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PARAMETERS ESTIMATION FOR CONSTANT-STRESS PARTIALLY ACCELERATED LIFE TESTS OF GENERALIZED HALF-LOGISTIC DISTRIBUTION BASED ON PROGRESSIVE TYPE-II CENSORING

Author: Abdullah M. Almarashi
– Statistics Department, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
aalmarashi@kau.edu.sa

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Abstract:

- In product-life testing experiments, the accelerated life testing (ALT) is applied to reduce the time and cost of tests. We consider the constant-stress partially ALT model when the lifetime of units under normal conditions follow the generalized half-logistic lifetime distribution based on progressive Type-II censored schemes. The likelihood functions of the parameters are derived and solved to present the maximum likelihood estimators of the model parameters. The approximate and two bootstrap confidence intervals are also proposed. The performance of the different methods were measured and compared through Monte Carlo simulation study. Finally, the results of a numerical example are discussed.

Keywords:

- *constant-stress partially accelerated life tests; generalized half-logistic distribution; maximum likelihood estimation; bootstrap confidence intervals.*

1. INTRODUCTION

According to [22, 18, 3, 4], there are different methods of accelerated life testing (ALT): the constant-stress ALT, in which the stress on the life test product remains at a constant level, the progressive-stress ALT, in which the stress applied to the product units in the test increases with time [7], and the step-stress ALT, in which the test condition changes for a given time or a specified number of failures [21, 7]. For more recent research on the constant-stress partially ALT, see [2, 1].

In product-life test experiments, censoring has played an important role. Different types of censoring are available. Type-I and Type-II censoring schemes (CSs) are commonly applied, both of which do not allow the removal of any units other than at the terminal point of the test. General CSs that allow units to be removed at any point during the test are called progressive Type-II right censoring. For important reviews of the literature on progressive censoring, see [9].

Let n be the number of units tested in a product-life testing experiment and T_1, T_2, ..., T_n, be the corresponding lifetimes. Assume that the T_i, $i = 1, 2, ..., n$ are independent and identically distributed (i.i.d.) with probability density function (PDF) $f(.)$ and cumulative distribution function (CDF) $F(.)$. In the progressive Type-II CS prior to the experiment, the effective sample size m and the corresponding CS $\mathbf{R} = \{R_1, R_2, ..., R_m\}$ are determined; then $T^{\mathbf{R}}_{i;m,n}$, $i = 1, 2, ..., m$ is the corresponding random variable of the progressive Type-II censored sample.

The joint likelihood function of the observed progressive Type-II censored sample $\underline{t} = (t^{\mathbf{R}}_{1;m,n}, t^{\mathbf{R}}_{2;m,n}, ..., t^{\mathbf{R}}_{m;m,n})$ is given by

$$f(\underline{t}, \theta) = Q \prod_{i=1}^{m} f(t^{\mathbf{R}}_{i;m,n})[1 - F(t^{\mathbf{R}}_{i;m,n})]^{Ri}, \tag{1.1}$$

where the observed progressive Type-II censored sample $\underline{t}$ satisfies $0 < t_{1;m,n} < t_{2;m,n} < ... < t_{m;m,n} < \infty$,and

$$Q = \prod_{i=0}^{m-1} \left(n - \sum_{j=0}^{i} R_j - i \right), \quad R_0 = 0. \tag{1.2}$$

Balakrishnan [8] has considered the half-logistic distribution as the distribution of the absolute standard logistic variate. Important properties of a generalized version of the logistic distribution are discussed by Balakrishnan and Hossain [10]. The point estimation of the stress–strength reliability of generalized half-logistic distribution (GHLD) is presented by Ramakrishnan [23]. The shape parameter of the GHLD was estimated under Type-I progressive censoring in Arora et al. [5]. The Bayesian approach with a GHLD was discussed in Kim et al. [20]. Recently, testing procedures for the reliability functions of the GHLD were considered in Chaturvedi et al. [14] and in a Type-I generalized half-logistic survival model in Awodutire et al. [6].

Let T be a random variable of a GHLD with shape parameter β; the PDF and CDF are given respectively by

$$f(t) = \frac{\beta}{1+\exp(-t)}\left(\frac{2\exp(-t)}{1+\exp(-t)}\right)^{\beta},\ t>0,\ \beta>0, \tag{1.3}$$

and

$$F(x)=1-\left(\frac{2\exp(-t)}{1+\exp(-t)}\right)^{\beta}, \tag{1.4}$$

The reliability function $S(t)$ and the hazard rate function $H(t)$ are expressed as

$$S(t) = \left(\frac{2\exp(-t)}{1+\exp(-t)}\right)^{\beta}, t>0,\ \beta>0, \tag{1.5}$$

and

$$H(t) = \frac{\beta}{1+\exp(-t)}. \tag{1.6}$$

This GHLD is considered as a special probability distribution with a location parameter and a scale parameter, defined by $F(x)=1-\left(\frac{2\exp(\frac{-t}{\sigma})}{1+\exp(\frac{-t}{\sigma})}\right)^{\beta}$ with $\sigma=1$. The best linear unbiased estimator of the location and scale parameters as well as the values of the variance and covariance of these estimators is presented in [11]. Ref. [13] discusses the estimator as an approximation of the likelihood functions based on a Type-II censoring sample. The estimation of the parameter of the half-logistic distribution under progressive Type-II censored sample is presented in [19].

The aim of this paper is to estimate the GHLD under constant-stress partially ALT with progressive Type-II CS. The maximum likelihood estimator (MLE) and the bootstrap estimator of each unknown GHLD parameter and the acceleration factor are presented. The point estimates of the MLE and bootstrap estimator mainly assess and compare their biases and mean-squared errors (MSE's), as well as the approximate interval estimation and bootstrap confidence intervals (CIs), with respect to coverage percentage and the mean of interval lengths using extensive simulation studies.

In this article, the assumptions and model are described in Section 2. The MLEs and the corresponding approximate confidence intervals (ACIs) are given in Section 3. Two bootstrap CIs are discussed in Section 4. We assess and compare the results of Monte Carlo studies in Section 5. A numerical example of a simulated data set is presented in Section 6. Finally, some comments about the results of the simulation studies are presented in Section 7.

2. ASSUMPTIONS AND MODEL

In the experiment design for the constant-stress partially ALTs, n_1 units from n testing units are randomly chosen to be tested under normal conditions; the remaining units $n_2 = n - n_1$ are tested under accelerated conditions. The model for the progressive Type-II censoring with constant-stress partially ALTs is described as follows. The subscript label $j = 1, 2$ signify the two conditions, normal and accelerated; when the first failure $T^{\mathbf{R}j}_{j1;m_j,n_j}$ is recorded, R_{j1} units are randomly removed from the remaining $n_j - 1$ surviving units. Also at the second failure, $T^{\mathbf{R}j}_{j2;m_j,n_j}$ is recorded and R_{j2} units from the remaining n_j − 2-R_{j1} units are randomly removed. The test continues until the m_j -th $T^{\mathbf{R}j}_{jm_j;m_j,n_j}$ failure and the remaining $R_{jmj} = n_j$-$m_j - \sum_{k=1}^{m_j-1} R_{jk}$ units are removed, for $j = 1, 2$. In this model, each of the R_{ji} and $m_j < n_j$ are fixed prior to beginning the test. If the times of failure of the n_j units originally in the test are from a continuous population with a distribution function $F_j(t)$ and probability density function $f_j(t)$, the joint probability density function for $T^{\mathbf{R}j}_{j1;m_j,n_j} < T^{\mathbf{R}j}_{j2;m_j,n_j} < ... < T^{\mathbf{R}j}_{jm_j;m_j,n_j}$ and $j = 1, 2$ is given as follows.

The joint likelihood function for $\underline{t} = (T^{\mathbf{R}j}_{j1;m_j,n_j}, T^{\mathbf{R}j}_{j2;m_j,n_j}, ..., T^{\mathbf{R}j}_{jm_j;m_j,n_j})$ for $j = 1, 2$, is given by

$$L(\beta, \lambda|\underline{t}) = \prod_{j=1}^{2} Q_j \left\{ \prod_{i=1}^{m_j} f_j(t^{\mathbf{R}j}_{ji;m_j,n_j}) \left(S_j(t^{\mathbf{R}j}_{ji;m_j,n_j}) \right)^{R_{ji}} \right\}, \tag{2.1}$$

where $Q_j = \prod_{i=0}^{m_j-1} \left(n_j - \sum_{l=0}^{i} R_{lj} - i \right)$, $R_{0j} = 0$. In the accelerated lifetime model, assuming that $S_2(t) = S_1(\lambda t)$. Let T be a random variable under normal conditions, then the lifetime of the unit under accelerated conditions can be defined by $Y = \frac{T}{\lambda}$, where λ is the acceleration factor. Hence, the probability density and cumulative distribution functions of the GHLD with observed lifetime under the accelerated condition are given by

$$f_2(y) = \frac{\lambda\beta}{1 + \exp(-\lambda y)} \left(\frac{2\exp(-\lambda y)}{1 + \exp(-\lambda y)} \right)^{\beta}, y > 0, \ \beta, \ \lambda > 0. \tag{2.2}$$

and

$$F_2(y) = 1 - \left(\frac{2\exp(-\lambda y)}{1 + \exp(-\lambda y)} \right)^{\beta}. \tag{2.3}$$

Also, the reliability function $S(y)$ and hazard rate function $H(y)$ are given, respectively, by

$$S_2(y) = \left(\frac{2\exp(-\lambda y)}{1 + \exp(-\lambda y)} \right)^{\beta}, \tag{2.4}$$

and

$$H_2(y) = \frac{\lambda\beta}{1 + \exp(-\lambda y)}. \tag{2.5}$$

3. MAXIMUM LIKELIHOOD ESTIMATION

3.1. Point estimation

Let $\underline{T} = \left(T^{\mathbf{R}j}_{j1;m_j,n_j} < T^{\mathbf{R}j}_{j2;m_j,n_j} < ... < T^{\mathbf{R}j}_{jm_j;m_j,n_j}\right)$, $j = 1,2$ denote two progressively Type-II censored samples from two populations for which the PDFs and CDFs are as given in (1.3), (1.4), (2.2), and (2.3), with $\mathbf{R}_j = (R_{j1},\ R_{j2},\ ...,\ R_{j1})$. The log-likelihood function $\ell(\beta,\lambda|\underline{t}) = \log L(\beta,\lambda|\underline{t})$ without normalized constant is then given by

$$\begin{aligned}\ell(\beta,\lambda|\underline{t}) &= (m_1+m_2)\log\beta + m_2\log\lambda + n\beta\log 2 - \sum_{i=1}^{m_1}\log\left[1+\exp(-t_{1i})\right)] \\ &- \sum_{i=1}^{m_2}\log\left[1+\exp(-\lambda t_{2i}))\right] - \beta\sum_{i=1}^{m_1}(R_{1i}+1)\log\left(1+\exp(t_{1i})\right) \\ &- \beta\sum_{i=1}^{m_2}(R_{2i}+1)\log\left(1+\exp(\lambda t_{2i})\right).\end{aligned} \tag{3.1}$$

The likelihood equation is obtained by calculating the first partial derivatives of (3.1) with respect to β and λ, and then equating each to zero:

$$\begin{aligned}\frac{\partial\ell(\beta,\lambda|\underline{t})}{\partial\beta} &= \frac{m_1+m_2}{\beta} + n\log 2 - \sum_{i=1}^{m_1}(R_{1i}+1)\log\left(1+\exp(t_{1i})\right) \\ &- \sum_{i=1}^{m_2}(R_{2i}+1)\log\left(1+\exp(\lambda t_{2i})\right) = 0,\end{aligned} \tag{3.2}$$

giving

$$\begin{aligned}\beta(\lambda) &= -(m_1+m_2)\left[\sum_{i=1}^{m_1}(R_{1i}+1)\log\left(1+\exp(t_{1i})\right)\right. \\ &\left. + \sum_{i=1}^{m_2}(R_{2i}+1)\log\left(1+\exp(\lambda t_{2i})\right) - n\log 2\right]^{-1},\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\frac{\partial\ell(\beta,\lambda|\underline{t})}{\partial\lambda} &= \frac{m_2}{\lambda} + \sum_{i=1}^{m_2} t_{2i}\left(1+\exp(\lambda t_{2i})\right)^{-1} + \beta\sum_{i=1}^{m_2}(R_{2i}+1) \\ &\times t_{2i}\left(1+\exp(-\lambda t_{2i})\right)^{-1} = 0,\end{aligned} \tag{3.4}$$

giving

$$\frac{m_2}{\lambda} + \sum_{i=1}^{m_2} t_{2i}\left(1+\exp(-\lambda t_{2i})\right)^{-1} + \beta\sum_{i=1}^{m_2}(R_{2i}+1)t_{2i}\left(1+\exp(-\lambda t_{2i})\right)^{-1} = 0\,. \tag{3.5}$$

The likelihood equation reduce to the single nonlinear equation (3.5), which can be solved numerically using the fixed point method or the quasi-Newton Raphson to obtain the MLE of λ say $\hat{\lambda}$, and hence $\hat{\beta}$ using (3.3).

3.2. Approximate interval estimation

The asymptotic normality theory is applied to construct asymptotic CIs of the MLEs. The Fisher information matrix requires the second partial derivatives of (3.1) with respect to β and λ:

$$\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \beta^2} = \frac{-(m_1+m_2)}{\beta}, \tag{3.6}$$

$$\frac{\partial^2 \ell(\beta,\lambda|\underline{t})}{\partial \beta \partial \lambda} = \frac{\partial^2 \ell(\beta,\lambda|\underline{t})}{\partial \lambda \partial \beta} = -\beta \sum_{i=1}^{m_2} (R_{2i}+1) t_{2i} \left(1+\exp(-\lambda t_{2i})\right)^{-1}, \tag{3.7}$$

and

$$\begin{aligned} \tfrac{\partial^2 \ell(\beta,\lambda|\underline{t})}{\partial \lambda^2} = \tfrac{-m_2}{\lambda^2} - \sum_{i=1}^{m_2} t_{2i}^2 \left(1+\exp(-\lambda t_{2i})\right)^{-2} + \beta \sum_{i=1}^{m_2} (R_{2i}+1) t_{2i}^2 \exp(\lambda t_{2i}) \\ \times \left(1+\exp(\lambda t_{2i})\right)^{-2}. \end{aligned} \tag{3.8}$$

Then, the expectation of the difference of equations (3.6) and (3.8) is defined as the Fisher information matrix $I(\beta,\lambda)$. The MLEs $(\hat{\beta},\hat{\lambda}\ )$ with some mild regularity conditions follows the approximately bivariate normal distribution with mean (β,λ) and covariance matrix $[I(\beta,\lambda)]^{-1}$. Usually, in practice, the estimate of $[I(\beta,\lambda)]^{-1}$ is used by $\left[I_0\left(\hat{\beta},\hat{\lambda}\right)\right]^{-1}$. A simpler and equally valid procedure is to use the approximation

$$(\hat{\beta},\hat{\lambda}) \sim N\left((\beta,\lambda), \left[I_0(\hat{\beta},\hat{\lambda})\right]^{-1}\right), \tag{3.9}$$

where $I_0(\beta,\lambda)$ is the observed information matrix

$$\begin{bmatrix} -\frac{\partial^2 \ell(\beta,\lambda|\underline{x})}{\partial \beta^2} & -\frac{\partial^2 \ell(\beta,\lambda|\underline{x})}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ell(\beta,\lambda|\underline{x})}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ell(\beta,\lambda|\underline{x})}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\hat{\beta},\hat{\lambda})}. \tag{3.10}$$

The approximate CIs for the parameters β and λ are obtained from the bivariate normal distribution with mean $(\beta,\ \lambda)$ and covariance matrix $\left[I_0(\hat{\beta},\hat{\lambda})\right]^{-1}$. Thus, the $100(1-2\alpha)\%$ ACIs for β and λ are

$$\hat{\beta} \mp z_\alpha \sqrt{v_{11}} \text{ and } \hat{\lambda} \mp z_\alpha \sqrt{v_{22}}, \tag{3.11}$$

respectively, where v_{11} and v_{22} are the elements on the diagonal of the covariance matrix $I_0^{-1}(\hat{\beta},\hat{\lambda})$ and z_α is the percentile of the standard normal distribution with the right-tail probability α.

4. BOOTSTRAP CONFIDENCE INTERVALS

In some cases, if the objective of the study is to determine the estimators, CIs, bias, and variance of an estimator or to calibrate hypothesis tests, then the bootstrap technique plays an important role. Different types of bootstrap techniques are available, such as those called parametric [15] and nonparametric [17]. In this section the parametric bootstrap technique is adopted to construct the percentile bootstrap CI (PBCI) (see [16] for more details) and the bootstrap-t CI (BTCI) (see [15]). The following algorithm is used to differentiate the two types of bootstrap techniques:

1. Based on the observed original progressively Type-II sample, $(t_{j1;m_j,n_j} < t_{j2;m_j,n_j} < ... < t_{jm_j;m_j,n_j})$, obtain $\hat{\beta}$, and $\hat{\lambda}$, $j = 1, 2$.
2. Based on the values of n_j and m_j $(1 < m_j < n_j)$ with the same values of R_{ji}, $(i = 1, 2, ..., m_j)$, $j = 1, 2$, generate two independent random samples of sizes m_1 and m_2 from the GHLD, $\underline{t}^* = (t^*_{j1;m_j,n_j} < t^*_{j2;m_j,n_j} < ... < t^*_{jm_j;m_j,n_j})$ using the algorithm described in [12].
3. As in step 1 based on t^* compute the bootstrap sample estimates of $\hat{\beta}$, and $\hat{\lambda}$ denoted here as $\hat{\beta}^*$ and $\hat{\lambda}^*$.
4. Steps 2 and 3 are repeated N times, thereby N different bootstrap samples are represented. The value of N may be taken as 1000.
5. The values of $\hat{\beta}^*$ and $\hat{\lambda}^*$ are arranged all in ascending order to obtain the bootstrap sample $(\hat{\theta}_l^{*[1]}, \hat{\theta}_l^{*[2]}, ..., \hat{\theta}_l^{*[N]})$, $l = 1, 2$ where $(\theta^*_1 = \beta^*, \theta^*_2 = \lambda^*)$.

Percentile bootstrap CIs

For given $H(y) = P(\hat{\theta}^*_k \leqslant y)$ the cumulative distribution function of $\hat{\theta}^*_k$. Define $\hat{\theta}^*_{kboot} = H^{-1}(y)$ for given y. The approximate bootstrap $100(1-2\alpha)\%$ CI of $\hat{\theta}^*_l$ is given by

$$\left[\hat{\theta}^*_{l\text{boot}}(\alpha), \hat{\theta}^*_{l\text{boot}}(1-\alpha)\right]. \tag{4.1}$$

Bootstrap-t CIs

First, we present the order statistics $\omega_k^{*[1]} < \omega_k^{*[2]} < ... < \omega_k^{*[N]}$,

$$\omega_k^{*[j]} = \frac{\hat{\theta}_l^{*[j]} - \hat{\theta}_l}{\sqrt{\text{var}\left(\hat{\theta}_l^{*[j]}\right)}}, \quad j = 1, 2, ..., N, \quad l = 1, 2, \tag{4.2}$$

where $\hat{\theta}_1 = \hat{\beta}$, $\hat{\theta}_2 = \hat{\lambda}$.

For given $H(y) = P(\omega^*_l < y)$ the cumulative distribution function of ω^*_l, and given y, is defined as

$$\hat{\theta}_{l\text{boot}-t} = \hat{\theta}_l + \sqrt{\text{Var}(\hat{\theta}_l)}H^{-1}(y). \tag{4.3}$$

The approximate $100(1-2\alpha)\%$ CIs of $\hat{\theta}_k$ is given by

$$\left(\hat{\theta}_{l\text{boot}-t}(\alpha), \hat{\theta}_{l\text{boot}-t}(1-\alpha)\right). \tag{4.4}$$

5. SIMULATION STUDIES

We now adopted undertake simulation studies with the help of the Mathematica program Ver. 8.0 to illustrate the theoretical results of the estimation problem. The performance of the different point estimators of the shape parameter of the GHLD and the acceleration factor are measured and compared with the average of the estimates (AVG), absolute relative bias (RAB), and mean square error (MSE); specifically,

$$\text{AVG}(\hat{\theta}_l) = \frac{1}{M}\sum_{i=1}^{M} \hat{\theta}_l^{(i)}, \ (\theta_1 = \beta, \theta_2 = \lambda), \tag{5.1}$$

$$\text{RAB}(\hat{\theta}_l) = \frac{|\overline{\hat{\theta}_l} - \theta_l|}{\theta_l}, \tag{5.2}$$

and

$$\text{MSE}(\hat{\theta}_l) = \frac{1}{M}\sum_{i=1}^{M} \left(\hat{\theta}_l^{(i)} - \theta_l\right)^2. \tag{5.3}$$

For each of the CIs, the ACIs and the different bootstrap CIs can be measured and compared using the average confidence lengths (AC) as well as the coverage percentages (CP). For the generated sample, we computed the 90% CIs, recorded AC, and checked whether the true value lay within the interval (CP). In simulation studies, this step is repeated 1000 times. The estimated CP was computed as the number of CIs that covered the true values divided by 1000 whereas the estimated expected width of the CI was computed as the sum of the lengths for all intervals divided by 1000. Now, we present the definitions of the different CSs that are used in our simulation studies:

CS I: $R_{ji} = 0$ for $i < m$ and $R_{jm} = n - m$,

CS II: $R_{ji} = 0$ for $i > 1$ and $R_{j1} = n - m$,

CS III: for odd m, $R_{ji} = 0$ for $i > \frac{m+1}{2}$ or $i < \frac{m+1}{2}$ and $R_{j\frac{m+1}{2}} = n - m$.

Also, for even m, $R_{ji} = 0$ for $i > \frac{m}{2}$ or $i < \frac{m}{2}$ and $R_{j\frac{m}{2}} = n - m$:

CS IV: $R_{j\frac{2m-n}{2}+1} = ... = R_{j\frac{n}{2}} = 1$, other $R_{ji} = 0$.

In our simulation studies, we consider two separate cases:

(1) The model parameter values $(\beta = 0.5, \lambda = 2.0)$, the sample sizes $(n_1 = n_2 = \mathbf{n})$ and observed failure times $(m_1 = m_2 = \mathbf{m})$; results are listed in Tables 1 and 2.

(2) The model parameter values $(\beta = 2.5, \lambda = 1.5)$, the sample sizes $(n_2 = 2n_1 = 2\mathbf{n})$ and observed failure times $(m_2 = 2m_1 = 2\mathbf{m})$; results are listed in Tables 3 and 4.

Table 1: AVG and RABs (MSEs) of ML and Bootstrap estimates for the parameters ($\beta = 0.5$ and $\lambda = 2.0$).

$(\mathbf{n}, \mathbf{m})$	CS	MLE		Bootstrap	
		β	λ	β	λ
(30,15)	I	0.5370 0.0507 (0.147)	1.8242 0.098 (0.471)	0.5410 0.055 (0.248)	1.8109 0.145 (0.645)
	II	0.5300 0.0481 (0.126)	1.8950 0.079 (0.410)	0.5312 0.049 (0.210)	1.8229 0.140 (0.584)
	III	0.5361 0.0497 (0.133)	1.8720 0.090 (0.425)	0.5347 0.053 (0.229)	1.8198 0.142 (0.609)
	IV	0.5457 0.0487 (0.123)	1.8889 0.087 (0.419)	0.5317 0.049 (0.219)	1.8301 0.142 (0.601)
(30,25)	I	0.5204 0.0413 (0.101)	1.889 0.052 (0.394)	0.5229 0.043 (0.131)	1.8740 0.085 (0.451)
	II	0.5154 0.039 (0.099)	1.9241 0.048 (0.289)	0.5201 0.040 (0.120)	1.8654 0.074 (0.325)
	III	0.5224 0.042 (0.102)	1.9094 0.049 (0.317)	0.5244 0.041 (0.135)	1.8741 0.081 (0.377)
	IV	0.5208 0.041 (0.100)	1.9107 0.050 (0.314)	0.5232 0.040 (0.124)	1.8841 0.080 (0.364)
(50,25)	I	0.5215 0.041 (0.098)	1.920 0.050 (0.378)	0.5240 0.045 (0.128)	1.9014 0.083 (0.440)
	II	0.5109 0.031 (0.081)	1.951 0.045 (0.326)	0.5217 0.041 (0.119)	1.9241 0.079 (0.420)
	III	0.5122 0.035 (0.093)	1.936 0.044 (0.331)	0.5217 0.040 (0.131)	1.9288 0.074 (0.426)
	IV	0.5220 0.034 (0.090)	1.944 0.043 (0.338)	0.5200 0.039 (0.130)	1.9233 0.071 (0.415)
(50,40)	I	0.5100 0.022 (0.052)	1.9821 0.033 (0.208)	0.5107 0.031 (0.101)	1.9621 0.036 (0.401)
	II	0.5102 0.020 (0.040)	1.9800 0.022 (0.109)	0.5099 0.022 (0.081)	1.9751 0.027 (0.265)
	III	0.5133 0.022 (0.042)	1.9741 0.025 (0.119)	0.5118 0.024 (0.094)	1.9751 0.029 (0.377)
	IV	0.5201 0.023 (0.041)	1.9788 0.024 (0.112)	0.5122 0.021 (0.090)	1.9788 0.030 (0.372)

Table 2: The (AC) and (CP) of 90% CIs (β, λ) at (0.5, 2.0).

$(\mathbf{n}, \mathbf{m})$	CS	MLE		Boot-P		Boot-t	
		β	λ	β	λ	β	λ
(30,15)	I	2.1214 (0.88)	3.2145 (0.87)	3.1354 (0.87)	5.2336 (0.86)	2.1019 (0.89)	3.2100 (0.88)
	II	2.1110 (0.88)	3.1177 (0.89)	3.1123 (0.93)	5.2210 (0.88)	2.1007 (0.89)	3.2006 (0.91)
	III	2.1133 (0.87)	3.1224 (0.88)	3.1209 (0.92)	5.2319 (0.88)	2.1016 (0.89)	3.2055 (0.90)
	IV	2.1125 (0.88)	3.1233 (0.88)	3.1212 (0.88)	5.2400 (0.87)	2.1109 (0.89)	3.2107 (0.91)
(30,25)	I	2.1009 (0.89)	3.2010 (0.88)	3.1210 (0.87)	5.2221 (0.88)	2.1000 (0.89)	3.2009 (0.90)
	II	2.0789 (0.92)	3.0166 (0.91)	3.1000 (0.92)	4.6215 (0.93)	1.9524 (0.91)	3.1612 (0.89)
	III	2.1087 (0.89)	3.0198 (0.89)	3.1017 (0.89)	5.1017 (0.92)	2.0041 (0.90)	3.2008 (0.89)
	IV	2.1108 (0.91)	3.1010 (0.90)	3.1205 (0.92)	5.1003 (0.89)	2.0000 (0.90)	3.2107 (0.919)
(50,25)	I	2.1023 (0.89)	3.1077 (0.88)	3.1187 (0.89)	5.2119 (0.88)	2.0139 (0.88)	3.1748 (0.90)
	II	2.0742 (0.93)	3.0142 (0.89)	2.9811 (0.92)	4.7217 (0.88)	1.9541 (0.90)	3.1752 (0.92)
	III	2.1102 (0.88)	3.1100 (0.89)	3.1107 (0.91)	5.1009 (0.89)	2.0051 (0.91)	3.2012 (0.89)
	IV	2.1111 (0.88)	3.1009 (0.91)	3.1217 (0.91)	5.1014 (0.89)	2.0021 (0.90)	3.2112 (0.89)
(50,40)	I	1.9821 (0.89)	3.0087 (0.89)	3.0584 (0.92)	5.0472 (0.88)	1.7742 (0.89)	3.1010 (0.910)
	II	1.7490 (0.88)	2.9874 (0.89)	2.6511 (0.89)	4.1145 (0.93)	1.7120 (0.89)	3.0770 (0.91)
	III	1.8890 (0.89)	3.1120 (0.89)	2.6742 (0.89)	4.1246 (0.92)	1.7331 (0.90)	3.1070 (0.90)
	IV	1.8741 (0.91)	3.10820 (0.92)	2.6662 (0.89)	4.1195 (0.92)	1.7320 (0.91)	3.1040 (0.89)

Table 3: AVG and RABs (MSEs) of ML and Bootstrap estimates for the parameters ($\beta = 2.5$ and $\lambda = 1.5$).

$(\mathbf{n}, \mathbf{m})$	CS	MLE		Bootstrap	
		β	λ	β	λ
(20,10)	I	2.5390 0.120 (0.521)	1.4522 0.109 (0.471)	2.5561 0.1324 (0.641)	1.4522 0.111 (0.499)
	II	2.5211 0.115 (0.446)	1.4745 0.087 (0.406)	2.5423 0.125 (0.549)	1.4642 0.099 (0.408)
	III	2.5341 0.120 (0.498)	1.4624 0.109 (0.450)	2.5450 0.129 (0.587)	1.4602 0.101 (0.470)
	IV	2.5327 0.118 (0.487)	1.4631 0.105 (0.450)	2.5462 0.131 (0.591)	1.4611 0.105 (0.465)
(20,15)	I	2.5220 0.101 (0.521)	1.4842 0.087 (0.328)	2.5325 0.101 (0.554)	1.4740 0.084 (0.332)
	II	2.5201 0.087 (0.421)	1.4892 0.060 (0.301)	2.5288 0.099 (0.511)	1.4884 0.050 (0.311)
	III	2.5213 0.099 (0.460)	1.4811 0.080 (0.317)	2.5485 0.110 (0.522)	1.4811 0.070 (0.328)
	IV	2.5217 0.097 (0.455)	1.4804 0.082 (0.322)	2.5477 0.108 (0.518)	1.4814 0.069 (0.331)
(30,20)	I	2.5198 0.100 (0.515)	1.4811 0.086 (0.312)	2.5311 0.099 (0.44)	1.4720 0.081 (0.311)
	II	2.5190 0.060 (0.400)	1.4893 0.055 (0.280)	2.5288 0.070 (0.500)	1.4870 0.046 (0.287)
	III	2.5196 0.090 (0.454)	1.4814 0.076 (0.312)	2.5462 0.101 (0.511)	1.4900 0.065 (0.314)
	IV	2.5211 0.097 (0.455)	1.4774 0.079 (0.318)	2.5477 0.106 (0.519)	1.4855 0.062 (0.325)
(30,25)	I	2.5101 0.089 (0.256)	1.4954 0.050 (0.214)	2.5210 0.060 (0.265)	1.4894 0.042 (0.266)
	II	2.5121 0.051 (0.202)	1.4998 0.020 (0.148)	2.5109 0.052 (0.215)	1.4899 0.040 (0.200)
	III	2.5111 0.060 (0.215)	1.4974 0.023 (0.201)	2.5109 0.069 (0.261)	1.4864 0.045 (0.212)
	IV	2.5113 0.059 (0.212)	1.4982 0.021 (0.212)	2.5110 0.067 (0.242)	1.4870 0.046 (0.209)

Table 4: The (AC) and (CP) of 90% CIs (β, λ) at (2.5, 1.5).

$(\mathbf{n}, \mathbf{m})$	CS	MLE		Boot-t		Boot-P	
		β	λ	β	λ	β	λ
(20,10)	I	4.1147 (0.87)	3.1231 (0.88)	5.2414 (0.85)	3.5421 (0.86)	4.1009 (0.89)	3.1037 (0.89)
	II	3.9544 (0.88)	3.0032 (0.88)	3.9881 (0.87)	3.2131 (0.88)	3.7542 (0.89)	3.0011 (0.91)
	III	3.9654 (0.88)	3.0172 (0.89)	3.9991 (0.92)	3.2321 (0.88)	3.8045 (0.90)	3.0712 (0.92)
	IV	3.9622 (0.88)	3.0161 (0.89)	3.9970 (0.93)	3.2300 (0.92)	3.8039 (0.91)	3.0702 (0.91)
(20,15)	I	3.7541 (0.91)	3.1001 (0.89)	3.7865 (0.88)	3.1124 (0.89)	3.7111 (0.89)	3.0099 (0.91)
	II	3.1542 (0.89)	2.8570 (0.88)	3.7742 (0.89)	2.899 (0.89)	3.1421 (0.90)	2.8110 (0.91)
	III	3.1588 (0.88)	2.8598 (0.89)	3.7760 (0.91)	2.9200 (0.88)	3.1441 (0.91)	2.8132 (0.91)
	IV	3.1570 (0.89)	2.8592 (0.90)	3.7755 (0.92)	2.9136 (0.91)	3.1432 (0.92)	2.8127 (0.90)
(30,20)	I	3.7531 (0.92)	3.0991 (0.89)	3.7854 (0.89)	3.1118 (0.89)	3.7101 (0.90)	3.0088 (0.92)
	II	3.1522 (0.90)	2.8550 (0.88)	3.7720 (0.91)	2.8965 (0.89)	3.1400 (0.91)	2.8094 (0.91)
	III	3.1573 (0.89)	2.8585 (0.89)	3.7750 (0.88)	2.9199 (0.88)	3.1432 (0.92)	2.8124 (0.91)
	IV	3.1555 (0.89)	2.8580 (0.88)	3.7742 (0.92)	2.9127 (0.93)	3.1421 (0.91)	2.8118 (0.91)
(30,25)	I	3.7014 (0.91)	3.0665 (0.89)	3.7116 (0.89)	3.0772 (0.92)	3.6542 (0.91)	3.0545 (0.92)
	II	3.5124 (0.901)	3.0256 (0.89)	3.5198 (0.91)	3.0281 (0.89)	3.5111 (0.90)	3.0231 (0.91)
	III	3.5321 (0.88)	3.0290 (0.89)	3.5221 (0.89)	3.0321 (0.88)	3.5185 (0.91)	3.0287 (0.91)
	IV	3.5314 (0.89)	3.0282 (0.898)	3.5214 (0.90)	3.0307 (0.92)	3.5172 (0.901)	3.0281 (0.92)

6. NUMERICAL EXAMPLE

For demonstration purposes, the estimation procedure described in the previous section is applied to the set of simulated progressive Type-II censoring data under the constant-stress partially ALT. The MLEs and the two bootstrap CIs are computed for model parameters β and λ with the real parameters are equal to 1.5 and 2.0, respectively. In this example, we simulate samples of size ($m_1 = m_2 = 15$ of $n_1 = n_2 = 30$) from the GHLD under the two progressive CSs $R_1 = R_2 = \{1, 0, 0, 0, 2, 0, 0, 2, 0, 0, 1, 0, 1, 0, 0, 0, 0, 2, 1, 0\}$ using the algorithm described in Balakrishnan and Sandhu [12]. The simulated data are presented in Table 5.

Table 5: Simulated progressively censored samples with constant PALTs.

Normal conditions	0.13901	0.22961	0.26912	0.47032	0.51005	0.52645	0.53583
	0.56987	0.65999	0.79289	0.80636	0.89349	1.56115	1.63822
	1.66079						
Accelerated conditions	0.00274	0.02767	0.06181	0.06717	0.12004	0.14341	0.25042
	0.27614	0.31457	0.42484	0.54109	0.54112	0.75652	1.13610
	1.41038						

In Figure 1, the two probability density functions show the effect of an acceleration factor.

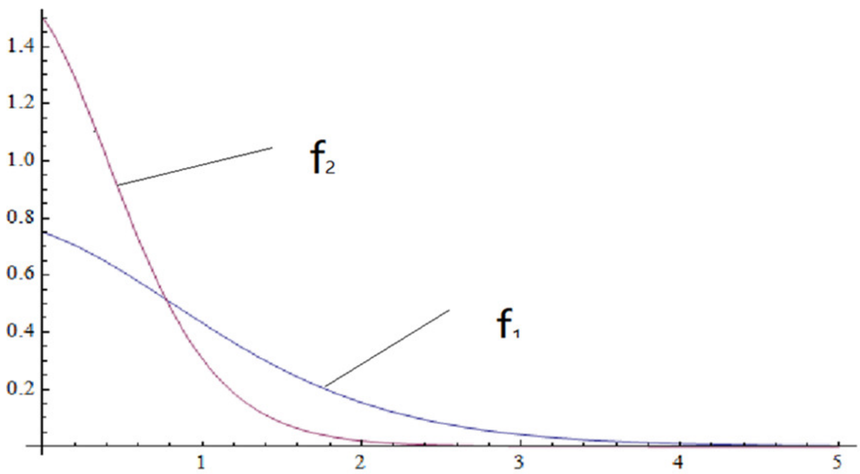

Figure 1: Probability density under normal and accelerated condition.

The iteration procedure of the MLE needs the initial value of parameter obtained from the profile log-likelihood function (Figure 2) as 1.8. The point estimates and related RABs and MSEs of the parameters as well as the 90% and 95% ACIs are listed in Table 6. Also, the point estimates and the relate RABs and MSEs of the parameters as well as the 90% and 95% PBCIs and BTCIs are presented in Table 7. We observed that the BTCIs and approximate MLE intervals are narrower than the PBCIs and always include the population parameter values.

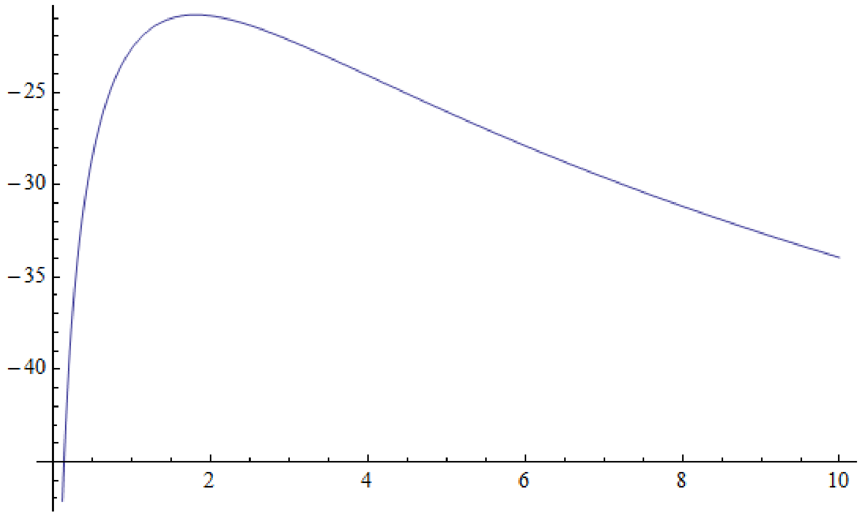

Figure 2: Profile log likelihood function of λ.

Table 6: MLEs, MSEs, RABs and (90%-95%) approximate confidence intervals.

$(.)_{\text{ML}}$	RAB	MSE	90%	95%
1.5495	0.0330	0.0495	(0.7769, 2.3221)	(0.9011, 2.1979)
1.8034	0.0983	0.1966	(0.7231, 2.8837)	(0.8968, 2.7100)

Table 7: Percentile bootstrap CIs and Bootstrap-t CIs based on 500 replications.

$(.)_{\text{Boot}}$	RAB	MSE	90%		95%	
			BPCI	BTCI	BPCI	BTCI
1.7421	0.1614	0.2421	(0.3241, 3.1205)	(0.7981, 2.2954)	(0.6581, 2.6325)	(0.8881, 2.1472)
2.3415	0.1707	0.3415	(0.5213, 3.2140)	(0.7751, 2.7098)	(0.4578, 2.6590)	(0.7922, 2.5213)

7. CONCLUDING REMARKS

In product-life testing experiments, reducing the time and cost, especially for units with high reliability, illustrates the importance of ALTs. Different types of ALTs are available, one of the types most suitable for different situations is the constant-stress partially ALTs. Also, the experimenter in some situations is unable to obtain complete information of failure times for all experimental units or is in need to remove some units other than the final point of the experiment. The conventional Type-I and Type-II CSs do not have the flexibility of allowing to remove any units at points other than the final point of the experiment.

Hence, in this paper, we adopted a more general CS with the constant-stress partially ALT, known as progressive Type-II censoring. Simulation studies were presented to assess and compare the performance of the proposed methods. From the results, we observed the following:

1. For fixed values of sample size **n** and with increasing effected sample size **m**, the MSEs and RABs of the considered parameters decrease.

2. For fixed values of the sample and failure time sizes, CS II, in which the censoring occurs after the first observed failure, gives more accurate results through the MSEs and RABs than the other schemes..

3. Results for the CS III and CS IV are more similar.

4. The bootstrap-t credible intervals give more accurate results than the ACIs than the bootstrap CIs because the lengths of the former are less than the lengths of the latter, for different sample sizes, observed failures, and schemes.

5. For fixed sample sizes and observed failures, CS II moreover gives lower lengths for the three methods to obtain the CIs compared with the other three schemes.

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