} + x + 1 \right) e^{-\theta x}. \end{aligned}$$

For the purpose of this study, the corresponding distribution is called the 2S-Lindley distribution (2S for Sum of 2 random variables). To the best of our knowledge, there is no work on the theoretical and practical aspect of this distribution, which motivates a part of this study.

As a first approach, some possible shapes of the pdf of the 2S-Lindley distribution are shown in Figure 1.

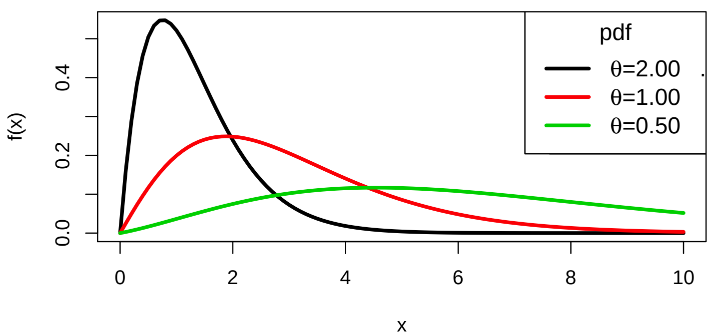

Figure 1: Plots of the pdf of the 2S-Lindley distribution for different values of θ.

2.2. Probability functions

First of all, after some algebraic manipulations, the cdf of the 2S-Lindley distribution is given by

$$F(x) = 1 - \frac{1}{6(1+\theta)^2}\left[\theta^3 x(x^2+6x+6) + 3\theta^2(x^2+4x+2) + 6\theta(x+2) + 6\right] e^{-\theta x},$$
$$x > 0.$$

The corresponding survival function (sf) is given by

$$\begin{aligned} S(x) &= 1 - F(x) \\ &= \frac{1}{6(1+\theta)^2}\left[\theta^3 x(x^2+6x+6) + 3\theta^2(x^2+4x+2) + 6\theta(x+2) + 6\right] e^{-\theta x}, \\ & x > 0. \end{aligned}$$

The corresponding hazard rate function (hrf) is given by

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta^4 x(x^2+6x+6)}{\theta^3 x(x^2+6x+6) + 3\theta^2(x^2+4x+2) + 6\theta(x+2) + 6}, \quad x > 0.$$

Also, the corresponding cumulative hazard rate function is given by

$$\begin{aligned} \Omega(x) &= -\log[S(x)] \\ &= \log(6) + 2\log(1+\theta) + \theta x \\ &- \log\left[\theta^3 x(x^2+6x+6) + 3\theta^2(x^2+4x+2) + 6\theta(x+2) + 6\right], \quad x > 0. \end{aligned}$$

The corresponding quantile function (qf), say $Q(u)$, can be obtained by solving the following equation: $F(Q(u)) = Q(F(u))$, $u \in (0,1)$. It can not be presented analytically but can be determined numerically for a given θ. Also, shapes of the hrf of the 2S-Lindley distribution are shown in Figure 2.

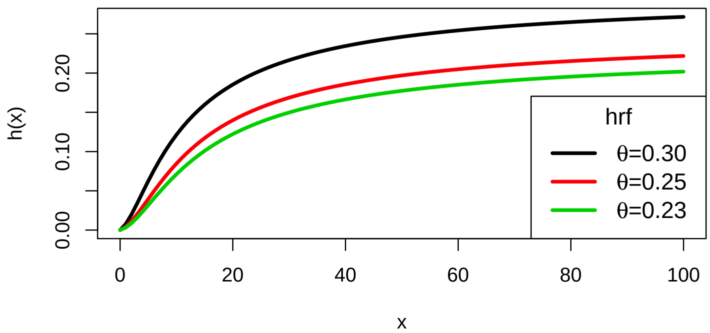

Figure 2: Plots of the hrf of the 2S-Lindley distribution for different values of θ.

2.3. Moments

The (ordinary) moments of the 2S-Lindley distribution are expressed in the following result.

Proposition 2.1. *Let $r \in \mathbb{N}$ and Z a random variable following the 2S-Lindley distribution with parameter θ. Then, the r-th moment of Z is given by*

$$\mu_r^* = E(Z^r) = \frac{1}{6\theta^r}\frac{1}{(1+\theta)^2}(r+1)!\left[6\theta^2 + 6\theta(r+2) + r^2 + 5r + 6\right].$$

Proof: Let us introduce the gamma function defined by $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$, $x > 0$. By using the pdf of Z given by (2.1), we have

$$\begin{aligned}
\mu_r^* = E(Z^r) &= \int_{-\infty}^{+\infty} x^r f(x)dx \\
&= \frac{\theta^4}{(1+\theta)^2}\left[\frac{1}{6}\int_0^{+\infty} x^{r+3}e^{-\theta x}dx + \int_0^{+\infty} x^{r+2}e^{-\theta x}dx + \int_0^{+\infty} x^{r+1}e^{-\theta x}dx\right] \\
&= \frac{\theta^4}{(1+\theta)^2}\left[\frac{1}{6}\frac{1}{\theta^{r+4}}\Gamma(r+4) + \frac{1}{\theta^{r+3}}\Gamma(r+3) + \frac{1}{\theta^{r+2}}\Gamma(r+2)\right] \\
&= \frac{1}{6\theta^r}\frac{1}{(1+\theta)^2}(r+1)!\left[6\theta^2 + 6\theta(r+2) + r^2 + 5r + 6\right].
\end{aligned}$$

This ends the proof of Proposition 2.1. □

An alternative proof of Proposition 2.1 using the Lindley distribution as baseline is given below. Let us recall that, for any $r \in \mathbb{N}$ and a random variable X following the Lindley distribution with parameter θ, the r-th moment of X is given by

$$\mu_r' = E(X^r) = \frac{r!(\theta + r + 1)}{\theta^r(1+\theta)}.$$

Therefore, by $Z = X + Y$ and the binomial formula, the r-th moment of Z is given by

$$\begin{aligned}
\mu_r^* = E((X+Y)^r) &= \sum_{k=0}^{r}\binom{r}{k}\mu_{r-k}'\mu_k' \\
&= \sum_{k=0}^{r}\binom{r}{k}\frac{(r-k)!(\theta+r-k+1)}{\theta^{r-k}(1+\theta)}\frac{k!(\theta+k+1)}{\theta^k(1+\theta)} \\
&= \frac{1}{\theta^r}\frac{1}{(1+\theta)^2}r!\sum_{k=0}^{r}(\theta+r-k+1)(\theta+k+1) \\
&= \frac{1}{6\theta^r}\frac{1}{(1+\theta)^2}(r+1)!\left[6\theta^2 + 6\theta(r+2) + r^2 + 5r + 6\right].
\end{aligned}$$

Also, owing to Proposition 2.1, we have

$$\mu_1^* = \frac{2(\theta+2)}{\theta(1+\theta)}, \quad \mu_2^* = \frac{6\theta(\theta+4)+20}{\theta^2(1+\theta)^2}, \quad \mu_3^* = \frac{24(\theta^2+5\theta+5)}{\theta^3(1+\theta)^2}$$

and

$$\mu_4^* = \frac{120(\theta^2 + 6\theta + 7)}{\theta^4(1+\theta)^2}.$$

In particular, the mean of Z is given by $\mu = \mu_1^*$ and the variance of Z is given by

$$\sigma^2 = \mu_2^* - \mu^2 = \frac{2(\theta^2 + 4\theta + 2)}{\theta^2(1+\theta)^2}.$$

Other important quantities can be defined via the moments as, for instance, the skewness and kurtosis coefficients of Z, respectively given by

$$\sqrt{\beta_1} = \frac{1}{\sigma^3} E\left[(Z-\mu)^3\right] = \frac{1}{\sigma^3} \sum_{k=0}^{3} \binom{3}{k} (-1)^{3-k} \mu_k^* \mu^{3-k}$$

and

$$\beta_2 = \frac{1}{\sigma^4} E\left[(Z-\mu)^4\right] = \frac{1}{\sigma^4} \sum_{k=0}^{4} \binom{4}{k} (-1)^{4-k} \mu_k^* \mu^{4-k}.$$

Table 1 indicates numerical values for the quantities above, i.e., μ_1^*, μ_2^*, μ_3^*, μ_4^*, σ^2, $\sqrt{\beta_1}$ and β_2, for selected values for θ.

Table 1: Numerical values of some measures of the 2S-Lindley distribution for selected values of parameter θ.

θ	μ_1^*	μ_2^*	μ_3^*	μ_4^*	σ^2	$\sqrt{\beta_1}$	β_2
0.002	1998.004	4992018	14970071832	5.23803e+13	999998	14.9442	4.5000
0.02	198.0392	49217.61	14707036	5132929642	9998.0787	14.4597	4.500
0.2	18.3333	434.7222	12583.33	429166.7	98.6111	10.8801	4.5420
0.1	38.1818	1856.198	109289.3	7547107	398.3471	12.6300	4.5124
1	3.0000	12.5000	66.0000	420.0000	3.5000	4.3525	4.8980
2	1.3333	2.5556	6.3333	19.1666	0.7778	0.4860	5.2347
5	0.4667	0.3222	0.2933	0.3305	0.1044	8.2294	5.6713
10	0.2182	0.0711	0.0307	0.0166	0.0235	22.0892	5.8688
20	0.1048	0.0164	0.0035	0.0009	0.0054	50.0478	5.9566
100	0.0202	0.0006	2.4715e-05	1.2477e-06	0.0002	275.9059	5.9978

2.4. Incomplete moments

A result on the incomplete moments for the 2S-Lindley distribution is given below.

Proposition 2.2. *Let r be a positive integer and Z a random variable following the 2S-Lindley distribution with parameter θ. Let us introduce the lower gamma function defined by $\gamma(x,y) = \int_0^y t^{x-1}e^{-t}dt$, $x > 0$ and $y \geq 0$. Then, the r-th incomplete moment of Z is given by*

$$\mu_r^*(t) = E(Z^r \mathbf{1}_{\{Z\leq t\}}) = \frac{1}{(1+\theta)^2\theta^r}\left[\frac{1}{6}\gamma(r+4,\theta t) + \theta\gamma(r+3,\theta t) + \theta^2\gamma(r+2,\theta t)\right].$$

Proof: By using the pdf of Z given by (2.1), we have

$$\begin{aligned}\mu_r^*(t) &= \int_{-\infty}^{t} x^r f(x)dx \\ &= \frac{\theta^4}{(1+\theta)^2}\left[\frac{1}{6}\int_0^t x^{r+3}e^{-\theta x}dx + \int_0^t x^{r+2}e^{-\theta x}dx + \int_0^t x^{r+1}e^{-\theta x}dx\right] \\ &= \frac{\theta^4}{(1+\theta)^2}\left[\frac{1}{6}\frac{1}{\theta^{r+4}}\gamma(r+4,\theta t) + \frac{1}{\theta^{r+3}}\gamma(r+3,\theta t) + \frac{1}{\theta^{r+2}}\gamma(r+2,\theta t)\right] \\ &= \frac{1}{(1+\theta)^2\theta^r}\left[\frac{1}{6}\gamma(r+4,\theta t) + \theta\gamma(r+3,\theta t) + \theta^2\gamma(r+2,\theta t)\right].\end{aligned}$$

This ends the proof of Proposition 2.2. □

The incomplete mean given by $\mu_1^*(t)$ deserves a particular focus, because it allows to express several important quantities, as the mean deviation of Z about the mean given by $\delta_1 = E(|Z-\mu|) = 2\mu F(\mu) - 2\mu_1^*(\mu)$, the mean residual life of Z given by $m_*(t) = E(Z - t \mid Z > t) = [1-\mu_1^*(t)]/[1-F(t)] - t$ and the mean waiting time of Z given by $M_*(t) = E(t - Z \mid Z < t) = t - \mu_1^*(t)/F(t)$, among others.

2.5. Characteristic function

The characteristic function of the 2S-Lindley distribution is provided in the following result.

Proposition 2.3. *Let Z be a random variable following the 2S-Lindley distribution with parameter θ. Then, the characteristic function of Z is given by*

$$\varphi(t) = \frac{\theta^4(\theta - it + 1)^2}{(1+\theta)^2(\theta - it)^4}, \quad t \in \mathbb{R}.$$

Proof: Let us recall that, for any $t \in \mathbb{R}$ and a random variable X following the Lindley distribution with parameter θ, the characteristic function of X is given by

$$\varphi_*(t) = E(e^{itX}) = \frac{\theta^2(\theta - it + 1)}{(1+\theta)(\theta - it)^2}.$$

Hence, using the representation $Z = X + Y$ with X and Y independent and identically distributed, the characteristic function for Z is given by

$$\varphi(t) = [\varphi_*(t)]^2 = \frac{\theta^4(\theta - it + 1)^2}{(1+\theta)^2(\theta - it)^4}.$$

This ends the proof of Proposition 2.3. □

2.6. Stochastic ordering

A result on stochastic ordering related to the 2S-Lindley distribution is now presented. Before that, some basics are recalled. Let X_1 and X_2 be two random variables having pdfs given by $f_1(x)$ and $f_2(x)$, respectively. Then, X_1 is said to be smaller than X_2 in the likelihood ratio order, denoted by $X_1 \leq_{lr} X_2$, if $f_1(x)/f_2(x)$ is decreasing in x. This property has important consequence in terms of distribution comparisons. We refer to [23] for the technical details.

Proposition 2.4. *Let X_1 be random variable following the 2S-Lindley distribution with parameter θ_1 and X_2 be a random variable following the 2S-Lindley distribution with parameter θ_2. Then, if $\theta_1 \geq \theta_2$, we have $X_1 \leq_{lr} X_2$.*

Proof: Let $f_1(x)$ and $f_2(x)$ be the pdfs of X_1 and X_2 given by (2.1) with $\theta = \theta_1$ and $\theta = \theta_2$, respectively. Then, for $x > 0$, we have

$$\frac{f_1(x)}{f_2(x)} = \frac{\theta_1^4(1+\theta_2)^2}{\theta_2^4(1+\theta_1)^2} e^{-(\theta_1-\theta_2)x},$$

which is clearly decreasing if $\theta_1 \geq \theta_2$, implying the desired result. This ends the proof of Proposition 2.4. □

2.7. Extreme order statistics

Let us consider a random sample $X_1, ..., X_n$ of size n from the 2S-Lindley distribution with parameter θ. Let $X_{1:n} = \min(X_1, ..., X_n)$ be the sample minima and $X_{n:n} = \max(X_1, ..., X_n)$ be the sample maxima. Then, we have the following limit results:

$$\lim_{t\to 0} \frac{F(xt)}{F(t)} =$$
$$\lim_{t\to 0} \frac{1 - \frac{1}{6(1+\theta)^2}\left[\theta^3 xt(x^2t^2+6xt+6) + 3\theta^2(x^2t^2+4xt+2) + 6\theta(xt+2) + 6\right] e^{-\theta xt}}{1 - \frac{1}{6(1+\theta)^2}\left[\theta^3 t(t^2+6t+6) + 3\theta^2(t^2+4t+2) + 6\theta(t+2) + 6\right] e^{-\theta t}}$$
$$= x.$$

Thus, [13, Theorem 1.6.2] ensures the existence of a_n and b_n such that

$$\lim_{n\to\infty} P(a_n(X_{1:n} - b_n) \leq x) = 1 - e^{-x}.$$

We recognize the cdf of the exponential distribution with parameter 1, showing that $a_n(X_{1:n} - b_n)$ can be approximated by this distribution.

Moreover, we have

$$\lim_{t\to+\infty} \frac{1 - F(x+t)}{1 - F(t)} =$$
$$\lim_{t\to+\infty} \frac{\theta^3(x+t)((x+t)^2 + 6(x+t) + 6) + 3\theta^2((x+t)^2 + 4(x+t) + 2) + 6\theta((x+t)+2) + 6}{\theta^3 t(t^2+6t+6) + 3\theta^2(t^2+4t+2) + 6\theta(t+2) + 6} e^{-\theta x}$$
$$= e^{-\theta x}.$$

Thus, [13, Theorem 1.6.2] ensures the existence of a_n and b_n such that

$$\lim_{n\to\infty} P(a_n(X_{n:n} - b_n) \leq x) = \exp\left(-e^{-\theta x}\right).$$

We recognize the cdf of the Gumbel distribution with parameters 1 and $1/\theta$, showing that $a_n(X_{n:n} - b_n)$ can be approximated by this distribution.

The form of the norming constants can also be determined using [13, Corollary 1.6.3].

2.8. Maximum likelihood estimator

Let $x_1, ..., x_n$ be n observations of a random variable Z following the 2S-Lindley distribution with parameter θ. Then, the likelihood and log-likelihood functions are, respectively, defined by

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = \frac{\theta^{4n}}{(1+\theta)^{2n}} \left[\prod_{i=1}^{n} x_i\right] \left[\prod_{i=1}^{n} \left(\frac{x_i^2}{6} + x_i + 1\right)\right] e^{-\theta \sum_{i=1}^{n} x_i}$$

and

$$\ell(\theta) = \log[L(\theta)] = 4n\log(\theta) - 2n\log(1+\theta) - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(x_i)$$
$$+ \sum_{i=1}^{n} \log\left(\frac{x_i^2}{6} + x_i + 1\right).$$

The maximum likelihood estimator (MLE) of θ, denoted by $\widehat{\theta}$, is defined by the θ maximizing $L(\theta)$ or $\ell(\theta)$. Thus, it can be obtained by solving the following equation: $\partial\ell(\theta)/\partial\theta = 0$, i.e.,

$$\frac{4n}{\theta} - \frac{2n}{1+\theta} - \sum_{i=1}^{n} x_i = 0.$$

After some algebra, we have

$$\widehat{\theta} = \frac{-\sum\limits_{i=1}^{n} x_i + \sqrt{\left(2n - \sum\limits_{i=1}^{n} x_i\right)^2 + 16n\sum\limits_{i=1}^{n} x_i} + 2n}{2\sum\limits_{i=1}^{n} x_i}.$$

Hence, $\widehat{\theta}$ has a simple expression. As any MLE, it enjoys desirable properties of convergence, guarantied by the well-established theory of the maximum likelihood method.

2.9. Simulation study

In this section, we present some simulation results to examine the finite sample behavior of the MLE proposed in previous section in the case of the 2S-Lindley distribution.

The simulation study is repeated for $N = 1000$ iterations each with sample size $n = 25$, 50, 150 and 300 from the 2S-Lindley distribution. The 2S-Lindley random number generation was performed using the sum of *rlindley()* function from **LindleyR** package [17] and the parameters are estimated by using the method of MLE by using the package nlm in R. The evaluation of the assessment is based on two quantities such as the bias and the mean squared errors (MSE), as follows:

1) bias of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R),$$

2) mean square error of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)^2,$$

where R is the true value of parameters θ. The results of our simulation study are summarized in Table 2. Based on the table, notice that the MLEs are close to the true parameter values for the current sample sizes, which means that the maximum likelihood method can be used effectively for estimating θ. Also, we can see that the bias and MSE of the MLEs converge to zero when the sample size is increased, as expected.

Table 2: Bias and MSE of $\hat{\theta}$ for the 2S-Lindley distribution.

	$\theta = 0.3$		$\theta = 0.5$		$\theta = 1.0$		$\theta = 1.2$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
25	0.0031	0.0009	0.0055	0.0027	0.0166	0.0124	0.0145	0.0178
50	0.0023	0.0004	0.0032	0.0013	0.0075	0.0055	0.0077	0.0079
100	0.0011	0.0002	0.0010	0.0007	0.0053	0.0029	0.0027	0.0041
200	6.7e-05	0.0001	0.0006	0.0003	0.0034	0.0015	0.0027	0.0022
300	4.6e-05	7.7e-05	5.1e-05	0.0002	0.0014	0.0010	0.0020	0.0014

2.10. Applications

Here, we use four data sets to illustrate the power of the proposed 2S-Lindley distribution. We compare the proposed distribution with the Lindley and exponential distributions. The first real data set corresponds to arose in tests on endurance of deep groove ball bearings from [12] on the number of million revolutions before failure for each of the 23 ball bearings in the life tests. The data are given below:

17.88 28.92 33.00 41.52 42.12 45.60 48.80 51.84 51.96 54.12 55.56 67.80 68.44 68.64 68.88 84.12 93.12 98.64 105.12 105.84 127.92 128.04 173.40 .

The second real data set is from [28]. It represents the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England. The data are given below:

0.55 0.93 1.25 1.36 1.49 1.52 1.58 1.61 1.64 1.68 1.73 1.81 2.00 0.74 1.04 1.27 1.39 1.49 1.53 1.59 1.61 1.66 1.68 1.76 1.82 2.01 0.77 1.11 1.28 1.42 1.50 1.54 1.60 1.62 1.66 1.69 1.76 1.84 2.24 0.81 1.13 1.29 1.48 1.50 1.55 1.61 1.62 1.66 1.70 1.77 1.84 0.84 1.24 1.30 1.48 1.51 1.55 1.61 1.63 1.67 1.70 1.78 1.89

The third real data set is reported by [7]. It demonstrates the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic. The data are given below:

1.1 1.4 1.3 1.7 1.9 1.8 1.6 2.2 1.7 2.7 4.1 1.8 1.5 1.2 1.4 3 1.7 2.3 1.6 2

The fourth data set is taken from [4]. it gives the strength data of glass of the aircraft window. The data are given below:

18.83 20.8 21.657 23.03 23.23 24.05 24.321 25.5 25.52 25.8 26.69 26.77 26.78 27.05 27.67 29.9 31.11 33.2 33.73 33.76 33.89 34.76 35.75 35.91 36.98 37.08 37.09 39.58 44.045 45.29 45.381.

For comparing the goodness of fit of the models, we found the unknown parameters (by the maximum likelihood method), standard error (SE), −log likelihood (−logL), AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), corrected Akaike Information Criterion (AICc) and Kolmogorov-Smirnov (K-S) statistic, given by

$$-\text{LogL} = -\log(L), \quad \text{AIC} = -2\text{LogL} + 2k, \quad \text{BIC} = -2\text{LogL} + k\log(n),$$

$$\text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

and

$$\text{K-S} = \max\{|F(x_i) - \hat{F}(x_i)|, |F(x_i) - \hat{F}(x_{i-1})|\},$$

where L is the maximum value of the corresponding likelihood function, k is the number of parameters, n is the sample size, $F(x_i)$ denote the value of the cdf of the candidate distribution at x_i and $\hat{F}(x_i)$ denote the value of the empirical distribution function at x_i.

Table 3, Table 4 Table 5 and Table 6 summarize the results of the fitted 2S-Lindley, Lindley and exponential distributions for the four considered data sets.

Table 3: Estimated values, −logL, AIC, BIC, AICc and K-S statistics for the first data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc	K-S
2S-Lindley	0.0570 (0.0028)	113.0799	228.1598	229.2953	228.2254	0.10606
Lindley	0.0273 (0.0040)	115.7356	233.4713	234.6068	233.66	0.19299
exponential	0.0138 (0.0029)	121.4365	244.8731	246.0086	245.06	0.30677

Table 4: Estimated values, −logL, AIC, BIC, AICc and K-S statistics for the second data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc	K-S
2S-Lindley	1.8011 (0.1274)	62.2742	126.5484	128.6916	126.614	0.32852
Lindley	0.9961 (0.0948)	81.27844	164.5569	166.7	164.6225	0.38643
exponential	0.6636 (0.0836)	88.83032	179.6606	181.8038	179.7262	0.418

Table 5: Estimated values, −logL, AIC, BIC, AICc and K-S statistics for the third data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc	K-S
2S-Lindley	1.4775 (0.1822)	24.8511	51.70225	52.69799	51.92447	0.29271
Lindley	0.8161 (0.1361)	30.24955	62.4991	63.49483	62.72132	0.43951
exponential	0.5263 (0.1179)	32.83708	67.67416	68.66989	67.89638	0.43951

Table 6: Estimated values, −logL, AIC, BIC, AICc and K-S statistics for the fourth data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc	K-S
2S-Lindley	0.1227 (0.011)	117.8023	237.6046	239.0386	237.7425	0.26915
Lindley	0.0630 (0.008)	126.9942	255.9884	257.4224	256.1263	0.36548
exponential	0.0324 (0.0058)	137.2644	276.5289	277.9629	276.6668	0.4586

From these tables, it is obvious that the smallest −logL, AIC, BIC, AICc and K-S statistic are acquired for the 2S-Lindley distribution. In summary, we can conclude that the 2S-Lindley model can be adequate for modeling these data.

3. ON THE DIFFERENCE OF TWO INDEPENDENT LINDLEY DISTRIBUTION

This section now focuses on the properties of the difference of two independent random variables following the Lindley distribution with the same parameter.

3.1. Definition

We now consider the pdf given by

$$f(x) = \frac{\theta}{4(1+\theta)^2}\left[\theta(2\theta+1)|x| + 2\theta^2 + 2\theta + 1\right] e^{-\theta|x|}, \quad x \in \mathbb{R}, \theta > 0. \tag{3.1}$$

The feature of this pdf is described in the result below.

Proposition 3.1. *Let X and Y be two independent random variables both following the Lindley distribution with parameter θ. Then, the random variable $Z = X - Y$ has the pdf given by (3.1).*

Proof: First of all, since the support of X and Y is $(0, +\infty)$, the support of Z is $\mathbb{R}$. Now, let us notice that the cdf and pdf of $-Y$ are, respectively, given by

$$F_{**}(x) = \left[1 - \frac{\theta}{1+\theta}x\right] e^{\theta x}, \quad f_{**}(x) = \frac{\theta^2}{1+\theta}(1-x)e^{\theta x}, \quad x < 0.$$

Since X and $-Y$ are independent, the pdf of Z is given by the convolution product:

$$\begin{aligned}
f(x) &= (f_* \star f_{**})(x) = \int_{-\infty}^{+\infty} f_*(x-t) f_{**}(t) dt \\
&= \int_{-\infty}^{\inf(x,0)} \frac{\theta^2}{1+\theta}[1 + (x-t)]e^{-\theta(x-t)} \frac{\theta^2}{1+\theta}(1-t)e^{\theta t} dt \\
&= \frac{\theta^4}{(1+\theta)^2} e^{-\theta x} \left\{ \int_{-\infty}^{\inf(x,0)} (1-t)^2 e^{2\theta t} dt + x \int_{-\infty}^{\inf(x,0)} (1-t)e^{2\theta t} dt \right\} \\
&= \frac{\theta}{4(1+\theta)^2} e^{-\theta[x - 2\inf(x,0)]} \big[2\theta^2 \inf(x,0)^2 - 2\theta^2 \inf(x,0)x - 4\theta^2 \inf(x,0) + 2\theta^2 x \\
&\quad + \ 2\theta^2 - 2\theta \inf(x,0) + \theta x + 2\theta + 1\big].
\end{aligned}$$

When $x \geq 0$, we have $\inf(x, 0) = 0$ implying that

$$f(x) = \frac{\theta}{4(1+\theta)^2}\left[\theta(2\theta+1)x + 2\theta^2 + 2\theta + 1\right] e^{-\theta x}.$$

When $x < 0$, we have $\inf(x, 0) = x$, implying that

$$f(x) = \frac{\theta}{4(1+\theta)^2}\left[-\theta(2\theta+1)x + 2\theta^2 + 2\theta + 1\right] e^{\theta x}.$$

By putting the above results together, we obtain the desired result. This ends the proof of Proposition 3.1. □

For the purpose of this study, the corresponding distribution is called the 2D-Lindley distribution (2D for Difference of 2 random variables). To the best of our knowledge, there is no work on the theoretical and practical aspect of this distribution, which motivates a part of this study. Figure 3 shows the behavior the pdf of the 2D-Lindley distribution for selected values of parameter θ.

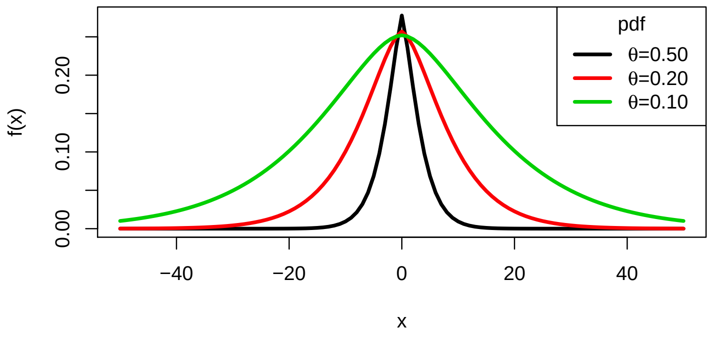

Figure 3: The pdf of the 2D-Lindley distribution for different values of θ.

3.2. Probability functions

The cdf of the 2D-Lindley distribution is presented in the proposition below.

Proposition 3.2. *The cdf of the 2D-Lindley distribution is given by*

$$F(x) = \begin{cases} \dfrac{1}{4(1+\theta)^2}\left[-\theta(2\theta+1)x + 2(1+\theta)^2\right] e^{\theta x} & \text{if } x < 0, \\ 1 - \dfrac{1}{4(1+\theta)^2}\left[\theta(2\theta+1)x + 2(1+\theta)^2\right] e^{-\theta x} & \text{if } x \geq 0. \end{cases} \tag{3.2}$$

Proof: For $x < 0$, by using (3.1), we have

$$\begin{aligned} F(x) = P(Z \leq x) &= \int_{-\infty}^{x} f(t)dt \\ &= \frac{\theta}{4(1+\theta)^2}\left[-\theta(2\theta+1)\int_{-\infty}^{x} te^{\theta t}dt + (2\theta^2+2\theta+1)\int_{-\infty}^{x} e^{\theta t}dt\right] \\ &= \frac{1}{4(1+\theta)^2}\left[-\theta(2\theta+1)x + 2(1+\theta)^2\right] e^{\theta x}. \end{aligned}$$

Since the distribution of Z is symmetric around 0, for $x \geq 0$, we have

$$F(x) = 1 - F(-x) = 1 - \frac{1}{4(1+\theta)^2}\left[\theta(2\theta+1)x + 2(1+\theta)^2\right] e^{-\theta x}.$$

We obtain the desired result by putting the above equalities together. This completes the proof of Proposition 3.2. □

By using Proposition 3.2, the corresponding survival function is given by

$$S(x) = \begin{cases} 1 - \dfrac{1}{4(1+\theta)^2}\left[-\theta(2\theta+1)x + 2(1+\theta)^2\right] e^{\theta x} & \text{if } x < 0, \\ \dfrac{1}{4(1+\theta)^2}\left[\theta(2\theta+1)x + 2(1+\theta)^2\right] e^{-\theta x} & \text{if } x \geq 0. \end{cases}$$

The corresponding hrf is given by

$$h(x) = \begin{cases} \dfrac{\theta\left[\theta(2\theta+1)|x| + 2\theta^2 + 2\theta + 1\right]}{4(1+\theta)^2 e^{-\theta x} + \theta(2\theta+1)x - 2(1+\theta)^2} & \text{if } x < 0, \\ \dfrac{\theta\left[\theta(2\theta+1)x + 2\theta^2 + 2\theta + 1\right]}{\theta(2\theta+1)x + 2(1+\theta)^2} & \text{if } x \geq 0. \end{cases}$$

Also, the corresponding chrf is given by

$$\Omega(x) = \begin{cases} -\log\left[1 - \dfrac{1}{4(1+\theta)^2}\left[-\theta(2\theta+1)x + 2(1+\theta)^2\right] e^{\theta x}\right] & \text{if } x < 0, \\ \log(4) + 2\log(1+\theta) + \theta x - \log\left[\theta(2\theta+1)x + 2(1+\theta)^2\right] & \text{if } x \geq 0. \end{cases}$$

The corresponding qf, say $Q(u)$, can be obtained by solving the following equation: $F(Q(u)) = Q(F(u))$, $u \in (0,1)$. It can not be presented analytically but can be determined numerically for a given θ. Further, Figure 4 depicts the behavior the hrf of the 2D-Lindley distribution for selected values of parameter θ.

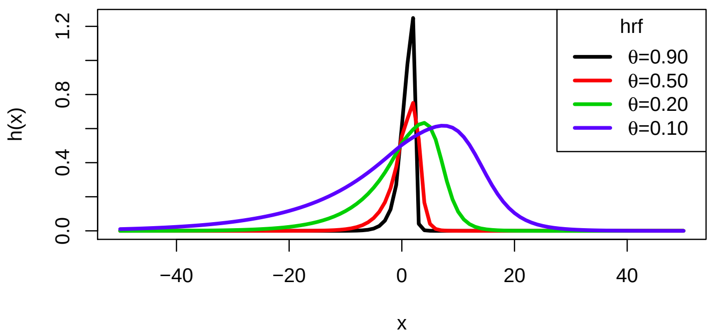

Figure 4: The hrf of the 2D-Lindley distribution for different values of θ.

3.3. Mixture

The 2D-Lindley distribution can be viewed as a particular mixture of distributions, as described below.

Proposition 3.3. *Let U, V and W be three random variables following the Laplace distribution with parameter θ and A a random variable following the Bernoulli distribution with parameter $\theta^2/(1+\theta)^2$, all these random variables are independent. Let Z be a random variable following the 2D-Lindley distribution with parameter θ. Then, we have the following stochastic representation:*

$$Z \overset{(d)}{=} AU + (1-A)(V+W).$$

Proof: It is enough to remark that we can write $f(x)$ given by (3.1) as

$$\begin{aligned} f(x) &= \frac{\theta^2}{(1+\theta)^2}\left[\frac{\theta}{2}e^{-\theta|x|}\right] + \frac{1+2\theta}{(1+\theta)^2}\left[\frac{\theta}{4}(1+\theta|x|)e^{-\theta|x|}\right] \\ &= pf_1(x) + (1-p)f_2(x), \end{aligned}$$

where

$$p = \frac{\theta^2}{(1+\theta)^2}, \quad f_1(x) = \frac{\theta}{2}e^{-\theta|x|}, \quad f_2(x) = \frac{\theta}{4}(1+\theta|x|)e^{-\theta|x|}.$$

One can notice that $f_1(x)$ is the pdf of the Laplace distribution with parameter θ and $f_2(x)$ is the pdf of the sum of two independent random variables both following the Laplace distribution with parameter θ as common distribution, see, [9, Section 2.3]. This ends the proof of Proposition 3.3. □

3.4. Moments

The moments of the 2D-Lindley distribution are described below.

Proposition 3.4. *Let $r \in \mathbb{N}$ and Z a random variable following the 2D-Lindley distribution with parameter θ. Then, the r-th moment of Z is given by*

$$\mu_r^* = E(Z^r) = [1+(-1)^r]\frac{1}{2\theta^r}r!\left[1+\frac{1+2\theta}{2(1+\theta)^2}r\right].$$

Proof: Since the distribution of Z is symmetric around 0 and the integral is well defined, for any $m \in \mathbb{N}$, we have $\mu_{2m+1}^* = 0$. By the use of the gamma function, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} \mu_{2m}^* = E(Z^{2m}) &= \int_{-\infty}^{+\infty} x^{2m} f(x)dx \\ &= \frac{\theta^2}{(1+\theta)^2}\int_{-\infty}^{+\infty} x^{2m}\frac{\theta}{2}e^{-\theta|x|}dx + \frac{1+2\theta}{(1+\theta)^2}\int_{-\infty}^{+\infty} x^{2m}\frac{\theta}{4}(1+\theta|x|)e^{-\theta|x|}dx \\ &= \frac{\theta^2}{(1+\theta)^2}\frac{1}{\theta^{2m}}\Gamma(2m+1) + \frac{1+2\theta}{(1+\theta)^2}\frac{1}{2}\frac{1}{\theta^{2m}}\left[\Gamma(2m+1)+\Gamma(2m+2)\right] \\ &= \frac{1}{\theta^{2m}}(2m)!\left[1+\frac{1+2\theta}{(1+\theta)^2}m\right]. \end{aligned}$$

By distinguishing the odd and even integer, we prove the desired result, ending the proof of Proposition 3.4. □

Owing to Proposition 3.4, we have

$$\mu_1^* = 0, \quad \mu_2^* = \frac{2(\theta^2+4\theta+2)}{\theta^2(1+\theta)^2}, \quad \mu_3^* = 0, \quad \mu_4^* = \frac{24[\theta(\theta+6)+3]}{\theta^4(1+\theta)^2}.$$

In particular, the mean of Z is given by $\mu = 0$ and the variance of Z is given by

$$\sigma^2 = \mu_2^* = \frac{2(\theta^2+4\theta+2)}{\theta^2(1+\theta)^2}.$$

Without surprise, the variance of the 2S and 2D Lindley distributions are the same.

The skewness of Z is equal to 0 and the kurtosis of Z is given by

$$\beta_2 = \frac{1}{\sigma^4} E\left[(Z-\mu)^4\right] = \frac{6(\theta^2+6\theta+3)(1+\theta)^2}{(\theta^2+4\theta+2)^2}.$$

Table 7 indicates numerical values for the quantities above, that is, μ_2^*, μ_4^*, σ^2 and β_2, for selected values for θ.

Table 7: Numerical values of some measures of the 2D-Lindley distribution for selected values of parameter θ.

θ	μ_2^*	μ_4^*	σ^2	β_2
0.02	9998.078	449884660	9998.078	4.5006
0.01	39998.04	7199529458	39998.04	4.5001
0.1	398.3471	716033.1	398.3471	4.5124
1	3.500	60.00	3.5000	4.8980
2	0.7778	3.1667	0.7778	5.2347
5	0.1044	0.0619	0.1044	5.6713
10	0.0235	0.0032	0.0235	5.8688
20	0.0055	0.0002	0.0055	5.9565
100	0.0002	2.494e-07	0.0002	5.9978

3.5. Characteristic function

The characteristic function of the 2D-Lindley distribution is presented below.

Proposition 3.5. *Let Z be a random variable following the 2D-Lindley distribution with parameter θ. Then, the characteristic function of Z is given by*

$$\varphi(t) = \frac{\theta^4[(1+\theta)^2+t^2]}{(1+\theta)^2(\theta^2+t^2)^2}, \quad t \in \mathbb{R}.$$

Proof: Let us recall that, for any $t \in \mathbb{R}$ and a random variable X following the Lindley distribution with parameter θ, the characteristic function of X is given by

$$\varphi_*(t) = E(e^{itX}) = \frac{\theta^2(\theta - it + 1)}{(1+\theta)(\theta - it)^2}.$$

Hence, using the representation $Z = X - Y$ with X and Y independent and identically distributed, the characteristic function for Z is given by

$$\varphi(t) = \varphi_*(t)\varphi_*(-t) = \frac{\theta^2(\theta - it + 1)}{(1+\theta)(\theta - it)^2} \times \frac{\theta^2(\theta + it + 1)}{(1+\theta)(\theta + it)^2} = \frac{\theta^4[(1+\theta)^2+t^2]}{(1+\theta)^2(\theta^2+t^2)^2}.$$

This ends the proof of Proposition 3.5. □

Let us mention that can prove Proposition 3.3 by using the characteristic function. It is enough to observe that we can write $\varphi(t)$ as

$$\varphi(t) = \frac{\theta^2}{(1+\theta)^2}\frac{\theta^2}{\theta^2+t^2} + \left(1 - \frac{\theta^2}{(1+\theta)^2}\right)\left[\frac{\theta^2}{\theta^2+t^2}\right]^2,$$

which is exactly the characteristic function of $AU + (1-A)(V+W)$, implying the desired result.

3.6. Maximum likelihood estimator

Let $x_1, ..., x_n$ be n observations of a random variable Z following the 2D-Lindley distribution with parameter θ. Then, the likelihood and log-likelihood functions are, respectively, defined by

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = \frac{\theta^n}{4^n(1+\theta)^{2n}}\left\{\prod_{i=1}^{n}\left[\theta(2\theta+1)|x_i| + 2\theta^2 + 2\theta + 1\right]\right\} e^{-\theta\sum_{i=1}^{n}|x_i|}$$

and

$$\begin{aligned}\ell(\theta) &= \log[L(\theta)] \\ &= n\log(\theta) - n\log(4) - 2n\log(1+\theta) - \theta\sum_{i=1}^{n}|x_i| \\ &+ \sum_{i=1}^{n}\log\left[\theta(2\theta+1)|x_i| + 2\theta^2 + 2\theta + 1\right].\end{aligned}$$

The MLE of θ can be obtained by solving the following equation: $\partial\ell(\theta)/\partial\theta = 0$, i.e.,

$$\frac{n}{\theta} - \frac{2n}{1+\theta} - \sum_{i=1}^{n}|x_i| + \sum_{i=1}^{n}\frac{(4\theta+1)|x_i| + 4\theta + 2}{\theta(2\theta+1)|x_i| + 2\theta^2 + 2\theta + 1} = 0.$$

This equation can not be solved analytically. However, some numerical algorithm allows to approach the solution in a precise way.

3.7. Simulation study

In this section, the simulation study is repeated for $N = 1000$ iterations from the 2D-Lindley distribution. For each replication, a random sample of size $n = 25$, 50, 100, 200 and 300 is drawn from the 2D-Lindley distribution. The 2D-Lindley random number generation was performed using difference of *rlindley()* function from **LindleyR** package [17] and the parameters are estimated by using the method of MLE by using the package nlm in R. The initial values of parameter are $\theta = 0.3, 0.5, 1.0$ and 1.5. The bias and MSE are presented in Table 8. From the table, we can observe that the bias and MSE of the MLEs converge to zero when the sample size is increased. This shows that the estimates are precise and accurate, hence, consistent and (asymptotically) unbiased.

Table 8: Bias and MSE of $\hat{\theta}$ for the 2D-Lindley distribution.

	$\theta = 0.3$		$\theta = 0.5$		$\theta = 1$		$\theta = 1.2$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
25	0.0066	0.0029	0.0055	0.0107	0.0288	0.0291	0.0307	0.0358
50	0.0048	0.0012	0.0090	0.0042	0.0133	0.0154	0.0154	0.0156
100	0.0020	0.0007	0.0027	0.0020	0.0057	0.0070	0.0074	0.0077
200	0.0012	0.0003	0.0010	0.0010	0.0017	0.0037	0.0040	0.0040
300	0.0008	0.0002	0.0007	0.0006	0.0016	0.0024	0.0022	0.0022

3.8. Applications

In this section, we analyze three data sets in order to illustrate the good performance of the 2D-Lindley distribution to compare with the Laplace and normal distributions, both with parameters standardly denoted by μ and σ. Here, we consider an extended form of the 2D-Lindley distribution by adding the location parameter μ in the pdf of the 2D-Lindley distribution. Thus, the related pdf is given by

$$f(x) = \frac{\theta}{4(1+\theta)^2}\left[\theta(2\theta+1)|x-\mu| + 2\theta^2 + 2\theta + 1\right] e^{-\theta|x-\mu|} \quad x, \mu \in \mathbb{R}, \theta > 0.$$

3.8.1. Comparison with the Laplace distribution

The first two data sets correspond to the age of the propellant and the tensile strength of kraft paper, respectively, reported in [21]. The data of the first set are given below:

15.5 23.75 8.0 17.0 5.5 19.0 24.0 2.5 7.5 11.0 13.0 3.75 25.0 9.75 22.0 18.0 6.0 12.5 2.0 21.5

The data of the second set are given below:

6.3 11.1 20.0 24.0 26.1 30.0 33.8 34.0 38.1 39.9 42.0 46.1 53.1 52.0 52.5 48.0 42.8 27.8 21.9

The third data set representing lung cancer rates data for 44 US states is given by `www.calvin.edu/stob/data/cigs.csv`. The data are given below:

17.05 19.8 15.98 22.07 22.83 24.55 27.27 23.57 13.58 22.8 20.3 16.59 16.84 17.71 25.45 20.94 26.48 22.04 22.72 14.2 15.6 20.98 19.5 16.7 23.03 25.95 14.59 25.02 12.12 21.89 19.45 12.11 23.68 17.45 14.11 17.6 20.74 12.01 21.22 20.34 20.55 15.53 15.92 25.88.

Table 9, Table 10 and Table 11 list the values of estimate, −logL, AIC, BIC and AICc, for the considered data sets.

Table 9: Estimated values, −logL, AIC, BIC and AICc for the first data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc
2D-Lindley	$\hat{\theta}$ **= 0.2335 (0.0447)**	**70.13335**	**144.2667**	**146.2582**	**144.9726**
	$\hat{\mu}$ **= 13.0205 (2.0049)**				
Laplace	$\hat{\mu}$ = 12.8350 (0.60)	71.33741	146.6748	148.6663	147.3807
	$\hat{\sigma}$ = 6.512916 (0.002)				

Table 10: Estimated values, −logL, AIC, BIC and AICc for the second data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc
2D-Lindley	$\hat{\theta}$ **= 0.1355 (0.0268)**	**77.22743**	**146.6748**	**148.6663**	**147.3807**
	$\hat{\mu}$ **= 34.7542 (3.4269)**				
Laplace	$\hat{\mu}$ = 34.00 (0.1062)	78.13456	160.2691	162.158	161.0191
	$\hat{\sigma}$ = 11.2337 (2.5791)				

Table 11: Estimated values, −logL, AIC, BIC and AICc for the third data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc
2D-Lindley	$\hat{\theta}$ **= 0.4182 (0.0536)**	**128.1709**	**260.3419**	**263.9103**	**260.6496**
	$\hat{\mu}$ **= 19.9190 (0.7894)**				
Laplace	$\hat{\mu}$ =20.3182 (0.27)	129.9786	263.9572	267.5255	264.2649
	$\hat{\sigma}$ = 3.5289 (0.03)				

From the tables, it may be noticed that the proposed 2D-Lindley model present the smallest values of the −logL, AIC, BIC and AICc and hence should be chosen as the best model for these datasets.

3.8.2. Comparison with the Normal distribution

Here we consider the data set artificially created from the standard Laplace distribution (with parameters 0 and 1) and truncated at the second decimal places which has been studied by [11]. The fourth data are given below:

-1.28 0.36 -1.29 -0.80 0.28 -0.06 -1.53 0.28 -0.54 0.17 0.59 6.22 2.41 0.33 -1.51 0.25 2.33 2.81 -0.92 2.12 -1.01 1.35 -0.37 -0.39 -4.39 -2.39 0.97 -0.58 -2.24 -0.05.

Table 12 shows the values of estimate, −logL, AIC, BIC and AICc, for the data set above.

Table 12: Estimated values, −logL, AIC, BIC and AICc for the fourth data set.

Distribution	Estimates (SE)	−logL	AIC	BIC	AICc
2D-Lindley	$\hat{\theta}$ =**1.0299 (0.1593)**	**59.4520**	**122.9040**	**125.7064**	**123.3484**
	$\hat{\mu}$ = **0.0295 (0.4476)**				
Normal	$\hat{\mu}$ = 0.1228 (0.3445)	61.61703	127.2341	130.0365	127.6785
	$\hat{\sigma}$ = 1.8870 (0.2436)				

From the Table 12, we can see that the 2D-Lindley model present the smallest values of the −logL, AIC, BIC and AICc, which confirm the suitability behavior of the 2D-Lindley distribution.

4. CONCLUDING REMARKS

In this paper, we have derived single representations for the exact distribution of the sum and difference of independent Lindley random variables. We referred to the distributions of sum and difference of two independent Lindley random variables as the 2S-Lindley and 2D-Lindley distributions, respectively. Statistical properties such as moments, incomplete moments, characteristic function, stochastic ordering and extreme order statistics of the 2S-Lindley distribution have been provided. At the same time, a comprehensive study of statistical properties of the 2D-Lindley distribution also has been discussed. The model parameters are estimated by maximum likelihood method for both cases. From simulation studies, the performance of the maximum likelihood estimators has been assessed. The new models provide consistently better fit than some classical models available in the literature. In conclusion, proposed model with their attracting properties should have a promising future in distribution theory.

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