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AVERAGES FOR MULTIVARIATE RANDOM VECTORS WITH RANDOM WEIGHTS: DISTRIBUTIONAL CHARACTERIZATION AND APPLICATION

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Abstract:

- We consider a random weights average of n independent continuous random vectors $\mathbf{X}_1, ..., \mathbf{X}_n$, where random weights are cuts of $[0, 1]$ by an increasing sequence of the order statistics of a random sample from a uniform $[0, 1]$. We employ the multivariate Stieltjes transform and Watson [15] celebrated formula involving the multivariate B-spline functions for distributional identification of multivariate random weights averages. We show that certain classes of Dirichlet and random scale stable random vectors are random weightes averages.

Keywords:

- *multivariate weighted average with random weights; multivariate Cauchy Stieltjes transform; Dirichlet distribution; multivariate stable distributions.*

AMS Subject Classification:

- 62H05, 46F12, 65R10.

1. INTRODUCTION

An average $UX_1+(1-U)X_2$ of two independent continuous random variables X_1 and X_2, $U\sim$ uniform$(0,1)$, is the subject of Johnson and Kotz [5] expository article on the work of Van Assche [14]. Indeed, Johnson and Kotz noticed that the random variable *uniformly distributed between two random variables*, named by Van Assche, is a random weightes average, RWA in short. Soltani and Roozegar [13] consider RWA of a finite number of independent continuous random variables, where the weights are cuts of $(0,1)$ by an increasing selection of the order statistics of a uniform $(0,1)$ random sample. In his work, Van Assche [14] noticed that the Stieltjes transform is an appropriate tool for the distributional identification of random weightes averages, as the Fourier transform is for averages with non-random weights.

In this paper, we consider RWA of a number of independent continuous random vectors with values in $\mathbb{R}^d$, the d dimensional Euclidian space. The random weights are as in Soltani and Roozegar [13]: the cuts of $(0,1)$ by an increasing sequence $U_{(k_1)}, U_{(k_2)}, ..., U_{(k_{m-1})}$ of the order statistics $U_{(1)}, ..., U_{(n-1)}$ from a uniform $(0,1)$ sample $U_1, ..., U_{n-1}$; $1\le k_1<k_2<...<k_{m-1}<k_m=n$, $U_{(n)}=1$. We employ the multivariate Stieltjes, also called Cauchy-Stieltjes, transform (MCST in short) for the distributional identification of multivariate randomly weighted averages, MRWA. In this article, we prove that the MCST of order n, $\mathcal{S}[\mathbf{F};n](\mathbf{z})$, of the distribution $\mathbf{F}$, the distribution of random weights averages of independent d-dimensional continuous random vectors $\mathbf{X}_1\sim\mathbf{F}_1, ..., \mathbf{X}_m\sim\mathbf{F}_m$, is equal to the product of the corresponding MCST of $\mathbf{F}_1, ..., \mathbf{F}_m$, namely,

$$\mathcal{S}[\mathbf{F};n](\mathbf{z})=\mathcal{S}[\mathbf{F_1};k_1](\mathbf{z})\mathcal{S}[\mathbf{F_2};k_2-k_1](\mathbf{z})...\mathcal{S}[\mathbf{F_m};n-k_{m-1}](\mathbf{z}),\quad z\in\mathbb{C}^d. \tag{1.1}$$

Our approach is somewhat new and different from those applied in the references cited above. Van Assche [14] applies certain techniques from the differentiation of Schwartz distributions. Soltani and Roozegar [13] apply the divided differences and the theory of knots. In this article, we apply the pioneering formula of Watson [15] involving B-splines, discussed in Karlin, Micchelli and Rinott [6]. This approach is more direct and easily applied. It can be applied to the univariate RWA as well.

The notion of random weights averages in the literature may be attributed to the interesting observation of Galton, the founder of regression. He observed that, on average, a child's height is more mediocre (average) than his or her parent's height. Plausibly, the child's height is a RWA of his or her parents' heights. In contrast to the univariate RWA, multivariate RWA can be used for modeling when a finite number of characteristics are considered simultaneously.

Univariate and multivariate RWA have appeared in certain areas, such as sampling, density estimation, Bayesian and distributional characterizations, among others. In theory, general regression and neural networks, multivariate kernel density estimations and multivariate kernel regressions are all randomly weighted averages, see Nadaraya [8] and Watson [16]. An interesting example of averages of multivariate quantities with random weights is the random vector of the serial correlation coefficients, introduced by Watson [15], $\mathbf{r}=(r_1,r_2,...,r_k)$, where

$$r_j=\frac{\lambda_0^{(j)}W_0+\lambda_1^{(j)}W_1+\cdots+\lambda_m^{(j)}W_m}{W_0+\cdots+W_m},\qquad j=1,2,...,k, \tag{1.2}$$

and $W_0, W_1, ..., W_m$ are independent gamma variables of integer order $\alpha_0, \alpha_1, ..., \alpha_m$, and $\boldsymbol{\lambda}_\ell = (\lambda_\ell^{(1)}, ..., \lambda_\ell^{(k)})$, $\ell = 0, ..., m$ are k-dimensional knots. In addition, Zeng [17] characterizes the multivariate stable distributions through the independence of the linear statistic $\mathbf{U} = \sum_{i=1}^n Y_i \mathbf{X}_i$ and the vector of random coefficients $\mathbf{Y} = (Y_1, ..., Y_n)^T \in \mathbb{R}^n$, where $\mathbf{X}_1, ..., \mathbf{X}_n$ are independent and identically distributed random vectors in $\mathbb{R}^k$, and are independent of $\mathbf{Y}$. A special form of $\mathbf{U}$ is a RWA of random vectors.

This article is organized as follows. We give preliminaries and the proof of (1.1) in Section 2. We proceed on to introduce and study interesting classes of distributions that are RWA of continuous random vectors. In particular, we prove that the RWA of independent Dirichlet random vectors is Dirichlet, and that the RWA of independent and identically symmetric stable random vectors is randomly scaled stable. We devote Section 3 to this issue.

2. PRELIMINARIES AND MAIN RESULT

Let us denote the RWA of m independent and continuous random vectors $\mathbf{X}_1, ..., \mathbf{X}_m$ in $\mathbb{R}^d$ by

$$\mathbf{S}_{m:n} = R_{1:n}\mathbf{X}_1 + R_{2:n}\mathbf{X}_2 + \cdots + R_{m:n}\mathbf{X}_m, \qquad m \geq 2, \tag{2.1}$$

where the random weights $R_{j:n}$ are assumed to be the m cuts of $[0, 1]$ by an increasing ordered array $U_{(k_1)}, ...U_{(k_{m-1})}$ of $U_{(1)}, ..., U_{(n-1)}$, the ordered statistics of $n-1$ independent and identically uniformly distributed random variables $U_1, ..., U_{n-1}$ on $[0, 1]$;

$$R_{j:n} = U_{(k_j)} - U_{(k_{j-1})}, \; j = 1, 2, \cdots, m, \;\; m \leq n,$$

where $k_0 = 0 < k_1 < ... < k_{m-1} < k_m = n$ are in $\{1, ..., n\}$ and $U_{(n)} = 1$.

The conditional density of $\mathbf{S}_{m:n}$ given $\mathbf{X}_1 = \mathbf{x}_1$, ..., $\mathbf{X}_m = \mathbf{x}_m$ is denoted by $M(\mathbf{t}|\mathbf{x}_1, ..., \mathbf{x}_m)$, $\mathbf{t} \in \mathbb{R}^d$. In the numerical analysis context, this density function is called "the Multivariate B-spline with knots $\{\mathbf{x}_1, ..., \mathbf{x}_m\}$", Karlin, Micchelli and Rinott [6]. The random vectors $\mathbf{X}_1, ..., \mathbf{X}_m \in \mathbb{R}^d$ have a convex hull with positive volume in $\mathbb{R}^d$. Our derivations very much rely on the fundamental result by Watson [15]:

$$\int_{\mathbb{R}^d} M(\mathbf{t}|\mathbf{x}_1, ..., \mathbf{x}_m) \frac{d\mathbf{t}}{(1- < \mathbf{t}, \mathbf{x} >)^{\sum_{i=1}^m r_i}} = \prod_{i=1}^m (1- < \mathbf{x}, \mathbf{x}_i >)^{-r_i}, \tag{2.2}$$

for $\max_i | < \mathbf{x}, \mathbf{x}_i > | < 1$, Karlin, Micchelli and Rinott [6].

The multivariate Cauchy-Stieltjes (or Stieltjes) transform (MCST) of a distribution $\mathbf{H}$ is defined by

$$\mathcal{S}[\mathbf{H}](\mathbf{z}) = \int_{\mathbb{R}^d} \frac{1}{1- < \mathbf{z}, \mathbf{x} >} \mathbf{H}(d\mathbf{x}), \qquad \mathbf{z} \in \mathbb{C}^d \cap (\operatorname{supp} \mathbf{H})^c, \tag{2.3}$$

for $| < \mathbf{z}, \mathbf{x} > | < 1$, $< \mathbf{a}, \mathbf{b} >= \sum_{i=1}^k a_i b_i$, $\mathbb{C}$ is the set of complex numbers and $\operatorname{supp} \mathbf{H}$ stands for the support of $\mathbf{H}$, Kerov and Tsilevich [7] and Cuyt, Golub, Milanfar and Verdonk [1]. Similarly the MCST of order n of a distribution $\mathbf{H}$ is defined by

$$\mathcal{S}[\mathbf{H}; n](\mathbf{z}) = \int_{\mathbb{R}^d} \frac{1}{[1- < \mathbf{z}, \mathbf{t} >]^n} \mathbf{H}(d\mathbf{t}), \qquad \mathbf{z} \in \mathbb{C}^d \cap (\operatorname{supp} \mathbf{H})^c, \tag{2.4}$$

for $| < \mathbf{z}, \mathbf{t} > | < 1$.

For $d = 1$, the MST is also called Markov transform, denoted by $\mathcal{M}_1[H](z)$. There is a relation between Markov transform and Stieltjes transform of a distribution H:

$$\mathcal{M}_1[H](z) = \frac{1}{z}\, \mathcal{S}[H](\frac{1}{z}),$$

where

$$\mathcal{S}[H](z) = \int_{\mathbb{R}} \frac{1}{z-x}\, H(dx),$$

for z in the set of complex numbers $\mathbb{C}$ which does not belong to the support of H, $z \in \mathbb{C} \cap (\operatorname{supp} H)^c$. For more on the Stieltjes transform see Debnath and Bhatta [2].

The following theorem is our main result in this section.

Theorem 2.1. *Let $\mathbf{X}_1, ..., \mathbf{X}_m$ be $m > 1$ independent and continuous random vectors in $\mathbb{R}^d$. Let $\mathbf{S}_{m:n}$ be the corresponding MRWA given by (2.1). Then*

$$\mathcal{S}[F_{\mathbf{S}_{m:n}}; n](\mathbf{z}) = \prod_{i=1}^{m} \mathcal{S}[F_i; r_i](\mathbf{z}), \qquad \mathbf{z} \in \mathbb{C}^d \bigcap_{i=1}^{m} (\operatorname{supp} F_i)^c, \tag{2.5}$$

where $r_i = k_i - k_{i-1}$; $\sum_{i=1}^{m} r_i = n$.

Proof: We note that

$$\begin{aligned}
F_{\mathbf{S}_{m:n}}(\mathbf{t}) &= E\left(I[\mathbf{S}_{m:n} \le \mathbf{t}]\right) \\
&= E\left(E\left(I[\mathbf{S}_{m:n} \le \mathbf{t}] | \mathbf{X}_1, ..., \mathbf{X}_m\right)\right) \\
&= \int_{\mathbb{R}^{md}} E\left(I[\mathbf{S}_{m:n} \le \mathbf{t}] | \mathbf{x}_1, ..., \mathbf{x}_m\right) \prod_{i=1}^{m} F_i(d\mathbf{x}_i) \\
&= \int_{\mathbb{R}^{md}} \int_{[\mathbf{s}<\mathbf{t}]} M(\mathbf{s}|\mathbf{x}_1, \cdots, \mathbf{x}_m) d\mathbf{s} \prod_{i=1}^{m} F_i(d\mathbf{x}_i) \\
&= \int_{[\mathbf{s}<\mathbf{t}]} \left\{ \int_{\mathbb{R}^{md}} M(\mathbf{s}|\mathbf{x}_1, \cdots, \mathbf{x}_m) \prod_{i=1}^{m} F_i(d\mathbf{x}_i) \right\} d\mathbf{s},
\end{aligned}$$

giving that

$$dF_{\mathbf{S}_{m:n}}(\mathbf{t}) = \int_{\mathbb{R}^{md}} M(\mathbf{t}|\mathbf{x}_1, \cdots, \mathbf{x}_m) \prod_{i=1}^{m} F_i(d\mathbf{x}_i).$$

Therefore,

$$\begin{aligned}
\mathcal{S}[F_{\mathbf{S}_{m:n}}; n](\mathbf{z}) &= \int_{\mathbb{R}^d} \frac{1}{[1- < \mathbf{z}, \mathbf{t} >]^n} dF_{\mathbf{S}_{m:n}}(\mathbf{t}) \\
&= \int_{\mathbb{R}^{md}} \left\{ \int_{\mathbb{R}^d} \frac{1}{[1- < \mathbf{z}, t >]^n} M(\mathbf{t}|\mathbf{x}_1, \cdots, \mathbf{x}_m) d\mathbf{t} \right\} \prod_{i=1}^{m} F_i(d\mathbf{x}_i) \\
&= \int_{\mathbb{R}^{md}} \prod_{i=1}^{m} [1- < \mathbf{z}, \mathbf{x_i} >]^{-r_i} \prod_{i=1}^{m} F_i(d\mathbf{x}_i) \qquad (2.6) \\
&= \prod_{i=1}^{m} \int_{\mathbb{R}^d} \frac{1}{[1- < \mathbf{z}, \mathbf{x}_i >]^{r_i}} F_i(d\mathbf{x}_i) \\
&= \prod_{i=1}^{m} \mathcal{S}[F_i; r_i](\mathbf{z}),
\end{aligned}$$

the third equality in (2.6) follows from (2.2). □

3. SOME CLASSES OF RWA OF RANDOM VECTORS

In this section we introduce two important classes of RWA of random vectors, Theorems 3.1 and 3.2.

In Theorem 3.1 below we assume $m = n$, $r_j = 1$ for $j = 1, 2, ..., m$.

Theorem 3.1. *Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be independent random vectors such that $\mathbf{X}_i$ has a Dirichlet distribution with parameters $\alpha_i = (\alpha_{1i}, \alpha_{2i}, ..., \alpha_{di})'$, $\sum_{j=1}^{d} \alpha_{ji} = 1$, $i = 1, 2, ..., n$. Then the MRWA $S_{n:n}$ given by (2.1) has a Dirichlet distribution with parameters*

$$\sum_{i=1}^{n} \alpha_i = (\sum_{i=1}^{n} \alpha_{1i}, \sum_{i=1}^{n} \alpha_{2i}, ..., \sum_{i=1}^{n} \alpha_{di}).$$

Proof: The density and the Stieltjes transform of a Dirichlet distribution with parameters $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)'$ are given by

$$dF(\mathbf{x}) = f(x_1, x_2, ..., x_d) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_d)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_d)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \quad \mathbf{x} \in \Delta^d, \tag{3.1}$$

and

$$\mathcal{S}[F](\mathbf{z}) = \int_{\Delta^d} \frac{F(d\mathbf{x})}{[1- < \mathbf{z}, \mathbf{x} >]^{\sum_{i=1}^{d} \alpha_i}} = \prod_{j=1}^{d} \frac{1}{(1 - z_j)^{\alpha_j}}, \ \mathbf{z} = (z_1, ..., z_d)$$

respectively, $\Delta^d = \{(x_1, ..., x_d) \in \mathbb{R}^d, \ x_i > 0, \forall i, \sum_{i=1}^{d} x_i = 1\}$, Kerov and Tsilevich [7]. Let $\mathbf{X}_i \sim F_i$, $i = 1, 2, ..., n$. Then it follows from Theorem 2.1 that

$$\mathcal{S}[F_{\mathbf{S}_{n:n}}; n](\mathbf{z}) = \prod_{i=1}^{n} \prod_{j=1}^{d} \frac{1}{(1 - z_j)^{\alpha_{ji}}} = \prod_{j=1}^{d} \frac{1}{(1 - z_j)^{\sum_{i=1}^{n} \alpha_{ji}}}.$$

It is plain to show this function is the MCST, of order n, of a Dirichlet distribution with parameters $\sum_{i=1}^{n} \alpha_i = (\sum_{i=1}^{n} \alpha_{1i}, \sum_{i=1}^{n} \alpha_{2i}, ..., \sum_{i=1}^{n} \alpha_{di})$. Indeed for F', a Dirichlet distribution with parameters $(b_1, b_2, ..., b_d)$, with $\sum_{j=1}^{d} b_j = n$, we have

$$\mathcal{S}[F'; n](\mathbf{z}) = C(n; b_1, ..., b_d) \int_{\Delta^d} \frac{x_1^{b_1-1} x_2^{b_2-1} ... x_{d-1}^{b_{d-1}-1} (1 - x_1 - ... - x_{d-1})^{b_d-1}}{[1- < \mathbf{z}, \mathbf{x} >]^n} dx_1 ... dx_{d-1},$$

where $C(n; b_1, ..., b_d) = \frac{\Gamma(n)}{\Gamma(b_1)\Gamma(b_2)...\Gamma(b_d)}$. Let $b_j = \sum_{i=1}^{n} \alpha_{ji}$, $j = 1, 2, ..., d$. Then the Euler type integral representation for the Lauricella function gives that

$$\frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_k)}{\Gamma(b_1 + b_2 + \cdots + b_k)} F_D^{(k)}(a, b_1, ..., b_k; b_1 + \cdots + b_k; z_1, ..., z_k) = \tag{3.2}$$

$$= \int \ldots \int_{\Delta^k} \frac{x_1^{b_1-1} x_2^{b_2-1} \cdots x_{k-1}^{b_{k-1}-1} (1 - x_1 - \cdots - x_{k-1})^{b_k-1}}{[1- < \mathbf{z}, \mathbf{x} >]^a} dx_1 \cdots dx_{k-1},$$

where

$$F_D^{(k)}(a, b_1, ..., b_k; c; z_1, ..., z_k) = \sum_{m_1, ..., m_k \geq 0} \frac{(a)_{m_1+\cdots+m_k} (b_1)_{m_1} \cdots (b_k)_{m_k}}{(c)_{m_1+\cdots+m_k}} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_k^{m_k}}{m_k!} \tag{3.3}$$

is the Lauricella function, $(a)_m = a(a+1)...(a+m-1)$ the Pochhammer symbol and $c = \sum_{j=1}^{k} b_j$; Exton [3, 2.1.4, 2.3.5]. Therefore,

$$\mathcal{S}[F'; n](\mathbf{z}) = F_D^{(d)}(n, b_1, ..., b_k; n; z_1, ..., z_d),$$

and

$$\begin{aligned} \mathcal{S}[F'; n](\mathbf{z}) &= \sum_{m_1,...,m_d \geq 0} (b_1)_{m_1} \cdots (b_d)_{m_d} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_d^{m_d}}{m_d!} \\ &= \prod_{j=1}^{d} \frac{1}{(1-z_j)^{b_j}} \\ &= \prod_{j=1}^{d} \frac{1}{(1-z_j)^{\sum_{i=1}^{n} \alpha_{ji}}} \\ &= \prod_{i=1}^{n} \prod_{j=1}^{d} \frac{1}{(1-z_j)^{\alpha_{ji}}}. \end{aligned}$$

The proof of the theorem is complete. □

Theorem 3.2. *Let $\mathbf{X}_1, ..., \mathbf{X}_m$ be independent and identically distributed random vectors in $\mathbb{R}^d$ having a symmetric multivariate stable distribution of exponent $0 < \alpha \leq 2$. Let $\mathbf{S}_{m:n}$ be the corresponding MRWA given in (2.1). Then $\mathbf{S}_{m:n} \stackrel{d}{=} V_\alpha \mathbf{X}_1$, where $V_\alpha = (\sum_{j=1}^{m} R_{j:n}^{\alpha})^{1/\alpha}$.*

Proof: It is well known that if $\mathbf{X}_1, ..., \mathbf{X}_m$ are independent, identical and symmetrically distributed stable random vectors of exponent α, then $\sum_{j=1}^{m} a_j \mathbf{X}_j \stackrel{d}{=} (\sum_{j=1}^{m} a_j^{\alpha})^{1/\alpha} \mathbf{X}_1$, for any set of univariate positive constants $a_1, ..., a_m$, see Samorodnitsky and Taqqu [12]. Let $\mathcal{S}_{m:n}(\mathbf{z})$ stand for the Stieltjes transform of $\mathbf{S}_{m:n}$, then

$$\begin{aligned} \mathcal{S}_{m:n}(\mathbf{z}) &= E\left(\frac{1}{1- < \mathbf{S}_{m:n}, \mathbf{z} >}\right) \\ &= E\left(E\left(\frac{1}{1- < \sum_{j=1}^{m} R_{j:n}\mathbf{X}_j, \mathbf{z} >} | R_{m:1}, ..., R_{m:n}\right)\right) \\ &= E\left(E\left(\frac{1}{1- < (\sum_{j=1}^{m} R_{j:n}^{\alpha})^{1/\alpha}\mathbf{X}_1, \mathbf{z} >} | R_{m:1}, ..., R_{m:n}\right)\right) \\ &= E\left(\frac{1}{1- < (\sum_{j=1}^{m} R_{j:n}^{\alpha})^{1/\alpha}\mathbf{X}_1, \mathbf{z} >}\right) \\ &= \mathcal{S}\{V_\alpha \mathbf{X}_1\}(\mathbf{z}), \end{aligned}$$

giving the result, where $\mathcal{S}\{V_\alpha \mathbf{X}_1\}(\mathbf{z})$ stands for the Stieltjes transform of $V_\alpha \mathbf{X}_1$. □

Remark 3.1. Interestingly, it follows from Theorem 3.2 that the RWA of independently and identically distributed stable random vectors is not stable (unless $\alpha = 1$), but it is a certain *randomly scaled stable* random vector. Moreover it follows from the inequality $(a+b)^p < a^p + b^p$, $0 < p < 1$, that for $1 < \alpha \leq 2$, $V_\alpha < 1$. Consequently, the RWA $\mathbf{S}_{n:n}$ exhibits smaller variation than $\mathbf{X}$s.

Remark 3.2. We note that for $\alpha = 1$, $V_1 = 1$, and consequently $\mathbf{S}_{n:n} \stackrel{d}{=} \mathbf{X}_1$. If $\mathbf{h}(\mathbf{x})$ is the density function of a multivariate stable distribution of exponent 1, *call it multivariate Cauchy distribution*, then it follows from Theorems 2.1, and 3.2 that

$$\int_{\mathbb{R}^d} \frac{1}{[1- < \mathbf{z}, \mathbf{x} >]^n} \mathbf{h}(\mathbf{x}) d\mathbf{x} = \left[\int_{\mathbb{R}^d} \frac{1}{1- < \mathbf{z}, \mathbf{x} >} \mathbf{h}(\mathbf{x}) d\mathbf{x} \right]^n, \text{ for every } n \geq 1.$$

The density function $\mathbf{h}(\mathbf{x})$, in the context of stable random vectors with exponent $\alpha = 1$, in general does not assume a close formulation. The density function of a special class of multivariate Cauchy random vectors, called "multivariate Cauchy of order one" assumes the following formulation, given in Press [9], namely,

$$\mathbf{h}(\mathbf{x}) = K|\Sigma|^{-\frac{1}{2}}[1 + (\mathbf{x} - \mathbf{a})^T \Sigma^{-1} (\mathbf{x} - \mathbf{a})]^{-\frac{1+d}{2}},$$

where $K = \Gamma(\frac{1+d}{2})\pi^{-\frac{1+d}{2}}$, $\mathbf{a} \in \mathbb{R}^d$ and the $d \times d$ matrix Σ is positive definite.

Let us also record the following interesting symmetrical property of MRWA.

Theorem 3.3. *Let every $\mathbf{X}_i$ be symmetric about $\mathbf{a}_i$, for $i = 1, ..., m$. Then the MRWAs $\mathbf{D}_{m:n} - \mathbf{S}_{m:n}$ and $\mathbf{S}_{m:n} - \mathbf{D}_{m:n}$ have the same distribution, where $\mathbf{D}_{m:n} = \sum_{j=1}^{m} R_{j:n}\mathbf{a}_j$. In particular, if every $\mathbf{X}_i$ is symmetric about $\mathbf{a}$, then $\mathbf{S}_{m:n}$ will be symmetric about $\mathbf{a}$.*

The proof is straightforward, so it is omitted.

Let us call $\mathbf{D}_{m:n} = \sum_{j=1}^{m} R_{j:n}\mathbf{a}_j$ the centroid for MRWA $\mathbf{S}_m$. This is interesting; indeed it follows from this theorem that the centroid is random regresses of $\mathbf{a}_1, ..., \mathbf{a}_m$. According to Galton, see Hansen [4, page 40], the projected height of child on parent is a weighted average of the population mean height and the parents height with weights $(1/3, 2/3)$. Indeed if we let $ER_1 = 1/3$, then $E[S_2|X_2] = ER_1 EX_1 + ER_2 X_2 = (1/3)\mu + (2/3)X_2$; the right side is the equation reported in Hansen [4].

Conclusion. Averages for multivariate random vectors with random weights where the weights are spacings corresponding to a uniform $(0, 1)$ sample are introduced and studied in this article. Certain techniques for the their distributional studies are introduced. This study gives rise to new families of multivariate distributions. The statistics literature is quite rich about the sample mean and its applications. The topics that are studied for the sample mean, such as strong law of large numbers, asymptotic theory and its applications in inference, would be interesting subjects for further research work on randomly weighted average of random vectors. For further references, see also Roozegar and Soltani [10, 11].

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