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THE BETA MARSHALL–OLKIN LOMAX DISTRIBUTION

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Abstract:

- Compounding distributions is the most common method in lifetime analysis to obtain more flexible families of distributions. Based on the beta Marshall–Olkin generated family, we present a new four-parameter distribution, so-called the beta Marshall–Olkin Lomax, for lifetime applications. We obtain some of its properties from those of well-established distributions. We provide a simulation study to illustrate the performance of the maximum likelihood estimates. An application to uncensored data is carried out and we use some goodness-of-fit statistics to study the flexibility of the new distribution, proving empirically that this model can be appropriate for lifetime applications.

Key-Words:

- *beta-G family; generalized distributions; lifetime analysis; Lomax distribution; Marshall–Olkin extended family.*

AMS Subject Classification:

- 62E15, 62N02, 62N99, 65C05.

1. INTRODUCTION

In applications involving lifetime models such as survival analysis, demography, reliability, actuarial study and others, the distributions with positive real supports play a fundamental role. Because of this, in recent years, there is a growing interest in constructing new distributions to model aging phenomena [15, 14]. The method that has received most attention by researchers to generate new models is that one by compounding existing distributions, usually referred to as generalized G families of distributions [28]. The principal reason for this is the ability of these generalized distributions to be more flexible than the baseline G distribution to provide better fits to skewed data and good control of the tails [23]. The second reason is the powerful computational and analytical facilities available in several software packages, which facilitate handling and computing complex mathematical expressions. Some of the generalized G families best known are: the Marshall–Olkin extended (MOE) family [18], the exponentiated-generated (exp-G) family [13, 8], the beta-generated (beta-G) family [9], the Kumaraswamy-generated (Kw-G) family [7], the gamma-generated (gamma-G) families [29, 25, 22] and the McDonald-generated (Mc-G) family [2]. A detailed compilation of these families is given in [28].

In this paper, we adopt the beta Marshall–Olkin generated (BMO-G) family proposed by Alizadeh *et al.* [3] to define the new *beta Marshall–Olkin Lomax* (BMOL) distribution obtained by taking the Lomax distribution [17] as the baseline G model. Given that the proposed distribution has positive real support, our objective is to define a wide flexible distribution for real lifetime applications.

The paper unfolds as follows. In Section 2, we describe some preliminaries and introduce the BMOL distribution. In Section 3, we plot its density and hazard rate functions for some parameter values. In Section 4, we obtain an expansion for the BMOL density function as a linear combination of exp-Lomax and Lomax densities. In Sections 5–10, we present explicit expressions for the quantile function (qf), moments, generating function, mean deviations, Bonferroni and Lorenz curves, Shannon entropy and order statistics. Section 11 is devoted to the maximum likelihood estimates (MLEs) for complete samples and, in Section 12, we carry out a simulation study to study the performance of these estimates. In Section 13, we consider an application of the BMOL distribution and compare it with others related distributions and with the exponentiated Weibull (EW) distribution [20] based on some goodness-of-fit statistics. Finally, Section 14 concludes the paper.

2. THE NEW DISTRIBUTION

Marshall & Olkin [18] pioneered a method of introducing an additional parameter to a distribution. If $G(x;\boldsymbol{\xi})$ is a baseline distribution with parameter vector $\boldsymbol{\xi}$, then the cumulative distribution function (cdf) given by

$$F(x;c,\boldsymbol{\xi}) = \frac{G(x;\boldsymbol{\xi})}{c+(1-c)G(x;\boldsymbol{\xi})}, \quad c>0, \tag{2.1}$$

defines a new distribution with an extra shape parameter c. As commented by Marshall & Olkin [18], "By various methods, new parameters can be introduced to expand families of distributions for added flexibility or to construct covariate models".

The cdf of the beta-G family (for $a,b>0$) is defined by

$$F(x;a,b,\boldsymbol{\xi}) = \frac{B(G(x;\boldsymbol{\xi});a,b)}{B(a,b)} = \frac{1}{B(a,b)}\int_0^{G(x;\boldsymbol{\xi})} w^{a-1}(1-w)^{b-1}\,dw, \tag{2.2}$$

where $B(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}\,dw$ is the beta function and $B(z;a,b)=\int_0^z w^{a-1}(1-w)^{b-1}\,dw$ is the incomplete beta function. In this case, the generated distribution $F(x;a,b,\boldsymbol{\xi})$ has two extra shape parameters a and b. The beta G family was introduced by Eugene *et al.* [9], who studied the properties of the beta-normal distribution. If the baseline $G(x;\boldsymbol{\xi})$ in (2.2) is the Lomax distribution, we obtain the beta-Lomax (BL) distribution as defined in [24].

A generalization of these concepts, introduced in [4], follows by considering the $T-X$ method. Let $R(x;\boldsymbol{\gamma})$ be a cdf with support $[d,e]$ and density $r(x;\boldsymbol{\gamma})$. For a given baseline distribution $G(x;\boldsymbol{\xi})$, let $W(\cdot)$ be a function satisfying the following properties

$$\begin{cases} W[G(x;\boldsymbol{\xi})] \in [d,e], \\ W[G(x;\boldsymbol{\xi})] \text{ is differentiable and monotonically non-decreasing}, \\ \lim_{x\to-\infty} W[G(x;\boldsymbol{\xi})] = d, \;\; \lim_{x\to\infty} W[G(x;\boldsymbol{\xi})] = e. \end{cases}$$

Then, the cdf

$$F(x;\boldsymbol{\delta},\boldsymbol{\gamma},\boldsymbol{\xi}) = \int_d^{W[G(x;\boldsymbol{\xi})]} r(t;\boldsymbol{\gamma})\,dt \tag{2.3}$$

defines a new distribution, where the link function $W(\cdot)=W(\cdot;\boldsymbol{\delta})$ possibly depends on a parameter vector $\boldsymbol{\delta}$. We say that the distribution $R(x;\boldsymbol{\gamma})$ is 'transformed' by the 'transformer' $W[G(x;\boldsymbol{\xi})]$.

Following this idea, Alizadeh *et al.* [3] introduced the BMO-G family by considering in (2.3) the function $W(z)=z/[c+(1-c)z]$, $c>0$, and the beta distribution as the 'transformed' distribution $R(x;\boldsymbol{\gamma})$. Notice that, in this case, the 'transformer' $W[G(x;\boldsymbol{\xi})]$ is given by (2.1).

In this paper, we study the BMOL distribution by considering the baseline $G(x;\boldsymbol{\xi})$ in (2.3) as the Lomax distribution [17], which has cdf given by

$$G(x;\alpha,\lambda) = 1 - \Big(1+\frac{x}{\lambda}\Big)^{-\alpha}, \quad x \geq 0,\ \alpha > 0,\ \lambda > 0 \tag{2.4}$$

and probability density function (pdf)

$$g(x;\alpha,\lambda) = \frac{\alpha}{\lambda}\Big(1+\frac{x}{\lambda}\Big)^{-(\alpha+1)}. \tag{2.5}$$

For the sake of simplicity, we will write sometimes the Lomax distribution with cdf $G(x)$ and pdf $g(x)$, respectively, without explicit mention to the parameters α and λ.

It is clear that a generalized G distribution has more parameters than the baseline G distribution. Generally, the use of four parameters should be sufficient for most practical purposes. In addition, notice that if $X \sim \text{Lomax}(\alpha,\lambda)$, then $X/\lambda \sim \text{Lomax}(\alpha,1)$ and, consequently, λ is just a scale parameter. Henceforth, we consider the BMOL distribution with only four parameters by taking, without loss of generality, $\lambda = 1$ in equations (2.4) and (2.5). Thus, if $\boldsymbol{\theta} = (a,b,c,\alpha)^\top$ is the parameter vector, we define the BMOL cdf by

$$F(x;\boldsymbol{\theta}) = \frac{B(W[G(x)];a,b)}{B(a,b)} = \frac{1}{B(a,b)}\int_0^{W[G(x)]} w^{a-1}(1-w)^{b-1}\,dw, \tag{2.6}$$

where $W[G(x)]$ is given by (2.1). From equations (2.1) and (2.4) (with $\lambda = 1$), we have

$$W[G(x)] = \frac{(1+x)^\alpha - 1}{(1+x)^\alpha + c - 1}. \tag{2.7}$$

The BMOL pdf follows from (2.6) as

$$f(x;\boldsymbol{\theta}) = \frac{1}{B(a,b)}\, g(x)\, w[G(x)]\, \{W[G(x)]\}^{a-1}\, \{1 - W[G(x)]\}^{b-1}, \tag{2.8}$$

where $w(z) = W'(z) = c/\,[c + (1-c)z]^2$. Thus, we obtain the BMOL pdf from (2.4), (2.7) and (2.8) as

$$f(x;\boldsymbol{\theta}) = \frac{\alpha\, c^b\, (1+x)^{-b\alpha-1}\big[1-(1+x)^{-\alpha}\big]^{a-1}}{\big\{c + (1-c)\,\big[1-(1+x)^{-\alpha}\big]\big\}^{a+b}\, B(a,b)}. \tag{2.9}$$

Hereafter, a random variable X with density function (2.9) will be denoted by $X \sim \text{BMOL}(a,b,c,\alpha)$.

In lifetime analysis, a very useful function is the hazard rate function (hrf) $h(x)$. Therefore, the hrf of $X \sim \mathrm{BMOL}(a,b,c,\alpha)$ is given by

$$h(x) = \frac{\alpha\, c^b\, (1+x)^{-b\alpha-1}\left[1-(1+x)^{-\alpha}\right]^{a-1}}{\left\{c+(1-c)\left[1-(1+x)^{-\alpha}\right]\right\}^{a+b}\left[B(a,b)-B(W[G(x)],a,b)\right]}. \tag{2.10}$$

A random variable X with pdf (2.9) is easily simulated as follows:
if $U \sim \mathrm{Beta}(a,b)$, then

$$X = \left[\left(\frac{1-(1-c)\,U}{1-U}\right)^{1/\alpha} - 1\right] \sim \mathrm{BMOL}(a,b,c,\alpha).$$

For specific values of the parameters a, b and c, some known sub-models of the BMOL distribution are given in Table 1.

Table 1: Some BMOL sub-models. MOEL: Marshall–Olkin extended Lomax, Kw-GL: Kumaraswamy-Generalized Lomax, BL: beta Lomax.

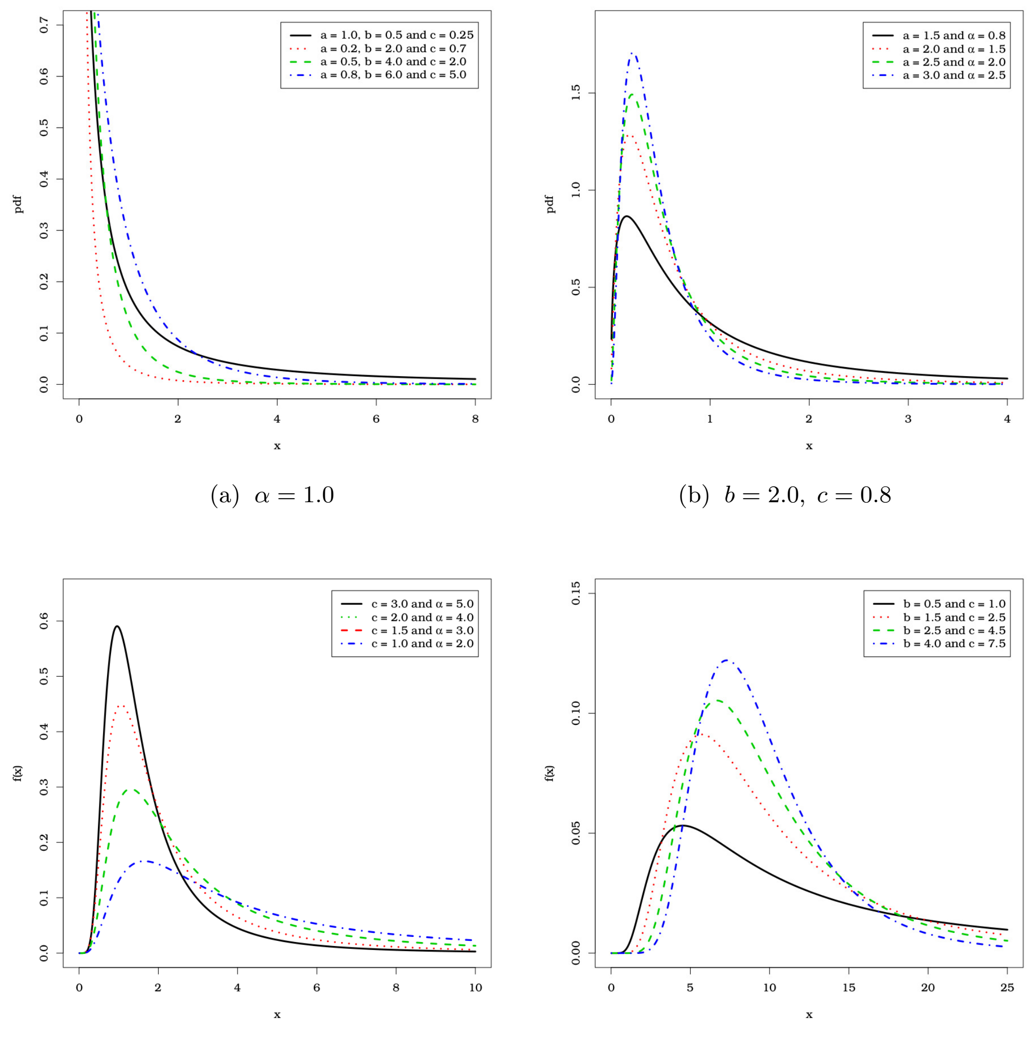
a	b	c	Model	Reference
1	1	1	$\mathrm{Lomax}(\alpha,1)$	[17]
1	1	—	$\mathrm{MOEL}(c,1,\alpha)$	[10]
1	—	1	$\mathrm{Kw{-}GL}(1,b,\alpha,1)$	[27]
—	—	1	$\mathrm{BL}(a,b,\alpha,1)$ (with $\mu=0$)	[24]

3. SHAPES OF THE DENSITY AND HAZARD RATE FUNCTIONS

The shapes of the pdf (2.9) can be described analytically by examining the roots of the equation $f'(x)=0$ and analyzing its limits in (2.9) when $x \to 0$ or $x \to \infty$. Clearly, since $f(x) \geq 0$ is integrable, then $\lim_{x\to\infty} f(x) = 0$. The behavior of $f(x)$ when $x \to 0$ is governed by the parameter a, which is inherited from the properties of the beta distribution. For $a \leq 1$, we have that $f(x)$ is convex and strictly decreasing. For $a=1$, $\lim_{x\to 0} f(x) = b\,\alpha/c$ and, for $a<1$, $\lim_{x\to 0} f(x) = \infty$. For $a>1$, $f(0)=0$ and it is unimodal with mode at

$$x_0 = -1 + \left\{\frac{A_{a,b,c,\alpha} + \left[A^2_{a,b,c,\alpha} - 4\,(c-1)\,(\alpha-1)\,(b\,\alpha+1)\right]^{1/2}}{2\,(b\,\alpha+1)}\right\}^{1/\alpha},$$

where $A_{a,b,c,\alpha} = 2 - c - \alpha + b\,\alpha + a\,c\,\alpha$. All parameters allow extensive control on the right tail, providing, when $a > 1$, more light or heavy tails, according to the parameters decrease or increase, respectively, and conversely when $a \leq 1$. Some plots in Figure 1 display possible shapes of the pdf for selected parameter values. These plots confirm the above analysis.



(a) $\alpha = 1.0$

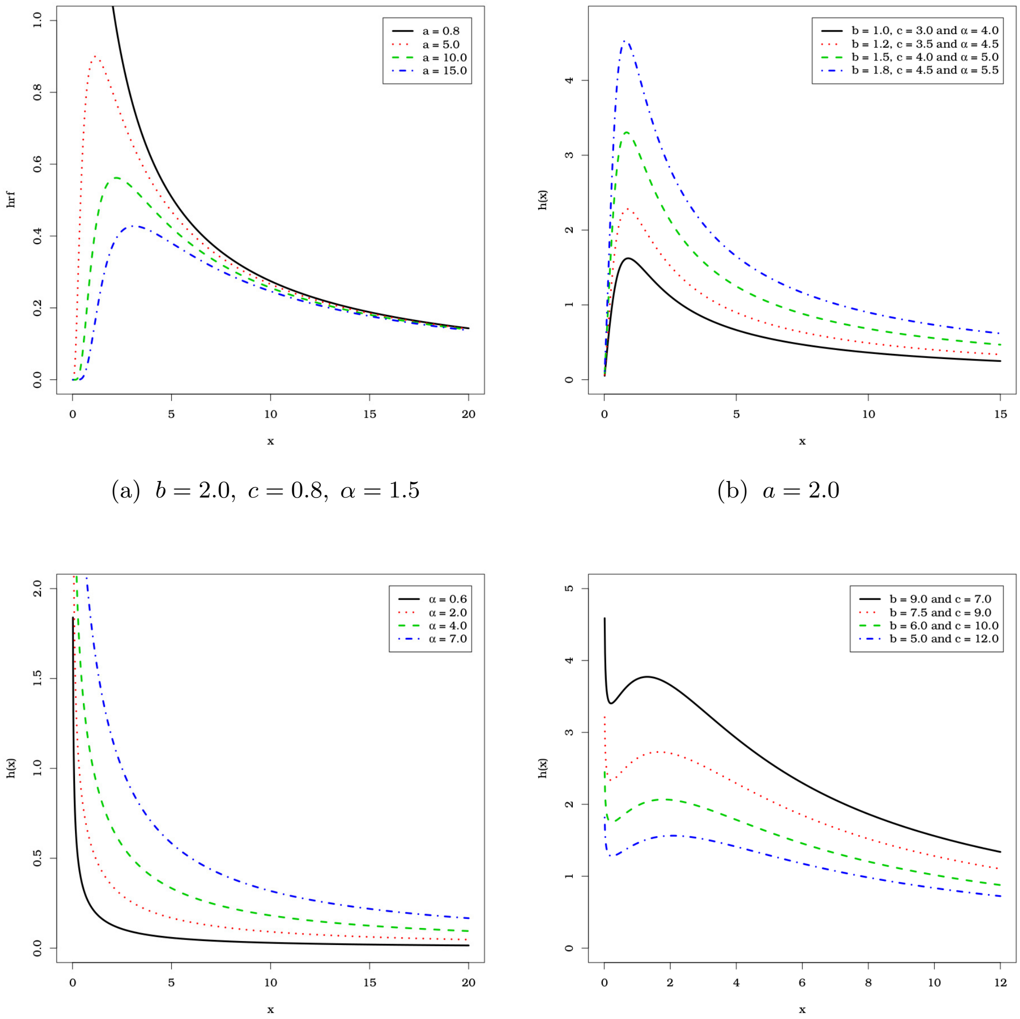
(b) $b = 2.0,\ c = 0.8$

(c) $a = 7.0,\ b = 0.5$

(d) $a = 15.0,\ \alpha = 1.5$

Figure 1: Plots of the pdf (2.9) for selected parameters.

The corresponding hrf can have the classical shapes such as decreasing or unimodal, as shown in Figure 2. Therefore, the new distribution can be appropriate for different applications in lifetime analysis.



(a) $b = 2.0,\ c = 0.8,\ \alpha = 1.5$ (b) $a = 2.0$

(c) $a = 0.5,\ b = 0.5,\ c = 1.0$ (d) $a = 0.8,\ \alpha = 2.0$

Figure 2: Plots of the hrf (2.10) for selected parameters.

4. USEFUL REPRESENTATION

Using the generalized binomial expansion, Alizadeh *et al.* [3] reveal that the cdf (2.6) admits the following power series

$$F(x) = \sum_{k=0}^{\infty} s_k\, G^k(x), \tag{4.1}$$

where $G(x)$ is the baseline cdf (2.4) (with $\lambda = 1$) and, for $k \geq 0$,

$$s_k = \sum_{i,j=0}^{\infty} \sum_{l=k}^{\infty} \frac{(-1)^{i+l+k}(1-c)^i \binom{b-1}{i} \binom{-a-i}{j} \binom{a+i+j}{l} \binom{l}{k}}{c^{a+i+j}\,(a+i)\,B(a,b)}. \tag{4.2}$$

We note that (4.2) is valid only for $c > 1$, it does not converge for $c < 1$ and it is not applicable for $c = 1$. Differentiating (4.1) term by term, we obtain

$$f(x) = \sum_{k=0}^{\infty} s_{k+1}\, h_{k+1}(x), \tag{4.3}$$

where $h_{k+1}(x) = (k+1)\, g(x)\, G^k(x)$ denotes the exp-G density function with power parameter $k+1$. Therefore, from (4.3), several properties of the new model can be derived from those exp-G properties [13].

It is possible to go a step further in (4.1). Using the binomial expansion in (4.1) gives

$$F(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j}\, s_k\, (1+x)^{-j\alpha}.$$

By exchanging the indices j and k in the sums, we can write

$$F(x) = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} (-1)^j \binom{k}{j}\, s_k\, (1+x)^{-j\alpha}. \tag{4.4}$$

Finally, differentiating (4.4) term by term, we obtain

$$f(x) = \sum_{j=0}^{\infty} p_j\, g(x; (j+1)\,\alpha, 1), \tag{4.5}$$

where $g(x; (j+1)\,\alpha, 1)$ is given in (2.5) and, for $j = 0, 1, ...,$

$$p_j = \sum_{k=j+1}^{\infty} (-1)^j \binom{k}{j+1}\, s_k. \tag{4.6}$$

From equation (4.5), we note that $f(x)$ is given by a linear combination of Lomax densities. Therefore, several properties of the BMOL distribution can be obtained from those of the Lomax distribution [17].

5. QUANTILE FUNCTION

Let $Q_{a,b}(z)$ denote the qf of the beta distribution with parameters a and b. Then, the qf of the BMOL distribution is given by

$$Q(z) = \left[\frac{1-(1-c)\,Q_{a,b}(z)}{1-Q_{a,b}(z)}\right]^{1/\alpha} - 1. \tag{5.1}$$

An expansion up to third order about $z = 0$ for the beta qf is given by

$$Q_{a,b}(z) = \sum_{i=1}^{3} q_i \, z^{i/a} + \mathcal{O}(z^{4/a}),$$

where $q_i = d_i \, [a\, B(a,b)]^{i/a}$, $i = 1, 2, 3$, with $d_1 = 1$,

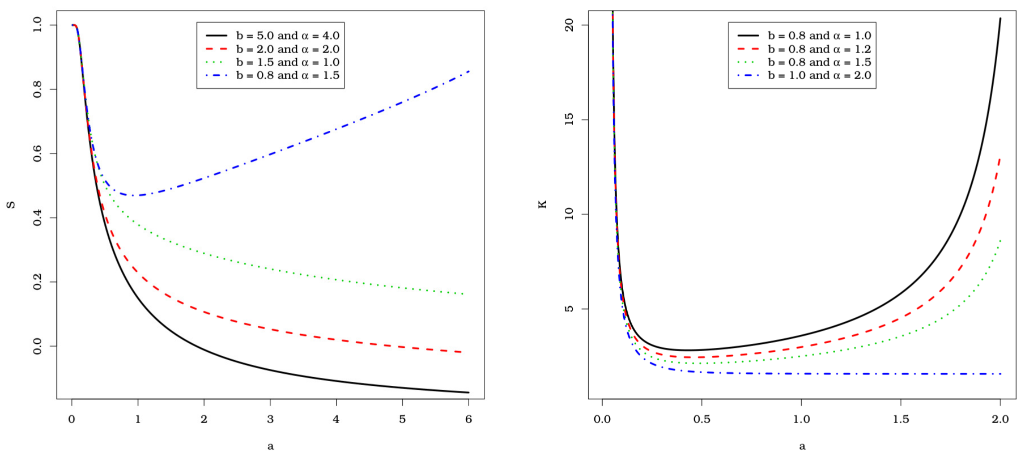
$$d_2 = \frac{b-1}{a+1}, \qquad d_3 = \frac{(b-1)\,(a^2 + 3ab - a + 5b - 4)}{2(a+1)^2(a+2)}.$$

The skewness and kurtosis measures are determined by $\alpha_3 = \mu_3/\sigma^3$ and $\alpha_4 = \mu_4/\sigma^4$, respectively, where μ_j is the j-th central moment and σ is the standard deviation. For some generalized distributions obtained by the T–X method defined by (2.3), as noted by Alzaatreh *et al.* [4], it could be difficult to determine the third and fourth moments. Alternative measures for the skewness and kurtosis based on the qf are sometimes more appropriate. The measures of skewness S of Bowley [12] and kurtosis K of Moors [19] are defined by

$$S = \frac{Q(6/8) + Q(2/8) - 2Q(4/8)}{Q(6/8) - Q(2/8)}, \tag{5.2}$$

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{5.3}$$

These measures are more robust and they exist even for distributions without moments.



(a) Skewness (b) Kurtosis

Figure 3: Plots of S of Bowley skewness (5.2) and of K of Moors kurtosis (5.3) measures for selected parameters ($c = 2.0$).

The plots in Figure 3 display the skewness (5.2) and kurtosis (5.3) as functions of the parameter a for some values of the parameters b, c and α. Note that, as pointed in Section 3, the BMOL pdf does not have mode when $a \leq 1$, which implies a greater skewness for these values of the parameter a, as illustrated in Figure 3(a). Similarly, note that the skewness increases when $b > 1$, obtaining negative values when $b, \alpha > 2$. In addition, note that the kurtosis decreases when the values of the parameters b and α increase, as illustrated in Figures 3(b), 1(c) and 1(d).

6. MOMENTS

The moments of X with cdf given by (2.6) can be expressed from the (r, k)-th probability weighted moment (PWM) of a random variable Y with baseline cdf $G(x)$ and pdf $g(x)$, which is defined, for $r, k = 0, 1, ...$, by

$$\omega_{r,k} = \mathbb{E}\Big[Y^r\, G^k(Y)\Big] = \int_0^\infty y^r\, G^k(y)\, g(y)\, dy.$$

Setting $u = G(y)$, we obtain

$$\omega_{r,k} = \int_0^1 Q_G^r(u)\, u^k\, du, \tag{6.1}$$

where $Q_G(u)$ is the qf of $G(x)$.

The r-th ordinary moment of X, with $r \in \mathbb{N}$, follows from (4.3), for $c > 1$, as

$$\mu'_r = \mathbb{E}(X^r) = \sum_{k=0}^{\infty} \int_0^\infty x^r\, s_{k+1}\, h_{k+1}(x)\, dx,$$

where it is possible to exchange the infinite sum and the integral using the dominated convergence theorem. By using (6.1) and $h_{k+1}(x) = (k+1)\, g(x)\, G^k(x)$, we obtain

$$\mu'_r = \sum_{k=0}^{\infty} (k+1)\, s_{k+1} \int_0^1 Q_G^r(u)\, u^k\, du = \sum_{k=0}^{\infty} (k+1)\, s_{k+1}\, \omega_{r,k}. \tag{6.2}$$

Equation (6.2) reveals that the moments of the BMOL distribution can be expressed as an infinite weighted sum of the baseline PWMs.

If $G(x)$ is the Lomax cdf (with $\lambda = 1$), we obtain, using the binomial expansion,

$$Q_G^r(z) = \left[\frac{1}{(1-z)^{1/\alpha}} - 1\right]^r = \sum_{j=0}^{r} \binom{r}{j} \frac{(-1)^{r+j}}{(1-z)^{j/\alpha}}$$

and therefore, from equation (6.1),

$$\omega_{r,k} = \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} \int_0^1 \frac{u^k}{(1-u)^{j/\alpha}}\, du. \tag{6.3}$$

As a result, from (6.2) and (6.3), we obtain that $\mu'_r < \infty$ for $r < \alpha$ and $\mu'_r = \infty$ for $0 < \alpha \le r$, a condition that also holds for the Lomax distribution.

We can express the r-th ordinary moment of X as a linear combination of the r-th ordinary moments of Lomax random variables. In fact, for $j = 0, 1, ...$, let $\alpha_j = (j+1)\,\alpha$. By applying the dominated convergence theorem and using equation (4.5), we can write, for $c > 1$,

$$\mu'_r = \sum_{j=0}^{\infty} p_j \int_0^{\infty} x^r g(x; \alpha_j, 1)\, dx = \sum_{j=0}^{\infty} p_j\, \mathbb{E}(Y_j^r),$$

where $Y_j \sim \mathrm{Lomax}(\alpha_j, 1)$.

From the equality $\mathbb{E}(Y_j^r) = \Gamma(r+1)\Gamma(\alpha_j - r)/\Gamma(\alpha_j)$, for $r < \alpha_j$, (see [16]), we obtain

$$\mu'_r = \Gamma(r+1) \sum_{j=0}^{\infty} p_j\, \frac{\Gamma(\alpha_j - r)}{\Gamma(\alpha_j)}, \qquad r < \alpha_j,\ \forall j. \tag{6.4}$$

Equations (6.2) and (6.4) are the main results of this section. However, the moments of X can be determined from (6.4) more easily than from (6.2).

7. GENERATING FUNCTION

A formula for the moment generating function (mgf) $M(t) = \mathbb{E}(\mathrm{e}^{\mathrm{tX}})$ of $X \sim \mathrm{BMOL}(a, b, c, \alpha)$ follows from (4.3) as

$$M(t) = \sum_{k=0}^{\infty} (k+1)\, s_{k+1}\, \rho_k(t), \tag{7.1}$$

where

$$\rho_k(t) = \int_0^{\infty} \mathrm{e}^{tx}\, g(x)\, G^k(x)\, dx.$$

We can obtain an expansion for $\rho_k(t)$, with $t < 0$ and $\alpha \in \mathbb{N}$, using the upper incomplete gamma function, which is defined as

$$\Gamma(\upsilon, z) = \int_z^{\infty} x^{\upsilon - 1}\, \mathrm{e}^{-x}\, dx, \quad \upsilon \in \mathbb{R},\ z > 0. \tag{7.2}$$

In fact, setting $w = 1 + x$, we have

$$\rho_k(t) = \int_1^\infty e^{t(w-1)}\, g(w-1)\, G^k(w-1)\, dw = \alpha \int_1^\infty e^{t(w-1)}\, w^{-\alpha-1}\, (1 - w^{-\alpha})^k\, dw.$$

Using the binomial expansion, we have

$$(1 - w^{-\alpha})^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} w^{-j\alpha},$$

which leads to

$$\rho_k(t) = \alpha \sum_{j=0}^{k} (-1)^{\alpha_j - j + 1} \binom{k}{j} t^{\alpha_j + 1}\, e^{|t|} \int_1^\infty e^{-|t|w}\, (|t|w)^{-\alpha_j - 1}\, dw, \quad t < 0,\ \alpha \in \mathbb{N},$$

Since $\alpha \in \mathbb{N}$, then $\alpha_j = (j+1)\,\alpha \in \mathbb{N}$ for all j, which ensures that the quantity $(-1)^{\alpha_j - j + 1}$ in the above expression is a real number. Finally, using (7.2), we obtain

$$\rho_k(t) = \alpha \sum_{j=0}^{k} (-1)^{\alpha_j - j} \binom{k}{j} t^{\alpha_j}\, e^{|t|}\, \Gamma(-\alpha_j, |t|), \quad t < 0,\ \alpha \in \mathbb{N}. \tag{7.3}$$

Equations (7.1) and (7.3) are the main results of this section.

8. MEAN DEVIATIONS AND BONFERRONI AND LORENZ CURVES

As before, for $j = 0, 1, ...,$ let $Y_j \sim \text{Lomax}(\alpha_j, 1)$. The mean deviations of $X \sim \text{BMOL}(a, b, c, \alpha)$ about the mean, $\delta_1 = \mathbb{E}|X - \mu_1'|$ (with $1 < \alpha_j,\ \forall j$), and about the median, $\delta_2 = \mathbb{E}|X - M|$, can be expressed as

$$\delta_1 = 2\,\mu_1'\, F(\mu_1') - 2\, m_X^{(1)}(\mu_1'), \quad \delta_2 = \mu_1' - 2\, m_X^{(1)}(M),$$

where μ_1' is the first ordinary moment of X given by (6.4), $m_X^{(1)}(z) = \int_0^z x f(x)\, dx$ denotes the first incomplete moment of X, $M = Q(0.5)$ is the median of X and $Q(\cdot)$ is given by (5.1). The mean deviations δ_1 and δ_2 are used frequently as dispersion measures.

Using (4.5), we can write

$$m_X^{(1)}(z) = \sum_{j=0}^{\infty} p_j \int_0^z x\, g(x; \alpha_j, 1)\, dx = \sum_{j=0}^{\infty} p_j\, m_{Y_j}^{(1)}(z), \tag{8.1}$$

where $m_{Y_j}^{(1)} = \int_0^z x\, g(x; \alpha_j, 1)\, dx$ denotes the first incomplete moment of Y_j and p_j is given by (4.6). For computing δ_1 and δ_2, we use (2.6), (6.4) and (8.1).

The incomplete moments can be applied to obtain the Bonferroni and Lorenz curves [1], which are useful in several areas. The Bonferroni and Lorenz curves are defined, respectively, by

$$B(\pi) = \frac{m_X^{(1)}(q)}{\pi \mu_1'}, \quad L(\pi) = \pi\, B(\pi),$$

where $q = Q(\pi)$ is evaluated from (5.1) for $0 < \pi < 1$.

9. ENTROPY

Entropy is a measure of disorder or uncertainty. Two variants of entropy are generally used, the Shannon and Rényi entropies [5]. The latter is a generalization of the first.

For a random variable $X \sim \text{BMOL}(a, b, c, \alpha)$, it is easier to obtain an explicit expression for the Shannon entropy than for the Rényi entropy. The Shannon entropy of an absolutely continuous random variable X with pdf $f(x)$ is defined by

$$\eta_X = \mathbb{E}_f\{-\log[f(X)]\} = -\int_0^\infty \log[f(x)]\, f(x)\, dx.$$

Considering that $W[G(x)]$ is an absolutely continuous distribution with density $g(x)\, w[G(x)]$, where $G(x)$ is the baseline distribution and $w(z) = W'(z)$ (see Section 2), it can be proved that the density $f(x)$ satisfies

$$\mathbb{E}_f\{\log\left(W[G(X)]\right)\} = -\xi(a, b),$$

$$\mathbb{E}_f\{1 - \log\left(W[G(X)]\right)\} = -\xi(b, a),$$

$$\mathbb{E}_f\{\log\left(w[G(X)]\right)\} + \mathbb{E}_f\{\log[g(X)]\} - \mathbb{E}_U\{\log[w(U)]\} - \mathbb{E}_U\{\log\left(g[Q_G(U)]\right)\} = 0,$$

where $\xi(a, b) = -\frac{\partial}{\partial a}\log[B(a, b)] = \psi(a + b) - \psi(a)$, $\psi(\cdot)$ denotes the digamma function and $U \sim \text{Beta}(a, b)$.

From the equalities $w(z) = c/[c + (1 - c)z]^2$ (with $c \neq 1$) and $g(Q_G(u)) = \alpha(1 - u)^{(\alpha+1)/\alpha}$, we obtain

$$\mathbb{E}_U\{\log[w(U)]\} = \log c - 2\, \mathbb{E}_U\{\log[c + (1 - c)\, U]\},$$

$$\mathbb{E}_U\{\log(g[Q_G(U)])\} = \log \alpha + \frac{\alpha + 1}{\alpha}\, \mathbb{E}_U[\log(1 - U)].$$

Further, we have

$$\begin{aligned}
\mathbb{E}_U\{\log(1-U)\} &= \frac{1}{B(a,b)}\int_0^1 \log(1-u)\, u^{a-1}\,(1-u)^{b-1}\,du \\
&= -\xi(b,a), \\
\mathbb{E}_U\{\log[c+(1-c)\,U]\} &= \frac{1}{B(a,b)}\int_0^1 \log[c+(1-c)u]\, u^{a-1}\,(1-u)^{b-1}\,du \\
&= \log c - I_{a,b,c}\ {}_3F_2(1,1,1+a;2,1+a+b;\frac{c-1}{c}),
\end{aligned}$$

where $I_{a,b,c} = \frac{a\,(c-1)}{c\,(a+b)}$ and ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ is the generalized hypergeometric function.

Thus, we can write

$$\begin{aligned}
\eta_X = \log[B(a,b)] + (a-1)\,\xi(a,b) + \left(b-1+\frac{\alpha+1}{\alpha}\right)\,\xi(b,a) + \log c - \log\alpha \\
- 2\,I_{a,b,c}\ {}_3F_2(1,1,1+a;2,1+a+b;\frac{c-1}{c}).
\end{aligned}$$

The Shannon entropy is relevant because it is related to other notions of entropy in various areas such as probability theory, computer sciences, dynamical systems and statistical physics.

10. ORDER STATISTICS

Let $X_1,...,X_n$ be a random sample of size n from a distribution $F(x)$. Then, the pdf of the m-th order statistic, $X_{(m)}$, is given by [26, p. 218]

$$f_{(m)}(x) = K\,F^{m-1}(x)(1-F(x))^{n-m}f(x),$$

where $K = n!/[(m-1)!\,(n-m)!]$.

For $1\le m\le n$, we obtain

$$f_{(m)}(x) = K\,f(x)\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}F^{m+j-1}(x).$$

Based on (4.1) and (4.2) and using an expansion for power series raised to positive integer powers [11, p. 17], we have, for $c>1$,

$$F^{m+j-1}(x) = \left(\sum_{k=0}^{\infty} s_k\,G^k(x)\right)^{m+j-1} = \sum_{k=0}^{\infty} v_{j,k}\,G^k(x),$$

where $G(x)$ is the baseline distribution given in (2.4) (with $\lambda = 1$), $v_{j,0} = s_0^{m+j-1}$ and, for $i \geq 1$,

$$v_{j,i} = \frac{1}{i\, s_0} \sum_{l=1}^{i} [(m+j)l - i]\, s_l\, v_{j,i-l}.$$

Therefore, we obtain

$$f_{(m)}(x) = K f(x) \sum_{j=0}^{n-m} \sum_{k=0}^{\infty} (-1)^j \binom{n-m}{j} v_{j,k}\, G^k(x),$$

where the density $f(x)$ is given in (2.9).

Considering the BMO-G family, Alizadeh *et al.* [3] propose other expansion for $f_{(m)}(x)$ given by

$$f_{(m)}(x) = \sum_{r,k=0}^{\infty} p_{r,k}\, h_{r+k+1}(x), \tag{10.1}$$

where $h_{r+k+1}(x)$ denotes the exp-G density function with parameter $r + k + 1$,

$$p_{r,k} = \frac{n!\,(r+1)\,(m-1)!\, s_{r+1}}{r+k+1} \sum_{j=0}^{n-m} \frac{(-1)^j\, v_{j,k}}{(n-m-j)!\, j!},$$

and s_r is given in (4.2) for $c > 1$.

Equation (10.1) reveals that, for the BMO-G family, the density function $f_{(m)}(x)$ of the m-th order statistic $X_{(m)}$ can be expressed as a linear combination of exp-G densities. Therefore, some structural properties of $X_{(m)}$ can be obtained from those of the exp-G distribution [13].

11. MAXIMUM LIKELIHOOD ESTIMATION

Several approaches for parameter estimation were proposed in the statistical literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties for constructing confidence intervals. In this section, we estimate the parameters of the BMOL distribution by maximum likelihood for complete data sets. Let $\boldsymbol{x} = (x_1, ..., x_n)^\top$ be a sample of size n from $X \sim \text{BMOL}(a, b, c, \alpha)$ and $\boldsymbol{\theta} = (a, b, c, \alpha)^\top$ the parameter vector. The log-likelihood for $\boldsymbol{\theta}$ corresponding to the sample $\boldsymbol{x}$, denoted by $\ell_f(\boldsymbol{\theta}; \boldsymbol{x})$, is given by

$$\begin{aligned}\ell_f(\boldsymbol{\theta}; \boldsymbol{x}) = &-n \log[B(a,b)] + (a-1) \sum_{i=1}^{n} \log\{W[G(x_i)]\} \\ &+ (b-1) \sum_{i=1}^{n} \log\{1 - W[G(x_i)]\} + \sum_{i=1}^{n} \log\{w[G(x_i)]\} + \ell_g(\alpha; \boldsymbol{x}),\end{aligned}$$

where $\ell_g(\alpha;\boldsymbol{x}) = \sum_{i=1}^{n} \log[g(x_i)]$ is the log-likelihood for the Lomax parameters (with $\lambda = 1$). From (2.4) and (2.7), we can write

$$\log\{W[G(x_i)]\} = \log\left[\frac{(1+x)^\alpha - 1}{(1+x)^\alpha + c - 1}\right],$$
$$\log\{1 - W[G(x_i)]\} = \log\left[\frac{c}{(1+x)^\alpha + c - 1}\right],$$
$$\log\{w[G(x_i)]\} = \log\left\{\frac{c\,(1+x)^{2\alpha}}{[(1+x)^\alpha + c - 1]^2}\right\}.$$

Then,

$$\begin{aligned}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) =& -n\log[B(a,b)] + (a-1)\sum_{i=1}^{n}\log\left[\frac{(1+x_i)^\alpha - 1}{(1+x_i)^\alpha + c - 1}\right] \\ &+ (b-1)\sum_{i=1}^{n}\log\left[\frac{c}{(1+x_i)^\alpha + c - 1}\right] \\ &+ \sum_{i=1}^{n}\log\left\{\frac{c\,(1+x_i)^{2\alpha}}{[(1+x_i)^\alpha + c - 1]^2}\right\} + \ell_g(\alpha;\boldsymbol{x}).\end{aligned} \tag{11.1}$$

The MLE $\hat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ can be obtained by maximizing (11.1) directly by using `SAS` (`PROC NLMIXED`), `R` (`optim` and `MaxLik` functions) or `Ox` program (sub-routine `MaxBFGS`). Details for fitting univariate distributions using maximum likelihood in `R` for censored or non censored data can be obtained at

`http://www.inside-r.org/packages/cran/fitdistrplus/docs/mledist`
[Accessed 28 02 2017].

Alternatively, we can obtain the components of the score vector $\boldsymbol{U}_{\boldsymbol{\theta}} = (U_a, U_b, U_c, U_\alpha)^\top$ and set them to zero. They are given by

$$U_a = \frac{\partial}{\partial a}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) = n[\psi(a+b) - \psi(a)] + \sum_{i=1}^{n}\log\left[\frac{(1+x_i)^\alpha - 1}{(1+x_i)^\alpha + c - 1}\right],$$
$$U_b = \frac{\partial}{\partial b}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) = n[\psi(a+b) - \psi(b)] + \sum_{i=1}^{n}\log\left[\frac{c}{(1+x_i)^\alpha + c - 1}\right],$$
$$U_c = \frac{\partial}{\partial c}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) = \frac{1}{c}\sum_{i=1}^{n}\frac{b\,[(1+x_i)^\alpha - 1] - c - a + 1}{(1+x_i)^\alpha + c - 1},$$
$$\begin{aligned}U_\alpha = \frac{\partial}{\partial \alpha}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) =& \frac{n}{\alpha} - \sum_{i=1}^{n}\log(1+x_i) \\ &+ \sum_{i=1}^{n}\frac{\log(1+x_i)}{(1+x_i)^\alpha + c - 1}\,[2(c-1) - (b-1)(1+x_i)^\alpha] \\ &+ c(a-1)\sum_{i=1}^{n}\frac{(1+x_i)^\alpha\,\log(1+x_i)}{[(1+x_i)^\alpha - 1]\,[(1+x_i)^\alpha + c - 1]}.\end{aligned}$$

The MLE $\hat{\boldsymbol{\theta}}_n$ is obtained by solving the equations $U_a = U_b = U_c = U_\alpha = 0$ simultaneously. Because they can not be solved in closed-form, numerical iterative Newton–Raphson type algorithms can be applied.

Under general regularity conditions, we have $(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \overset{a}{\sim} N_4(\mathbf{0}, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the 4×4 expected information matrix and $\overset{a}{\sim}$ denotes asymptotic distribution. For n large, $K(\boldsymbol{\theta})$ can be approximated by the observed information matrix. This normal approximation for the MLE $\hat{\boldsymbol{\theta}}_n$ can be used for construing approximate confidence intervals and for testing hypotheses on the parameters a, b, c and α.

Suppose that the parameter vector is partitioned as $\boldsymbol{\theta} = (\boldsymbol{\psi}_1^\top, \boldsymbol{\psi}_2^\top)^\top$, where $\dim(\boldsymbol{\psi}_1) + \dim(\boldsymbol{\psi}_2) = \dim(\boldsymbol{\theta})$. The likelihood ratio (LR) statistic for testing the null hypothesis $\mathcal{H}_0 : \boldsymbol{\psi}_1 = \boldsymbol{\psi}_1^{(0)}$ against the alternative hypothesis $\mathcal{H}_1 : \boldsymbol{\psi}_1 \neq \boldsymbol{\psi}_1^{(0)}$ is given by $\mathrm{LR}_n = 2\,\{\ell_f(\hat{\boldsymbol{\theta}}_n) - \ell_f(\tilde{\boldsymbol{\theta}}_n)\}$, where $\hat{\boldsymbol{\theta}}_n = (\hat{\boldsymbol{\psi}}_1^\top, \hat{\boldsymbol{\psi}}_2^\top)^\top$, $\tilde{\boldsymbol{\theta}}_n = (\boldsymbol{\psi}_1^{(0)\top}, \tilde{\boldsymbol{\psi}}_2^\top)^\top$, $\hat{\boldsymbol{\psi}}_i$ and $\tilde{\boldsymbol{\psi}}_i$ are the MLE's under the alternative and null hypotheses, respectively, and $\boldsymbol{\psi}_1^{(0)}$ is a specified parameter vector. Based on the first-order asymptotic theory, we know that $LR \overset{a}{\sim} \chi_k^2$, where $k = \dim(\boldsymbol{\psi}_1)$. Thus, we can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain LR statistics for testing some sub-models of the BMOL distribution (see Table 1).

12. SIMULATION STUDY

In this section, we perform a Monte Carlo simulation experiment to evaluate the behavior of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{a}_n, \hat{b}_n, \hat{c}_n, \hat{\alpha}_n)$ in finite samples and estimate the relative biases and mean squared errors (MSEs) of the estimates for different sample sizes n. We consider 10,000 Monte Carlo replications and use the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm with analytical derivatives to maximize the log-likelihood function (11.1). We set the parameter values $a = 0.5$, $c = 0.25$ and vary b and α. All computations are performed using the `C` programming language and the `GNU` Scientific Library (version 2.1).

The results given in Table 2 reveal that, generally, the relative bias and MSE values decrease when n increases, which is to be expected since the MLEs are asymptotically unbiased. The minimum absolute values for the relative biases and MSEs are equal to 0.003. In counterpart, the maximum absolute values for the relative biases and MSEs are, respectively, 0.930 and 2.182. Further, it can be noted in Table 2 that the parameter c was underestimated in some cases (negative relative bias).

Table 2: Relative bias and MSE values of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{a}_n, \hat{b}_n, \hat{c}_n, \hat{\alpha}_n)$ (with $a = 0.5$ and $c = 0.25$).

b	α	n	relative bias				MSE			
			$\hat{a}_n$	$\hat{b}_n$	$\hat{c}_n$	$\hat{\alpha}_n$	$\hat{a}_n$	$\hat{b}_n$	$\hat{c}_n$	$\hat{\alpha}_n$
0.5	0.5	100	0.115	0.170	−0.003	0.326	0.035	0.184	0.037	0.268
		200	0.052	0.119	−0.017	0.165	0.008	0.099	0.016	0.118
		300	0.034	0.081	−0.013	0.114	0.004	0.059	0.010	0.075
	0.75	100	0.113	0.180	0.015	0.297	0.040	0.187	0.045	0.544
		200	0.051	0.118	−0.013	0.161	0.008	0.096	0.016	0.260
		300	0.033	0.084	−0.011	0.110	0.004	0.060	0.010	0.167
0.75	0.5	100	0.093	0.092	0.080	0.598	0.035	0.382	0.170	0.602
		200	0.044	0.060	−0.015	0.333	0.006	0.161	0.016	0.257
		300	0.029	0.059	−0.024	0.219	0.004	0.124	0.009	0.149
	0.75	100	0.090	0.113	0.118	0.544	0.036	0.424	0.241	1.166
		200	0.042	0.065	−0.009	0.312	0.006	0.160	0.016	0.533
		300	0.029	0.060	−0.021	0.209	0.004	0.122	0.009	0.323
1.0	0.5	100	0.089	0.032	0.183	0.930	0.027	0.577	0.175	1.131
		200	0.046	0.010	0.011	0.551	0.005	0.256	0.022	0.504
		300	0.031	0.010	−0.008	0.388	0.003	0.191	0.012	0.295
	0.75	100	0.083	0.071	0.251	0.835	0.024	1.279	0.404	2.182
		200	0.044	0.023	0.018	0.506	0.005	0.260	0.024	0.999
		300	0.030	0.015	−0.006	0.363	0.003	0.186	0.012	0.609

13. APPLICATION

In this section, the potentiality of the BMOL distribution is proved empirically by means of one lifetime application. We use an uncensored data set corresponding to 84 observations on service times for failured windshields [21, Table 16.11] and fit the BMOL distribution and its sub-models (see Table 1) to these data. All computations are performed using the `R` software (version 3.0.2, `AdequacyModel` package). The descriptive statistics for the current data are given in Table 3.

Table 3: Descriptive statistics for the service times data.

min.	1st quantile	median	mean	3rd quantile	max.
0.040	1.839	2.354	2.557	3.393	4.663

For maximizing the log-likelihood function (11.1), we use the BFGS algorithm with numerical derivatives. The MLEs are given in Table 4 (with standard errors in parentheses). For purposes of comparison, we compute some goodness-of-fit statistics: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), Cramér–von Mises Criterion (W*) and Anderson–Darling Criterion (A*) [6]. In general, small values of these statistics suggest a better fit. We also include in the comparison the exponentiated-Weibull (EW) distribution [20], since it is a widely used lifetime model. Its cdf and pdf are given, respectively, by

$$R(x) = \left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\eta} \quad \text{and} \quad r(x) = \frac{\beta\eta}{\alpha}\,\left(\frac{x}{\alpha}\right)^{\beta-1}\left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\eta-1}\mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}},$$

where $x \geq 0$ and $\alpha, \beta, \eta > 0$.

Table 4: MLEs and standard errors for the service times data.

Distribution	MLE					$\hat{\eta}$
	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$	
Lomax$(\alpha, 1)$	—	—	—	0.824 (0.090)	—	—
MOEL$(c, 1, \alpha)$	—	—	441.875 (242.694)	4.957 (0.424)	—	—
BL$(a, b, \alpha, 1)$	6.664 (1.055)	38.687 (79.332)	—	0.133 (0.254)	—	—
Kw−GL$(a, b, \alpha, 1)$	4.378 (0.517)	244.216 (213.820)	—	0.254 (0.083)	—	—
BMOL(a, b, c, α)	1.377 (0.356)	6.243 (5.526)	209.269 (143.799)	2.954 (0.627)	—	—
EW(α, β, η)	—	—	—	3.972 (0.136)	5.958 (0.255)	0.271 (0.036)

The goodness-of-fit values for the fitted distributions are listed in Table 5.

Based on the figures in Table 5, we note that the EW distribution presents the smaller values of the AIC, BIC and HQIC statistics. On the other hand, the BMOL distribution presents the smaller values of the W* and A* statistics. Since the BMOL and EW distributions are non-embedded models, a comparison between them is more appropriate by means of these statistics. Also, note that the BMOL model presents the smaller value of the AIC statistic among all its sub-models and the smaller values of the BIC and HQIC statistics comparatively with the Lomax, BL and Kw-GL distributions. Therefore, we can conclude that the BMOL distribution gives the best fit to the current data. If a minimum number of parameters is taken into account, the MOEL or EW distributions can be chosen, since these also has less parameters.

Table 5: Goodness-of-fit statistics for the service times data.

Distribution	Statistic				
	AIC	BIC	HQIC	W*	A*
Lomax$(\hat{\alpha}, 1)$	406.442	408.873	407.419	0.562	3.786
MOEL$(\hat{c}, 1, \hat{\alpha})$	266.987	271.849	268.942	0.068	0.650
BL$(\hat{a}, \hat{b}, \hat{\alpha}, 1)$	312.806	320.098	315.737	0.553	3.737
Kw$-$GL$(\hat{a}, \hat{b}, \hat{\alpha}, 1)$	282.938	290.230	285.869	0.175	1.463
BMOL$(\hat{a}, \hat{b}, \hat{c}, \hat{\alpha})$	265.694	275.417	269.602	**0.048**	**0.487**
EW$(\hat{\alpha}, \hat{\beta}, \hat{\eta})$	**261.208**	**268.501**	**264.140**	0.129	0.831

To analyze how significant are the parameters of the BMOL distribution in modeling the current data, we use the LR statistic, as discussed in Section 11, for testing the BMOL model versus its sub-models listed in Table 1. The results are given in Table 6. Based on the figures in this table, we note that the rejection of the null hypotheses for the Lomax, MOEL, BL and Kw-GL models (at the 10% significance level) is significant. So, we have evidence of the potential need for including the parameters a, b and c to model the current data.

Table 6: LR tests for the service times data.

Models	Hypotheses	LR statistic	p-value
Lomax vs. BMOL	$\mathcal{H}_0$: $a=b=c=1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	146.748	1.33×10^{-31}
MOEL vs. BMOL	$\mathcal{H}_0$: $a=b=1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	5.294	7.09×10^{-2}
BL vs. BMOL	$\mathcal{H}_0$: $c=1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	49.112	2.42×10^{-12}
Kw$-$GL vs. BMOL	$\mathcal{H}_0$: $a=c=1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	19.244	6.63×10^{-5}

The plots of the estimated densities for the EW, MOEL and BMOL distributions are displayed in Figure 4. Based on these plots, it is possible to assess the best overall fit of the BMOL distribution to the current data.

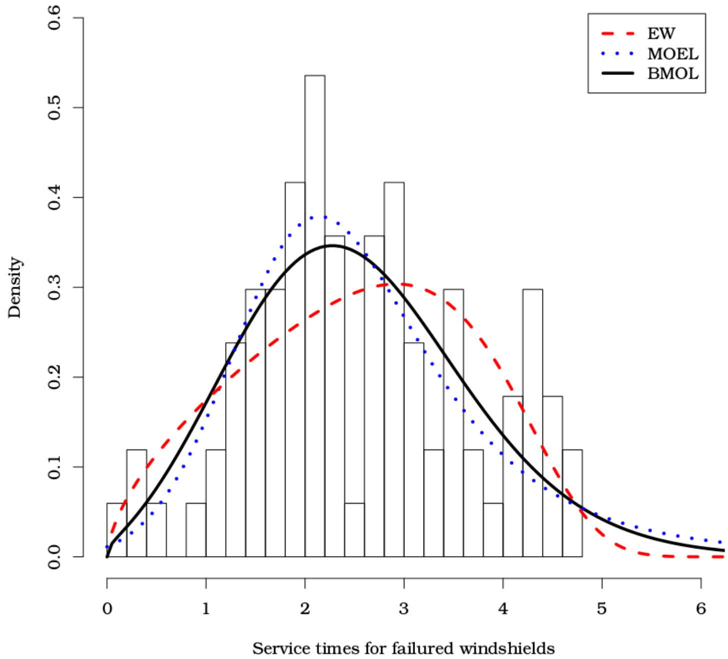


Figure 4: Comparison of the EW, MOEL and BMOL estimated densities for the service times data.

14. CONCLUSION AND FINAL REMARKS

In this chapter, we introduce a new four-parameter model, called the beta Marshall–Olkin Lomax (BMOL) distribution, as a member of the beta Marshall–Olkin generated (BMO-G) family [3] when the parent model is the Lomax distribution [17] (with $\lambda = 1$). Some sub-models of the BMOL distribution are presented. The new distribution has simple expressions for the cumulative and density functions. We study some of its mathematical and statistical properties. We demonstrate that the BMOL density can be expressed as linear combinations of Lomax and exponentiated-Lomax densities and therefore some of its structural properties can be obtained from those of these models. We present explicit expressions for the quantile function, moments, generating function, mean deviations, Bonferroni and Lorenz curves, Shannon entropy and order statistics. We obtain the maximum likelihood estimates for complete samples and perform a Monte Carlo simulation in order to evaluate the behavior of these estimates in finite samples. We compare the performance of the new model with other related distributions including the exponentiated Weibull model using classical goodness-of-fit statistics. The results confirm that the BMOL distribution is very appropriate for lifetime applications.

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