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ON THE q-GENERALIZED EXTREME VALUE DISTRIBUTION

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Abstract:

- Asymmetrical models such as the Gumbel, logistic, Weibull and generalized extreme value distributions have been extensively utilized for modeling various random phenomena encountered for instance in the course of certain survival, financial or reliability studies. We hereby introduce q-analogues of the generalized extreme value and Gumbel distributions, the additional parameter q allowing for increased modeling flexibility. These extended models can yield several types of hazard rate functions, and their supports can be finite, infinite as well as bounded above or below. Closed form representations of some statistical functions of the proposed distributions are provided. It is also shown that they compare favorably to three related distributions in connection with the modeling of a certain hydrological data set. Finally, a simulation study confirms the suitability of the maximum likelihood method for estimating the model parameters.

Key-Words:

- *extreme value theory; generalized extreme value distribution; goodness-of-fit statistics; Gumbel distribution; moments; Monte Carlo simulations; q-analogues.*

AMS Subject Classification:

- 62E15, 62F10.

1. INTRODUCTION

Extreme value theory deals with the asymptotic behavior of extreme observations in a sample of realizations of a random variable. This theory can be applied to the prediction of the occurrence of rare events such as high flood levels, large jumps in the stock markets and sizeable insurance claims. It is based on the extremal types theorem which states that exactly three types of distributions, namely the Gumbel, Fréchet and Weibull models, referred to as types I, II and III extreme value distributions, can model the limiting distribution of properly normalized maxima (or minima) of sequences of independent and identically distributed random variables. As the *generalized extreme value* ($\mathcal{GEV}$) distribution, also called the Fisher–Tippett [10] distribution, encompasses all three types, it can be utilized as an approximation to model the maxima of long (finite) sequences of random variables. The $\mathcal{GEV}$ and Gumbel distributions are widely utilized in finance, actuarial science, hydrology, economics, material sciences, telecommunications, engineering, time series modelling, risk management, reliability analysis as well as several other fields of scientific investigation involving extreme events. For informative scholarly works on extreme value distributions and related results, the reader is referred to [5], [12], [15] and [7].

Being a limiting distribution, the $\mathcal{GEV}$ model may prove somewhat inadequate in practice, and generalizations thereof ought to provide greater flexibility for modeling purposes.The extended models being proposed in this paper, namely, the q-generalized extreme value and q-Gumbel distributions, are in fact q-analogues of the distributions of origin which are re-expressed in terms of an additional parameter denoted by q.

Mathai [17] developed a pathway model involving superstatistics, which arise in statistical mechanics in connection with the study of nonlinear and non-equilibrium systems. As explained for example in [8, 28], such systems exhibit spatio-temporal dynamics that are inhomogeneous and can be described by a "superposition of several statistics on different scales". The non-equilibrium steady-state macroscopic systems being considered are assumed to be made up of a large number of smaller cells that are temporarily in local equilibrium; moreover, each of these cells can take on a given value x of the variable of interest with probability density function $g(x)$ wherefrom one can determine the generalized Boltzmann factor, $B(\epsilon) = \int_0^\infty \mathrm{e}^{-\epsilon x} g(x)\,\mathrm{d}x$, ϵ denoting the energy of a microstate occurring within each cell. Such distributions are related to Tsallis statistics [27] which find applications in statistical mechanics, turbulence studies and Monte Carlo computational methods. Recently, several q-type superstatistical distributions such as the q-exponential, q-Weibull and q-logistic were developed in the context of statistical mechanics, information theory and reliability modelling, as discussed for instance in [30, 31, 20, 18, 14] and [21].

The cumulative distribution function (cdf) and probability density function (pdf) of the $\mathcal{GEV}$ distribution, including the Gumbel distribution as a limiting case wherein $\xi \to 0$, are respectively given by

$$F_1(x) = F_1(x;\mu,\sigma,\xi) = \begin{cases} \exp\left[-\left(1+\xi\left(\dfrac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right], & \xi \neq 0\,, \\[2ex] \exp\left[-\exp\left(-\left(\dfrac{x-\mu}{\sigma}\right)\right)\right], & \xi \to 0\,, \end{cases} \tag{1.1}$$

and

$$f_1(x) = f_1(x;\mu,\sigma,\xi) = \begin{cases} \dfrac{1}{\sigma}\left(1+\xi\left(\dfrac{x-\mu}{\sigma}\right)\right)^{(-1/\xi)-1} \\ \qquad \times \exp\left[-\left(1+\xi\left(\dfrac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right], & \xi \neq 0\,, \\[2ex] \dfrac{1}{\sigma}\exp\left[-\exp\left(-\left(\dfrac{x-\mu}{\sigma}\right)\right)\right] \\ \qquad \times \exp\left(-\left(\dfrac{x-\mu}{\sigma}\right)\right), & \xi \to 0\,, \end{cases}$$

where μ is a location parameter, σ is a positive scale parameter and ξ is the shape parameter. The support of the distribution is

$$x \in \begin{cases} (\mu-\sigma/\xi, \infty)\,, & \xi > 0\,, \\ (-\infty,\infty)\,, & \xi \to 0\,, \\ (-\infty, \mu-\sigma/\xi)\,, & \xi < 0\,. \end{cases} \tag{1.2}$$

On reparameterizing the $\mathcal{GEV}$ distribution by setting $m = \mu/\sigma$ and $s = \sigma^{-1}$ in (1.1) and (1.2), one has the following representations of the cdf and pdf:

$$F_2(x;s,m,\xi) = \begin{cases} \exp\left[-\big(1+\xi(sx-m)\big)^{-1/\xi}\right], & \xi \neq 0\,, \\[1ex] \exp\left[-\exp\big(-(sx-m)\big)\right], & \xi \to 0\,, \end{cases} \tag{1.3}$$

and

$$f_2(x;s,m,\xi) = \begin{cases} s\big(1+\xi(sx-m)\big)^{(-1/\xi)-1} \\ \qquad \times \exp\left[-\big(1+\xi(sx-m)\big)^{-1/\xi}\right], & \xi \neq 0\,, \\[1ex] s\exp\left[-\exp\big(-(sx-m)\big)\right] \\ \qquad \times \exp\big(-(sx-m)\big), & \xi \to 0\,. \end{cases}$$

The support then becomes

$$x \in \begin{cases} \left(\dfrac{m}{s} - \dfrac{1}{\xi s}, \infty\right), & \xi > 0\,, \\ (-\infty, \infty)\,, & \xi \to 0\,, \\ \left(-\infty, \dfrac{m}{s} - \dfrac{1}{\xi s}\right), & \xi < 0\,. \end{cases} \tag{1.4}$$

Paralleling the pathway approach advocated by Mathai [17], we now introduce the q-analogues of the $\mathcal{GEV}$ and Weibull distributions, namely, the *q-generalized extreme value* (q-$\mathcal{GEV}$) and q-Gumbel distributions. The cdf and pdf of the q-$\mathcal{GEV}$ and q-Gumbel (obtained by letting $\xi \to 0$ in the q-$\mathcal{GEV}$ model) distributions are respectively given by

$$F(x) = F(x; s, m, \xi, q) = \begin{cases} \left[1 + q\big(\xi(sx - m) + 1\big)^{-\frac{1}{\xi}}\right]^{-\frac{1}{q}}, & \xi \neq 0\,, \quad q \neq 0\,, \\ \left[1 + q\,\mathrm{e}^{-(sx-m)}\right]^{-\frac{1}{q}}, & \xi \to 0\,, \quad q \neq 0\,, \end{cases} \tag{1.5}$$

and

$$f(x) = f(x; s, m, \xi, q) = \begin{cases} s\big(1 + \xi(sx - m)\big)^{-\frac{1}{\xi}-1} \\ \quad \times \left[1 + q\big(\xi(sx - m) + 1\big)^{-\frac{1}{\xi}}\right]^{-\frac{1}{q}-1}, & \xi \neq 0\,, \quad q \neq 0\,, \\ s\,\mathrm{e}^{m-sx}\left[1 + q\,\mathrm{e}^{m-sx}\right]^{-\frac{1}{q}-1}, & \xi \to 0\,, \quad q \neq 0\,, \end{cases}$$

where the support of the distributions is as follows:

$$x \in \begin{cases} \left(\dfrac{m}{s} - \dfrac{1}{\xi s}, \infty\right), & q > 0\,, \quad \xi > 0\,, \\ \left(-\infty, \dfrac{m}{s} - \dfrac{1}{\xi s}\right), & q > 0\,, \quad \xi < 0\,, \\ \left(\dfrac{(-q)^{\xi} - 1}{\xi s} + \dfrac{m}{s}, \infty\right), & q < 0\,, \quad \xi > 0\,, \\ \left(\dfrac{(-q)^{\xi} - 1}{\xi s} + \dfrac{m}{s}, \dfrac{m}{s} - \dfrac{1}{\xi s}\right), & q < 0\,, \quad \xi < 0\,, \\ (-\infty, \infty)\,, & \xi \to 0\,, \quad q > 0\,, \\ \left(\dfrac{m + \ln(-q)}{s}, \infty\right), & \xi \to 0\,, \quad q < 0\,. \end{cases} \tag{1.6}$$

The intervals specifying the supports of these distributions are such that the terms being raised to non-integer powers remain positive for the respective domains of q and ξ.

The effects of the parameters q and ξ on the shape of the distributions are illustrated graphically in Figures 1 to 5. Plots of the hazard rates of X are displayed in Figures 6 and 7 for certain parameter values. These plots illustrate the impressive versatility of the proposed models.

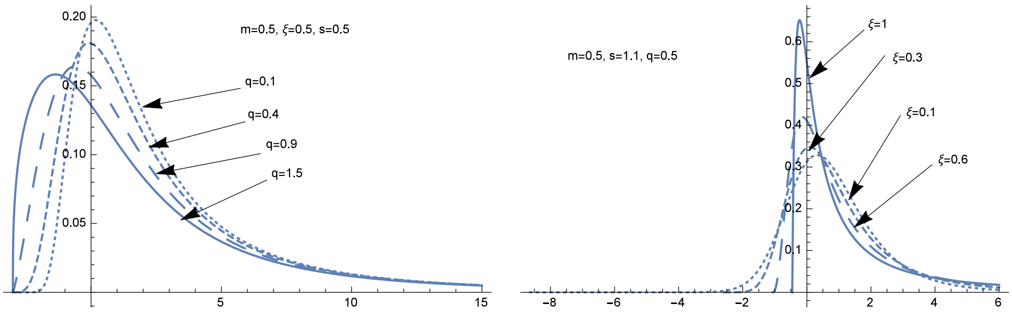

Figure 1: Plots of the q-$\mathcal{GEV}$ density function for certain parameter values ($q > 0,\ \xi > 0$).

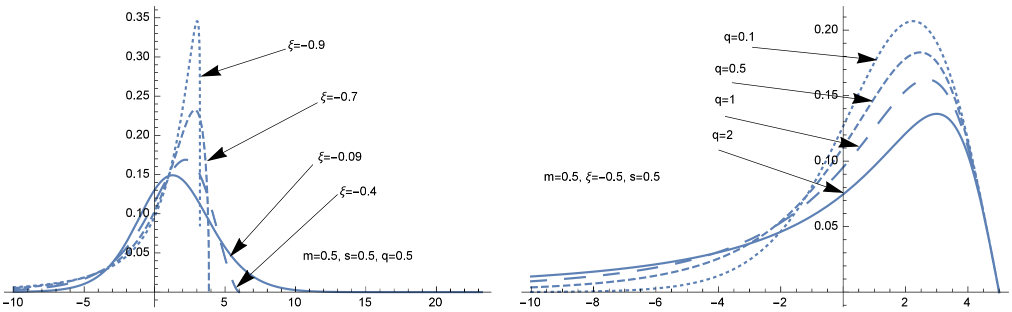

Figure 2: Plots of the q-$\mathcal{GEV}$ density function for certain parameter values ($q > 0,\ \xi < 0$).

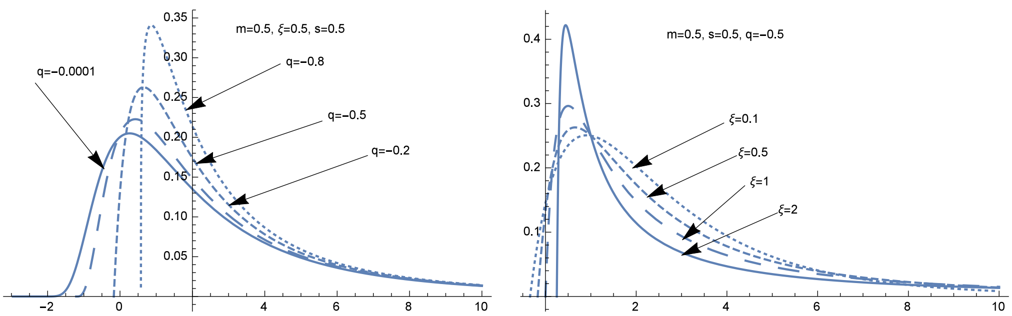

Figure 3: Plots of the q-$\mathcal{GEV}$ density function for certain parameter values ($q < 0,\ \xi > 0$).

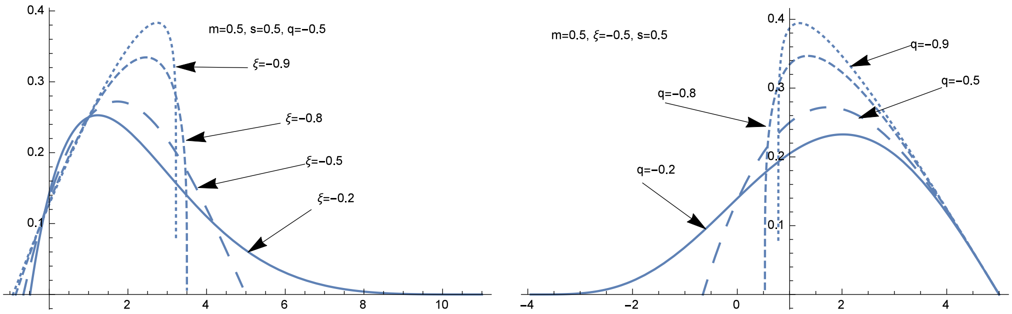

Figure 4: Plots of the q-$\mathcal{GEV}$ density function for certain parameter values ($q < 0,\ \xi < 0$).

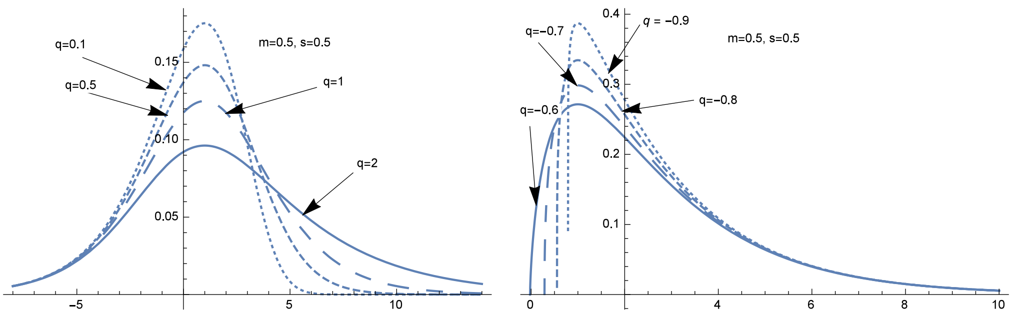

Figure 5: Plots of the q-Gumbel density function for certain parameter values. Right panel: $q > 0$; Left panel: $q < 0$.

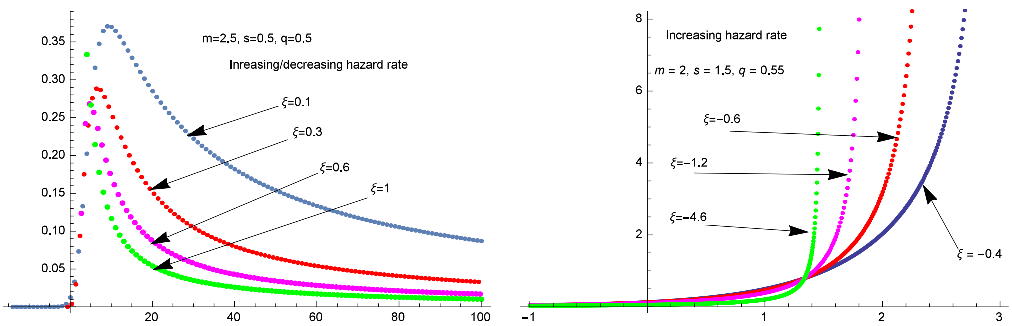

Figure 6: Plots of the q-$\mathcal{GEV}$ hazard rates for certain parameter values. Right panel: $\xi < 0$, $q > 0$; Left panel: $\xi > 0$, $q > 0$.

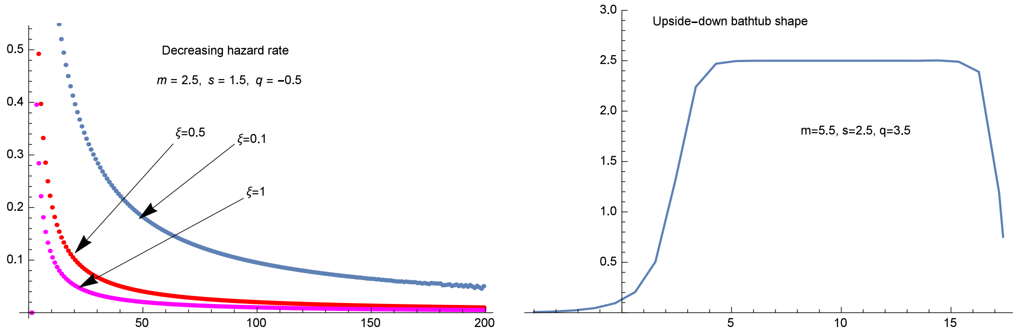

Figure 7: Plots of the q-$\mathcal{GEV}$ (left panel) and q-Gumbel (right panel) hazard rates for certain parameter values.

Remark 1.1. The $\mathcal{GEV}$ and Gumbel distributions are respectively obtained as limiting cases of the q-$\mathcal{GEV}$ and q-Gumbel distributions by letting q approach zero.

The paper is organized as follows. Section 2 contains computable representations of certain statistical functions of the q-$\mathcal{GEV}$ and q-Gumbel distributions. Section 3 explains how to determine the maximum likelihood estimators of the model parameters. In Section 4, the proposed distributions as well as three related models are fitted to an actual data set, and several statistics are utilized to assess goodness of fit. A Monte Carlo simulation study is carried out in Section 5 to verify the accuracy of the maximum likelihood estimates. Finally, some concluding remarks are included in the last section.

2. CERTAIN STATISTICAL FUNCTIONS

This section includes certain computable representations of the ordinary moments and the L-moments of the q-Gumbel (s, m, q) and q-$\mathcal{GEV}(s, m, \xi, q)$ random variables, which were obtained by making use of the symbolic computation package *Mathematica*. Closed form representations of their quantile functions as well as the moment-generating function of the q-Gumbel distribution are also provided. Whenever such closed form representations could be determined, the numerical results were found to agree to at least five decimals with those evaluated by numerical integration. Thus, numerical integration can arguably be employed to evaluate any required statistical function with great accuracy.The following identity can be particularly useful for evaluating the expected value of an integrable function of a continuous random variable denoted by $W(X)$:

$$E[W(X)] = \int_{-\infty}^{\infty} W(x)\, f(x)\, \mathrm{d}x = \int_{0}^{1} W(Q_X(p))\, \mathrm{d}p\,,$$

where $f(x)$ is the pdf of X and $Q_X(p)$ denotes the quantile function of X as defined in Section 2.1.

2.1. The quantile function

The quantile function is frequently utilized for determining confidence intervals or eliciting certain properties of a distribution. In order to obtain the quantile function of a random variable X, that is,

$$Q_X(p) = \inf\Big\{x \in \mathbb{R}\colon p \le F(x)\Big\}\,, \qquad p \in (0,1)\,,$$

one has to solve the equation $F(x) = p$ with respect to x for some fixed $p \in (0,1)$, where $F(x)$ denotes the cdf of X.

The following quantile functions of the q-$\mathcal{GEV}$ ($\xi \neq 0$) and q-Gumbel ($\xi \to 0$) can be readily obtained from their cdf's as specified by Equation (1.5):

$$(2.1)\; x_p \equiv Q_X(p) = F^{-1}(p) = \begin{cases} \dfrac{m}{s} + \dfrac{1}{s\xi}\left[\left(\dfrac{p^{-q}-1}{q}\right)^{-\xi} - 1\right], & \xi \neq 0\,, \\[2ex] \dfrac{m}{s} - \dfrac{1}{s}\ln\left(\dfrac{p^{-q}-1}{q}\right), & \xi \to 0\,. \end{cases}$$

2.2. Moments

Many key characteristics of a distribution can be inferred from its central moments. We first determine conditions under which the integer moments of the q-$\mathcal{GEV}$ distribution are finite. In light of the relationship given in the introduction of this section and the representation of quantile function of the q-$\mathcal{GEV}$ distribution specified by Equation (2.1), the k^{th} moment of this distribution can be evaluated as

$$\int_0^1 \left(\frac{1}{\xi}\right)^k \left(\left(\frac{p^{-q}-1}{q}\right)^{-\xi} - 1\right)^k \mathrm{d}p\,.$$

It is assumed without any loss of generality that $m = 0$ and $s = 1$. On applying the binomial expansion to $\big(\big(\frac{p^{-q}-1}{q}\big)^{-\xi} - 1\big)^k$, the k^{th} moment is expressible as a linear combination of the integrals

$$\int_0^1 \left(\frac{p^{-q}-1}{q}\right)^{j(-\xi)} \mathrm{d}p\,, \qquad j = 0,1,...,k\,.$$

Letting $\tau = \xi j$ and integrating, *Mathematica* provides the following condition for the existence of the integral when q is positive: $-\frac{1}{q} < \tau < 1$ or $\xi j < 1$ and $\xi j > -1/q$.
If q is negative, the condition for the existence of the k^{th} moment is $\tau < 1$, that is, $\xi j < 1$, $j = 1,...,k$. Thus the conditions for the existence of the positive integer

moments of the q-$\mathcal{GEV}$ distribution are as follows: $\xi < \frac{1}{k}$ whenever $q > 0$ and $\xi > 0$; $\xi > -1/(kq)$ whenever $q > 0$ and $\xi < 0$; $\xi < 1/k$ whenever $q < 0$ and $\xi > 0$; no requirement being necessary when q and ξ are both negative.
Moreover, as in the case of the Gumbel distribution, the positive integer moments of the q-Gumbel distribution are finite whether q is positive or negative.
As determined by symbolic computations, the n^{th} ordinary moment of the q-$\mathcal{GEV}$ distribution can be expressed as follows:

$$\begin{aligned} E(X^n) &= \frac{(-1)^n}{\xi^n} - \frac{\sum_{i=0}^{n-1}(-1)^{i+1}\binom{n}{i}\Gamma\big(1-(n-i)\,\xi\big)\left(\frac{1}{q}\right)^{1-\xi(n-i)}}{\xi^n\,\Gamma\Big(1+\frac{1}{q}\Big)} \\ &\quad \times\ \Gamma\bigg((n-i)\,\xi+\frac{1}{q}\bigg), \qquad q>0\,, \\ &= \frac{1}{s^n}\left[\frac{(m\xi-1)^n}{\xi^n} - \Gamma\bigg(\frac{q-1}{q}\bigg)\sum_{i=0}^{n-1}\frac{c_i\,(m\xi-1)^i\,(-q)^{\xi(n-i)}}{\xi^{n-1}\,\Gamma\big(-(n-i)\xi-\frac{1}{q}+1\big)}\right. \\ &\quad \left.\times\,\big(I_{i\neq 0}-(1/\xi)\,I_{i=0}\big)\,\Gamma\big(I_{i=0}-(n-i)\,\xi\big)\right], \qquad q<0\,, \end{aligned}$$

where I denotes the indicator function and the c_i's are such that $c_i = 1$ if $i = 0$, $c_i = n!/(i!\,(n-i-1)!)$ if $1 \le i \le (n-1)/2$ and $c_i = n!/i!$ if $i > (n-1)/2$.

A necessary condition for the existence of the n^{th} moment of X is $\xi < 1/n$. The representation obtained for $q < 0$ also requires that $q\xi$ be greater than $-(1/n)$. As previously pointed out, numerical integration will provide accurate results when a closed form representation is unavailable.

It should be noted that, for instance, letting Y have a q-Gumbel distribution with pdf $f(y; 1, 0, q)$, is straightforward to determine the h^{th} moment of $X = (m + Y)/s$ — whose pdf is $f(x; s, m, q)$ — in terms of the first h moments of Y since

$$E(X^h) \;=\; \frac{1}{s^h}\sum_{j=0}^{h}\binom{h}{j}\,m^{h-j}\,E(Y^j)\,.$$

When q is positive, the h^{th} moment of the q-Gumbel distribution whose parameters m and s are respectively 0 and 1, is given by

$$\begin{aligned} E(X^h) \;&=\; h!\left[{}_{h+2}F_{h+1}\bigg(1,...,1,\frac{1}{q}+1;\,2,...,2;\,-q\bigg)\right. \\ &\quad \left. +\,(-1)^h\,q_{h+1}^{h-\frac{1}{q}}\,F_h\bigg(\frac{1}{q},...,\frac{1}{q};\,\frac{1}{q}+1,...,\frac{1}{q}+1;\,-\frac{1}{q}\bigg)\right], \end{aligned} \tag{2.2}$$

where the generalized hypergeometric function ${}_pF_q(a; b; z)$ admits the power series $\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k\,k!}\,z^k$.

The following closed form representation of the moment-generating function of the q-Gumbel distribution wherein $m = 0$, $s = 1$ was obtained assuming that $q < 0$:

$$M(t) \;=\; \frac{\Gamma\Big(\frac{q-1}{q}\Big)\,\Gamma(1-t)\,(-q)^t}{\Gamma\Big(-t-\frac{1}{q}+1\Big)}\,.$$

The h^{th} moment of this distribution when its parameter q is negative can then be obtained by differentiating $M(t)$. For instance when $q < 0$, the first and second moments of the q-Gumbel distribution are

$$E(X) \;=\; H_{-\frac{1}{q}} + \log(-q)$$

and

$$E(X^2) \;=\; \Big(H_{-\frac{1}{q}} + \log(-q)\Big)^2 - \psi^{(1)}\left(\frac{q-1}{q}\right) + \frac{\pi^2}{6}\,,$$

where H_δ denotes the Harmonic function $\int_0^1 \frac{1-x^\delta}{1-x}\,\mathrm{d}x$ and $\psi^{(1)}(\cdot)$ is the digamma function.

2.3. L-Moments

Unlike the conventional moments, the L-moments of a random variable whose mean is finite always exist, which explains their frequent use in extreme value theory. Since L-moments can be evaluated as linear combinations of probability weighted moments, which are defined for instance in [4], we first determine the latter.

The m^{th} order probability weighted moment of the q-Gumbel distribution is given by

$$\begin{aligned}
\beta_m \;&=\; \int_{-\infty}^{\infty} y\,F(y)^r\,\mathrm{d}F(y)\\
&=\; \frac{1}{s}\Bigg[\mathrm{e}^m\,{}_3F_2\left(1,1,\frac{k}{q}+\frac{1}{q}+1;2,2;-q\,\mathrm{e}^m\right)\\
&\qquad - \frac{q\,(q+\mathrm{e}^{-m})^{\frac{k+1}{q}}\,\big(q\,(\mathrm{e}^m q+1)\big)^{-\frac{k+1}{q}}}{(k+1)^2}\\
&\qquad \times\,{}_2F_1\left(\frac{k+1}{q},\frac{k+1}{q};\frac{k+q+1}{q};-\frac{\mathrm{e}^{-m}}{q}\right)\Bigg]\,, \qquad q>0\,, \qquad (2.3)\\
&=\; \Re\left(\frac{\mathrm{e}^{\frac{i(k+1)\pi}{q}}(k+1)\,\pi\csc\Big(\frac{(k+1)\pi}{q}\Big) - \Big(\frac{1}{q}\Big)^{\frac{k-q+1}{q}}\,{}_2F_1\Big(\frac{k+1}{q},\frac{k+1}{q};\frac{k+q+1}{q};-\frac{1}{q}\Big)}{(k+1)^2}\right.\\
&\qquad \left. +\;{}_3F_2\left(1,1,\frac{k}{q}+\frac{1}{q}+1;2,2;-q\right)\right)\,, \qquad q<0\,,
\end{aligned}$$

where m is a nonnegative integer, $i=\sqrt{-1}$ and $\Re(s)$ denotes the real part of s. The first four L-moments of the q-Gumbel distribution are then obtained as follows: $\lambda_1=\beta_0$, $\lambda_2=2\beta_1-\beta_0$, $\lambda_3=6\beta_2-6\beta_1+\beta_0$ and $\lambda_4=20\beta_3-30\beta_2+12\beta_1-\beta_0$.

The L-moments of the q-$\mathcal{GEV}$ distribution, as well as other statistical functions of either of the newly introduced distributions, such as incomplete moments and mean deviations, can readily and accurately be evaluated by numerical integration. All the expressions included in this section were verified numerically for several values of the parameters, the code being available upon request.

3. MAXIMUM LIKELIHOOD ESTIMATION AND GOODNESS-OF-FIT STATISTICS

The parameters of the q-$\mathcal{GEV}$ and q-Gumbel distributions are estimated by making use of the maximum likelihood method. As well, several goodness-of-fit statistics to be utilized in Section 4 are defined in this section.

3.1. Maximum Likelihood Estimation

In order to estimate the parameters of the q-$\mathcal{GEV}$ and q-Gumbel distributions whose density functions are as specified in Equation (1.6), one has to maximize their respective log-likelihood functions with respect to the model parameters. Given the observations x_i, $i=1,...,n$, the log-likelihood functions of the q-$\mathcal{GEV}$ and q-Gumbel models are respectively given by

$$\begin{aligned}\ell(s,m,\xi,q) &= n\log(s)+\left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\log\Big(q\big(\xi\,(s\,x_i-m)+1\big)^{-1/\xi}+1\Big)\\ &\quad+\left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n}\log\big(\xi\,(s\,x_i-m)+1\big)\,,\end{aligned} \tag{3.1}$$

whenever $\xi\neq 0$ and

$$\ell(s,m,q) = n\log(s)+\sum_{i=1}^{n}\log(s\,x_i-m)+\left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\log(1+q\,\mathrm{e}^{m-s\,x_i})$$

as $\xi\to 0$.

The associated log-likelihood system of equations are respectively

$$
\begin{aligned}
\frac{\partial \ell(s,m,\xi,q)}{\partial s} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n} -\frac{q\,x_i\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}-1}}{q\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}}+1} \\
&\quad + \left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n}\frac{\xi\,x_i}{\xi\,(s\,x_i-m)+1}+\frac{n}{s} \;=\; 0\,, \\[2ex]
\frac{\partial \ell(s,m,\xi,q)}{\partial m} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\frac{q\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}-1}}{q\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}}+1} \\
&\quad + \left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n} -\frac{\xi}{\xi\,(s\,x_i-m)+1} \;=\; 0\,, \\[2ex]
\frac{\partial \ell(s,m,\xi,q)}{\partial \xi} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\frac{q\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}}}{q\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}}+1} \\
&\quad \times\left(\frac{\log\big(\xi\,(s\,x_i-m)+1\big)}{\xi^2}-\frac{s\,x_i-m}{\xi\,\big(\xi\,(s\,x_i-m)+1\big)}\right) \\
&\quad + \frac{\sum_{i=1}^{n}\log\big(\xi\,(s\,x_i-m)+1\big)}{\xi^2} \\
&\quad + \left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n}\frac{s\,x_i-m}{\xi\,(s\,x_i-m)+1} \;=\; 0\,, \qquad (3.2) \\[2ex]
\frac{\partial \ell(s,m,\xi,q)}{\partial q} &= \frac{\sum_{i=1}^{n}\log\Big(q\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}}+1\Big)}{q^2} \\
&\quad + \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\frac{\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}}}{q\big(\xi\,(s\,x_i-m)+1\big)^{-\frac{1}{\xi}}+1} \;=\; 0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \ell(s,m,q)}{\partial s} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n} -\frac{q\,x_i\,\mathrm{e}^{m-s\,x_i}}{q\,\mathrm{e}^{m-s\,x_i}+1}-\sum_{i=1}^{n}x_i+\frac{n}{s} \;=\; 0\,, \\[2ex]
\frac{\partial \ell(s,m,q)}{\partial m} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\frac{q\,\mathrm{e}^{m-s\,x_i}}{q\,\mathrm{e}^{m-s\,x_i}+1}+n\,, \qquad (3.3) \\[2ex]
\frac{\partial \ell(s,m,q)}{\partial q} &= \frac{\sum_{i=1}^{n}\log(q\,\mathrm{e}^{m-s\,x_i}+1)}{q^2}+\left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\frac{\mathrm{e}^{m-s\,x_i}}{q\,\mathrm{e}^{m-s\,x_i}+1}\,.
\end{aligned}
$$

Solving the nonlinear systems specified by the sets of equations (3.2) and (3.3) respectively yields the maximum likelihood estimates ($\mathcal{MLE}$'s) of the parameters of the q-$\mathcal{GEV}$ and q-Gumbel distributions. Since these equations cannot be solved analytically, iterative methods such as the Newton–Raphson technique are required. For both distributions, all the second order log-likelihood derivatives exist. In order to determine approximate confidence intervals for the parameters of the q-$\mathcal{GEV}$ and q-Gumbel distributions, one needs the 4×4 and 3×3 observed information matrices which are obtained by taking the opposite of the matrices of the second derivatives of the loglikelihood functions wherein the parameters are replaced by the $\mathcal{MLE}$'s, these matrices being denoted by $J(\upsilon_1)=\{J(\upsilon_1)_{rt}\}$ for $r,t=s,m,\xi,q$, where υ_1 denotes the vector of the parameters s,m,ξ,q, and $J(\upsilon_2)=\{J(\upsilon_2)_{rt}\}$ for $r,t=s,m,q$, where υ_2 is a vector whose components are s,m,q. Under standard regularity conditions, $(\upsilon_1-\widehat{\upsilon}_1)$ asymptotically follows the multivariate normal distribution $\mathcal{N}_4(O,-J(\widehat{\upsilon}_1)^{-1})$ and the asymptotic distribution of $(\upsilon_2-\widehat{\upsilon}_2)$ is $\mathcal{N}_3(O,-J(\widehat{\upsilon}_2)^{-1})$. These distributions can be utilized to construct approximate confidence intervals for the model parameters. Thus, denoting for example the total observed information matrix evaluated at $\widehat{\upsilon}_1$, that is, $-J(\widehat{\upsilon}_1)$, by $-\widehat{J}$, one would have the following approximate $100(1-\alpha)\%$ confidence intervals for the parameters of the q-$\mathcal{GEV}$ distribution

$$\widehat{s} \pm z_{\phi/2}\sqrt{(-\widehat{J}^{-1})_{ss}}\,, \qquad \widehat{m} \pm z_{\phi/2}\sqrt{(-\widehat{J}^{-1})_{mm}}\,,$$

$$\widehat{\xi} \pm z_{\phi/2}\sqrt{(-\widehat{J}^{-1})_{\xi\xi}}\,, \qquad \widehat{q} \pm z_{\phi/2}\sqrt{(-\widehat{J}^{-1})_{qq}}\,,$$

where $z_{\alpha/2}$ denotes the $100(1-\alpha/2)^{\rm th}$ percentile of the standard normal distribution. The observed information matrices for the q-$\mathcal{GEV}$ and q-Gumbel models are provided in Appendices A and B.

One can determine the global maximum of the log-likelihood functions by setting certain initial values for the parameters in the maximizing routine being used. To that end, one could for instance make use of estimates of the parameters obtained for a sub-model such as those of the $\mathcal{GEV}$ distribution when assigning initial values to the parameters s,m,ξ of the q-$\mathcal{GEV}$ distribution. While Park and Sohn [23] obtained parameter estimates for the $\mathcal{GEV}$ distribution by making use of generalized weighted least squares and estimates of the three parameters are given in Chapter 30 of [4] in terms of probability weighted moments, Prescott and Walden [24] advocated the use of the maximum likelihood approach. It should be noted that, for both distributions under consideration, the $\mathcal{MLE}$'s do not appear to be particularly sensitive to the initial parameter values.

3.2. Goodness-of-fit statistics

In order to assess the relative adequacy of competing models, one has to rely on certain goodness-of-fit statistics. These may include the log-likelihood function evaluated at the $\mathcal{MLE}$'s denoted by $\widehat{\ell}$, Akaike's information criterion (AIC), the corrected Akaike information criterion (CAIC), as well as the modified Anderson–Darling (A^*), the modified Cramér–von Mises (W^*) and the Kolmogrov–Smirnov (K–S) statistics. The smaller these statistics are, the better the fit. The AIC and AICC statistics are respectively given by

$$\text{AIC} = -2\ell(\widehat{\theta}) + 2p \qquad \text{and} \qquad \text{AICC} = \text{AIC} + \frac{2p\,(p+1)}{n-p-1}\,,$$

where $\ell(\widehat{\theta})$ denotes the log-likelihood function evaluated at the $\mathcal{MLE}$'s, p is the number of estimated parameters and n, the sample size.

The Anderson–Darling and Cramér–von Mises statistics can be evaluated by means of the following formulae:

$$A^* = \left(\frac{2.25}{n^2} + \frac{0.75}{n} + 1\right)\left[-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\bigl(z_i\,(1 - z_{n-i+1})\bigr)\right],$$

and

$$W^* = \left(\frac{0.5}{n} + 1\right)\left[\sum_{i=1}^{n}\left(z_i - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}\right],$$

where $z_i = \text{cdf}(y_i)$, the y_i's denoting the ordered observations.

As for the Kolmogrov–Smirnov statistic, it is defined by

$$\text{K–S} = \text{Max}\left[\frac{i}{n} - z_i,\, z_i - \frac{i-1}{n}\right].$$

As is explained in [2], unlike the asymptotic distributions of the AIC and AICC statistics, those of the A^* and W^* statistics have complicated forms requiring numerical techniques for determining specific percentiles.

4. APPLICATIONS

4.1. A hydrological data set

In this section, we fit five models to a rainfall precipitation data set which is freely available on the Korea Meteorological Administration (KMA) website `http://www.kma.go.kr` and represent the annual maximum daily rainfall amounts in millimeters in Seoul, Korea during the period 1961–2002. The selected models are the three-parameter $\mathcal{GEV}$, the Kumaraswamy generalized extreme value (Kum$\mathcal{GEV}$) [9], the exponentiated generalized Gumbel (EGGu) [3], and the newly introduced q-$\mathcal{GEV}$ and q-Gumbel distributions. Then, five statistics are employed in order to assess goodness of fit. Table 1 displays certain descriptive statistics associated with the set of observations under consideration.

Table 1: Descriptive statistics for the Seoul rainfall data.

Mean	Median	SD	Kurtosis	Skewness	MD – mean	MD – median	Entropy
144.599	131.6	66.1781	3.80435	0.940673	48.7761	33.2	4.61435

MD := Mean deviation

The Kum$\mathcal{GEV}$ and EGGu density functions are respectively given by

$$f(x;a,b,\xi,\sigma,\mu) \;=\; \frac{1}{\sigma}\,abu\,\exp(-au)\,\big[1-\exp(-au)\big]^{b-1}\,,$$

where $u=\{1+\xi(x-\mu)/\sigma\}^{-1/\xi}$ with x such that $(1+\xi(x-\mu)/\sigma)>0$; $a>0$, $b>0$, $\xi\in\mathbb{R}$, $\sigma>0$ and $\mu\in\mathbb{R}$, and

$$f(x;\sigma,\mu,\alpha,\beta) \;=\; \frac{\alpha\beta}{\sigma}\,\mathrm{e}^{-\left(\frac{x-\mu}{\sigma}+\mathrm{e}^{\frac{\mu-x}{\sigma}}\right)}\left(1-\mathrm{e}^{-\mathrm{e}^{\frac{\mu-x}{\sigma}}}\right)^{\alpha-1}\left[1-\left(1-\mathrm{e}^{-\mathrm{e}^{\frac{\mu-x}{\sigma}}}\right)^{\alpha}\right]^{\beta-1},$$

where $x\in\mathbb{R}$, $\xi\in\mathbb{R}$, $\sigma>0$, $\mu\in\mathbb{R}$, $\alpha>0$ and $\beta>0$.

The $\mathcal{MLE}$'s of the parameters are included in Table 2 for each of the fitted distributions. It can be seen from the values of the goodness–of–fit statistics appearing Table 3 that the two proposed distributions provide the most adequate models. The plots of the cdf's that are superimposed on the empirical cdf in the right panel of Figure 8 also suggest that they better fit the data. Additionally, asymptotic confidence intervals for the model parameters are included in Table 4.

Table 2: $\mathcal{MLE}$'s of the parameters (standard errors in parentheses) for the Seoul rainfall data.

Distribution	Estimates				
$\mathcal{GEV}(s,m,\xi)$	0.0212 (0.0015)	2.3781 (0.1666)	0.0028 (0.0570)		
Kum$\mathcal{GEV}(a,b,\xi,\sigma,\mu)$	18.289 (5.652)	15.412 (13.558)	21.175 (9.868)	1.1934 (0.440)	2.1339 (11.002)
EGGu$(\sigma,\mu,\alpha,\beta)$	85.686 (206.89)	−18.428 (509.13)	1.7687 (4.4618)	18.593 (201.49)	
q-$\mathcal{GEV}(s,m,\xi,q)$	0.0303 (0.0085)	4.1082 (1.6329)	0.1973 (0.0922)	1.1225 (1.266)	
q-Gumbel(s,m,q)	0.02045 (0.0026)	2.4323 (0.4135)	0.1129 (0.1746)		

Table 3: Goodness-of-fit statistics for the Seoul rainfall data.

Distribution	AIC	AICC	A^*	W^*	K–S	p-value (K–S)
$\mathcal{GEV}(s, m, \xi)$	1169.63	1169.87	0.9583	0.1325	0.0892	0.3725
Kum$\mathcal{GEV}(a, b, \xi, \sigma, \mu)$	1174.726	1175.33	0.8566	0.1505	0.0889	0.3767
EGGu$(\sigma, \mu, \alpha, \beta)$	1169.16	1169.56	0.6566	0.1099	0.0872	0.4007
q-$\mathcal{GEV}(s, m, \xi, q)$	1168.64	1169.04	0.4638	0.0678	0.0716	0.6535
q-Gumbel(s, m, q)	1166.94	1167.18	0.6279	0.1021	0.0862	0.4157

Table 4: Confidence intervals for the parameters of the q-Gumbel and q-$\mathcal{GEV}$ models (Seoul rainfall data).

CI (q-Gumbel)	s	m	q
95%	[0, 0.025546]	[2.4272, 2.4373]	[−0.229316, 0.4551]
99%	[0.01374, 0.027158]	[2.4255, 2.4390]	[−0.337568, 0.5633]

CI (q-$\mathcal{GEV}$)	s	m	ξ	q
95%	[0, 0.04096]	[0.9078, 7.3086]	[0.01698, 0.3776]	[−1.257536, 3.7075]
99%	[0, 0.02496]	[−0.1046, 8.3210]	[−0.040576, 0.43517]	[−2.042828, 4.4928]

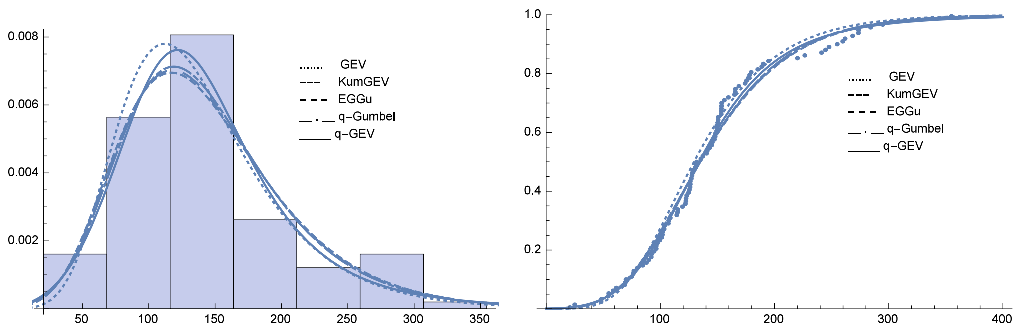

Figure 8: The $\mathcal{GEV}$, Kum$\mathcal{GEV}$, EGGu, q-Gumbel and q-$\mathcal{GEV}$ estimated pdf's superimposed on the histogram of the data (left panel); the estimated cdf's and empirical cdf (right panel).

4.2. Return level

A return period (sometimes referred to as recurrence interval) is an estimate of the likelihood of an event, such as a certain rainfall precipitation level or a given river discharge flow level. It is a statistical measure that is based on historical data, which proves especially useful in risk analysis as it represents the average recurrence interval over an extended period of time. In fact, the return period is the inverse of the probability that the level will be exceeded in any one year — or, equivalently, the expected waiting time or mean number of years it will take for an exceeding level to occur. For example, a rainfall precipitation return level x_5 has a 20% (or one fifth) probability of being exceeded in any one year, which of course, does not mean that such a rainfall level will happen regularly every 5 years or only once in a five-year period, despite what the phrase "return period" might suggest.

Based on these considerations and assuming that the event components are independently distributed, the probability that an exceeding event will occur for the first time in t years is $p(1-p)^{t-1}$, $t = 1, 2, ...,$ which is the geometric probability mass function ([25]) whose mean is equal to $T = 1/p$, when the yearly exceedance probability $p = P(X \geq x_T)$ is assumed to remain constant throughout the future years of interest ([1] and [22]) . The probability of exceeding x_T can be estimated by the survival probability, $1 - F(x_T)$, the return period T then being equal to $1/P(X \geq x_T)$. Thus, for a given return period T, the corresponding return level can be obtained as follows:

$$x_T = F^{-1}(1 - 1/T) ,$$

which yields

$$x_T = \frac{1}{s} \left\{ m - \log\left(-\frac{1-(1-1/t)^{-q}}{q} \right) \right\}$$

for the q-Gumbel model and

$$x_T = -\frac{1}{\xi s} \left\{ \left(-\frac{1-(1-1/t)^{-q}}{q} \right)^{-\xi} \left(-m\xi \left(-\frac{1-(1-1/t)^{-q}}{q} \right)^{\xi} - 1 \right) \right\}$$

for the q-$\mathcal{GEV}$ model, where $x_T > 0$ and $T > 1$. When unknown, the parameters are replaced by their $\mathcal{MLE}$'s. The estimates of the return levels x_T obtained from the q-$\mathcal{GEV}$ distribution for the return periods, T=2, 5, 10, 20, 50, 100 years, which appear in Table 5, apply to the previously analyzed Seoul rainfall precipitation data.

Table 5: Return level estimates $\widehat{x}_T$ for given values of T (Seoul rainfall data).

T	$\widehat{x}_T$ (q-$\mathcal{GEV}$ model)
2	133.964
5	187.515
10	225.94
20	267.07
30	293.139
50	328.625
100	382.323

5. SIMULATION STUDY

The suitability of the maximum likelihood approach for estimating the parameters of the q-Gumbel and q-$\mathcal{GEV}$ distributions is assessed in this section. Samples of sizes 50, 100, 300 and 500 were generated from the quantile functions of these distributions by Monte Carlo simulations for several values of the parameters. The biases and mean squared errors (MSE's) of the resulting $\mathcal{MLE}$'s were determined for each combination of sample sizes and assumed parameter values on the basis of 5,000 replications.

The simulations results that were obtained for the q-Gumbel and q-$\mathcal{GEV}$ are respectively reported in Tables 6 and 7. As expected, the biases and MSE's generally decrease as the sample sizes increase. It should be noted that the $\mathcal{MLE}$'s remain fairly accurate even for moderately sized samples. Those results corroborate the appropriateness of the maximum likelihood methodology — as described in Section 3.1 — for estimating the parameters of the proposed models.

Table 6: Monte Carlo simulation results: biases and MSE's for the q-Gumbel model.

n	Actual values			Bias			MSE		
	q	s	m	$\widehat{q}$	$\widehat{s}$	$\widehat{m}$	$\widehat{q}$	$\widehat{s}$	$\widehat{m}$
50	0.5	1.0	0.0	−0.0015	0.0424	0.0109	0.2461	0.0749	0.1535
	1.5	2.0	1.0	0.1779	0.1933	0.2060	1.1077	0.5524	0.9587
	3.0	2.0	1.0	0.3521	0.1904	0.2552	2.3258	1.1566	1.7621
	−0.5	1.0	0.0	−0.1432	−0.0315	−0.0965	0.0863	0.0434	0.0698
	−1.5	2.0	1.0	0.0121	−0.0007	0.0114	0.0008	0.0013	0.0004
	−3.0	2.0	1.0	0.0109	−0.0036	−0.0001	0.0006	0.0001	0.0000
100	0.5	1.0	0.0	−0.0104	0.0160	−0.0008	0.0808	0.0240	0.0507
	1.5	2.0	1.0	0.0644	0.0775	0.0791	0.3300	0.1625	0.2677
	3.0	2.0	1.0	0.2278	0.1351	0.1708	1.5085	0.3438	0.7338
	−0.5	1.0	0.0	−0.0704	−0.0196	−0.0495	0.0282	0.0183	0.0258
	−1.5	2.0	1.0	0.0075	0.0053	0.0101	0.0002	0.0004	0.0002
	−3.0	2.0	1.0	0.0031	−0.001	0.0000	0.0001	0.0000	0.0000
300	0.5	1.0	0.0	−0.0020	0.0052	−0.0003	0.0243	0.0072	0.0148
	1.5	2.0	1.0	0.0192	0.0246	0.0246	0.0851	0.0411	0.0684
	3.0	2.0	1.0	0.0715	0.0404	0.0516	0.3001	0.0617	0.1298
	−0.5	1.0	0.0	−0.0275	−0.0099	−0.0201	0.0058	0.0052	0.0065
	−1.5	2.0	1.0	0.0039	0.0059	0.0070	0.0000	0.0001	0.0001
	−3.0	2.0	1.0	−0.0003	0.0001	0.0000	0.0000	0.0000	0.0000
500	0.5	1.0	0.0	−0.0013	0.0032	0.0002	0.0142	0.0041	0.0089
	1.5	2.0	1.0	0.0148	0.0175	0.0169	0.0483	0.0236	0.0384
	3.0	2.0	1.0	0.0421	0.0243	0.0322	0.1764	0.0355	0.0742
	−0.5	1.0	0.0	−0.0180	−0.0066	−0.0141	0.0030	0.0031	0.0036
	−1.5	2.0	1.0	0.0035	0.0057	0.0065	0.0000	0.0001	0.0001
	−3.0	2.0	1.0	−0.0002	0.0001	0.0000	0.0000	0.0000	0.0000

Table 7: Monte Carlo simulation results: biases and MSE's for the q-$\mathcal{GEV}$ model.

n	Actual values				Bias				MSE			
	q	s	m	ξ	$\widehat{q}$	$\widehat{s}$	$\widehat{m}$	$\widehat{\xi}$	$\widehat{q}$	$\widehat{s}$	$\widehat{m}$	$\widehat{\xi}$
50	0.5	1.0	0.0	0.5	0.9087	0.0420	0.3640	−0.0684	1.9675	0.0743	0.3694	0.0412
	1.5	1.0	1.0	0.5	0.5282	0.0210	0.1514	0.0031	0.7358	0.0522	0.2084	0.0096
	1.5	2.0	1.0	0.5	0.5474	0.0369	0.1576	0.0027	0.7963	0.2146	0.2457	0.0089
	1.5	2.0	1.0	1.5	0.1072	0.0004	0.0007	−0.0005	0.0271	0.0001	0.0000	0.0000
	−0.5	1.0	0.0	−0.5	−0.2784	−0.2644	−0.1517	−0.2271	0.1553	0.1932	0.0781	0.0966
	−1.5	2.0	1.0	−0.5	0.0026	0.0086	0.0115	−0.0164	0.0012	0.0012	0.0005	0.0027
	−1.5	2.0	1.0	−1.5	−0.0025	0.0023	0.0019	−0.0023	0.0000	0.0000	0.0000	0.0001
100	0.5	1.0	0.0	0.5	0.6083	0.0439	0.2602	−0.0343	0.9423	0.0289	0.1939	0.0115
	1.5	1.0	1.0	0.5	0.3917	0.0092	0.1279	−0.0088	0.4535	0.0223	0.1105	0.0056
	1.5	2.0	1.0	0.5	0.4033	0.0071	0.1314	−0.0084	0.4327	0.0902	0.1029	0.0053
	1.5	2.0	1.0	1.5	0.0827	−0.0002	0.0001	0.0000	0.0223	0.0000	0.0000	0.0000
	−0.5	1.0	0.0	−0.5	−0.1429	−0.1471	−0.0842	−0.1121	0.0514	0.0725	0.0291	0.0312
	−1.5	2.0	1.0	−0.5	−0.0003	0.0118	0.0096	−0.0076	0.0005	0.0004	0.0003	0.0008
	−1.5	2.0	1.0	−1.5	−0.0009	0.0009	0.0007	−0.0008	0.0000	0.0000	0.0000	0.0000
300	0.5	1.0	0.0	0.5	0.2501	0.0220	0.1144	−0.0132	0.2391	0.0087	0.0578	0.0026
	1.5	1.0	1.0	0.5	0.1988	0.0088	0.0822	−0.0118	0.1599	0.0066	0.0403	0.0024
	1.5	2.0	1.0	0.5	0.1974	0.0180	0.0801	−0.0117	0.1590	0.0252	0.0396	0.0023
	1.5	2.0	1.0	1.5	0.0352	0.0000	0.0000	0.0000	0.0133	0.0000	0.0000	0.0000
	−0.5	1.0	0.0	−0.5	−0.0491	−0.0539	−0.0320	−0.0363	0.0090	0.0163	0.0073	0.0050
	−1.5	2.0	1.0	−0.5	0.0005	0.0092	0.0069	−0.0019	0.0001	0.0002	0.0001	0.0001
	−1.5	2.0	1.0	−1.5	−0.0002	0.0002	0.0001	−0.0002	0.0000	0.0000	0.0000	0.0000
500	0.5	1.0	0.0	0.5	0.1581	0.0136	0.0734	0.0010	0.1418	0.0015	0.0347	0.0011
	1.5	1.0	1.0	0.5	0.1289	0.0051	0.0558	0.0005	0.0853	0.0015	0.0227	0.0005
	1.5	2.0	1.0	0.5	0.1330	0.0127	0.0566	0.0002	0.0873	0.0015	0.0227	0.0000
	1.5	2.0	1.0	1.5	0.0199	0.0000	0.0000	0.0000	0.0100	0.0000	0.0000	0.0000
	−0.5	1.0	0.0	−0.5	−0.0334	−0.0378	−0.0229	0.0025	0.0047	0.0025	0.0041	0.0025
	−1.5	2.0	1.0	−0.5	0.0011	0.0072	0.0057	0.0001	0.0000	0.0001	0.0001	0.0001
	−1.5	2.0	1.0	−1.5	−0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

6. CONCLUDING REMARKS

The q-generalized extreme value and the q-Gumbel distributions introduced herein are truly versatile: they can be positively or negatively skewed; they can give rise to increasing, decreasing and upside-down bathtub shaped hazard rate functions, and their supports can be finite, bounded above or below, or infinite. The flexibility of these models was further confirmed by applying them to fit a certain data set consisting of annual maximum daily precipitations, and comparing them to three other models by means of several goodness-of-fit statistics. As well, the model parameters were successfully estimated by the method of maximum likelihood, the suitability of this approach having been supported by a simulation study. Moreover, we observed that numerical integration produces highly accurate results when evaluating various statistical functions of the q-analogues of the $\mathcal{GEV}$ and Gumbel random variables. In practice, the q-generalized extreme value model ought to be more realistic and useful than its original counterpart, which is actually a limiting distribution, and the proposed extended models should lead to further advances in risk theory, biostatistics, hydrology, meteorology, survival analysis and engineering, among several other fields of research that have already benefited from the utilization of existing related models.

APPENDIX A

The 4×4 total observed information matrix associated with the q-$\mathcal{GEV}$ distribution is given by $-J(\upsilon_1)$ wherein the parameters are replaced by their $\mathcal{MLE}$'s where

$$J(\upsilon_1) = \begin{pmatrix} J(\upsilon_1)_{ss} & J(\upsilon_1)_{sm} & J(\upsilon_1)_{s\xi} & J(\upsilon_1)_{sq} \\ J(\upsilon_1)_{ms} & J(\upsilon_1)_{mm} & J(\upsilon_1)_{m\xi} & J(\upsilon_1)_{mq} \\ J(\upsilon_1)_{\xi s} & J(\upsilon_1)_{\xi m} & J(\upsilon_1)_{\xi\xi} & J(\upsilon_1)_{\xi q} \\ J(\upsilon_1)_{qs} & J(\upsilon_1)_{qm} & J(\upsilon_1)_{q\xi} & J(\upsilon_1)_{qq} \end{pmatrix},$$

with

$$\begin{aligned} J(\upsilon_1)_{ss} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(-\frac{q^2x_i^2\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{2}{\xi}-2}}{\left(q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1\right)^2}\right. \\ &\quad \left. -\frac{\left(-\frac{1}{\xi}-1\right)\xi q x_i^2\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}-2}}{q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1}\right) \\ &\quad + \left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n}-\frac{\xi^2x_i^2}{\left(\xi\left(sx_i-m\right)+1\right)^2}-\frac{n}{s^2}, \end{aligned}$$

$$\begin{aligned} J(\upsilon_1)_{sm} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left[-\frac{q^2x_i\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{2}{\xi}-2}}{\left(q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1\right)^2}\right. \\ &\quad \left. +\frac{\left(-\frac{1}{\xi}-1\right)\xi q x_i\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}-2}}{q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1}\right] \\ &\quad + \left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n}\frac{\xi^2x_i}{\left(\xi\left(sx_i-m\right)+1\right)^2}, \end{aligned}$$

$$\begin{aligned} J(\upsilon_1)_{s\xi} &= \sum_{i=1}^{n}\frac{\xi x_i}{\left(\xi\left(sx_i-m\right)+1\right)\xi^2} \\ &\quad + \left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n}\left(\frac{x_i}{\xi\left(sx_i-m\right)+1}-\frac{\xi x_i\left(sx_i-m\right)}{\left(\xi\left(sx_i-m\right)+1\right)^2}\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left[\frac{q^2 x_i\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{2}{\xi}-1}}{\left(q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1\right)^2}\right.\\
&\times\left(\frac{\log\left(\xi\left(sx_i-m\right)+1\right)}{\xi^2}-\frac{sx_i-m}{\xi\left(\xi\left(sx_i-m\right)+1\right)}\right)\\
&\left.-\frac{qx_i\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}-1}\left(\frac{\log\left(\xi\left(sx_i-m\right)+1\right)}{\xi^2}+\frac{\left(-\frac{1}{\xi}-1\right)\left(sx_i-m\right)}{\xi\left(sx_i-m\right)+1}\right)}{q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1}\right],
\end{aligned}
$$

$$
\begin{aligned}
J(\upsilon_1)_{sq} &= \frac{\sum_{i=1}^{n}-\frac{qx_i\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}-1}}{q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1}}{q^2}\\
&+\left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(\frac{qx_i\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{2}{\xi}-1}}{\left(q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1\right)^2}\right.\\
&\left.-\frac{x_i\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}-1}}{q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1}\right),
\end{aligned}
$$

$$
\begin{aligned}
J(\upsilon_1)_{mm} &= \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(-\frac{q^2\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{2}{\xi}-2}}{\left(q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1\right)^2}\right.\\
&\left.-\frac{\left(-\frac{1}{\xi}-1\right)\xi q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}-2}}{q\left(\xi\left(sx_i-m\right)+1\right)^{-\frac{1}{\xi}}+1}\right)\\
&+\left(-\frac{1}{\xi}-1\right)\sum_{i=1}^{n}-\frac{\xi^2}{\left(\xi\left(sx_i-m\right)+1\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
J(\upsilon_1)_{m\xi} &= \sum_{i=1}^{n}-2\log(x_i)\,x_i^{\xi}+\sum_{i=1}^{n}\left(-\frac{\mathrm{e}^{sx_i+mx_i^{\xi}}(-1+q)\log(x_i)\,x_i^{\xi}}{\left(-\mathrm{e}^{sx_i+mx_i^{\xi}}(-1+q)+2q\right)}\right.\\
&-\frac{\mathrm{e}^{2sx_i+2mx_i^{\xi}}(-1+q)^2 m\log(x_i)\,x_i^{2\xi}}{\left(-\mathrm{e}^{sx_i+mx_i^{\xi}}(-1+q)+2q\right)^2}\\
&\left.-\frac{\mathrm{e}^{sx_i+mx_i^{\xi}}(-1+q)m\log(x_i)\,x_i^{2\xi}}{\left(-\mathrm{e}^{sx_i+mx_i^{\xi}}(-1+q)+2q\right)}\right)\\
&+\sum_{i=1}^{n}\left(\frac{x_i^{\xi}}{sx_i+\xi mx_i^{\xi}}+\frac{\xi\log(x_i)\,x_i^{\xi}}{sx_i+\xi mx_i^{\xi}}-\frac{\xi x_i^{\xi}\left(mx_i^{\xi}+\xi m\log(x_i)\,x_i^{\xi}\right)}{\left(sx_i+\xi mx_i^{\xi}\right)^2}\right),
\end{aligned}
$$

$$J(\upsilon_1)_{mq} = \sum_{i=1}^{n}\left(\frac{e^{sx_i+mx_i^{\xi}}\left(2-e^{sx_i+mx_i^{\xi}}\right)(-1+q)\,x_i^{\xi}}{\left(-e^{sx_i+mx_i^{\xi}}(-1+q)+2q\right)^2} - \frac{e^{sx_i+mx_i^{\xi}}\,x_i^{\xi}}{-e^{sx_i+mx_i^{\xi}}(-1+q)+2q}\right),$$

$$\begin{aligned}
J(\upsilon_1)_{\xi\xi} = & \sum_{i=1}^{n}\Bigg(-\frac{(q-1)\,m^2\,x_i^{2\xi}\log^2(x_i)\,e^{mx_i^{\xi}+sx_i}}{2q-(q-1)e^{mx_i^{\xi}+sx_i}} \\
& - \frac{(q-1)^2 m^2 x_i^{2\xi}\log^2(x_i)\,e^{2mx_i^{\xi}+2sx_i}}{\left(2q-(q-1)e^{mx_i^{\xi}+sx_i}\right)^2} - \frac{(q-1)\,mx_i^{\xi}\log^2(x_i)\,e^{mx_i^{\xi}+sx_i}}{2q-(q-1)e^{mx_i^{\xi}+sx_i}}\Bigg) \\
& + \sum_{i=1}^{n}\left(\frac{\xi m x_i^{\xi}\log^2(x_i)+2mx_i^{\xi}\log(x_i)}{\xi m x_i^{\xi}+sx_i} - \frac{\left(mx_i^{\xi}+\xi m x_i^{\xi}\log(x_i)\right)^2}{\left(\xi m x_i^{\xi}+sx_i\right)^2}\right) \\
& + \sum_{i=1}^{n} -2mx_i^{\xi}\log^2(x_i)\,,
\end{aligned}$$

$$\begin{aligned}
J(\upsilon_1)_{\xi q} = & \sum_{i=1}^{n}\Bigg(\frac{(q-1)\,mx_i^{\xi}\log(x_i)\,e^{mx_i^{\xi}+sx_i}\left(2-e^{mx_i^{\xi}+sx_i}\right)}{\left(2q-(q-1)e^{mx_i^{\xi}+sx_i}\right)^2} \\
& - \frac{mx_i^{\xi}\log(x_i)\,e^{mx_i^{\xi}+sx_i}}{2q-(q-1)e^{mx_i^{\xi}+sx_i}}\Bigg)
\end{aligned}$$

and

$$J(\upsilon_1)_{qq}(s,m,\xi,q) = \sum_{i=1}^{n} -\frac{\left(2-e^{sx_i+mx_i^{\xi}}\right)^2}{\left(-e^{sx_i+mx_i^{\xi}}(-1+q)+2q\right)^2}\,.$$

APPENDIX B

The 3×3 total observed information matrix associated with the q-Gumbel distribution is given by $-J(\upsilon_2)$ wherein the parameters are replaced by their $\mathcal{MLE}$'s where

$$J(\upsilon_2) = \begin{pmatrix} J(\upsilon_2)_{ss} & J(\upsilon_2)_{sm} & J(\upsilon_2)_{sq} \\ J(\upsilon_2)_{ms} & J(\upsilon_2)_{mm} & J(\upsilon_2)_{mq} \\ J(\upsilon_2)_{qs} & J(\upsilon_2)_{qm} & J(\upsilon_2)_{qq} \end{pmatrix},$$

with

$$J(\upsilon_2)_{ss} = \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(\frac{q\,x_i^2\,e^{m-sx_i}}{q\,e^{m-sx_i}+1} - \frac{q^2\,x_i^2\,e^{2m-2sx_i}}{(q\,e^{m-sx_i}+1)^2}\right) - \frac{n}{s^2}\,,$$

$$J(\upsilon_2)_{sm} = \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(\frac{q^2\,x_i\,e^{2m-2sx_i}}{(q\,e^{m-sx_i}+1)^2} - \frac{q\,x_i\,e^{m-sx_i}}{q\,e^{m-sx_i}+1}\right),$$

$$
\begin{aligned}
J(\upsilon_2)_{sq} &= \frac{\sum_{i=1}^{n} -\frac{q\,x_i\,\mathrm{e}^{m-sx_i}}{q\,\mathrm{e}^{m-sx_i}+1}}{q^2} \\
&\quad + \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(\frac{q\,x_i\,\mathrm{e}^{2m-2sx_i}}{(q\,\mathrm{e}^{m-sx_i}+1)^2}-\frac{x_i\,\mathrm{e}^{m-sx_i}}{q\,\mathrm{e}^{m-sx_i}+1}\right),
\end{aligned}
$$

$$
J(\upsilon_2)_{mm} = \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(\frac{q\,\mathrm{e}^{m-sx_i}}{q\,\mathrm{e}^{m-sx_i}+1}-\frac{q^2\,\mathrm{e}^{2m-2sx_i}}{(q\,\mathrm{e}^{m-sx_i}+1)^2}\right),
$$

$$
\begin{aligned}
J(\upsilon_2)_{mq} &= \frac{\sum_{i=1}^{n} \frac{q\,\mathrm{e}^{m-sx_i}}{q\,\mathrm{e}^{m-sx_i}+1}}{q^2} \\
&\quad + \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{m-sx_i}}{q\,\mathrm{e}^{m-sx_i}+1}-\frac{q\,\mathrm{e}^{2m-2sx_i}}{(q\,\mathrm{e}^{m-sx_i}+1)^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
J(\upsilon_2)_{qq} &= -\frac{2\sum_{i=1}^{n}\log(q\,\mathrm{e}^{m-sx_i}+1)}{q^3} + \frac{2\sum_{i=1}^{n}\frac{\mathrm{e}^{m-sx_i}}{q\,\mathrm{e}^{m-sx_i}+1}}{q^2} \\
&\quad + \left(-\frac{1}{q}-1\right)\sum_{i=1}^{n} -\frac{\mathrm{e}^{2m-2sx_i}}{(q\,\mathrm{e}^{m-sx_i}+1)^2}\,.
\end{aligned}
$$

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