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A UNIFICATION OF FAMILIES OF BIRNBAUM–SAUNDERS DISTRIBUTIONS WITH APPLICATIONS

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Abstract:

- This paper considers an extension for the skew-elliptical Birnbaum–Saunders model by considering the power-normal model. Some properties of this family are studied and it is shown, in particular, that the range of asymmetry and kurtosis surpasses that of the ordinary skew-normal and power-normal models. Estimation is dealt with by using the maximum likelihood approach. Observed and expected information matrices are derived and it is shown to be nonsingular at the vicinity of symmetry. The applications illustrate the better performance of the new distribution when compared with other recently proposed alternative models.

Key-Words:

- *elliptical Birnbaum–Saunders distribution; maximum likelihood; power-normal distribution.*

1. INTRODUCTION

Vilca-Labra and Leiva-Sánchez ([30]) extended the ordinary Birnbaum–Saunders (BS) distribution by considering the generalized Birnbaum–Saunders skew-elliptical distribution which is based on replacing the normal distribution by the elliptical family of distributions of which the normal distribution is a special case. Such general family of distributions is very successful in dealing with data sets with high degrees of asymmetry and kurtosis.

In this paper, we consider an extension of the generalized BS (GBS) model proposed in Díaz-García and Leiva-Sánchez ([9]) to the case of elliptical distributions. A comprehensive review of the GBS model can be found in Sanhueza *et al.* ([29]). Another important feature of this distribution is related to robustness with respect to parameter estimation which was studied in Barros *et al.* ([4]). The generalized Birnbaum–Saunders skew-elliptical distribution represents an important extension of the ordinary BS distribution to the case of symmetrical and asymmetrical distributions, which can be appropriate for applications in life data and material fatigue data.

The family of elliptical distributions has proved to be an important alternative to the normal distribution. The distributions in this family are symmetric and include distributions with greater and smaller kurtosis than the normal distribution. The normal distribution is an important member of the family. The elliptical family of distributions has been studied by many authors including Fang and Zang ([12]), Fang *et al.* ([11]), Gupta and Varga ([13]), Arellano-Valle and Bolfarine ([2]), among others.

A random variable X is distributed according to the elliptical distribution with location parameter ξ and scale parameter η if its pdf can be written as

$$f(x) = \frac{c}{\eta} g\left(\left(\frac{x-\xi}{\eta}\right)^2\right), \tag{1.1}$$

for some nonnegative function $g(u)$, $u>0$, such that $\int_0^\infty u^{-\frac{1}{2}} g(u)du = 1/c$, where c is a normalizing constant. The function $g(\cdot)$ is known as the density generator function. If X is elliptically distributed with location-scale parameters ξ and η and generator function g, denoted $X \sim EC(\xi,\eta;g)$. If $\xi=0$ and $\eta=1$, then X has spherical distribution, denoted as $X \sim EC(0,1;g)$. Properties of this family can be studied in Kelker ([15]), Cambanis *et al.* ([5]), Fang *et al.* ([11]), Arellano-Valle and Bolfarine ([2]) and Gupta and Varga ([13]) among others. Particular cases of the $X \sim EC(0,1;g)$ distribution are the Pearson type VII distribution, the type Kotz distribution, the Student-t (t_ν) distribution, the Cauchy distribution and the normal distribution, among others.

Díaz-García and Leiva-Sánchez ([9]) present the GBS distribution, by assuming that

$$Z = \frac{1}{\gamma}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim EC(0,1;g).$$

where $\gamma>0$ is the shape parameter and $\beta>0$ is the scale parameter and the distribution median. Then, from

$$T = \frac{\beta}{4}\left[\gamma Z + \sqrt{\gamma^2 Z^2 + 4}\right]^2,$$

the GBS distribution follows, which we denote by $T \sim GBS(\gamma, \beta; g)$. The pdf for the random variable T is given by

$$f_{GBS}(t) = cg\left(\frac{1}{\gamma^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)\frac{t^{-3/2}(t+\beta)}{2\gamma\beta^{1/2}}, \qquad t > 0, \tag{1.2}$$

where c is a normalizing constant and g is the generator function. Moreover, letting

$$a_t(\gamma, \beta) = a_t = \frac{1}{\gamma}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right), \tag{1.3}$$

it follows that

$$A_t(\gamma, \beta) = \frac{d}{dt}a_t(\gamma, \beta) = \frac{t^{-3/2}(t+\beta)}{2\gamma\beta^{1/2}},$$

so that (1.2) can be written as

$$f_{GBS}(t) = f(a_t(\gamma, \beta))A_t(\gamma, \beta),$$

where f is given in (1.1).

An extension of the elliptical model to the asymmetric case was given in Vilca-Labra and Leiva-Sánchez ([30]), where it is defined the standard elliptical asymmetric or skew-elliptical (SE) model as

$$f_Y(y; \lambda) = 2f(y)F(\lambda y); \quad y, \lambda \in \mathbb{R},$$

where f is given in (1.1), F is its respective cumulative distribution function (cdf) and λ is an asymmetry parameter. We use the notation $Y \sim SE(0, 1; g, \lambda)$. The cumulative distribution function for this model is given by

$$F_Y(y) = 2\int_{-\infty}^{y} f(t)F(\lambda t)dt. \tag{1.4}$$

A particular case of model (1.4) is the skew-normal (SN) distribution (see Azzalini, ([3])) with $f = \phi$ and $F = \Phi$ with pdf and cdf given, respectively, by

$$\phi_{SN}(y) = 2\phi(y)\Phi(\lambda y), \qquad y \in \mathbb{R}, \tag{1.5}$$

$$\Phi_{SN}(y) = \Phi(y) - 2T(y; \lambda), \qquad y \in \mathbb{R},$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of $N(0, 1)$ (the standard normal distribution), respectively and $T(\cdot; \cdot)$ is Owen's ([25]) function.

Extensions of the BS model to elliptical distributions were studied in Vilca-Labra and Leiva-Sánchez ([30]), namely, skew-elliptical Birnbaum–Saunders (SEBS) distribution. Model construction is based on the condition that

$$Z = \frac{1}{\gamma}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim SE(0, 1; g, \lambda).$$

We use the notation $SEBS(\gamma, \beta; g, \lambda)$. The case of model SEBS based on SN distribution, we denote $SNBS(\gamma, \beta, \lambda)$. Additional references on the BS distribution can be found in the recent book by Leiva ([18]).

An alternative asymmetric distribution is studied in Durrans ([10]), by introducing the fractional order statistical model, with pdf given by

$$\varphi_H(z;\alpha) = \alpha h(z)\{H(z)\}^{\alpha-1}, \qquad z \in \mathbb{R}, \tag{1.6}$$

where H is an absolutely continuous cumulative distribution function with pdf h and $\alpha > 0$ is a parameter that controls the distributional shape. The case $H = \Phi$ is called the power-normal (PN) distribution, with pdf given by

$$\varphi_\Phi(z;\alpha) = \alpha\phi(z)\{\Phi(z)\}^{\alpha-1}, \qquad z \in \mathbb{R},$$

denoted $Z \sim PN(\alpha)$. This model is an alternative to adjust data with asymmetry and kurtosis above (or below) the expected for the normal distribution.

In this paper we extend the SEBS model considered in Vilca-Labra and Leiva-Sánchez ([30]), using the fractionary order statistical model of Durrans ([10]). This generalization leads to a more flexible model in what concerns asymmetry and kurtosis, that the SEBS model, given that those models are special cases (hence also the ordinary BS model). It than can used for fitting fatigue data as well as life data.

The paper is organized as follows. Section 2 is devoted to study extensions of the GBS elliptical model by using the fractionary order statistical model in Durrans ([10]). Some properties of this family are studied and it is shown, in particular, that the range of asymmetry and kurtosis surpasses that of the ordinary skew-normal and power-normal models. Maximum likelihood estimation for the model proposed is implemented in Section 3. Observed and expected information matrices are derived and it is shown to be nonsingular at the vicinity of symmetry. Results of three real data application is presented in Section 4. The main conclusion is that the model proposed offers a viable alternative to others considered in the literature.

2. POWER SKEW-ELLIPTICAL BIRNBAUM–SAUNDERS DISTRIBUTIONS

We start by extending the model (1.6) assuming that the pdf h it is as follows

$$h(y;\lambda) = 2f(y)F(\lambda y); \quad y, \lambda \in \mathbb{R}, \tag{2.1}$$

where f is given in (1.1), F is its respective cumulative distribution function and λ is an asymmetry parameter. We call it the power skew-elliptical(PSE) model with pdf given by

$$\varphi_{PSE}(z;\lambda,\alpha) = \alpha h(z;\lambda)\{H(z;\lambda)\}^{\alpha-1}, \quad z \in \mathbb{R}. \tag{2.2}$$

We use the notation $Z \sim PSE(0,1;g,\lambda,\alpha)$.

Moments of the random variable Z have no closed form, but under a variable change the r-th moment of the random variable Z can be written as

$$E(Z^r) = \alpha\int_0^1 [H^{-1}(z;\lambda)]^r z^{\alpha-1}dx,$$

where H^{-1} is the inverse of the function H.

If the pdf h follows model (1.5), then, we have the power skew-normal (PSN) model with parameters λ and α introduced in Martínez-Flórez *et al.* ([23]). This model we denote by $PSN(\lambda, \alpha)$.

Special cases of model PSN occur with $\alpha = 1$, so that the skew-normal model $\phi_{SN}(x)$, follows. On the other hand, with $\lambda = 0$ the model with pdf $\varphi_\Phi(x)$, that is, Durrans generalized normal model follows. The ordinary standard normal model is also a special case which follows by taking $\alpha = 1$ and $\lambda = 0$, that is, $\varphi_{PSN}(x; 0, 1) = \phi(x)$. Notice from Figure 1 (a) and (b) below that α and λ affect both, distribution asymmetry and kurtosis and hence the model proposed seems more flexible than the models by Azzalini ([3]) and Durrans ([10]).

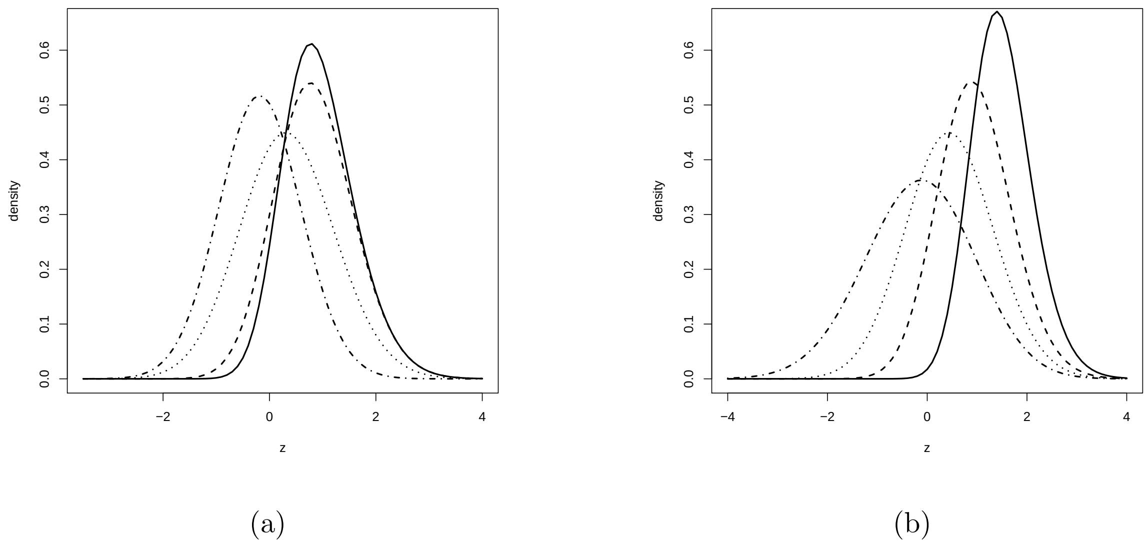


Figure 1: PSN model. (a) $\alpha = 1.5$ and λ =-0.75 (dotted dashed line), 0 (dotted line), 1 (dashed line) and 1.75 (solid line), (b) $\lambda =$ 0.70 and $\alpha = 0.50$ (dotted-dashed line), 1.0 (dotted line), 2.0 (dashed line) and 5.0 (solid line).

For some values of λ and $\alpha \in [0.1, 100]$, asymmetry and kurtosis coefficients namely $\sqrt{\beta_1}$ and β_2, for $Z \sim PSN(\lambda, \alpha)$, are in the intervals [-1.4676,0.9953) and [1.4672,5.4386] respectively, see Martínez-Flórez *et al.* ([23]). Such intervals contain the corresponding intervals for the skew-normal distribution, given by (-0.9953,0.9953) and [3,3.8692) respectively, and for the PN model, given by [-0.6115,0.9007] and [1.7170,4.3556], respectively, see Pewsey *et al.* ([26]). This illustrates the fact that the exponentiated skew-normal family contains models with greater (and smaller) asymmetry than both skew-normal (Azzalini, ([3])) and the power-normal (generalized normal) model (Durrans, ([10])). It then encompasses a family of distributions with more of both, platykurtic and leptokurtic, distributions. This illustrates the fact that the PSE model can be more flexible, respective to asymmetry and kurtosis, than the models characterized by density functions f_Y and φ_H.

We consider now an extension of the BS model to the case of exponentiated skew elliptical distributions. Assuming that

$$Z = \frac{1}{\gamma}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim PSE(0,1;g,\lambda,\alpha),$$

it follow that Z is distributed according to model (2.2). Therefore, trough a simple variable change, it can be shown that the random variable

$$T = \frac{\beta}{4}\left[\gamma Z + \sqrt{\gamma^2 Z^2 + 4}\right]^2, \tag{2.3}$$

is distributed according to the power skew-elliptical Birnbaum–Saunders (PSEBS) distribution, denoted by $T \sim PSEBS(\gamma,\beta;g,\lambda,\alpha)$.

The pdf for random variable (2.3) is given by

$$\varphi_{PSEBS}(t;\gamma,\beta,\lambda,\alpha) = \alpha h(a_t(\gamma,\beta);\lambda)\{H(a_t(\gamma,\beta);\lambda)\}^{\alpha-1}A_t(\gamma,\beta), \quad t \in \mathbb{R}^+. \tag{2.4}$$

This model provides then a generalization for the model introduced by Díaz-García and Leiva-Sánchez ([9]) and Vilca-Labra and Leiva-Sánchez ([30]). Notice that for $\alpha = 1$, the SEBS model (Vilca-Labra and Leiva-Sánchez ([30])) is obtained and for $\lambda = 0$ and $\alpha = 1$ we obtain the GBS model (Díaz-García and Leiva-Sánchez ([9])). The case $\lambda = 0$ constitutes an extension for the BS model since it contains the ordinary BS model. This model has been studied in Martínez-Flórez *et al.* ([22]), supposing that $Z \sim PN(\alpha)$ and is called the power normal Birnbaum–Saunders (PNBS) model, denoted $PNBS(\gamma,\beta,\alpha)$ for the case of the normal distribution. Some properties and moments of the PSEBS distribution represented by the random variable T in (2.3) are presented next. Properties are similar to the ones derived for the SEBS distribution by Vilca-Labra and Leiva-Sánchez ([30]), for T with $Z \sim SE(0,1;g,\lambda)$.

Theorem 2.1. *Let $T \sim PSEBS(\gamma,\beta;g,\lambda,\alpha)$. Then,*

1. $bT \sim PSEBS(\gamma,b\beta;g,\lambda,\alpha)$, $\;b > 0$ *and*
2. $T^{-1} \sim PSEBS(\gamma,\beta^{-1};g,-\lambda,\alpha)$.

Proof: **1**. Let $T \sim PSEBS(\gamma,\beta;g,\lambda,\alpha)$ and $Y = bT$ for $b > 0$ so that $T = \frac{Y}{b}$, where the Jacobian is $J = \frac{1}{b}$. Moreover, since $a_t(\gamma,\beta) = a_{y/b}(\gamma,\beta) = a_y(\gamma,b\beta)$ and $|J|\frac{d}{dt}a_t(\gamma,\beta) = |J|\frac{d}{dt}a_{y/b}(\gamma,\beta) = \frac{d}{dy}a_y(\gamma,b\beta) = A_y(\gamma,b\beta)$, so that, from the above transformations we have

$$\begin{aligned} f_Y(y) &= \alpha h(a_{y/b}(\gamma,\beta);\lambda)\left\{H(a_{y/b}(\gamma,\beta);\lambda)\right\}^{\alpha-1}\frac{d}{dt}a_{y/b}(\gamma,\beta)|J| \\ &= \alpha h(a_y(\gamma,b\beta);\lambda)\left\{H(a_y(\gamma,b\beta);\lambda)\right\}^{\alpha-1}A_y(\gamma,b\beta), \end{aligned}$$

so that $Y = bT \sim PSEBS(\gamma,b\beta;g,\lambda,\alpha)$.

2. Let $T \sim PSEBS(\gamma,\beta;g,\lambda,\alpha)$ and $Y = T^{-1}$ then $T = Y^{-1}$ the jacobian of the transformation is $J = Y^{-2}$. Moreover, $a_t(\gamma,\beta) = a_{y^{-1}}(\gamma,\beta) = -a_y(\gamma,\beta^{-1})$ and $|J|\frac{d}{dt}a_t(\gamma,\beta) = |J|\frac{d}{dt}a_{y^{-1}}(\gamma,\beta) = \frac{d}{dy}a_y(\gamma,\beta^{-1}) = A_y(\gamma,\beta^{-1})$.

Then, $h(a_t(\gamma,\beta);\lambda) = h(a_{y^{-1}}(\gamma,\beta);\lambda) = h(a_y(\gamma,\beta);-\lambda)$ and

$$\begin{aligned}
H(a_t(\gamma,\beta);\lambda) &= H(-a_y(\gamma,\beta^{-1});\lambda) \\
&= \int_{-\infty}^{-a_y(\gamma,\beta^{-1})} 2cg(x^2)F(\lambda x)dx \\
&= \int_0^y 2cg(a_x(\gamma,\beta^{-1})^2)F(-\lambda a_x(\gamma,\beta^{-1}))\frac{d}{dx}a_x(\gamma,\beta^{-1})dx \\
&= \int_{-\infty}^{a_y(\gamma,\beta^{-1})} h(x;-\lambda)dx \\
&= H(a_y(\gamma,\beta^{-1});-\lambda).
\end{aligned}$$

Using the above transformations, we have that

$$\begin{aligned}
f_Y(y) &= \alpha h(a_{y^{-1}}(\gamma,\beta);\lambda)\left\{H(a_{y^{-1}}(\gamma,\beta);\lambda)\right\}^{\alpha-1}\frac{d}{dt}a_{y^{-1}}(\gamma,\beta)|J| \\
&= \alpha h(a_y(\gamma,\beta^{-1});-\lambda)\left\{H(a_y(\gamma,\beta^{-1});-\lambda)\right\}^{\alpha-1}A_y(\gamma,\beta^{-1})
\end{aligned}$$

then we conclude that $Y = T^{-1} \sim PSEBS(\gamma,\beta^{-1};g,-\lambda,\alpha)$. □

Theorem 2.2. *Let $T \sim PSEBS(\gamma,\beta;g,\lambda,\alpha)$, $\mathbb{H}_T$ its cumulative distribution function and H the distribution function of $Z \sim PSE(0,1;g,\lambda,\alpha)$. Then,*

$$\mathbb{H}_T(t,\gamma,\beta;g,\lambda,\alpha) = \{H(a_t(\gamma,\beta);\lambda)\}^{\alpha}.$$

Proof: Let $a_x(\lambda,\beta)$, as defined above, so that

$$\begin{aligned}
\mathbb{H}_T(t,\gamma,\beta;g\lambda,\alpha) &= \int_0^t \alpha h(a_x(\gamma,\beta);\lambda)\{H(a_x(\gamma,\beta);\lambda)\}^{\alpha-1}A_x(\gamma,\beta)dx \\
&= \int_0^t \alpha h(a_x(\gamma,\beta);\lambda)\{H(a_x(\gamma,\beta);\lambda)\}^{\alpha-1}\frac{d}{dx}a_x(\gamma,\beta)dx \\
&= \int_{-\infty}^{a_t(\gamma,\beta)} \alpha h(x;\lambda)\{H(x;\lambda)\}^{\alpha-1}dx \\
&= \mathbb{F}_Z(a_t(\gamma,\beta);\lambda,\alpha).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{F}_Z(a_t(\gamma,\beta);\lambda,\alpha) &= \int_{-\infty}^{a_t(\gamma,\beta)} \alpha h(x;\lambda)\{H(x;\lambda)\}^{\alpha-1}dx \\
&= \int_{-\infty}^{a_t(\gamma,\beta)} \frac{d}{dx}\{H(x;\lambda)\}^{\alpha}dx \\
&= \{H(a_t(\gamma,\beta);\lambda)\}^{\alpha},
\end{aligned}$$

concluding the proof. □

Theorem 2.3. *The p-th percentile of the $PSEBS(\gamma,\beta;g,\lambda,\alpha)$, $t_p = \mathbb{H}^{-1}(p,\gamma,\beta;g\lambda,\alpha)$, is given by:*

$$t_p = \beta\left[\frac{\lambda}{2}z_p + \sqrt{\left(\frac{\lambda}{2}z_p\right)^2 + 1}\right]^2,$$

where z_p is the p-th percentile of the distribution of $PSE(0,1;g,\lambda,\alpha)$, given by $z_p = H^{-1}(p^{1/\alpha};\lambda)$.

Proof: For $p \in (0,1)$ as in Theorem 2.2, it follows that $p = \{H(a_t(\gamma,\beta);\lambda)\}^\alpha$ so that $a_T(\gamma,\beta) = Z_p = H^{-1}(p^{1/\alpha};\lambda) \sim PSE(0,1:g,\lambda,\alpha)$ where H^{-1} is the inverse of H. Therefore, result follows from (2.3). □

Theorem 2.4. *The survivor function, cumulative risk function, risk and inverted risk functions for model PSEBS are given, respectively, by:*

$$S(t) = 1 - \{H(a_t(\gamma,\beta);\lambda)\}^\alpha, \quad M(t) = -\log[S(t)],$$

$$r(t) = \alpha r_{SEBS}(t)\frac{\{H(a_t(\gamma,\beta);\lambda)\}^{\alpha-1} - \{H(a_t(\gamma,\beta);\lambda)\}^\alpha}{1 - \{H(a_t(\gamma,\beta);\lambda)\}^\alpha} \text{ and } R(t) = \alpha R_{SEBS}(t),$$

where $r_{SEBS}(t)$ and $R_{SEBS}(t)$ denote the risk and inverted risk for the skew-elliptical BS model.

Proof: Result follows directly from the definitions of survival function risk and inverse risk using the result in Theorem 2.2. □

From Theorem 2.4 we can conclude that the inverse risk rate is proportional to the risk rate for the SEBS distribution. Hence, the intervals where $R(t)$ is decreasing or increasing, are the same as the intervals where $R_{SEBS}(t)$ is decreasing or increasing.

The following two Theorem discuss the existence and the r-th moment of a random variable $T \sim PSEBS(\gamma,\beta;g,\lambda,\alpha)$.

Theorem 2.5. *Let $T \sim PSEBS(\gamma,\beta;g,\lambda,\alpha)$ and $Z \sim PSE(0,1;g,\lambda,\alpha)$. Hence, $E(T^r)$ exists if and only if,*

$$\mathbb{E}\left[\left(\frac{\gamma Z}{2}\right)^{k+l}\left(\left(\frac{\gamma Z}{2}\right)+1\right)^{\frac{k-l}{2}}\right] \tag{2.5}$$

exists $k = 1,2,...,r$ with $l = 0,1,...,k$.

Proof: Taking $Z \sim PSE(0,1;g,\lambda,\alpha)$ it follows that

$$\begin{aligned}\mathbb{E}\left\{\left[\frac{T}{\beta}\right]^n\right\} &= \mathbb{E}\left\{\left[\frac{\gamma}{2}Z + \sqrt{\left(\frac{\gamma}{2}Z\right)^2+1}\right]^2\right\}^n \\ &= \mathbb{E}\left\{\left[1+\left\{\frac{\gamma^2}{2}Z^2 + \gamma Z\sqrt{\left(\frac{\gamma}{2}Z\right)^2+1}\right\}\right]^n\right\}.\end{aligned}$$

Therefore, using the binomial expansion, we have

$$\mathbb{E}\left\{\left[\frac{T}{\beta}\right]^n\right\} = \sum_{k=0}^{n}\binom{n}{k}\mathbb{E}\left\{\left[\frac{\gamma^2}{2}Z^2 + \gamma Z\sqrt{\left(\frac{\gamma}{2}Z\right)^2+1}\right]^k\right\}$$

and doing another binomial expansion, we obtain

$$\mathbb{E}\left\{\left[\frac{T}{\beta}\right]^n\right\} = \sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}2^k\mathbb{E}\left\{\left[\left(\frac{\gamma}{2}Z\right)^{k+l}\left[\left(\frac{\gamma}{2}Z\right)^2+1\right]^{\frac{k-l}{2}}\right]\right\},$$

so that $\mathbb{E}\left\{\left[\frac{T}{\beta}\right]^n\right\}$ exists if, and only if, $\mathbb{E}\left\{\left[\left(\frac{\gamma}{2}Z\right)^{k+l}\left[\left(\frac{\gamma}{2}Z\right)^2+1\right]^{\frac{k-l}{2}}\right]\right\}$ exists, for $k = 0, 1, ..., n$ and $l = 0, 1, ..., k$. □

Theorem 2.6. *Let $T \sim PSEBS(\gamma, \beta; g, \lambda, \alpha)$ and $Z \sim PSE(0, 1; g, \lambda, \alpha)$. If $\mathbb{E}[Z^r]$ exists for $r = 1, 2, ...,$ then*

$$\mu_r = \mathbb{E}(T^r) = \beta^r \sum_{[0\le k\le r/2]}\binom{r}{2k}\left(\frac{1}{2}\right)^{2k}\sum_{j=0}^{2k}\binom{2k}{j}\mathbb{E}[(\gamma Z)^{4k-j}(\gamma^2Z^2+4)^{j/2}]$$
$$+ \beta^r \sum_{[0\le k< r/2]}\binom{r}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\sum_{j=0}^{2k+1}\binom{2k+1}{j}\mathbb{E}[(\gamma Z)^{4k+2-j}(\gamma^2Z^2+4)^{j/2}]$$

where $[\cdot]$ corresponds to the sum of the integer part of the subscripts.

Corollary 2.1. *For $r = 1, 2$ we have that*

$$\mathbb{E}(T) = \frac{\beta}{2}[2+\gamma^2\nu_2+\gamma\kappa_1] \text{ and } \mathbb{E}(T^2) = \frac{\beta^2}{2}[2+4\gamma^2\nu_2+\gamma^4\nu_4+2\gamma\kappa_1+\gamma^3\kappa_3],$$

where $\nu_k = \mathbb{E}[Z^k]$ and $\kappa_k = \mathbb{E}\left[Z^k\left(\gamma^2Z^2+4\right)^{1/2}\right]$. Then, the variance is given by

$$Var(T) = \mathbb{E}(T^2) - \mathbb{E}^2(T) = \frac{\gamma^2\beta^2}{4}[4\nu_2-\kappa_1^2+2\gamma\kappa_3-2\gamma\nu_2\kappa_1-\gamma^2\nu_2^2+2\gamma^2\nu_4].$$

The central moments, $\acute{\mu}_r = \mathbb{E}(T-\mathbb{E}(T))^r$, for $r = 2, 3, 4$ can be obtained using $\acute{\mu}_2 = \mu_2 - \mu_1^2$, $\acute{\mu}_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$ and $\acute{\mu}_4 = \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4$. Hence, variation coefficient, asymmetry and kurtosis can be obtained by using:

$$CV = \frac{\sqrt{\sigma_T^2}}{\mu_1}, \qquad \sqrt{\beta_1} = \frac{\acute{\mu}_3}{[\acute{\mu}_2]^{3/2}} \qquad \text{and} \qquad \beta_2 = \frac{\acute{\mu}_4}{[\acute{\mu}_2]^2}.$$

2.1. Power skew-normal Birnbaum–Saunders distribution

The power skew-normal Birnbaum–Saunders distribution is obtained by taking $H = \Phi_{SN}$ (and $h = \phi_{SN}$) in (2.4) and is denoted by PSNBS. It follows then that the density function is given by

$$\varphi_{PSNBS}(t; \gamma, \beta; \phi, \lambda, \alpha) = \alpha\phi_{SN}(a_t(\gamma, \beta))\{\Phi_{SN}(a_t(\gamma, \beta))\}^{\alpha-1}A_t(\gamma, \beta),$$

with a_t given in (1.3). Notice that the ordinary BS is a special case which follows by taking $F = \Phi$, $\lambda = 0$ and $\alpha = 1$. If $\alpha = 1$, the asymmetric BS model studied in Vilca-Labra and

Leiva-Sánchez ([30]) is derived and for $\lambda = 0$, we obtain the power-normal BS model studied in Martínez-Flórez *et al.* ([22]). Moreover, some properties of the BS distribution holds for the PSNBS distribution.

The cumulative distribution function for this model is given by

$$\mathbb{H}_{PSNBS}(t, \gamma, \beta; \lambda, \alpha) = \{\Phi(a_t(\gamma, \beta)) - 2T(a_t(\gamma, \beta); \lambda)\}^{\alpha}, \quad t > 0,$$

Figures 2 and 3 depicts the behavior of the PSNBS distribution for those values of α and λ.

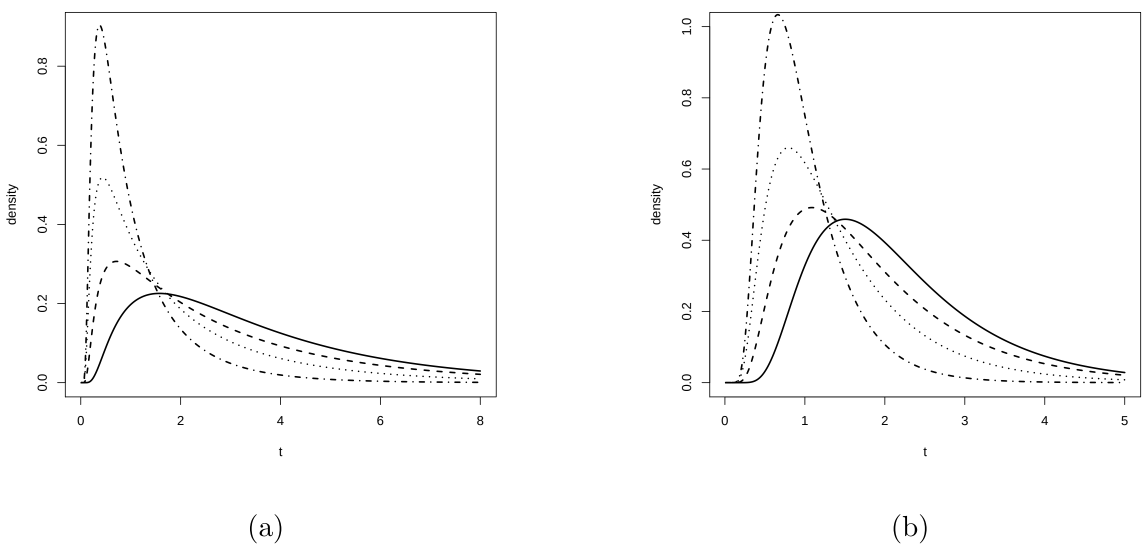


Figure 2: Plots for density function $\varphi_T(t; \gamma, \beta, \lambda, \alpha)$. (a) $(\gamma, \beta, \lambda, \alpha) =$ (0.75,1,-1,1.75) (dashed and dotted lines), $(\gamma, \beta, \lambda, \alpha) =$ (0.75,1,-0.25,1.75) (dotted line), $(\gamma, \beta, \lambda, \alpha) =$ (0.75,1,0.25,1.75) (dashed line) and $(\gamma, \beta, \lambda, \alpha) =$ (0.75,1,1,1.75) (solid line). (b)$(\gamma, \beta, \lambda, \alpha) =$ (1.25,1,-1,1.75) (dashed and dotted lines), $(\gamma, \beta, \lambda, \alpha) =$ (1.25,1,-0.25,1.75) (dotted line), $(\gamma, \beta, \lambda, \alpha) =$ (1.25,1,0.25,1.75) (dashed line) and $(\gamma, \beta, \lambda, \alpha) =$ (1.25,1,1,1.75) (solid line).

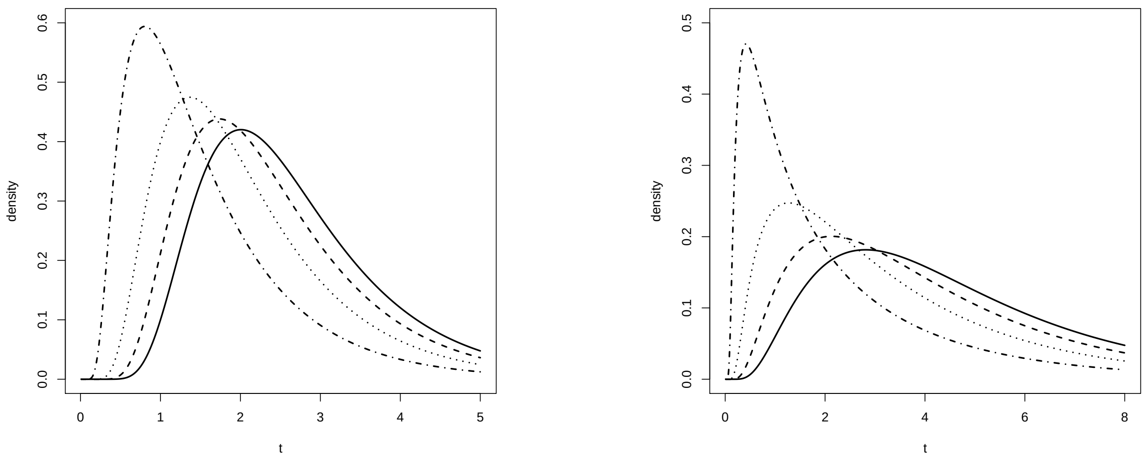
From Theorem 2.4, the survivor function, risk and inverted risk functions for model PSNBS are given, respectively, by

$$S(t) = 1 - \{\Phi_{SN}(a_t(\gamma, \beta))\}^{\alpha}, \quad M(t) = -\log[S(t)], \tag{2.6}$$

$$r(t) = \alpha r_{SNBS}(t) \frac{\{\Phi_{SN}(a_t(\gamma, \beta))\}^{\alpha-1} - \{\Phi_{SN}(a_t(\gamma, \beta))\}^{\alpha}}{1 - \{\Phi_{SN}(a_t(\gamma, \beta))\}^{\alpha}} \text{ and } R(t) = \alpha R_{SNBS}(t),$$

where $r_{SNBS}(t)$ and $R_{SNBS}(t)$ respectively denote the risk and inverted risk of the skew-normal Birnbaum–Saunders.

The following Theorem shows that for $t \to \infty$ the limit of the risk function of the PSNBS model coincides with the limit to infinity for the risk function of the SNBS model, result found by Leiva *et al.* ([20]).



a) b)

Figure 3: Plots for density function $\varphi_T(t;\gamma,\beta,\lambda,\alpha)$. a) $(\gamma,\beta,\lambda,\alpha) =$ (0.75,1,1,0.75) (dashed and dotted lines), $(\gamma,\beta,\lambda,\alpha) =$(0.75,1,1,1.5) (dotted line), $(\gamma,\beta,\lambda,\alpha) =$(0.75,1,1,2.25) (dashed line) and $(\gamma,\beta,\lambda,\alpha) =$(0.75,1,1,3) (solid line). b)$(\gamma,\beta,\lambda,\alpha) =$ (1.25,1,1,0.75) (dashed and dotted lines), $(\gamma,\beta,\lambda,\alpha) =$ (1.25,1,1,1.5) (dotted line), $(\gamma,\beta,\lambda,\alpha) =$ (1.25,1,1,2.25) (dashed line) and $(\gamma,\beta,\lambda,\alpha) =$ (1.25,1,1,3) (solid line).

Theorem 2.7.

$$\lim_{t\to\infty} r(t) = (1+\lambda^2)(2\gamma^2\beta)^{-1}.$$

Proof: Rewritting the risk function in the form

$$r(t) = \alpha r_{SNBS}(t)\{\Phi_{SN}(a_t(\gamma,\beta))\}^{\alpha-1}\frac{1-\Phi_{SN}(a_t(\gamma,\beta))}{1-\{\Phi_{SN}(a_t(\gamma,\beta))\}^{\alpha}},$$

and using L'Hôpital rule, we obtain

$$\lim_{t\to\infty}\frac{1-\Phi_{SN}(a_t(\gamma,\beta))}{1-\{\Phi_{SN}(a_t(\gamma,\beta))\}^{\alpha}} = \lim_{t\to\infty}\frac{-\phi_{SN}(a_t(\gamma,\beta))A_t(\gamma,\beta)}{-\alpha\{\Phi_{SN}(a_t(\gamma,\beta))\}^{\alpha-1}\phi_{SN}(a_t(\gamma,\beta))A_t(\gamma,\beta)} = \frac{1}{\alpha},$$

where $A_t(\gamma,\beta) = \frac{d}{dt}a_t(\gamma,\beta)$.

Therefore,

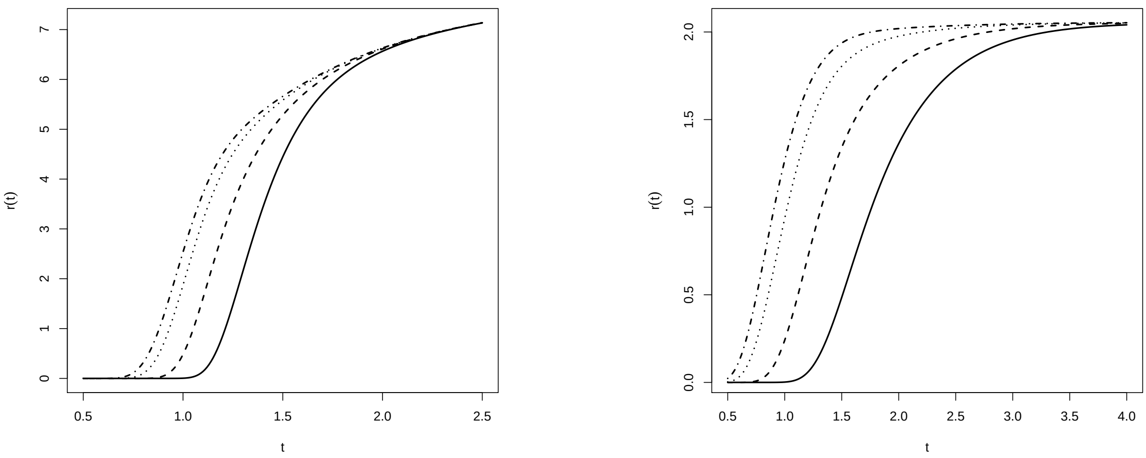
$$\lim_{t\to\infty} r(t) = \alpha\lim_{t\to\infty} r_{SNBS}(t)\frac{1}{\alpha} = \lim_{t\to\infty} r_{SNBS}(t) = (1+\lambda^2)(2\gamma^2\beta)^{-1}$$

where

$$\lim_{t\to\infty} r_{SNBS}(t) = (1+\lambda^2)(2\gamma^2\beta)^{-1},$$

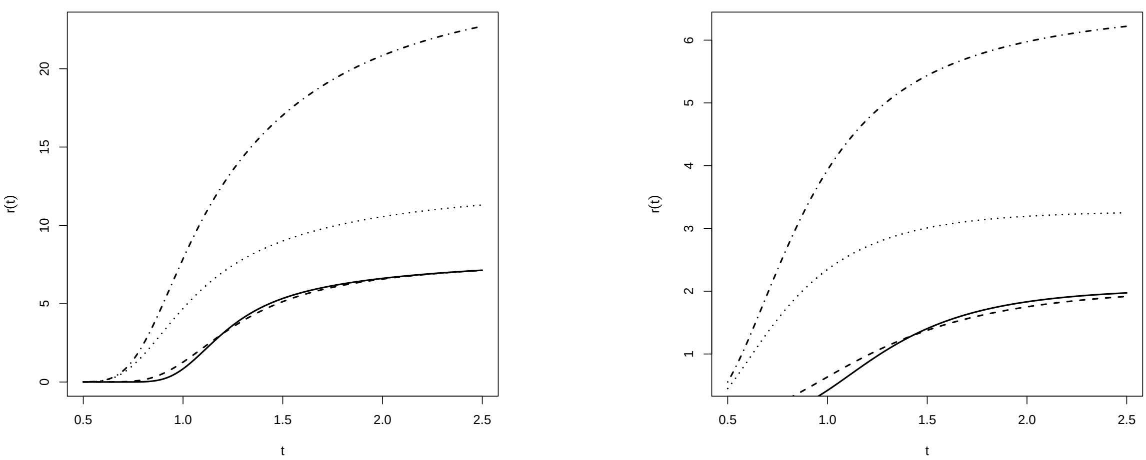
as shown in Leiva *et al.* ([20]). □

Figures 4 and 5 reveals the fact that the risk function is a non decreasing (and unimodal) function of t, but an increasing function of parameter α. Moreover, $r(t)$ is a non decreasing function for parameter γ.



(a) (b)

Figure 4: Function $r(t)$, for (a) $\gamma = 0.25$, $\beta = 1.0$, $\lambda = 2$ and $\alpha = 0.75$ (dashed and dotted line), α =1 (dotted line), α =2 (dashed line) and α =5 (solid line). (b)$\gamma = 0.5$, $\beta = 1.0$, $\lambda = 2$ and $\alpha = 0.75$ (dashed and dotted line), α =1 (dotted line), α =2 (dashed line) and α =5 (solid line).



(a) (b)

Figure 5: Function $r(t)$, for (a) $\gamma = 0.25$, $\beta = 1.0$, $\alpha = 1.75$ and λ =-1.5 (dashed and dotted lines), λ =-0.75 (dotted line), λ =0.75 (dashed line) and λ =1.5 (solid line). (b)$\gamma = 0.5$, $\beta = 1.0$, α = 1.75 and λ = -1.5 (dashed and dotted lines), λ =-0.75 (dotted line), λ =0.75 (dashed line) and λ =1.5 (solid line).

2.2. Inference for the PSNBS model

We present in this section the score functions and the observed and expected information matrices for the parameter $\theta = (\gamma, \beta, \lambda, \alpha)$. Given a random sample of size n, $\mathbf{t} = (t_1, ..., t_n)'$, from the distribution $PSNBS(\gamma, \beta, \lambda, \alpha)$, the log-likelihood function for $\theta = (\gamma, \beta, \lambda, \alpha)'$ can be written as

$$\begin{aligned}\ell(\theta; \mathbf{t}) \;&=\; n\left[\log(\alpha) - \log(\gamma) - \frac{1}{2}\log(\beta)\right] + \sum_{i=1}^{n}\log(t_i + \beta) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) \\ &\quad - \frac{1}{2\gamma^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right] + \sum_{i=1}^{n}\log(\Phi(\lambda a_{t_i})) + (\alpha - 1)\sum_{i=1}^{n}\log(\Phi_{SN}(a_{t_i})). \end{aligned} \tag{2.7}$$

The maximum likelihood (ML) estimators are obtained by maximizing the log-likelihood function given in (2.7). The score function, defined as the derivative of the likelihood function with respect to model parameters is denoted by $U(\theta) = (U(\gamma), U(\beta), U(\lambda), U(\alpha))'$ so that the score equations follow by equating the scores to zero, leading to the following equations

$$U(\gamma) = -\frac{1}{\gamma}\sum_{i=1}^{n}\left[1 - a_{t_i}^2 + a_{t_i}\left[\lambda w_i + (\alpha - 1)w_{1i}\right]\right] = 0,$$

$$U(\beta) = -\frac{n}{2\beta} + \sum_{i=1}^{n}\frac{1}{\beta + t_i} - \frac{1}{2\gamma^2}\sum_{i=1}^{n}\left[\frac{1}{t_i} - \frac{t_i}{\beta^2}\right] - \frac{1}{2\gamma\beta^{\frac{3}{2}}}\sum_{i=1}^{n}\frac{t_i + \beta}{t_i^{\frac{1}{2}}}\left[\lambda w_i + (\alpha - 1)w_{1i}\right] = 0,$$

$$U(\lambda) = \sum_{i=1}^{n}a_{t_i}\frac{\phi(\lambda a_{t_i})}{\Phi(\lambda a_{t_i})} - \sqrt{\frac{2}{\pi}}\frac{(\alpha - 1)}{1 + \lambda^2}\sum_{i=1}^{n}w_{2i} = 0, \qquad U(\alpha) = \frac{n}{\alpha} + \sum_{i=1}^{n}u_i = 0,$$

where $u_i = \log\{\Phi_{SN}(a_{t_i})\}$,

$$w_i = \frac{\phi(\lambda a_{t_i})}{\Phi(\lambda a_{t_i})}, \qquad w_{1i} = \frac{\phi_{SN}(a_{t_i})}{\Phi_{SN}(a_{t_i})},$$

and

$$w_{2i} = \frac{\phi\left(\sqrt{1 + \lambda^2}a_{t_i}\right)}{\Phi_{SN}(a_{t_i})}, \qquad i = 1, ..., n.$$

Numerical approaches are required for solving the above system of equations.

The elements of the observed information matrix are the negative of the second partial derivatives of the likelihood function with respect to the model parameters evaluated at the ML estimators. We use the notation $j_{\gamma\gamma}$, $j_{\beta\gamma}$, $j_{\lambda\gamma}$, $j_{\alpha\gamma}$, ..., $j_{\alpha\lambda}$, $j_{\alpha\alpha}$ so that, after extensive algebraic manipulations,

$$\begin{aligned} j_{\gamma\gamma} &= -\frac{n}{\gamma^2} + \frac{3}{\gamma^2}\sum_{i=1}^{n}a_{t_i}^2 + \frac{\lambda}{\gamma^2}\sum_{i=1}^{n}a_{t_i}^2 w_i\left[\lambda^2 a_{t_i} + \lambda w_i - 2\right] - \sqrt{\frac{2}{\pi}}\frac{\lambda(\alpha - 1)}{\gamma^2}\sum_{i=1}^{n}a_{t_i}^2 w_{2i} \\ &\quad - \frac{(\alpha - 1)}{\gamma^2}\sum_{i=1}^{n}a_{t_i}w_{1i}\left[2 + a_{t_i}^2 - a_{t_i}w_{1i}\right]. \end{aligned}$$

$$j_{\beta\gamma} = \frac{1}{\gamma^3}\sum_{i=1}^{n}\left[\frac{t_i}{\beta^2}-\frac{1}{t_i}\right] - \frac{\lambda}{2\gamma^2\beta^{3/2}}\sum_{i=1}^{n}\frac{t_i+\beta}{\sqrt{t_i}}w_i\left[1-\lambda a_{t_i}(\lambda a_{t_i}+w_i)\right]$$
$$-\frac{\alpha-1}{2\gamma^2\beta^{3/2}}\sum_{i=1}^{n}\frac{t_i+\beta}{\sqrt{t_i}}\left[\sqrt{\frac{2}{\pi}}\lambda a_{t_i}w_{2i}+w_{1i}(1+a_{t_i}^2-a_{t_i}w_{1i})\right],$$

$$j_{\lambda\gamma} = \frac{1}{\gamma}\sum_{i=1}^{n}a_{t_i}w_i\left[1-\lambda a_{t_i}w_i(\lambda a_{t_i}+w_i)\right] + \sqrt{\frac{2}{\pi}}\frac{\alpha-1}{\gamma}\sum_{i=1}^{n}a_{t_i}w_{2i}\left[a_{t_i}+\frac{1}{1+\lambda^2}w_{1i}\right],$$

$$j_{\beta\beta} = -\frac{n}{2\beta^2}+\sum_{i=1}^{n}\frac{1}{(t_i+\beta)^2}+\frac{1}{\gamma^2\beta^3}\sum_{i=1}^{n}t_i-\frac{1}{2\gamma\beta^{5/2}}\sum_{i=1}^{n}\frac{3t_i+\beta}{\sqrt{t_i}}[\lambda w_i+(\alpha-1)w_{1i}]$$
$$+\frac{1}{4\gamma^2\beta^3}\sum_{i=1}^{n}\frac{(t_i+\beta)^2}{t_i}\left[\lambda^2 w_i\left(\frac{\lambda(t_i-\beta)}{\gamma\beta^{1/2}t_i^{1/2}}+w_i\right)\right.$$
$$\left.+(\alpha-1)\left(\frac{t_i-\beta}{\gamma\beta^{1/2}t_i^{1/2}}w_{1i}+w_{1i}^2-\sqrt{\frac{2}{\pi}}\lambda w_{2i}\right)\right],$$

$$j_{\lambda\beta} = \frac{1}{2\gamma\beta^{3/2}}\sum_{i=1}^{n}\frac{t_i+\beta}{\sqrt{t_i}}w_i\left[1-\lambda a_{t_i}w_i(\lambda a_{t_i}+w_i)\right]$$
$$+\sqrt{\frac{2}{\pi}}\frac{\alpha-1}{2\gamma\beta^{3/2}}\sum_{i=1}^{n}\frac{t_i+\beta}{\sqrt{t_i}}w_{2i}\left[a_{t_i}+\frac{1}{1+\lambda^2}w_{1i}\right],$$

$$j_{\alpha\gamma} = \frac{1}{\gamma}\sum_{i=1}^{n}a_{t_i}w_{1i}, \qquad j_{\alpha\beta} = \frac{1}{2\gamma\beta^{3/2}}\sum_{i=1}^{n}\frac{t_i+\beta}{\sqrt{t_i}}w_{1i},$$

$$j_{\lambda\lambda} = \sum_{i=1}^{n}a_{t_i}^2 w_i(\lambda a_{t_i}+w_i)-\sqrt{\frac{2}{\pi}}\frac{2\lambda(\alpha-1)}{(1+\lambda^2)^2}\sum_{i=1}^{n}w_{2i}$$
$$+\sqrt{\frac{2}{\pi}}\frac{\alpha-1}{1+\lambda^2}\sum_{i=1}^{n}w_{2i}\left[-\lambda a_{t_i}^2+\sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}w_{2i}\right],$$

$$j_{\alpha\lambda} = \sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}\sum_{i=1}^{n}w_{2i}, \qquad j_{\alpha\alpha} = \frac{n}{\alpha^2}.$$

The elements of the Fisher information matrix are n^{-1} times the expected values of the elements of the matrix of second derivatives of the log-likelihood function.

Considering now $\lambda = 0$ and $\alpha = 1$ and using the approximation in Cribari-Neto and Branco ([8]), we can write the expected Fisher information matrix as

$$I_F(\theta) = \begin{pmatrix} \frac{1}{\gamma^2} & 0 & 0 & \frac{1}{4\gamma}\frac{\pi^2}{\sqrt{8+\pi^2}} \\ 0 & \frac{\sqrt{2\pi}+\gamma p(\gamma)}{\sqrt{2\pi}\gamma^2\beta^2} & A_1(\gamma,\beta) & A_2(\gamma,\beta) \\ 0 & A_1(\gamma,\beta) & \frac{2}{\pi} & \sqrt{\frac{1}{2}} \\ \frac{1}{4\gamma}\frac{\pi^2}{\sqrt{8+\pi^2}} & A_2(\gamma,\beta) & \sqrt{\frac{1}{2}} & 1 \end{pmatrix},$$

where $p(\gamma) = \gamma\sqrt{\frac{2}{\pi}} - \frac{\pi \exp(\frac{2}{\gamma^2})}{2} erfc(\frac{2}{\gamma})$, with $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)dt$ being the error function (see Prudnikov *et al.* ([27])), $A_1(\gamma,\beta) = \sqrt{\frac{2}{\pi}} \frac{1}{4\gamma^2\beta^2} \int_0^{\infty} \left(1 + \frac{\beta}{t}\right) \phi(a_t)dt$ and $A_2(\gamma,\beta) = \sqrt{\frac{2}{\pi}} \frac{1}{4\gamma^2\beta^2} \int_0^{\infty} \left(1 + \frac{\beta}{t}\right) \phi(2\sqrt{2}a_t/\pi)\Phi(-a_t)dt$.

The 2x2 superior submatrix of $I(\theta)$ is the Fisher information matrix for the ordinary BS distribution, as can be seen in Lemonte *et al.* ([21]). It can be verified that the columns (lines) of the matrix $I_F(\theta)$ are linearly independent and hence, it is invertible. Hence, for large samples, the MLE $\hat{\theta}$ of θ is asymptotically normal, that is,

$$\hat{\theta} \xrightarrow{A} N_4(\theta, I_F(\theta)^{-1}),$$

resulting that the asymptotic variance of the ML estimators $\hat{\theta}$ is the inverse of $I_F(\theta)$, which we denote by $\Sigma_{\hat{\theta}} = I_F(\theta)^{-1}$.

Approximation $N_4(\theta, \Sigma_{\hat{\theta}})$ can be used to construct confidence intervals for θ_r, which are given by $\hat{\theta}_r \mp z_{1-\rho/2}\sqrt{\hat{\sigma}(\hat{\theta}_r)}$, where $\hat{\sigma}(\cdot)$ corresponds to the r-th diagonal element of the matrix $\Sigma_{\hat{\theta}}$ and $z_{1-\rho/2}$ denotes $100(1-\rho/2)$-quantile of the standard normal distribution.
On the other hand, in presence of right-censoring we can adopt the following scheme. Assuming that for each individual the failure time is independent of the censoring time (say, Y_i and C_i for $i = 1, ..., n$ respectively). The observed times are given by $T_i = \min(Y_i, C_i)$ and the failure indicator is denoted as $\delta_i = I(Y_i \le C_i)$. Given a sample of observed times and failure indicators $(t_1, \delta_1), (t_2, \delta_2), ..., (t_n, \delta_n)$ and under the additional assumption of non-informative censoring, i.e., the distribution of failure times (Y_i) don't provide information about the censoring times (C_i) and viceversa (see Lagakos ([16])), the log-likelihood function for θ is given

$$l(\theta; \mathbf{t}) = \sum_{i=1}^{n} \left[\delta_i \log \varphi_{PSNBS}(t_i; \gamma, \beta; \phi, \lambda, \alpha) + (1 - \delta_i) \log S((t_i; \gamma, \beta; \phi, \lambda, \alpha))\right]. \tag{2.8}$$

For $\delta_i = 1$, $i = 1, ..., n$, equation (2.8) is reduced to (2.7). Finally, inference based on (2.8) can be performed in a similar manner as was done in the uncensored case, as described above.

3. RELATIONSHIP AMONG DISTRIBUTIONS OF THE FAMILY PSEBS

The pdf for the PSEBS model with t_ν distribution (denoted PSTBS) is given by:

$$\varphi_{PSTBS}(t;\xi) = \frac{\alpha\Gamma(\frac{\nu+1}{2})}{(\nu\pi)^{1/2}\Gamma(\frac{\nu}{2})} \left[1 + \frac{a_t^2}{\nu}\right]^{-\frac{\nu+1}{2}} F_{st}(\lambda a_t)\{H_{st}(a_t;\lambda)\}^{\alpha-1} A_t(\gamma,\beta), \tag{3.1}$$

where $\xi = (\gamma, \beta; \lambda, \alpha, \nu)$ and ν representing degrees of freedom and F_{st} is the cdf of the t_ν distribution (see Johnson *et al.* ([14])) and H_{st} is the cdf of the skew-t$_\nu$ distribution. The power skew-Cauchy Birnbaum–Saunders (PSCBS) model follows from pdf (3.1) by taking $\nu = 1$. Note that in the particular case that $\lambda = 0$ and $\alpha = 1$, the PSTBS coincides with the Birnbaum–Saunders-t$_\nu$ (BST) distribution studied in Díaz-García and Leiva-Sánchez ([9]) and for $\lambda = 0$ is obtained the Power Birnbaum–Saunders Student-t distribution studied in ([24]). Moreover, for $\alpha = 1$, we obtain the skew-t_ν-Birnbaum–Saunders (STBS) model, studied in Vilca-Labra and Leiva-Sánchez ([30]). The relationships among some of those models are presented in Figure 6.

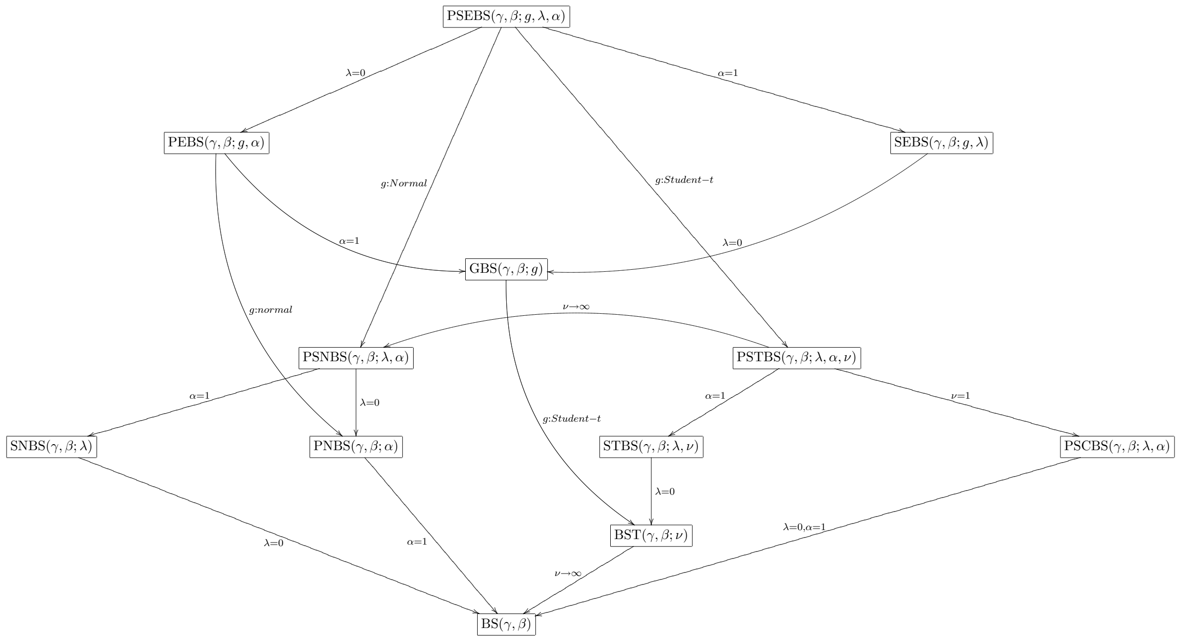


Figure 6: Relationship among distributions of the family PSEBS.

The density generator of the normal, Cauchy, t_ν, generalized t_ν, type I logistic, type II logistic and power exponential are, respectively, given by $g(u) = (2\pi)^{-1/2}\exp(-u/2)$, $g(u) = \{\pi(1+u)\}^{-1}$, $g(u) = \nu^{\nu/2}B(1/2,\nu/2)^{-1}(\nu+u)^{-(\nu+1)/2}$, where $\nu > 0$ and $B(\cdot,\cdot)$ is the beta function, $g(u) = s^{r/2}B(1/2,r/2)^{-1}(s+u)^{-(r+1)/2}$ $(s,r>0)$, $g(u) = c\exp(-u)(1+\exp(-u))^{-2}$, where $c \approx 1.484300029$ is the normalizing constant obtained from $\int_0^\infty u^{-1/2}g(u)du = 1$, $g(u) = \exp(-\sqrt{u})(1+\exp(-\sqrt{u}))^{-2}$ and $g(u) = c(k)\exp(-\frac{1}{2}u^{1/(1+k)})$, $-1 < k \le 1$, where $c(k) = \Gamma(1+(k+1)/2)2^{1+(1+k)/2}$.

4. APPLICATIONS

In this section, it is shown that the model discussed in the previous sections can give good feedback to understand relations between variables in applied problems. The first application considers the remission times (in months) of the bladder cancer patients. The second application presented is based on certain features of the trees in a forestry area, and the last applications is a censured data.

4.1. Application I

We consider an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients. These data were previously studied by Lee and Wang ([17]). Bladder cancer is a disease in which abnormal cells multiply without control in the bladder. The most common type of bladder cancer recapitulates the normal histology of the urothelium and is known as transitional cell carcinoma.

Descriptive statistics results are summarized in Table 1, where $\sqrt{b_1}$ and b_2 are sample asymmetry and sample kurtosis coefficients, respectively. There is indication of high kurtosis in this data set, which suggest that PSNBS model can be more appropriate than BS model. ML estimators were computed by maximizing log-likelihood using function "optim" in R Core Team ([28]). Table 2 shows the fitting of the BS, SNBS, PNBS and PSNBS models (standard error are in parenthesis). To compare the fitting of these models, we use Akaike ([1]) criterion, namely

$$AIC = -2\ell(\cdot;\mathbf{t}) + 2k,$$

we consider also the AICC (corrected Akaike information criterion), namely

$$AICC = AIC + \frac{2k(k+1)}{n-(k+1)},$$

where k is the number of parameters in the model. According to this criterion the model that best fits the data is the one with the lowest AIC or AICC. We also apply the formal goodness-of-fit tests in order to verify which distribution fits better to these data. We consider the Cramér-von Mises (W^*), Anderson-Darling (A^*) statistics, Kolmolgorov- Smirnov(K-S) test statistics and p-value. The statistics W^* and A^* are described in detail in Chen and Balakrishnan ([6]). In general, the smaller the values of the statistics W^* and A^*, the better the fit to the data.

Table 1: Descriptive statistics for the data set.

n	$\bar{t}$	s^2	$\sqrt{b_1}$	b_2
128	4.1293	9.3660	3.2480	15.1950

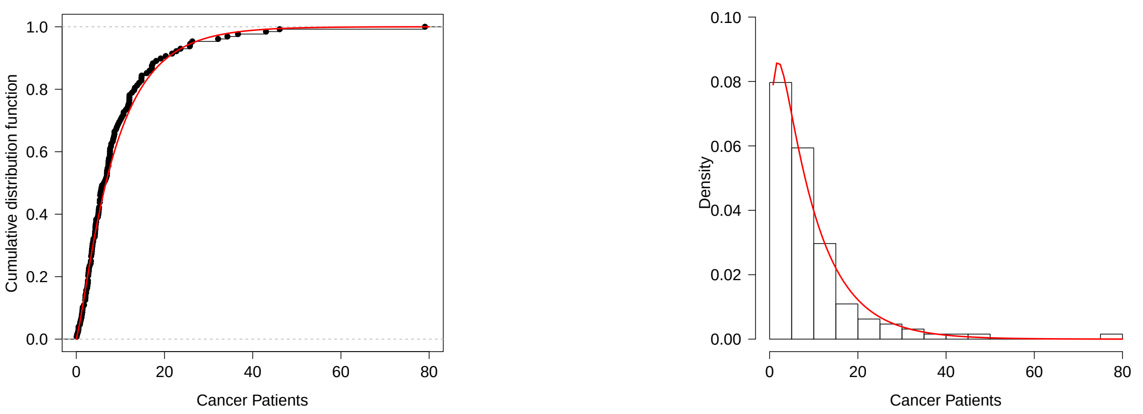
Table 2: ML estimates for BS, PNBS, SNBS and PSNBS models.

Parameters	γ	β	α	λ
BS	1.3740(0.0862)	4.5711(0.4461)	–	–
PNBS	3.2915(0.2856)	0.4227(0.6321)	5.1830(0.2051)	–
SNBS	2.3350(0.4131)	1.3566(0.3849)	–	1.9050(1.1294)
PSNBS	5.3315(3.0351)	0.1764(0.2060)	2.3024(0.4235)	2.5762(3.7211)

The values of these statistics for all models are given in Table 3. As expected, the values of AIC, AICC, W^*, A^*, K-S and p-value indicates better fit for the PSNBS model over the SNBS, PNBS and BS models. Figure 7 shows graphs for PSNBS model (a) empirical cdf (b) histogram and Figure 8 (a) and (b) shows the qq-plot for the models with better fit.

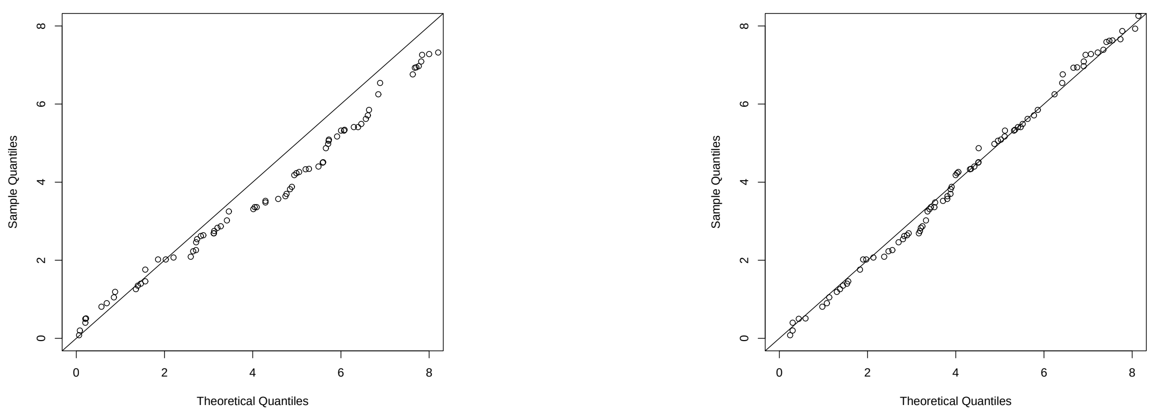
Table 3: AIC, AICC, W^*, A^*, K-S and p-value for the remission times of bladder cancer data for BS, PNBS, SNBS and PSNBS models.

	$\ell(\theta)$	AIC	AICC	W^*	A^*	K-S	P-value
BS	430.0420	864.0836	864.1898	0.4136	2.5615	0.1689	0.0013
PNBS	413.0645	832.1290	832.3433	0.1196	0.7219	0.0694	0.5680
SNBS	418.8570	843.7140	843.9283	0.1667	1.0930	0.1214	0.0459
PSNBS	411.8310	831.6620	832.0224	0.0829	0.5073	0.0623	0.7037



(a) (b)

Figure 7: Graphs for PSNBS model (a) empirical cdf (b) histogram.



(c) (d)

Figure 8: (a) qq-plot PNBS and (b) qq-plot PSNBS.

Note that the PSNBS model provides better fit to the data set analyzed. Therefore, the PSNBS model fits better than the other models, although it has one more parameter.

4.2. Application II

A major problem with forest areas is tree mortality due to various factors that can be seen as caused by stress through a phenomenon similar to material fatigue. In this context, two problems of great interest are tree mortality and the distribution of the diameter at the breast height (DBH). It has been observed that the BS distribution has a failure rate that can capture such features. As seen above, the ordinary BS is a particular case of the PSEBS distribution, so that the PSEBS is more flexible to explain skewness and kurtosis excess. Thus, we apply this distribution to explain the behavior of the variable DHB (in cm) in explaining forest mortality of Gray Birch (Betula populifolia Marshall) of a perennial with an average height of ten meters. The data basis consists of 160 trees and are available in Leiva *et al.* ([19]). Descriptive statistics results are summarized in Table 4. There is indication of high kurtosis in this data set, that suggest a more flexible model than the BS model, such as the PSTBS model. For this reason we implement the BS, BST, STBS and PSTBS models.

Table 4: Descriptive statistics for the data set.

n	$\bar{t}$	s^2	$\sqrt{b_1}$	b_2
160	14.5387	13.0510	2.8893	13.9716

Table 5 reports the estimates of the degrees of freedom, ν, for each model based on the t_ν distribution, which are obtained by maximizing the profile log-likelihood function. ML estimates (standard errors in parenthesis) are presented in Table 6.

Table 5: Estimation of ν for the BST, STBS and PSTBS models by maximizing the log-likelihood function.

	Log-likelihood	Log-likelihood	Log-likelihood
ν	BST	STBS	PSTBS
1	-406.4265	-402.8126	-390.2868
2	-392.7834	-387.5216	-383.0684
3	-389.9824	-383.4061	-381.0513
4	**−389.4381**	-381.8612	-380.0609
5	-389.5679	-381.1933	-379.6883
6	-389.9238	-380.8925	-379.4448
7	-390.3497	-380.8505	-379.4001
8	-390.7852	**−380.7285**	-379.0779
9	-391.2060	-381.0066	**−378.8113**
10	-391.6025	-383.8818	-379.3470

Table 6: ML estimates for BS, BST ($\nu = 4$), STBS ($\nu = 8$) and PSTBS ($\nu = 9$) models.

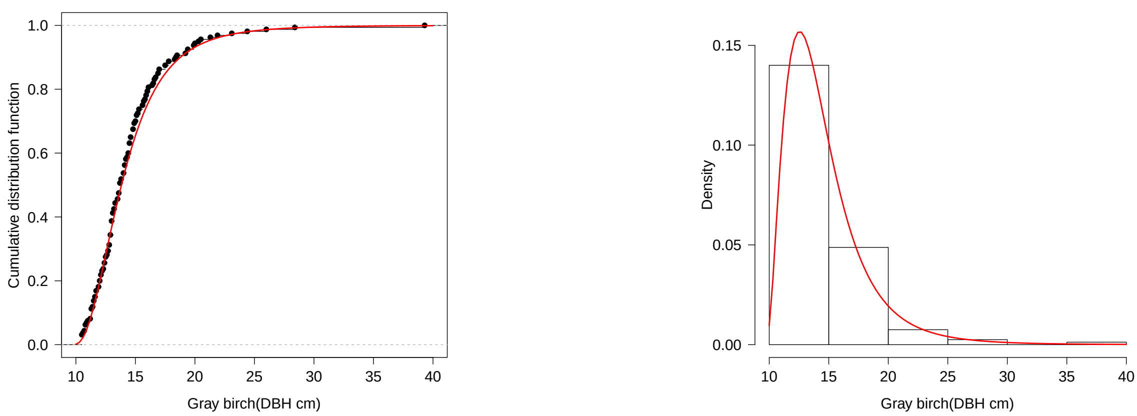
	γ	β	α	λ
BS	0.2083(0.0116)	14.2302(0.2331)	–	–
BST	0.151(0.074)	13.818(0.014)	–	–
STBS	0.2653(0.103)	11.346(0.025)	–	3.325(1.174)
PSTBS	0.2796(0.1135)	9.8844(0.1244)	2.3178(0.9654)	7.7185(11.4417)

According to the AIC and AICC criteria, W^*, A^*, K-S and p-value indicates better fit for the PSNBS model over the other models. See Table 7.

Table 7: AIC, AICC, W^*, A^*, K-S and p-value for the remission times of Gray birch data for BS, BST_4, STBS_8 and PSTBS_9 models.

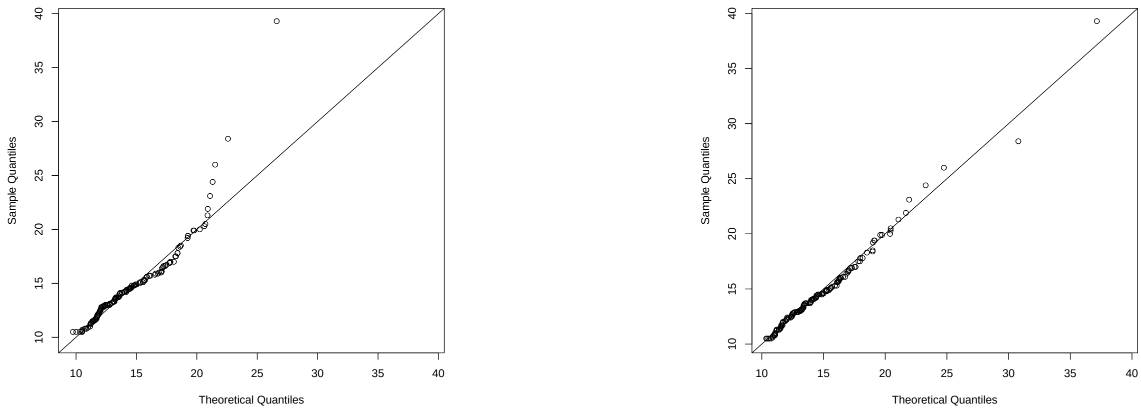
	$\ell(\theta)$	AIC	AICC	W^*	A^*	K-S	P-value
BS	399.7764	803.5528	803.6590	0.4396	2.7084	0.1066	0.0526
BST	389.4381	782.8762	782.9526	0.16602	1.1472	0.0707	0.4004
STBS	380.7285	767.4569	767.6108	0.04515	0.3166	0.0535	0.7506
PSTBS	378.8113	764.355	764.6131	0.040	0.2966	0.047	0.8614

Figure 9 shows graphs for PSTBS_9 model (a) empirical cdf (b) histogram and Figure 10 (a) and (b) shows the qq-plot for the models with better fit.



(a) (b)

Figure 9: Graphs for PSTBS_9 model (a) empirical cdf and (b) histogram.



(a) (b)

Figure 10: (a) qq-plot $STBS_8$ and (b) qq-plot $PSTBS_9$.

4.3. Application III (censored data)

The World Health Organization recommends breastfeeding exclusive for babies until 4 and 6 months. For this reason, an study from Universidade Federal de Minas Gerais UFMG main breastfeeding practice, as well as the possible factors of risk for an early weaning. The study consists of 150 mothers with children under 2 years of age. The response variable was the maximum time of breastfeeding, i.e., the time counted from birth to the weaning. More details on this data set can be found in Colosimo and Giolo ([7]). The values of the ML estimates for the BS, SNBS and PSNBS statistics for all models are given in Table 8. As expected, the values of AIC better fit for the PSNBS over other models, and the Figure 11 we can see that most babies stop having exclusive breastfeeding after 7 or 8 months.

Table 8: ML estimates for BS, SNBS and PSNBS models and AIC criteria.

	γ	β	α	λ	$\ell(\theta)$	AIC
BS	2.362 (0.268)	6.696(1.372)	–	–	-243.545	491.090
SNBS	5.380(1.538)	0.591(0.277)	–	4.015 (1.712)	-230.047	466.094
PSNBS	6.441(2.794)	0.252(0.227)	1.489(0.299)	3.761(2.287)	-228.357	464.715

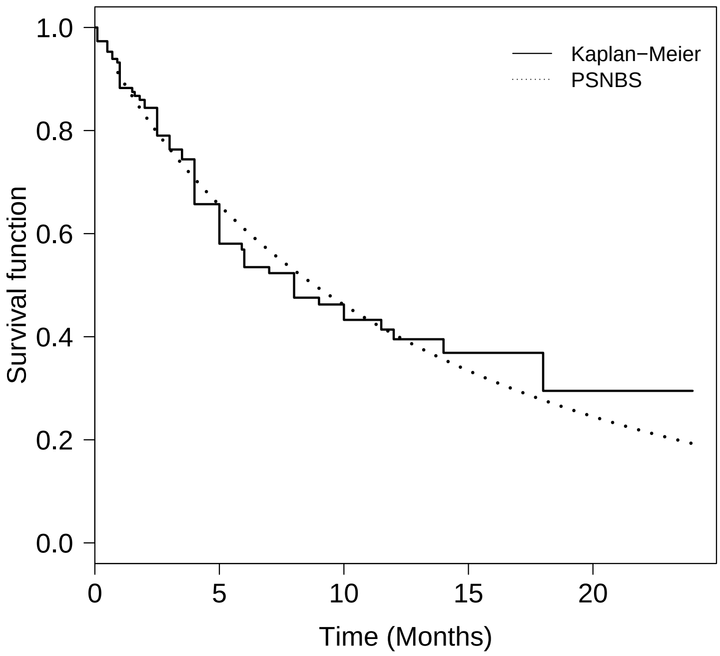


Figure 11: Estimated survival function for weaning study data under PSNBS model.

5. FINAL COMMENTS

This Paper proposes a flexible asymmetric BS distribution which contains previous ones as special cases and is able to surpass traditional models in terms of wider ranges of asymmetry and kurtosis. It is also shown that it is able to perform well in real applications, outperforming potential rival models. Maximum likelihood estimation is implemented and Fisher and observed information matrices are derived. It is shown that both are nonsingular. Some more features of this family of distributions are:

- The PSEBS model contains, as special cases, the SEBS model proposed by Vilca-Labra and Leiva-Sánchez ([30]) and the PEBS model proposed by Martínez-Flórez *et al.* ([22]).
- The proposed model it has a closed expression and presents more flexible asymmetry and kurtosis coefficients than PEBS and SEBS models.
- Some properties of the BS distribution were extended for the PSEBS model.
- The moments of the PSEBS family are finite.
- In the three applications it is shown that the PSEBS model fit better than the other models. This is confirmed by the different criteria used.

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