REVSTAT – Statistical Journal
Volume 18, Number 1, January 2020, 47–69

COMPOUND POWER SERIES DISTRIBUTION WITH NEGATIVE MULTINOMIAL SUMMANDS: CHARACTERISATION AND RISK PROCESS

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Received: March 2017 Revised: October 2017 Accepted: October 2017

Abstract:

- The paper considers a multivariate distribution whose coordinates are compounds. The number of the summands is itself also a multivariate compound with one and the same univariate Power series distributed number of summands and negative multinomially distributed summands. In the total claims amount process the summands are independent identically distributed random vectors. We provide the first full characterization of this distribution. We show that considered as a mixture this distribution would be Mixed Negative multinomial distribution having the possibly scale changed power series distributed THE first parameter. We provide an interesting application to risk theory.

Key-Words:

- *power series distributions; Multivariate Compound distribution; Negative multinomial distribution; risk process.*

AMS Subject Classification:

- 60E05, 62P05.

1. INTRODUCTION AND PRELIMINARY RESULTS

It seems that Bates and Neyman [3] were first to introduce Negative multinomial (NMn) distribution in 1952. They obtained it by considering a mixture of independent Poisson distributed random variables (r.vs) with one and the same Gamma distributed mixing variable. Their first parameter could be a real number. Wishart [25] considers the case when the first parameter could be only integer. He calls this distribution Pascal multinomial distribution. At the same time Tweedie [24] obtained estimators of the parameters. Sibuya *et al.* [18] make a systematic investigation of this distribution and note that the relation between Binomial distribution and Negative binomial (NBi) distribution is quite similar to that between the Multinomial distribution and NMn distribution. The latter clarifies the probability structure of the individual distributions. The bivariate case of the compound power series distribution with geometric summands (i.e. $n = 1$ and $k = 2$) is partially investigated in [12]. Another related work is [10].

A version of k-variate negative binomial distribution with respect to risk theory is considered in [2, 26]. The authors show that it can be obtained by mixing of iid Poisson random variables with a multivariate finite mixture of Erlang distributions with one and the same second parameter. Further on they interpret it as the loss frequencies and obtain the main characteristics. Due to covariance invariance property, the corresponding counting processes can be useful to model a wide range of dependence structures. See [2, 26] for examples. Using probability generating functions, the authors present a general result on calculating the corresponding compound, when the loss severities follow a general discrete distribution. The similarity of our paper and papers [2, 26] is that both consider the aggregate losses of an insurer that runs through several correlated lines of business. In (2.1) and (2.2) [2] consider Mixed k-variate Poisson distribution (with independent coordinates, given the mixing variable) and the mixing variable is Mixed Erlang distributed. More precisely the first parameter in the Erlang distribution is replaced with a random variable. The mixing variable is multivariate and the coordinates of the compounding vector are independent. In our case the mixing variable is one and the same and the coordinates of the counting vector are dependent.

Usually Negative Multinomial (NMn) distribution is interpreted as the one of the numbers of outcomes A_i, $i = 1, 2, ..., k$ before the n-th B, in series of independent repetitions, where A_i, $i = 1, 2, ..., k$ and B form a partition of the sample space. See e.g. Johnson *et al.* [7]. Let us recall the definition.

Definition 1.1. Let $n \in \mathbb{N}$, $0 < p_i$, $i = 1, 2, ..., k$ and $p_1 + p_2 + \cdots + p_k < 1$. A vector $(\xi_1, \xi_2, ..., \xi_k)$ is called Negative multinomially distributed with parameters $n, p_1, p_2, ..., p_k$, if its probability mass function (p.m.f.) is

$$P(\xi_1 = i_1,\, \xi_2 = i_2,\, ...,\, \xi_k = i_k) = \\ = \binom{n + i_1 + i_2 + \cdots + i_k - 1}{i_1, i_2, ..., i_k, n-1} p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k} (1 - p_1 - p_2 - \cdots - p_k)^n , \\ i_s = 0, 1, \ldots, \quad s = 1, 2, ..., k .$$

Briefly $(\xi_1, \xi_2, ..., \xi_k) \sim NMn(n; p_1, p_2, ..., p_k)$.

If $A_1, A_2, ..., A_k$ describe all possible mutually exclusive "successes" and the event $\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_k$ presents the "failure", then the coordinates ξ_i of the above vector can be interpreted as the number of "successes" of type A_i, $i = 1, 2, ..., k$ until n-th "failure".

This distribution is a particular case of Multivariate Power series distribution. (The definition is recalled below.) Considering this distribution for $k = 1$, we obtain a version of NBi distribution used in this paper. We denote the membership of a random variable ξ_1 to this class of distributions by $\xi_1 \sim NBi(n; 1 - p_1)$.

Notice that the marginal distributions of NMn distributed random vector are $NBi(n, 1 - \rho_i)$, $\rho_i = \frac{p_i}{1 - \sum_{j \neq i} p_j}$. More precisely their probability generating function (p.g.f.) is $G_\xi(z) = Ez^{\xi_i} = \left(\frac{1-\rho_i}{1-\rho_i z}\right)^n$, $|z| < \frac{1}{\rho_i}$, $i = 1, 2, ..., k$.

The distribution in Definition 1.1 is sometimes called Multivariate Negative Binomial distribution.

For $n = 1$ the NMn distribution is a Multivariate geometric distribution. Some properties of the bivariate version of this distribution are considered e.g. by Phatak *et al.* [15]. A systematic investigation of multivariate version could be found e.g. in Srivastava *et al.* [21].

If $(\xi_1, \xi_2, ..., \xi_k) \sim NMn(n; p_1, p_2, ..., p_k)$, its probability generating function (p.g.f.) is

$$G_{\xi_1,\xi_2,...,\xi_k}(z_1, z_2, ..., z_k) = \left\{ \frac{1 - p_1 - p_2 - \cdots - p_k}{1 - (p_1 z_1 + p_2 z_2 + \cdots + p_k z_k)} \right\}^n , \tag{1.1}$$

$|p_1 z_1 + p_2 z_2 + \cdots + p_k z_k| < 1$.

For $m = 2, 3, ..., k-1$, its finite dimensional distributions (f.d.ds) are, $(\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_m}) \sim NMn(n; \rho_{i_1}, \rho_{i_2}, ..., \rho_{i_m})$, with

$$\rho_{i_s} = \frac{p_{i_s}}{1 - \sum_{j \notin \{i_1, i_2, ..., i_m\}} p_j} , \qquad s = 1, 2, ..., m , \tag{1.2}$$

and for the set of indexes $\bar{i}_1, \bar{i}_1, ..., \bar{i}_{k-m}$ that complements $i_1, i_2, ..., i_m$ to the set $1, 2, ..., k$ its conditional distributions are

$$\begin{aligned} &\left(\xi_{\bar{i}_1}, \xi_{\bar{i}_2}, ..., \xi_{\bar{i}_{k-m}} \,\middle|\, \xi_{i_1} = n_1,\ \xi_{i_2} = n_2,\ ...,\ \xi_{i_m} = n_m\right) \sim \\ &\qquad \sim\ NMn\left(n + n_1 + n_2 + \cdots + n_m;\ p_{\bar{i}_1}, p_{\bar{i}_2}, ..., p_{\bar{i}_{k-m}}\right) . \end{aligned} \tag{1.3}$$

More properties of NMn distribution can be found in Bates and Neyman [3] or Johnson *et al.* [7].

The set of all NMn distributions with one and the same $p_1, p_2, ..., p_k$ is closed with respect to convolution.

Lemma 1.1. *If r.vs* $\mathbf{S}_i \sim NMn(n_i; p_1, p_2, ..., p_k)$, $i = 1, 2, ..., m$ *are independent, then the random vector*

$$\mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_m \sim NMn\left(n_1 + n_2 + \cdots + n_m;\ p_1, p_2, ..., p_k\right) . \tag{1.4}$$

One of the most comprehensive treatments with a very good list of references on Multivariate discrete distributions is the book of Johnson *et al.* [7].

The class of Power Series (PS) Distributions seems to be introduced by Noack (1950) [13] and Khatri (1959) [11]. A systematic approach on its properties could be found e.g. in Johnson *et al.* [8]. We will recall now only the most important for our work.

Definition 1.2. Let $\vec{a} = (a_0, a_1, ...)$, where $a_i \geq 0$, $i = 0, 1, ...$ and $\theta \in \mathbb{R}$ is such that

$$0 < g_{\vec{a}}(\theta) = \sum_{n=0}^{\infty} a_n \theta^n < \infty \,. \tag{1.5}$$

A random variable (r.v.) X is Power series distributed, associated with the function $g_{\vec{a}}$ and the parameter θ (or equivalently associated with the sequence $\vec{a}$ and the parameter θ), if it has p.m.f.

$$P(X = n) = \frac{a_n \theta^n}{g_{\vec{a}}(\theta)}\,, \qquad n = 0, 1, ... \tag{1.6}$$

Briefly $X \sim PS(a_1, a_2, ...; \theta)$ or $X \sim PS(g_{\vec{a}}(x); \theta)$. The radius of convergence of the series (1.5) determines the parametric space Θ for θ. Further on we suppose that $\theta \in \Theta$.

Notice that given a PS distribution and the function $g_{\vec{a}}$ the constants θ and $a_1, a_2, ...$ are not uniquely determined, i.e. it is an ill-posed inverse problem. However, given the constants θ and $a_0, a_1, ...$ (or the function $g_{\vec{a}}(x)$ and θ) the corresponding PS distribution is uniquely determined. In this case, it is well known that:

- The p.g.f. of X is

$$\mathbb{E} z^X = \frac{g_{\vec{a}}(\theta z)}{g_{\vec{a}}(\theta)}\,, \qquad z\theta \in \Theta\,. \tag{1.7}$$

- The type of all PS distributions is closed under convolution and more precisely if $X_1 \sim PS(g_1(x); \theta)$ and $X_2 \sim PS(g_2(x); \theta)$ are independent and $\theta \in \Theta_1 \cap \Theta_2$, then

$$X_1 + X_2 \sim PS\big(g_1(x)\, g_2(x); \theta\big)\,. \tag{1.8}$$

- The mean is given by

$$\mathbb{E}X = \theta\, \frac{g'_{\vec{a}}(\theta)}{g_{\vec{a}}(\theta)} = \theta \big[\log\big(g_{\vec{a}}(\theta)\big)\big]'. \tag{1.9}$$

  From now on we denote the first and the second derivative of $g(x)$ with respect to x briefly by $g'(x)$ and $g''(x)$.

- The variance of X has the form

$$\operatorname{Var} X = \theta^2 \big[\log\big(g_{\vec{a}}(\theta)\big)\big]'' + \mathbb{E}X\,; \tag{1.10}$$

- The Fisher index is given by

$$FIX = 1 + \theta\, \frac{\big[\log\big(g_{\vec{a}}(\theta)\big)\big]''}{\big[\log\big(g_{\vec{a}}(\theta)\big)\big]'}\,.$$

We show that the class of Compound Power Series Distributions with Negative Multinomial Summands is a particular case of Multivariate Power series distribution (MPSD) considered by Johnson *et al.* [7]. Therefore let us remind the definition and its main properties.

Definition 1.3. Let $\theta_j > 0,\ j = 1, 2, ..., k$ be positive real numbers and $a_{(i_1, i_2, ..., i_k)}$, $i_j = 0, 1, ...,$ be non-negative constants such that

$$A_{\vec{a}}(\theta_1, \theta_2, ..., \theta_k) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} a_{(i_1, i_2, ..., i_k)}\, \theta_1^{i_1} \theta_2^{i_2} \cdots \theta_k^{i_k} < \infty \,. \tag{1.11}$$

The distribution of the random vector $\vec{X} = (X_1, X_2, ..., X_k)$ with probability mass function

$$P\big(X_1 = n_1,\, X_2 = n_2,\, ...,\, X_k = n_k\big) = \frac{a_{(n_1, n_2, ..., n_k)}\, \theta_1^{n_1} \theta_2^{n_2} \cdots \theta_k^{n_k}}{A_{\vec{a}}(\theta_1, \theta_2, ..., \theta_k)}$$

is called Multivariate Power Series Distribution (MPSD) with parameters $A_{\vec{a}}(\vec{x})$, $a_{(i_1, i_2, ..., i_k)}$ and $\vec{\theta} = (\theta_1, \theta_2, ..., \theta_k)$. Briefly $\vec{X} \sim MPSD(A_{\vec{a}}(\vec{x}), \vec{\theta})$. As follows, Θ_k denotes the set of all parameters $\vec{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ that satisfy (1.11).

This class of distributions seems to be introduced by Patil (1965) [14] and Khatri (1959) [11]. A very useful necessary and sufficient condition that characterise this family is obtained by Gerstenkorn(1981). It is well known (see e.g. Johnson *et al.* [7]) that the p.g.f. of $\vec{X}$ is

$$\mathbb{E} z_1^{X_1} z_2^{X_2} \cdots z_k^{X_k} = \frac{A_{\vec{a}}(\theta_1 z_1, \theta_2 z_2, ..., \theta_k z_k)}{A_{\vec{a}}(\theta_1, \theta_2, ..., \theta_k)}\,, \qquad (\theta_1 z_1, \theta_2 z_2, ..., \theta_k z_k) \in \Theta_k \,. \tag{1.12}$$

Through the paper $k = 2, 3, ...,$ is fixed and it corresponds to the number of the coordinates. We denote by $\stackrel{d}{=}$ the coincidence in distribution, by "$\sim$" the fact that a r.v. belongs to a given class of distributions, by $G_{\xi_1, \xi_2, ..., \xi_k}(z_1, z_2, ..., z_k) = E(z_1^{\xi_1} \cdots z_k^{\xi_k})$, the joint p.g.f. of a random vector $(\xi_1, \xi_2, ..., \xi_k)$ and by $FI\xi$ the index of dispersion of the r.v. ξ (i.e. the variance of ξ divided by the corresponding mean).

One can consider the different concepts for compounds. We use the following one.

Definition 1.4. Let $\overrightarrow{\xi_i} = \big(\xi_i^{(1)}, \xi_i^{(2)}, ..., \xi_i^{(k)}\big)$, $i = 1, 2, ...,$ be i.i.d. random vectors and N be a discrete r.v. independent on them. We call compound, a random vector $\vec{X}_N = \big(X_N^{(1)}, X_N^{(2)}, ..., X_N^{(k)}\big)$, defined by

$$X_N^{(j)} = I_{\{N>0\}} \sum_{i=1}^{N} \xi_i^{(j)} = \begin{cases} \sum_{i=1}^{N} \xi_i^{(j)} & \text{if } N > 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad j = 1, 2, ..., k\,.$$

The distribution of ξ is called compounding distribution.

Further on we are going to use the following properties:

1. $G_{\vec{X}_N}(z) = G_N(G_{\vec{\xi}}(z))$.
2. If $EN < \infty$ and $E\vec{\xi} < \infty$, then $E\vec{X}_N = ENE\vec{\xi}$ (see [17], Cor. 4.2.1).

3. If $\operatorname{Var} N < \infty$ and coordinate-wise $\operatorname{Var} \vec{\xi} < \infty$, $\operatorname{Var} \vec{X}_N = \operatorname{Var} N (E\vec{\xi})^2 + EN \operatorname{Var} \vec{\xi}$ (see [17], Cor. 4.2.1).

4. $FI\vec{X}_N = FI\,N\,E\vec{\xi} + FI\vec{\xi}$.

Notice that properties 2. and 3. are particular cases of the well known Wald's equations.

Here we consider a multivariate distribution which coordinates are dependent compounds. In the notations of the Definition 1.4, N is PS distributed and $\vec{\xi}$ is NMn distributed. The cases when N is Poisson distributed is partially investigated in 1962, by G. Smith [20]. In Section 2, following the traditional approach about definition of distributions, first we define this distribution through its p.m.f., then we investigate its properties. We consider the case when the summands are NMn distributed. We obtain its main numerical characteristics and conditional distributions. Finally explain its relation with compounds and mixtures. We prove that the class of Compound Power Series Distributions with Negative Multinomial Summands is a particular case of Multivariate Power series distribution and find the explicit form of the parameters. We show that considered as a Mixture this distribution would be (possibly Zero-inflated) Mixed Negative Multinomial distribution with possibly scale changed Power series distributed first parameter. Using these relations we derive several properties and its main numerical characteristics. In Section 3 the risk process application is provided, together with simulations of the risk processes and estimation of ruin probabilities in a finite time interval.

2. DEFINITION AND MAIN PROPERTIES OF THE COMPOUND POWER SERIES DISTRIBUTION WITH NEGATIVE MULTINOMIAL SUMMANDS

Let us first define Compound Power series distribution with Negative multinomial summands and then to investigate its properties.

Definition 2.1. Let $\pi_j \in (0,1)$, $j = 1, 2, ..., k$, $\pi_0 := 1 - \pi_1 - \pi_2 - \cdots - \pi_k \in (0,1)$, $a_s \geq 0$, $s = 0, 1, ...$ and $\theta \in \mathbb{R}$ be such that

$$0 < g_{\vec{a}}(\theta) = \sum_{n=0}^{\infty} a_n \theta^n < \infty \,. \tag{2.1}$$

A random vector $\vec{X} = (X_1, X_2, ..., X_k)$ is called Compound Power series distributed with negative multinomial summands and with parameters $g_{\vec{a}}(x)$, θ; n, $\pi_1, ..., \pi_k$, if for $i = 1, 2, ..., k$, $m_i = 0, 1, 2, ...$, and $(m_1, m_2, ..., m_k) \neq (0, 0, ..., 0)$,

$$\begin{aligned} P\big(X_1 = m_1,\, X_2 = m_2,\, ...,\, X_k = m_k\big) \;&= \\ &= \frac{\pi_1^{m_1} \pi_2^{m_2} \cdots \pi_k^{m_k}}{g_{\vec{a}}(\theta)} \sum_{j=1}^{\infty} a_j\, \theta^j \binom{jn + m_1 + m_2 + \cdots + m_k - 1}{m_1, m_2, ..., m_k, jn - 1} \pi_0^{nj} \,, \\ P\big(X_1 = 0,\, X_2 = 0,\, ...,\, X_k = 0\big) \;&=\; \frac{g_{\vec{a}}(\theta \pi_0^n)}{g_{\vec{a}}(\theta)} \,. \end{aligned} \tag{2.2}$$

Briefly $\vec{X} \sim CPSNMn\big(g_{\vec{a}}(x),\, \theta;\, n,\, \pi_1, \pi_2, ..., \pi_k\big)$ or $\vec{X} \sim CPSNMn\big(\vec{a},\, \theta;\, n,\, \pi_1, \pi_2, ..., \pi_k\big)$. [1]

[1] For $n = 1$ and $k = 2$ see [12].

In the next theorem we show that this distribution is a particular case of $MPSD(A(\vec{x}), \vec{\theta})$ considered in Johnson *et al.* [7].

Theorem 2.1. *Suppose $\pi_i \in (0,1)$, $i = 1, 2, ..., k$, $\pi_0 := 1 - \pi_1 - \pi_2 - \cdots - \pi_k \in (0,1)$, $a_i \geq 0$, $i = 0, 1, ...$ and $\theta \in \mathbb{R}$ are such that (2.1) is satisfied. If*

$$\vec{X} \sim CPSNMn\big(g_{\vec{a}}(x), \theta; n, \pi_1, \pi_2, ..., \pi_k\big),$$

then:

1. $\vec{X} \sim MPSD(A(\vec{x}), \vec{\theta})$, *where* $\vec{\theta} = (\pi_1, \pi_2, ..., \pi_k)$, $a_{(0,...,0)} = g_{\vec{a}}(\theta \pi_0^n)$.
For $(i_1, i_2, ..., i_k) \neq (0, 0, ..., 0)$,

$$a_{(i_1,i_2,...,i_k)} = \sum_{j=1}^{\infty} a_j \, \theta^j \binom{jn + i_1 + i_2 + \cdots + i_k - 1}{i_1, i_2, ..., i_k, jn - 1} \pi_0^{nj},$$
$$A(x_1, x_2, ..., x_k) = g_{\vec{a}}\left\{\theta \frac{\pi_0^n}{[1 - (x_1 + x_2 + \cdots + x_k)]^n}\right\},$$

$x_i \in (0,1)$, $i = 1, 2, ..., k$ *and* $x_1 + x_2 + \cdots + x_k \in (0,1)$.

2. *For* $|\pi_1 z_1 + \pi_2 z_2 + \cdots + \pi_k z_k| < 1$,

$$G_{X_1,X_2,...,X_k}(z_1, z_2, ..., z_k) = \frac{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{1-(\pi_1 z_1 + \pi_2 z_2 + \cdots + \pi_k z_k)}\right)^n\right]}{g_{\vec{a}}(\theta)}$$
$$= \frac{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0 + \pi_1(1-z_1) + \pi_2(1-z_2) + \cdots + \pi_k(1-z_k)}\right)^n\right]}{g_{\vec{a}}(\theta)}.$$

3. *For all* $r = 2, 3, ..., k$,

$$(X_{i_1}, X_{i_2}, ..., X_{i_r}) \sim CPSNMn\Bigg(g_{\vec{a}}(x), \theta;\, n, \frac{\pi_{i_1}}{\pi_0 + \pi_{i_1} + \pi_{i_2} + \cdots + \pi_{i_r}}, \frac{\pi_{i_2}}{\pi_0 + \pi_{i_1} + \pi_{i_2} + \cdots + \pi_{i_r}}, ..., \frac{\pi_{i_r}}{\pi_0 + \pi_{i_1} + \pi_{i_2} + \cdots + \pi_{i_r}}\Bigg).$$

4. *For* $i = 1, 2, ..., k$,

$$X_i \sim CPSNBi\left(g_{\vec{a}}(x), \theta;\, n, \frac{\pi_i}{\pi_0 + \pi_i}\right),$$
$$G_{X_i}(z_i) = \frac{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0 + \pi_i(1-z_i)}\right)^n\right]}{g_{\vec{a}}(\theta)}, \qquad |\pi_i z_i| < \pi_0 + \pi_i,$$
$$EX_i = n\,\theta\big[\log\big(g_{\vec{a}}(\theta)\big)\big]' \frac{\pi_i}{\pi_0} = n\,\theta \frac{\pi_i}{\pi_0} \frac{g'_{\vec{a}}(\theta)}{g_{\vec{a}}(\theta)},$$
$$\mathrm{Var}\, X_i = n \frac{\pi_i \theta}{\pi_0^2}\Big[n\,\pi_i\,\theta\big[\log\big(g_{\vec{a}}(\theta)\big)\big]'' + \big[\log\big(g_{\vec{a}}(\theta)\big)\big]'\big(\pi_0 + \pi_i(n+1)\big)\Big],$$
$$FI\,X_i = 1 + \frac{\pi_i}{\pi_0}\left(n\,\theta \frac{\big[\log\big(g_{\vec{a}}(\theta)\big)\big]''}{\big[\log\big(g_{\vec{a}}(\theta)\big)\big]'} + n + 1\right).$$

5. *For* $i \neq j = 1, 2, ..., k,$

$$
\begin{aligned}
(X_i, X_j) &\sim CPSNMn\left(g_{\vec{a}}(x), \theta;\, n, \frac{\pi_i}{\pi_0 + \pi_i + \pi_j}, \frac{\pi_j}{\pi_0 + \pi_i + \pi_j}\right),\\
G_{X_i,X_j}(z_i, z_j) &= \frac{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0 + (1-z_i)\pi_i + (1-z_j)\pi_j}\right)^n\right]}{g_{\vec{a}}(\theta)}, \qquad |\pi_i z_i + \pi_j z_j| < \pi_0 + \pi_i + \pi_j\,,\\
\mathrm{cov}(X_i, X_j) &= \frac{n\pi_i\pi_j\theta}{\pi_0^2}\left\{n\,\theta\left[\log g_{\vec{a}}(\theta)\right]'' + (n+1)\left[\log g_{\vec{a}}(\theta)\right]'\right\},\\
\mathrm{cor}(X_i, X_j) &= \sqrt{\frac{(FIX_i - 1)\,(FIX_j - 1)}{FIX_i\, FIX_j}}\,.
\end{aligned}
$$

6. *For* $i, j = 1, 2, ..., k,\ j \neq i,$

(**a**) *For* $m_j \neq 0,$

$$
\begin{aligned}
P\big(X_i = m_i\,|\,X_j = m_j\big) &= \left(\frac{\pi_0 + \pi_j}{\pi_0 + \pi_i + \pi_j}\right)^{m_j} \frac{\pi_i^{m_i}}{m_i!\,(\pi_0 + \pi_i + \pi_j)^{m_i}}\\
&\quad \cdot \frac{\sum_{s=1}^{\infty} a_s\theta^s \frac{(sn+m_i+m_j-1)!}{(sn-1)!}\left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns}}{\sum_{s=1}^{\infty} a_s\theta^s \frac{(sn+m_j-1)!}{(sn-1)!}\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^{ns}}, \quad m_i = 0, 1, ...
\end{aligned}
$$

(**b**)
$$
\big(X_i\,|\,X_j = 0\big) \sim CPSNMn\left[a_s,\, \widetilde{\theta} = \theta\left(\frac{\pi_0}{\pi_0 + \pi_j}\right)^n;\, n, \frac{\pi_i}{\pi_0 + \pi_i + \pi_j}\right],
$$

$$
\begin{aligned}
P\big(X_i = m_i\,|\,X_j = 0\big) &= \frac{\pi_i^{m_i}}{m_i!\,(\pi_0 + \pi_i + \pi_j)^{m_i}}\\
&\quad \cdot \frac{\sum_{s=1}^{\infty} a_s \frac{(sn+m_i-1)!}{(sn-1)!}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]^s}{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]}, \quad m_i \in \mathbb{N},\\
P\big(X_i = 0\,|\,X_j = 0\big) &= \frac{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^n\right]}{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]}\,.
\end{aligned}
$$

(**c**) *For* $i, j = 1, 2, ..., k,\ j \neq i,\ m_j = 1, 2, ...,$

$$
\begin{aligned}
E\big(z_i^{X_i}\,|\,X_j = m_j\big) &= \left(\frac{\pi_0 + \pi_j}{\pi_0 + \pi_j + \pi_i - z_i\pi_i}\right)^{m_j}\\
&\quad \cdot \frac{\sum_{s=1}^{\infty} a_s \frac{(sn+m_i-1)!}{(sn-1)!}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j+\pi_i-z_i\pi_i}\right)^n\right]^s}{\sum_{s=1}^{\infty} a_s \frac{(sn+m_j-1)!}{(sn-1)!}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]^s},\\
E\big(z_i^{X_i}\,|\,X_j = 0\big) &= \frac{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j+(1-z_i)\pi_i}\right)^n\right]}{g_{\vec{a}}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]}\,.
\end{aligned}
$$

(**d**) For $z_j = 1, 2, \ldots,$

$$E\big(X_i \,|\, X_j = z_j\big) \;=\; \frac{\pi_i}{\pi_0+\pi_j}\, \frac{\sum_{s=1}^{\infty} a_s \frac{(sn+z_j)!}{(sn-1)!} \left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]^s}{\sum_{s=1}^{\infty} a_s \frac{(sn+z_j-1)!}{(sn-1)!} \left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]^s}\,,$$

$$\begin{aligned} E\big(X_i \,|\, X_j = 0\big) \;&=\; \frac{n\,\pi_i}{\pi_0+\pi_j}\, \frac{\sum_{s=1}^{\infty} s\, a_s \left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]^s}{\sum_{s=0}^{\infty} a_s \left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]^s} \\ &=\; \frac{n\,\theta\,\pi_0^n \pi_i}{(\pi_0+\pi_j)^{n+1}}\, \frac{g'_{\vec a}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]}{g_{\vec a}\left[\theta\left(\frac{\pi_0}{\pi_0+\pi_j}\right)^n\right]}\,. \end{aligned}$$

7. $X_1 + X_2 + \cdots + X_k \sim CPSNBi\big(g_{\vec a}(x), \theta; n, 1-\pi_0\big)$.

8. For $i = 1, 2, \ldots, k$

$$\big(X_i,\, X_1 + X_2 + \cdots + X_k - X_i\big) \;\sim\; CPSNMn\big(g_{\vec a}(x), \theta;\, n, \pi_i, 1-\pi_0-\pi_i\big)\,.$$

9. For $i = 1, 2, \ldots, k$, $m \in \mathbb{N}$

$$\big(X_i \,|\, X_1 + X_2 + \cdots + X_k = m\big) \;\sim\; Bi\left(m, \frac{\pi_i}{1-\pi_0}\right).$$

Sketch of the proof:

1. We substitute of the considered values and function A in the necessary and sufficient condition, given in p. 154, Johnson *et al.* [7], for MPSD and prove that the following two conditions are satisfied:

$$P\big(X_1 = 0,\, X_2 = 0,\, \ldots,\, X_k = 0\big) \;=\; \frac{a_{(0,0,\ldots,0)}}{A(\theta_1, \theta_2, \ldots, \theta_k)}\,,$$

$$\frac{P\big(X_1 = n_1+m_1,\, X_2 = n_2+m_2,\, \ldots,\, X_k = n_k+m_k\big)}{P\big(X_1 = n_1,\, X_2 = n_2,\, \ldots,\, X_k = n_k\big)} \;=$$

$$= \frac{a_{(n_1+m_1, n_2+m_2, \ldots, n_k+m_k)}}{a_{(n_1, n_2, \ldots, n_k)}}\; \theta_1^{m_1} \theta_2^{m_2} \cdots \theta_k^{m_k}\,, \qquad m_i, n_i = 0, 1, \ldots, \quad i = 1, 2, \ldots, k\,.$$

2.–3. Analogously to [12], who works in case $n = 1$ and $k = 2$. Here we have used the definition of p.g.f., the definition of $g_{\vec a}(x)$ and the formula

$$\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \frac{(i_1+i_2+\cdots+i_k+r-1)!}{i_1!\, i_2! \cdots i_k!\, (r-1)!}\; x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} \;=\; \frac{1}{(1-x_1-x_2-\cdots-x_k)^r}\,.$$

6.(a) For $m_i = 0, 1, \ldots$ we substitute the proposed parameters and function in the formula of p.m.f. of PS distribution and obtain the above formula

$$
\begin{aligned}
P\big(X_i = m_i \,|\, X_j = m_j\big) &= \frac{\sum_{s=1}^{\infty} a_s \theta^s \frac{(sn+m_i+m_j-1)!}{(sn-1)!m_i!} \left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns} \left(\frac{\pi_i}{\pi_0+\pi_i+\pi_j}\right)^{m_i}}{\sum_{k=0}^{\infty} \sum_{s=1}^{\infty} a_s \theta^s \frac{(sn+k+m_j-1)!}{(sn-1)!k!} \left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns} \left(\frac{\pi_i}{\pi_0+\pi_i+\pi_j}\right)^{k}} \\
&= \frac{\sum_{s=1}^{\infty} a_s \theta^s \frac{(sn+m_i+m_j-1)!}{(sn-1)!m_i!} \left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns} \left(\frac{\pi_i}{\pi_0+\pi_i+\pi_j}\right)^{m_i}}{\sum_{s=1}^{\infty} \frac{a_s}{(sn-1)!} \theta^s \left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns} \sum_{k=0}^{\infty} \frac{(sn+k+m_j-1)!}{k!} \left(\frac{\pi_i}{\pi_0+\pi_i+\pi_j}\right)^{k}} \\
&= \frac{\sum_{s=1}^{\infty} a_s \theta^s \frac{(sn+m_i+m_j-1)!}{(sn-1)!m_i!} \left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns} \left(\frac{\pi_i}{\pi_0+\pi_i+\pi_j}\right)^{m_i}}{\sum_{s=1}^{\infty} \frac{a_s}{(sn-1)!} \theta^s \left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns} \frac{(sn+m_j-1)!}{\left[1-\left(\frac{\pi_i}{\pi_0+\pi_i+\pi_j}\right)\right]^{sn+m_j}}} \\
&= \frac{1}{m_i!} \left(\frac{\pi_i}{\pi_0+\pi_i+\pi_j}\right)^{m_i} \left(\frac{\pi_0+\pi_j}{\pi_0+\pi_i+\pi_j}\right)^{m_j} \\
&\quad \cdot \frac{\sum_{s=1}^{\infty} a_s \theta^s \frac{(sn+m_i+m_j-1)!}{(sn-1)!} \left(\frac{\pi_0}{\pi_0+\pi_i+\pi_j}\right)^{ns}}{\sum_{s=1}^{\infty} \frac{a_s \theta^s (sn+m_j-1)!}{(sn-1)!} \left(\frac{\pi_0}{\pi_0+\pi_j}\right)^{ns}}.
\end{aligned}
$$

6.(c) and **6.(d)** are analogous to [12], who work in case $n = 1$ and $k = 2$.

9. For $i = 1, 2, \ldots, k,\ m \in \mathbb{N},$ we use 7., 8., the definitions about CPSNMn distribution and conditional probability, and obtain

$$
\begin{aligned}
P(X_i = s \,|\, X_1 + X_2 + \cdots + X_k = m) &= \frac{P(X_i = s, X_1 + X_2 + \cdots + X_k = m)}{P(X_1 + X_2 + \cdots + X_k = m)} \\
&= \frac{P(X_i = s, X_1 + X_2 + \cdots + X_k - X_i = m - s)}{P(X_1 + X_2 + \cdots + X_k = m)} \\
&= \frac{\pi_i^s (1-\pi_0-\pi_i)^{m-s} \sum_{j=1}^{\infty} a_j \theta^j \binom{jn+m-1}{s, m-s, jn-1} \pi_0^{nj}}{(1-\pi_0)^m \sum_{j=1}^{\infty} a_j \theta^j \binom{jn+m-1}{m, jn-1} \pi_0^{nj}} \\
&= \frac{\pi_i^s (1-\pi_0-\pi_i)^{m-s} \sum_{j=1}^{\infty} a_j \theta^j \frac{(jn+m-1)!}{s!(m-s)!(jn-1)!} \pi_0^{nj}}{(1-\pi_0)^m \sum_{j=1}^{\infty} a_j \theta^j \frac{(jn+m-1)!}{m!(jn-1)!} \pi_0^{nj}} \\
&= \binom{m}{s} \left(\frac{\pi_i}{1-\pi_0}\right)^s \left(1 - \frac{\pi_i}{1-\pi_0}\right)^{m-s}, \qquad s = 0, 1, \ldots, m\,.
\end{aligned}
$$

We use the definition of Binomial distribution and complete the proof.

Note 2.1. The conclusion 1. in this theorem states that also in the univariate case the CPSMNn distribution is just a particular case of PSD with more complicated coefficients.

The next theorem presents this distribution as a mixture.

Theorem 2.2. *Suppose* $n \in N$, $\pi_i \in (0,1)$, $i = 1,2,...,k$, $\pi_0 := 1 - \pi_1 - \pi_2 - \cdots - \pi_k \in (0,1)$, $a_j \geq 0$, $j = 0,1,...$, $\theta \in \mathbb{R}$ *are such that* (2.1) *is satisfied and* $\vec{X} \sim CPSNMn\big(g_{\vec{a}}(x), \theta; n, \pi_1, \pi_2, ..., \pi_k\big)$. *Then there exists a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$, *a r.v.* $M \sim PSD(g_{\vec{a}}(x), \theta)$ *and a random vector* $\vec{Y} = (Y_1, Y_2, ..., Y_k)$ *defined on it, such that* $\vec{Y}|M = m \sim NMn(nm, \pi_1, \pi_2, ..., \pi_k)$, $m = 1,2,...$,

$$P\big(Y_1 = 0, Y_2 = 0, ..., Y_k = 0 \,|\, M = 0\big) = 1\,,$$

and $\vec{X} \stackrel{d}{=} \vec{Y}$. *Moreover*

1. *For* $|\pi_1 z_1 + \pi_2 z_2 + \cdots + \pi_k z_k| < 1$,

$$G_{X_1,X_2,...,X_k}(z_1, z_2, ..., z_k) = G_M\left[\left(\frac{\pi_0}{\pi_0 + \pi_1(1-z_1) + \pi_2(1-z_2) + \cdots + \pi_k(1-z_k)}\right)^n\right].$$

2. *For* $i = 1,2,...,k$,

$$\begin{aligned} EX_i &= n\,EM\,\frac{\pi_i}{\pi_0}\,, \qquad i = 1,2,...,k\,, \\ \operatorname{Var} X_i &= \operatorname{Var} M\, n^2\,\frac{\pi_i^2}{\pi_0^2} + EM\,n\,\frac{\pi_i(\pi_0 + \pi_i)}{\pi_0^2} \\ &= n\,\frac{\pi_i}{\pi_0}\,EM\left[\frac{\pi_i}{\pi_0}\,(n\,FI\,M\, + 1) + 1\right], \\ FI\,X_i &= 1 + \frac{\pi_i}{\pi_0}\,(n\,FI\,M + 1)\,. \end{aligned}$$

3. *For* $i \neq j = 1,2,...,k$,

$$\begin{aligned} \operatorname{cov}(X_i, X_j) &= n\,\frac{\pi_i \pi_j}{\pi_0^2}\,\{n\,FI\,M + 1\}\,EM, \\ \operatorname{cor}(X_i, X_j) &= \sqrt{\frac{(FI\,Y_i - 1)\,(FI\,Y_j - 1)}{FI\,Y_i\;FI\,Y_j}}\,. \end{aligned}$$

Note 2.2. Following analogous notations of Johnson *et al.* [7], the above two theorems state that CPSNMn distribution coincides with

$$I_{\{M>0\}}\,NMn(nM, \pi_1, \pi_2, ..., \pi_k)\bigwedge_M PSD\big(g_{\vec{a}}(x); \theta\big),$$

where $I_{M>0}$ is a Bernoulli r.v. or indicator of the event "$M > 0$".

The following representation motivates the name of CPSNMn distribution.

Theorem 2.3. *Suppose* $\pi_i \in (0,1)$, $i = 1,2,...,k$, $\pi_0 = 1 - \pi_1 - \cdots - \pi_k \in (0,1)$, $a_k \geq 0$, $k = 0,1,...$, *and* $\theta \in \mathbb{R}$ *are such that such that* (2.1) *is satisfied. Let* $M \sim PS(g_{\vec{a}}(x); \theta)$ *and* $(Y^{(1)}, ..., Y^{(k)}) \sim NMn(n; \pi_1, ..., \pi_k)$ *be independent. Denote by* $I_{\{M>0\}}$ *the Bernoulli r.v. that is an indicator of the event* $\{M > 0\}$ *and defined on the same probability space. Define a random vector* $(T_M^{(1)}, T_M^{(2)}, ..., T_M^{(k)})$ *by*

$$\text{(2.3)} \qquad T_M^{(j)} = I_{\{M>0\}} \sum_{i=1}^{M} Y_i^{(j)} = \begin{cases} \sum_{i=1}^{M} Y_i^{(j)} & \text{if } M > 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad j = 1,2,...,k\,.$$

Then

1. *For* $m \in \mathbb{N}$, $\big(T_M^{(1)}, T_M^{(2)}, ..., T_M^{(k)} \,|\, M = m\big) \sim NMn\big(nm; \pi_1, \pi_2, ..., \pi_k\big)$;
2. $\big(T_M^{(1)}, T_M^{(2)}, ..., T_M^{(k)}\big) \sim CPSNMn\big(g_{\vec{a}}(x), \theta; n, \pi_1, \pi_2, ..., \pi_k\big)$;
3. $\big(T_M^{(1)}, T_M^{(2)}, ..., T_M^{(k)} \,|\, M > 0\big) \sim CPSNMn\big(g_{\tilde{\vec{a}}}(x), \theta; n, \pi_1, \pi_2, ..., \pi_k\big)$, *where* $\tilde{a}_0 = 0$, $\tilde{a}_i = a_i$, $i = 1, 2, ...$

Sketch of the proof: We apply (1.4) and Theorem 2.2. 2. is analogous to [12], who work in case $n = 1$ and $k = 2$.

If we have no weights at coordinate planes we need to consider the following distribution.

Definition 2.2. Let $\pi_j \in (0,1)$, $j = 1, 2, ..., k$, $\pi_0 := 1 - \pi_1 - \pi_2 - \cdots - \pi_k \in (0,1)$, $a_s \geq 0$, $s = 0, 1, ...$, and $\theta \in \mathbb{R}$ be such that

$$g_{\vec{a}}(\theta) = \sum_{n=0}^{\infty} a_n \theta^n < \infty \,.$$

A random vector $\vec{X} = (X_1, X_2, ..., X_k)$ is called Compound Power series distributed with negative multinomial summands on $\mathbb{N}^k$ and with parameters $g_{\vec{a}}(x)$, θ; n, $\pi_1, ..., \pi_k$, if for $i = 1, 2, ..., k$, $m_i = 1, 2, ...$,

$$\begin{aligned} P\big(X_1 = m_1, X_2 = m_2, ..., X_k = m_k\big) &= \\ &= \frac{1}{\rho}\, \frac{\pi_1^{m_1}\pi_2^{m_2}\cdots\pi_k^{m_k}}{g_{\vec{a}}(\theta\pi_0^n)} \sum_{j=1}^{\infty} a_j\, \theta^j \binom{jn + m_1 + m_2 + \cdots + m_k - 1}{m_1, m_2, ..., m_k, jn - 1} \pi_0^{nj}, \\ \rho &= 1 - \sum_{m=1}^{k} (-1)^{m+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} \frac{g_{\vec{a}}\big[\theta\pi_0^n(\pi_0 + \pi_{i_1} + \cdots + \pi_{i_m})^{-n}\big]}{g_{\vec{a}}(\theta)} \,. \end{aligned} \tag{2.4}$$

Briefly $\vec{X} \sim CPSNMn_{\mathbb{N}^k}\big(g_{\vec{a}}(x), \theta; n, \pi_1, \pi_2, ..., \pi_k\big)$.

The relation between *CPSNMn* and $CPSNMn_{\mathbb{N}^k}$ distributions is given in the following theorem.

Theorem 2.4. *If* $\vec{X} \sim CPSNMn\big(g_{\vec{a}}(x), \theta; n, \pi_1, \pi_2, ..., \pi_k\big)$, *then*

$$\big(X_1, X_2, ..., X_k \,|\, X_1 \neq 0, X_2 \neq 0, ..., X_k \neq 0\big) \sim CPSNMn_{\mathbb{N}^k}\big(g_{\vec{a}}(x), \theta; n, \pi_1, \pi_2, ..., \pi_k\big)\,.$$

3. APPLICATIONS TO RISK THEORY

In [9] we obtained the approximations of Compound Poisson risk process mixed with Pareto r.v. and provide a brief summary of previous results about risk process approximations. In this section we provide risk process application of the *CPSNMn*. Here $k, n \in \mathbb{N}$, $p_M \in (0,1)$, $\pi_i \in (0,1)$, $i = 1, 2, ..., k$, $\pi_1 + \cdots + \pi_k < 1$ and $\pi_0 = 1 - \pi_1 - \pi_2 - \cdots - \pi_k$.

3.1. The counting process

Here we consider a discrete time counting process, satisfying the following conditions:

C1. The insurance company have no claims at moment $t = 0$.

C2. In any other moments of time $t = 1, 2, ...$ a group of claims can arrive with probability p_M independently of others. We denote the number of groups of claims, arrived in the insurance company over an interval $[0, t]$ by $M(t)$ and by $0 < T_{G,1} < T_{G,2} < \cdots$ the moments of arrivals of the corresponding group, i.e. $T_{G,k}$ is the occurrence time of the k-th group. By definition $M(0) = 0$.

C3. The claims can be of one of k mutually exclusive and totally exhaustive different types $A_1, A_2, ..., A_k$, e.g. claims of one individual having several pension insurances.

C4. In any of the time points $0 < T_{G,1} < T_{G,2} < \cdots$, we denote the number of claims of type $i = 1, 2, ..., k$, arrived in the insurance company by $Y_{i,j}$, $j = 1, 2, ...$. We assume that the random vectors $(Y_{1,j}, Y_{2,j}, ..., Y_{k,j})$, $j = 1, 2, ...$ are i.i.d. and

$$(Y_{1,j}, Y_{2,j}, ..., Y_{k,j}) \sim NMn(n, \pi_1, \pi_2, ..., \pi_k)\,.$$

Note 3.1. Conditions C1–C2 means that the counting process of the groups of claims up to time $t > 0$ is a Binomial process. In case when the claim sizes are discrete they are considered e.g. in [5, 19, 4, 22]. The number of groups arrived up to time t is $M(t) \sim Bi(t, p_M)$ and the intervals $T_{G,1}, T_{G,2} - T_{G,1}, T_{G,3} - T_{G,2}, ...$ between the groups arrivals are i.i.d. Geometrically distributed on $1, 2, ...$, with parameter p_M.

C4 means that it is possible to have zero reported losses of one or of all k-types of insurance claims within one group. In that case there is a group arrived, however, the number of participants in the group is zero. This can happen e.g. when there is a claim, but it is not accepted, or it is estimated by zero value by the insurer.

Let us denote the number of claims of type $i = 1, 2, ..., k$, arrived in the company in the interval $[0, t]$ by $N_{i,t}$. Conditions C1–C4 imply that $(N_1(0), N_2(0), ..., N_k(0)) = (0, 0, ..., 0)$ and, for all $t = 1, 2, ...,$

$$N_i(t) = I\{M(t) > 0\} \sum_{j=1}^{M(t)} Y_{i,j}\,, \qquad j = 1, 2, ..., k\,.$$

Therefore

$$\big(N_1(t), N_2(t), ..., N_k(t)\big) \sim CPSNMn\bigg((1+x)^t, \frac{p_M}{1-p_M}; n, \pi_1, \pi_2, ..., \pi_k\bigg)$$

and

$$P\big(N_1(t) + N_2(t) + \cdots + N_k(t) = 0\big) = \frac{(1-p_M)^t}{(1-p_M\,\pi_0^n)^t}\,.$$

3.2. The total claim amount process and its characteristics

Consider the total claim amount process defined as

$$S(t) = I_{\{N_1(t)>0\}} \sum_{j_1=1}^{N_1(t)} Z_{1,j_1} + I_{\{N_2(t)>0\}} \sum_{j_2=1}^{N_2(t)} Z_{2,j_2} + \cdots + I_{\{N_k(t)>0\}} \sum_{j_k=1}^{N_k(t)} Z_{k,j_k}\,, \tag{3.1}$$

$t = 1, 2, ...,$ satisfying C1–C4.

We impose the following conditions on the claim sizes:

C5. In any of the time points $0 < T_{G,1} < T_{G,2} < \cdots$, we denote the claim sizes of the claims of type $i = 1, 2, ..., k$ by $Z_{i,j}$, $j = 1, 2, ...$. We assume that the random vectors $(Z_{1,j}, Z_{2,j}, ..., Z_{k,j})$, $j = 1, 2, ...,$ are i.i.d. and the coordinates of this vector are also independent, with absolutely continuous c.d.fs. correspondingly F_i, $i = 1, 2, ..., k$, concentrated on $(0, \infty)$.

C6. The claim arrival times and the claim sizes are assumed to be independent.

Proposition 3.1. *Consider the total claim amount process defined in (3.1) and satisfying conditions C1–C6.*

1. *If* $\mathbb{E}Z_{i,j} = \mu_i < \infty$, $i = 1, 2, ..., k$, *then*

$$\mathbb{E}S(t) = \frac{n\,t\,p_M(1-p_M)}{\pi_0}\,(\mu_1\pi_1 + \mu_2\pi_2 + \cdots + \mu_k\pi_k)\,. \tag{3.2}$$

2. *If additionally* $\operatorname{Var} Z_{i,j} = \sigma_i^2 < \infty$, $i = 1, 2, ..., k$, *then*

$$\begin{aligned} \operatorname{Var} S(t) &= n\,t\,\frac{p_M}{\pi_0}\left\{\sum_{i=1}^{k}\pi_i\,(\sigma_i^2+\mu_i^2) + \frac{(1-p_M)\,n+1}{\pi_0}\left(\sum_{i=1}^{k}\mu_i\,\pi_i\right)^2\right\},\\ FI\,S(t) &= \frac{\sum_{i=1}^{k}\pi_i(\sigma_i^2+\mu_i^2) + \frac{(1-p_M)n+1}{\pi_0}\left(\sum_{i=1}^{k}\mu_i\,\pi_i\right)^2}{(1-p_M)\,(\mu_1\pi_1+\mu_2\pi_2+\cdots+\mu_k\pi_k)}\,. \end{aligned} \tag{3.3}$$

Proof: [**1.**] is a consequence of the double expectation formula.

[**2.**] Using the double expectation formula, the facts that $EM(t) = tp_M$, $\operatorname{Var} M(t) = tp_M(1-p_M)$ and Theorem 2.2 we obtain:

$$\begin{aligned} \operatorname{Var} S(t) &= \sum_{i=1}^{k} ntp_M\frac{\pi_i}{\pi_0}\sigma_i^2 + ntp_M\sum_{i=1}^{k}\left[(1-p_M)n\,\frac{\pi_i^2}{\pi_0^2} + \frac{\pi_i(\pi_0+\pi_i)}{\pi_0^2}\right]\mu_i^2\\ &\quad + 2\sum_{1\le i<j\le k}\mu_i\mu_j\operatorname{cov}(N_i(t),N_j(t)) = \end{aligned}$$

$$
\begin{aligned}
&= nt\frac{p_M}{\pi_0}\left\{\sum_{i=1}^{k}\pi_i\sigma_i^2+\sum_{i=1}^{k}\left[(1-p_M)n\,\frac{\pi_i^2}{\pi_0}+\frac{\pi_i(\pi_0+\pi_i)}{\pi_0}\right]\mu_i^2\right\}\\
&\quad+2\sum_{1\le i<j\le k}\mu_i\mu_j\operatorname{cov}(N_i(t),N_j(t))\\
&= nt\frac{p_M}{\pi_0}\left\{\sum_{i=1}^{k}\pi_i(\sigma_i^2+\mu_i^2)+\sum_{i=1}^{k}\left[(1-p_M)n\,\frac{\pi_i^2}{\pi_0}+\frac{\pi_i^2}{\pi_0}\right]\mu_i^2\right\}\\
&\quad+2\sum_{1\le i<j\le k}\mu_i\mu_j\operatorname{cov}(N_i(t),N_j(t))\\
&= nt\frac{p_M}{\pi_0}\left\{\sum_{i=1}^{k}\pi_i(\sigma_i^2+\mu_i^2)+\frac{(1-p_M)n+1}{\pi_0}\sum_{i=1}^{k}\mu_i^2\pi_i^2\right\}\\
&\quad+2\sum_{1\le i<j\le k}\mu_i\mu_j\operatorname{cov}(N_i(t),N_j(t))\\
&= nt\frac{p_M}{\pi_0}\left\{\sum_{i=1}^{k}\pi_i(\sigma_i^2+\mu_i^2)+\frac{(1-p_M)n+1}{\pi_0}\sum_{i=1}^{k}\mu_i^2\pi_i^2\right\}\\
&\quad+2ntp_M\frac{[n(1-p_M)+1]}{\pi_0^2}\sum_{1\le i<j\le k}\mu_i\mu_j\pi_i\pi_j\\
&= nt\frac{p_M}{\pi_0}\left\{\sum_{i=1}^{k}\pi_i(\sigma_i^2+\mu_i^2)+\frac{(1-p_M)n+1}{\pi_0}\left(\sum_{i=1}^{k}\mu_i\pi_i\right)^2\right\},
\end{aligned}
$$

$$
FI\,S(t) = \frac{\operatorname{Var}S(t)}{\mathbb{E}\,S(t)} = \frac{\sum_{i=1}^{k}\pi_i(\sigma_i^2+\mu_i^2)+\frac{(1-p_M)n+1}{\pi_0}\left(\sum_{i=1}^{k}\mu_i\pi_i\right)^2}{(1-p_M)(\mu_1\pi_1+\mu_2\pi_2+\cdots+\mu_k\pi_k)}\,. \qquad \square
$$

3.3. The risk process and probabilities of ruin

Consider the following discrete time risk process

$$
(3.4)\quad R_u(t) = u+ct-I_{\{N_1(t)>0\}}\sum_{j_1=1}^{N_1(t)}Z_{1,j_1}-I_{\{N_2(t)>0\}}\sum_{j_2=1}^{N_2(t)}Z_{2,j_2}-\cdots-I_{\{N_k(t)>0\}}\sum_{j_k=1}^{N_k(t)}Z_{k,j_k}\,,
$$

$t=0,1,...,$ satisfying C1–C6. If we consider the claims in a group as one claim, we can see that it is a particular case of the Binomial risk process. [2]

The r.v. that describes the time of ruin with an initial capital $u\ge 0$ is defined as

$$
\tau_u = \min\{t>0\colon R_u(t)<0\}.
$$

The probability of ruin with infinite time and initial capital $u\ge 0$ will be denoted by $\Psi(u)=P(\tau_u<\infty)$. The corresponding probability to survive is $\Phi(u)=1-\Psi(u)$. Finally, $\Psi(u,t)=P(\tau_u\le t)$ is for the probability of ruin with finite time $t=1,2,\dots$.

[2] See e.g. [5, 19, 4, 22].

If we assume that $\mathbb{E}Z_{i,j} = \mu_i < \infty$, $i = 1, 2, ..., k$, and in a long horizon, the expected risk reserve for unit time is positive:

$$\lim_{t\to\infty} \frac{\mathbb{E}R_u(t)}{t} > 0\,.$$

The last is equivalent to

$$c > \lim_{t\to\infty} \frac{\mathbb{E}S(t)}{t}\,,$$

$$c > \frac{np_M(1-p_M)}{\pi_0}\,(\pi_1\mu_1 + \pi_2\mu_2 + \cdots + \pi_k\mu_k)\,.$$

Note that this condition does not depend on u and it means the incomes at any $t = 1, 2, ...$ to be bigger that the mean expenditures at that time:

$$\frac{c\pi_0}{np_M(1-p_M)(\pi_1\mu_1 + \pi_2\mu_2 + \cdots + \pi_k\mu_k)} > 1\,.$$

Therefore, the safety loading ρ should be defined as usually as the proportion between the expected risk reserve at time t with zero initial capital, i.e. $\mathbb{E}R_0(t)$, and the expected total claim amount at same moment of time, for any fixed $t = 1, 2, ...$:

$$\rho = \frac{c\pi_0}{np_M(1-p_M)(\pi_1\mu_1 + \pi_2\mu_2 + \cdots + \pi_k\mu_k)} - 1\,.$$

Thus the above condition is equivalent to the safety loading condition $\rho > 0$. If this condition is not satisfied, the probability of ruin in infinite time would be 1, for any initial capital u.

The proof of the next theorem is analogous to the corresponding one in the Cramer–Lundberg model[3] and in particular to those of the Polya–Aepply risk model[4].

Theorem 3.1. *Consider the Risk process defined in (3.4) and satisfying conditions C1–C6. Given the Laplace transforms $l_{Z_{i,1}}(s) = \mathbb{E}e^{-sZ_{i,1}}$, of $Z_{i,1}$, $i = 1, 2, ..., k$, are finite in $-s$,*

1. *The Laplace transform of the risk process is*

$$\mathbb{E}e^{-sR_0(t)} = e^{-g(s)t}\,, \qquad t = 0, 1, 2, \ldots\,,$$

where

$$g(s) = sc - \log\left\{1 - p_M + p_M\left[\frac{\pi_0}{1 - [\pi_1 l_{Z_{1,1}}(-s) + \cdots + \pi_k l_{Z_{k,1}}(-s)]}\right]^n\right\}.$$

2. *The process $R_0^*(t) = e^{-sR_0(t)+g(s)t}$, $t \geq 0$, is an $A_{R_0(\leq t)} = \sigma\{R_0(s),\ s \leq t\}$-martingale.*

3. $$\Psi(u,t) \leq e^{-su} \sup_{y\in[0,t]} e^{-yg(s)}\,, \qquad t = 1, 2, \ldots$$

4. $$\Psi(u) \leq e^{-su} \sup_{y\geq 0} e^{-yg(s)}\,.$$

5. *If the Lundberg exponent ε exists, it is a strictly positive solution of the equation*

$$g(s) = 0. \tag{3.5}$$

In that case, $\Psi(u) \leq e^{-\varepsilon u}$.

[3] See e.g. [1] or [6], p. 10, 11.
[4] [23], Proposition 6.3.

Proof: [1.]

$$\begin{aligned}
\mathbb{E}e^{-sR_0(t)} &= \mathbb{E}e^{-s\{ct-I_{\{N_1(t)>0\}}\sum_{j_1=1}^{N_1(t)} Z_{1,j_1}-\dots-I_{\{N_k(t)>0\}}\sum_{j_k=1}^{N_k(t)} Z_{k,j_k}\}}\\
&= e^{-sct}G_{N_1(t),N_2(t),\dots,N_k(t)}(l_{Z_{1,1}}(-s),l_{Z_{2,1}}(-s),\dots,l_{Z_{k,1}}(-s))\\
&= e^{-sct}G_{M(t)}\left[\left(\frac{\pi_0}{1-(\pi_1 l_{Z_{1,1}}(-s)+\pi_2 l_{Z_{2,1}}(-s)+\dots+\pi_k l_{Z_{k,1}}(-s))}\right)^n\right]\\
&= e^{-sct}\left\{1-p_M+p_M\left(\frac{\pi_0}{1-(\pi_1 l_{Z_{1,1}}(-s)+\pi_2 l_{Z_{2,1}}(-s)+\dots+\pi_k l_{Z_{k,1}}(-s))}\right)^n\right\}^t\\
&= e^{-sct}e^{t\log\left\{1-p_M+p_M\left(\frac{\pi_0}{1-(\pi_1 l_{Z_{1,1}}(-s)+\pi_2 l_{Z_{2,1}}(-s)+\dots+\pi_k l_{Z_{k,1}}(-s))}\right)^n\right\}}\\
&= e^{-t\left\{sc-\log\left[1-p_M+p_M\left(\frac{\pi_0}{1-(\pi_1 l_{Z_{1,1}}(-s)+\pi_2 l_{Z_{2,1}}(-s)+\dots+\pi_k l_{Z_{k,1}}(-s))}\right)^n\right]\right\}} = e^{-tg(s)}\,.
\end{aligned}$$

[2.] Consider $t = 0, 1, 2, ...$ and $y \le t$. Then, because the process $\{S(t),\ t = 0, 1, 2, ...\}$ has independent and time homogeneous additive increments,

$$\begin{aligned}
\mathbb{E}(R_0^*(t)|A_{R_0(\le y)}) &= \mathbb{E}(e^{-sct+sS(t)+g(s)t}|A_{R_0(\le y)})\\
&= \mathbb{E}(e^{-scy+sS(y)+g(s)y-sc(t-y)+s(S(t)-S(y))+g(s)(t-y)}|A_{R_0(\le y)})\\
&= \mathbb{E}(R_0^*(y)e^{-sc(t-y)+s(S(t)-S(y))+g(s)(t-y)}|A_{R_0(\le y)})\\
&= R_0^*(y)\mathbb{E}(e^{-sc(t-y)+sS(t-y)+g(s)(t-y)})\\
&= R_0^*(y)\mathbb{E}(e^{-sR_0(t-y)+g(s)(t-y)})\\
&= R_0^*(y)\mathbb{E}(e^{-sR_0(t-y)})e^{g(s)(t-y)}\\
&= R_0^*(y)e^{-g(s)(t-y)}e^{g(s)(t-y)} = R_0^*(y)\,.
\end{aligned}$$

[3.] Following the traditional approach we start with the definition of R_0^* and use that for $R_0^*(0) = 1$. Because τ_u is a random stopping time, by Doob's martingale stopping theorem, the stopped process $R_0^*(\min(\tau_u, t))$, is again a martingale. Therefore, for any $0 \le t < \infty$, by the double expectations formula,

$$\begin{aligned}
1 &= R_0^*(0) = \mathbb{E}R_0^*(0) = \mathbb{E}R_0^*(\min(\tau_u,t))\\
&= \mathbb{E}(R_0^*(\min(\tau_u,t))|\tau_u \le t)P(\tau_u \le t) + \mathbb{E}(R_0^*(\min(\tau_u,t))|\tau_u > t)P(\tau_u > t)\\
&\ge \mathbb{E}(R_0^*(\min(\tau_u,t))|\tau_u \le t)P(\tau_u \le t)\\
&= \mathbb{E}(e^{-sR_0(\min(\tau_u,t))+g(s)\min(\tau_u,t)}|\tau_u \le t)P(\tau_u \le t)\\
&= e^{su}\mathbb{E}(e^{g(s)\min(\tau_u,t)}|\tau_u \le t)P(\tau_u \le t)\\
&\ge e^{su}\mathbb{E}(e^{g(s)\tau_u}|\tau_u \le t)P(\tau_u \le t)\\
&= e^{su}\mathbb{E}(e^{g(s)\tau_u}|\tau_u \le t)\Psi(u,t) \ge e^{su}\inf_{y\in[0,t]} e^{g(s)y}\Psi(u,t)\,.
\end{aligned}$$

[4.] This is an immediate consequence of 3., when $t \to \infty$.

[5.] This is an immediate consequence of the inequality

$$1 \ \ge\ \mathbb{E}\big(e^{-\varepsilon R_0(\min(\tau_u,t))+g(\varepsilon)\min(\tau_u,t)}\,|\,\tau_u \le t\big)\,P(\tau_u \le t)\,,$$

applied for $t \to \infty$ and the fact that $R_0(s) = R_u(s) - u$. □

Remark 3.1. In general, to compute solution of the equation (3.5) is a difficult task and it can be done only numerically, since e.g. for exponential claims it involves roots of algebraic equations of high order. These solutions can be however also negative or/and complex conjugates. To illustrate the complexity of this setup, let us consider $k = 1$ and respective equation (for special choice of parameters) $s = \log\big(1 - 0.5 + 0.5\big((1-p)/(1-p/(1-s))\big)^n\big)$. Then, real solutions are plotted at Figure 1 for $n = 1, ..., 10$, $p \in (0, 1)$, therein we can see the complexity of such computations.

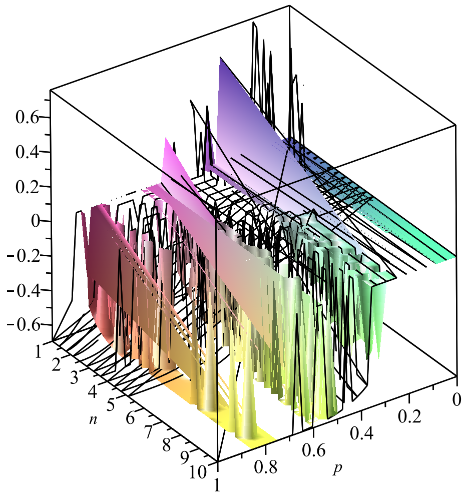

Figure 1: Real solutions of equation (3.5) for $\pi_M = \pi_0 = \pi_1 = 0.5$.

Theorem 3.2. *Consider the Risk process defined in (3.4) and satisfying conditions C1–C6. Suppose that it satisfies the net profit condition. Denote by*

$$\sigma_S = \sqrt{\operatorname{Var} S(1)} = \sqrt{n\frac{p_M}{\pi_0}\left\{\sum_{i=1}^{k}\pi_i(\sigma_i^2+\mu_i^2) + \frac{(1-p_M)n+1}{\pi_0}\left(\sum_{i=1}^{k}\mu_i\pi_i\right)^2\right\}}.$$

Define

$$R_m(t) = \frac{u_m + cmt - S(mt)}{\sigma_S\sqrt{m}},$$

where

$$\frac{u_m}{\sigma_S} \sim u_0\sqrt{m}, \qquad m \to \infty,$$

and

$$\frac{\rho_m\mu_S}{\sigma_S} \sim \frac{\rho_0}{\sqrt{m}}, \qquad m \to \infty,$$

$$\mu_S = \mathbb{E}S(1) = \frac{np_M(1-p_M)}{\pi_0}(\mu_1\pi_1 + \mu_2\pi_2 + \cdots + \mu_k\pi_k),$$

then

$$R_m(t) \Longrightarrow u_0 + \rho_0 t + W(p_M t), \qquad m \to \infty.$$

3.4. Simulations of the risk processes and estimation of the probabilities of ruin

In this subsection we provide a brief simulation study on probabilities of ruin in a finite time in the model (3.4). For any of them 10 000 sample paths were created and the relative frequencies of those which goes at least once below zero was determined. The number of groups is $k = 20$. The parameters of the NMn distribution are $n = 40$ and $p = (0.002, 0.004, 0.006, 0.008, 0.01, 0.012, 0.014, 0.016, 0.018, 0.02, 0.022, 0.024, 0.026, 0.028, 0.03, 0.032, 0.034, 0.036, 0.038, 0.04)$. Different parameters on different coordinates allow higher flexibility of the model. The probability of arrival of a group in a fixed time point is $p_M = 0.4$, and premium income rate is $c = 0.1$.

Example 3.1. Exponential claim sizes. For computations of ruin probabilities under exponential claims we consider parameter vector $\lambda = (10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200)$. The i-th coordinate describe the parameter of the Exponential distribution of the claim sizes within the i-th group. The resulting probabilities for ruin for different initial capitals u and time intervals $[0, t]$ are presented in the Table 1. The corresponding 10 000 sample paths of the risk process are depicted on Figure 2.

Table 1: Probabilities of ruin for exponential claims.

u	t					
	2	5	10	20	50	100
0	0.3224	0.5581	0.7037	0.8053	0.9016	0.9515
1	0.0001	0.0036	0.0370	0.1546	0.4234	0.6586
2	0	0	0.0005	0.0072	0.1131	0.3386
3	0	0	0	0.0001	0.0159	0.1366
4	0	0	0	0	0.0012	0.0364
5	0	0	0	0	0	0.0081

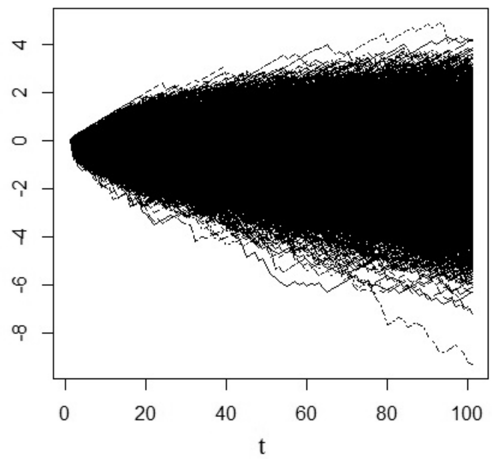

Figure 2: 10 000 sample paths of the risk process (3.4) for Exponential individual claim sizes, $t = 100$.

Example 3.2. Gamma claim sizes. Table 2 presents the probabilities for ruin in case when the claim sizes are Gamma distributed with parameters $\alpha = seq\,(from = 0.001,$ $to = 0.001 + (k-1) * 0.005,\ by = 0.005)$ and $\beta = seq\,(from = 1,\ to = 1 + (k-1) * 0.2,\ by = 0.2)$, where *seq* is the function for creating a sequence in R software, see [16]. The corresponding 10 000 sample paths of the risk process are depicted on Figure 3.

Table 2: Probabilities of ruin for gamma claims.

u	t					
	2	5	10	20	50	100
0	0.164	0.294	0.442	0.529	0.706	0.787
1	0.042	0.085	0.183	0.273	0.490	0.578
2	0.017	0.046	0.086	0.160	0.317	0.458
3	0.010	0.024	0.051	0.098	0.202	0.358
4	0.003	0.008	0.019	0.057	0.139	0.248
5	0.000	0.002	0.013	0.027	0.071	0.171

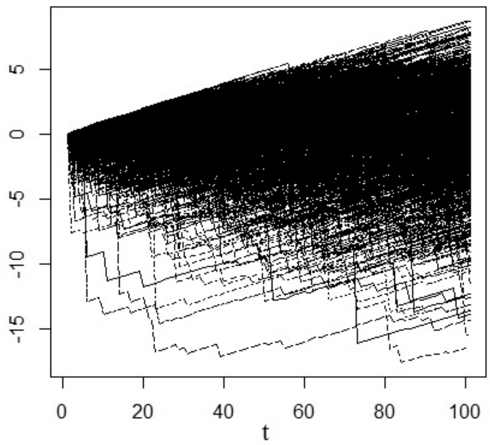

Figure 3: 10 000 sample paths of the risk process (3.4) for Gamma individual claim sizes, $t = 100$.

Example 3.3. Uniform claim sizes. The ruin probabilities presented in the Table 3 are calculated under assumption for uniform claim sizes with left and right bounds of the intervals, presented correspondingly via parameter vectors $Umin = seq\,(from = 0.0001,\ to =$ $0.0001 + (k-1) * 0.0001,\ by = 0.0001)$ and $Umax = Umin + 0.01$.

The corresponding 10 000 sample paths of the risk process are depicted on Figure 4.

Analogously, the probabilities for ruin in a finite time interval, for different claim sizes with finite variance, and related with the risk process (3.4) can be estimated. The corresponding confidence intervals can be calculated using the Central Limit Theorem, applied to relative frequencies.

Table 3: Probabilities of ruin for uniform claims.

u	t					
	2	5	10	20	50	100
0	0.133	0.253	0.361	0.402	0.392	0.412
1	0.000	0.000	0.000	0.001	0.001	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000

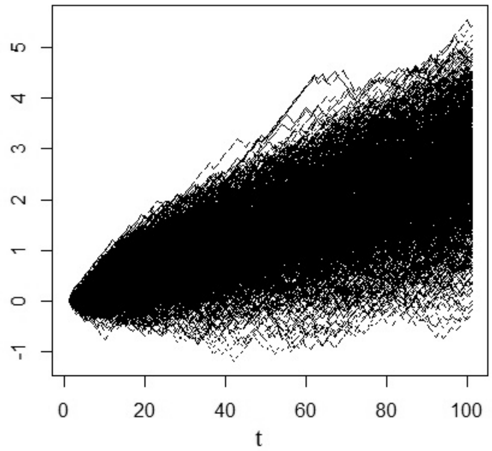

Figure 4: 10 000 sample paths of the risk process (3.4) for Uniform individual claim sizes, $t = 100$.

4. CONCLUSIONS

The paper shows that CPSMNn distribution is easy to work with, and it be can be very useful for modelling of the number of claims in Risk theory. Recently [26] and [2] have published another important application of multivariate negative binomial distribution in actuarial risk theory. Both models show that they are suitable for capturing the overdispersion phenomena. These distributions provide a flexible modelling of the number of claims that have appeared up to time t. The number of summands of the random sum reflects the number of groups of claims that have occurred up to this moment. The negative multinomial summands and their dependence structure describe types of claims within a group which are different from those given by [26] and [2]. From mathematical point of view our paper describes completely novel presentations of the CPSNMn distributions. Thus we can conclude by following conclusions:

- These distributions are a particular case of Multivariate PSD.
- Considered as a mixture, CPSMNn would be called (possibly Zero-inflated) Mixed NMn with scale changed PSD first parameter. More precisely,

$$I_{\{M>0\}} NMn\big(nM, \pi_1, \pi_2, ..., \pi_k\big) \bigwedge_M PSD\big(g_{\vec{a}}(x); \theta\big)\,,$$

  where $I_{M>0}$ is a Bernoulli r.v. or indicator of the event "$M > 0$".

- CPSMNn is particular case of compounds or random sums $\left(T_M^{(1)}, T_M^{(2)}, ..., T_M^{(k)}\right)$, where

$$T_M^{(j)} = I_{\{M>0\}} \sum_{i=1}^{M} Y_i^{(j)} = \begin{cases} \sum_{i=1}^{M} Y_i^{(j)} & \text{if } M > 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad j = 1, 2, ..., k\,.$$

These observations allow us to make the first complete characterization of Compound power series distribution with negative multinomial summands and to give an example of their application in modelling the main process in the Insurance risk theory.

ACKNOWLEDGMENTS

The work was supported by project Fondecyt Proyecto Regular No. 1151441, Project LIT-2016-1-SEE-023, and by the bilateral projects Bulgaria – Austria, 2016–2019, "Feasible statistical modeling for extremes in ecology and finance", BNSF, Contract number 01/8, 23/08/2017 and WTZ Project No. BG 09/2017, https://pavlinakj.wordpress.com/.

The authors are very grateful to the Editor, Associated Editor and the Reviewers for their valuable comments.

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