REVSTAT – Statistical Journal
Volume 15, Number 4, October 2017, 601–628

THE TRANSMUTED BIRNBAUM–SAUNDERS DISTRIBUTION

Authors: Marcelo Bourguignon
– Department of Statistics, Universidade Federal do Rio Grande do Norte, Brazil
m.p.bourguignon@gmail.com

Jeremias Leão
– Department of Statistics, Universidade Federal do Amazonas, Brazil
leaojeremiass@gmail.com

Víctor Leiva
– School of Industrial Engineering, Pontificia Universidad Católica de Valparaíso, Chile
victorleivasanchez@gmail.com, www.victorleiva.cl

Manoel Santos-Neto
– Department of Statistics, Universidade Federal de Campina Grande, Brazil
manoel.ferreira@ufcg.edu.br, www.santosnetoce.com.br

Received: February 2016 Revised: April 2016 Accepted: April 2016

Abstract:

- The Birnbaum–Saunders distribution has been largely studied and applied because of its attractive properties. We introduce a transmuted version of this distribution. Various of its mathematical and statistical features are derived. We use the maximum likelihood method for estimating its parameters and determine the score vector and Hessian matrix for inference and diagnostic purposes. We evaluate the performance of the maximum likelihood estimators by a Monte Carlo study. We illustrate the potential applications of the new transmuted Birnbaum–Saunders distribution by means of three real-world data sets from different areas.

Key-Words:

- *data analysis; likelihood methods; Monte Carlo simulations; Ox and R softwares; transmutation map.*

AMS Subject Classification:

- 62F10; 62J20; 62P99.

1. INTRODUCTION

The Birnbaum–Saunders (BS) distribution has been widely studied and applied due to its interesting properties. Some of its more relevant characteristics are the following:

- **(i)** it is a transformation of the normal distribution, inheriting several of its properties;
- **(ii)** it has two parameters, modifying its shape and scale;
- **(iii)** it has positive skewness, doing its probability density function (PDF) to be asymmetrical to the right, but due to its flexibility, symmetric data can also be modeled by the BS distribution;
- **(iv)** its PDF and failure rate (FR) are unimodal, but also other shapes for its FR may be modeled;
- **(v)** it belongs to the scale and closed under reciprocation families of distributions; and
- **(vi)** its scale parameter is also its median, so that the BS distribution can be seen as an analogue, but in an asymmetrical setting, of the normal distribution, which has the mean as one of its parameters.

For more details of the BS distribution, see Birnbaum and Saunders (1969b), Johnson *et al.* (1995, pp. 651–663), or the recent book by Leiva (2016). The BS distribution is a direct competitor of the gamma, inverse Gaussian (IG), lognormal and Weibull distributions; see details of these last distributions in Johnson *et al.* (1994).

The BS distribution has its genesis from fatigue of materials. Then, its natural applications have been mainly focussed on engineering and reliability. However, today they range diverse fields including business, environment and medicine. For some of its more recent applications, see Villegas *et al.* (2011), Marchant *et al.* (2013, 2016a,b), Saulo *et al.* (2013, 2017), Leiva *et al.* (2014a,d, 2015a, 2016c, 2015b, 2017), Rojas *et al.* (2015), Wanke and Leiva (2015), Desousa *et al.* (2017), Garcia-Papani *et al.* (2017), Leão *et al.* (2017a,b), Lillo *et al.* (2016) and Mohammadi *et al.* (2017). These and other applications, as well as several extensions and generalizations of the BS distribution, have been conducted by an international, transdisciplinary group of researchers. The first extension of the BS distribution is attributed to Volodin and Dzhungurova (2000), which established that the BS distribution is the mixture equally weighted of an IG distribution and its convolution with the chi-squared distribution with one degree of freedom. The authors provided a physical interpretation in terms of fatigue-life models and introduced a general family of distributions, with members such as the IG, normal and BS distributions, as well as others used in reliability applications. Then, Díaz-García and Leiva (2005) introduced the generalized BS (GBS) distribution; see also Azevedo *et al.* (2012). Owen (2006) proposed a three-parameter extension of the BS distribution. Vilca and Leiva (2006) derived a BS distribution based on skew-normal models (skew-BS). Gómez *et al.* (2009) extended the BS distribution from the slash-elliptic model. Guiraud *et al.* (2009) deducted a non-central version of the BS distribution. Leiva *et al.* (2009) provided a length-biased version of the BS distribution. Ahmed *et al.* (2010) analyzed a truncated version of the BS distribution. Kotz *et al.* (2010) performed mixture models related to the BS distribution. Vilca *et al.* (2010) and Castillo *et al.* (2011) developed the epsilon-skew BS distribution. Balakrishnan *et al.* (2011) considered mixture BS distributions. Cordeiro and Lemonte (2011) defined the beta-BS distribution. Leiva *et al.* (2011) modeled wind energy flux using a shifted BS distribution. Athayde *et al.* (2012) viewed the BS distributions as part of the Johnson system, allowing location-scale BS distributions to be obtained. Ferreira *et al.* (2012) and Leiva *et al.* (2016a) proposed an extreme value version of the BS distribution and its modeling. Santos-Neto *et al.* (2012, 2014, 2016) and Leiva *et al.* (2014c) reparameterized the BS distribution obtaining interesting properties and modeling. Saulo *et al.* (2012) presented the Kumaraswamy-BS distribution. Fierro *et al.* (2013) generated the BS distribution from a non-homogeneous Poisson process. Lemonte (2013) studied the Marshall–Olkin-BS (MOBS) distribution. Bourguignon *et al.* (2014) derived the power-series BS class of distributions. Martinez *et al.* (2014) introduced an alpha-power extension of the BS distribution. Leiva *et al.* (2016c) derived a zero-adjusted BS distribution.

The above mentioned review about extensions and generalizations of the BS distribution is in agreement with the important growing that the distribution theory has had in the last decades. This is because, although the Gaussian (or normal) distribution has dominated this theory during more than 100 years, many real-world applications cannot be well modeled by this distribution. Then, non-normal distributions which must be flexible in skewness and kurtosis are needed. The interested reader can find a good collection of non-normal distributions in Johnson *et al.* (1994, 1995). Several of these distributions were constructed using methods early proposed by Pearson (1895), Edgeworth (1917), Cornish and Fisher (1937) and Johnson (1949), based on differential equations, mathematical approximations and translation techniques; see more details in Johnson *et al.* (1994, pp. 15–62). A more recent proposal on non-normal distributions is attributed to Azzalini (1985). In the line of these works and motivated from financial mathematics, where applications in the calculation of value at risk and corrections to the Black–Scholes options need flexibility in skewness and kurtosis, Shaw and Buckley (2009) introduced new parametric families of distributions based on the transmutation method. This method modifies the skewness and/or kurtosis into symmetric and asymmetric distributions and generates a new distributional family known as transmuted (or changed in its shape) distributions. The transmutation method proposed by Shaw and Buckley (2009) carries out a function composition between the cumulative distribution function (CDF) of a distribution and the quantile function (QF) of another. Aryal and Tsokos (2009) defined the transmuted extreme value distribution. Aryal and Tsokos (2011) and Khan and King (2013) presented transmuted Weibull distributions. Aryal (2013) proposed the transmuted log-logistic distribution. Ashour and Eltehiwy (2013) analyzed the transmuted Lomax distribution. Mroz (2013a,b) studied the transmuted Lindley and Rayleigh distributions, whereas Sharma *et al.* (2014) derived a transmuted inverse Rayleigh distribution. Khan and King (2014) considered the transmuted inverse Weibull distribution. Merovci and Puka (2014) deduced the transmuted Pareto distribution. Tiana *et al.* (2014) developed the transmuted linear exponential distribution. Saboor *et al.* (2015) created a transmuted exponential-Weibull distribution. Louzada and Granzotto (2016) introduced the transmuted log-logistic regression model. To our best knowledge, no transmuted versions of the BS distribution exist. Therefore, the main objective of this paper is to propose and derive the transmuted BS (TBS) distribution, as well as a comprehensive treatment of its mathematical and statistical properties.

Section 2 presents the TBS distribution and derives some of its characteristics including its PDF, CDF and QF, as well as its FR, moments and a generator of random numbers. Section 3 provides the estimation of the TBS parameters using the maximum likelihood (ML) method, including the corresponding score vector and Hessian matrix for inferential and diagnostic purposes. In this section, the performance of the ML estimators is evaluated by means of Monte Carlo (MC) simulations. In addition, diagnostic tools are derived to detect influential data in the ML estimation. Section 4 illustrates the potential applications of the TBS distribution with three real-world data sets from different areas. Section 5 discusses the conclusions of this work and future research about the topic.

2. FORMULATION AND CHARACTERISTICS

In this section, we provide a background of the BS distribution, formulate the new distribution and obtain some of its more relevant characteristics.

2.1. The BS distribution

A random variable T_1 has a BS distribution with shape ($\alpha > 0$) and scale ($\beta > 0$) parameters if it can be represented by

$$T_1 = \beta\Big(\alpha Z/2 + \big((\alpha Z/2)^2 + 1\big)^{1/2}\Big)^2,$$

where $Z \sim \mathrm{N}(0,1)$. In this case, the notation $T_1 \sim \mathrm{BS}(\alpha,\beta)$ is used. The CDF of T_1 is

$$F_{\mathrm{BS}}(t;\alpha,\beta) = \Phi\big((1/\alpha)\,\rho(t/\beta)\big)\,, \qquad t > 0\,,$$

where $\rho(y) = y^{1/2} - y^{-1/2}$, for $y > 0$, and Φ denotes the standard normal CDF.

The PDF of T_1 is

$$f_{\mathrm{BS}}(t;\alpha,\beta) \,=\, \kappa(\alpha,\beta)\, t^{-3/2}(t+\beta)\, \exp\bigl(-\tau(t/\beta)/(2\alpha^2)\bigr)\,, \qquad t>0\,,$$

where $\kappa(\alpha,\beta)=\exp(1/\alpha^2)/(2\alpha\sqrt{2\pi\beta})$ and $\tau(y)=y+1/y$, for $y>0$. Note that the inverse function of the CDF of a random variable, also known as QF, is defined by $F^{-1}(y)=\inf_{x\in\mathbb{R}}\{F(x)\geq y\}$, for $y\in[0,1]$. Then, the QF of T_1 is

$$t_1(q;\alpha,\beta) \,=\, F_{\mathrm{BS}}^{-1}(q;\alpha,\beta) \,=\, \beta\Bigl(\alpha\, z(q)/2+\bigl((\alpha\, z(q)/2)^2+1\bigr)^{1/2}\Bigr)^2, \qquad 0<q<1\,,$$

where $z=\Phi^{-1}$ is the inverse function of the standard normal CDF (or QF), and F_{BS}^{-1} is the inverse function of F_{BS}. As mentioned, the BS distribution holds the following scale and reciprocation properties:

- **(i)** $b\,T_1 \sim \mathrm{BS}(\alpha, b\,\beta)$, for $b>0$, and
- **(ii)** $1/T_1 \sim \mathrm{BS}(\alpha, 1/\beta)$, respectively.

The rth moment of T_1 is

$$\mathrm{E}(T_1^r) \,=\, \frac{\beta^r\bigl(K_{r+1/2}(1/\alpha^2)+K_{r-1/2}(1/\alpha^2)\bigr)}{2\,K_{1/2}(1/\alpha^2)}\,,$$

with $K_\nu(u)$ denoting the modified Bessel function of the third kind of order ν and argument u given by

$$K_\nu(u) \,=\, \frac{1}{2}\Bigl(\frac{u}{2}\Bigr)^{\nu}\int_0^\infty w^{-\nu-1}\exp\Bigl(-w-\frac{u^2}{4\,w}\Bigr)\,\mathrm{d}w\,;$$

see Gradshteyn and Randzhik (2000, p. 907).

2.2. The TBS distribution

The TBS distribution that we propose is motivated by the work of Shaw and Buckley (2009). As mentioned, they introduced a class of generalized distributions based on the transmutation method, which is described next. Let F_1 and F_2 be the CDFs of two distributions with a common sample space and F_1^{-1} and F_2^{-1} be their inverse functions, that is, their QFs, respectively. The general rank transmutation map as given in Shaw and Buckley (2009) is defined by $G_{12}(u)=F_2(F_1^{-1}(u))$ and $G_{21}(u)=F_1(F_2^{-1}(u))$. The functions G_{12} and G_{21} both map the unit interval $[0,1]$ into itself. Under suitable assumptions, G_{12} and G_{21} satisfy $G_{ij}(0)=0$ and $G_{ij}(1)=1$, for $i,j=1,2$, with $i\neq j$. A quadratic rank transmutation map is defined as $G_{12}(u)=u+\lambda\,u(1-u)$, for $|\lambda|\leq 1$, from which follows that the CDF satisfies the relationship $F_2(x)=(1+\lambda)\,F_1(x)-\lambda(F_1(x))^2$. Then, by differentiation, it yields $f_2(x)=f_1(x)\,(1+\lambda-2\lambda\,F_1(x))$, where f_1 and f_2 are the corresponding PDFs associated with the CDFs F_1 and F_2, respectively.

For more details about the quadratic rank transmutation map, see Shaw and Buckley (2009). By using the BS CDF and PDF, we have the TBS CDF and PDF, with parameters α, β and λ, given respectively by

$$\begin{aligned} F_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) &= (1+\lambda)\,\Phi\big((1/\alpha)\,\rho(t/\beta)\big) - \lambda\Big(\Phi\big((1/\alpha)\,\rho(t/\beta)\big)\Big)^2, \\ f_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) &= \Big(1+\lambda-2\lambda\,\Phi\big((1/\alpha)\,\rho(t/\beta)\big)\Big)\,f_{\mathrm{BS}}(t;\alpha,\beta), \qquad t>0, \end{aligned} \tag{2.1}$$

where $|\lambda| \le 1$ is an additional skewness parameter, whose role is to introduce skewness and to vary the corresponding tail weights. Hereafter, a random variable T with CDF or PDF given as in (2.1) is denoted by $T \sim \mathrm{TBS}(\alpha,\beta,\lambda)$. Note that, at $\lambda = 0$, we have the BS distribution. It can also be shown that

$$\lim_{t\to 0} f_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) = \lim_{t\to\infty} f_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) = 0.$$

Observe that the PDF of T can be expressed as a finite linear combination of $\mathrm{BS}(\alpha,\beta)$ and skew-$\mathrm{BS}(\alpha,\beta,1)$ PDFs; see Vilca and Leiva (2006) for details on the skew-BS distribution and its features. Thus,

$$f_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) = (1+\lambda)\,f_{\mathrm{BS}}(t;\alpha,\beta) - \lambda f_{\text{skew-BS}}(t;\alpha,\beta,1),$$

where $f_{\text{skew-BS}}(t;\alpha,\beta,\eta) = 2\,\Phi\big((\eta/\alpha)\,\rho(t/\beta)\big)\,f_{\mathrm{BS}}(t;\alpha,\beta)$, for $\eta \in \mathbb{R}$. In addition, if $\lambda = -1$, then $T \sim$ skew-$\mathrm{BS}(\alpha,\beta,1)$. Figure 1 (first panel/row) displays several shapes of the PDF given in (2.1) for some parameter values. These shapes reveal that the TBS distribution is very versatile and that the additional skewness parameter λ has substantial effects on its skewness. Note that the shapes of the TBS distribution are much more flexible than those of the BS distribution.

2.3. Characteristics of the TBS distribution

Several of the mathematical properties of the TBS distribution can be obtained directly from the BS and skew-BS distributions. For example, the ordinary moments and moment generating function of the TBS distribution follow immediately from the moments of BS and skew-BS distributions. For more details of the skew-BS distribution, see Vilca and Leiva (2006) and Saulo *et al.* (2013). Some properties of the TBS distribution are as follow. If $T \sim \mathrm{TBS}(\alpha,\beta,\lambda)$, then:

- **(i)** $b\,T \sim \mathrm{TBS}(\alpha, b\,\beta, \lambda)$, for $b > 0$, that is, the TBS distribution is closed under scale transformations;
- **(ii)** $1/T \sim \mathrm{TBS}(\alpha, 1/\beta, -\lambda)$, that is, the TBS distribution is closed under reciprocation;
- **(iii)** $Y = (\alpha^2/\beta)T \sim \mathrm{TBS}(\alpha,\alpha^2,\lambda)$, that is, Y follows a two-parameter TBS distribution.

The FR of $T \sim \mathrm{TBS}(\alpha,\beta,\lambda)$ is

$$(2.2) \qquad h_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) = \frac{\Big(1+\lambda-2\lambda\,\Phi\big((1/\alpha)\,\rho(t/\beta)\big)\Big)\,h_{\mathrm{BS}}(t;\alpha,\beta)}{1-\lambda\,\Phi\big((1/\alpha)\,\rho(t/\beta)\big)}\,, \qquad t>0\,,$$

where $h_{\mathrm{BS}}(t) = f_{\mathrm{BS}}(t)/(1-F_{\mathrm{BS}}(t))$ is the FR of the BS distribution. Figure 1 (second panel/row) shows the FR of the TBS distribution for some parameter values.

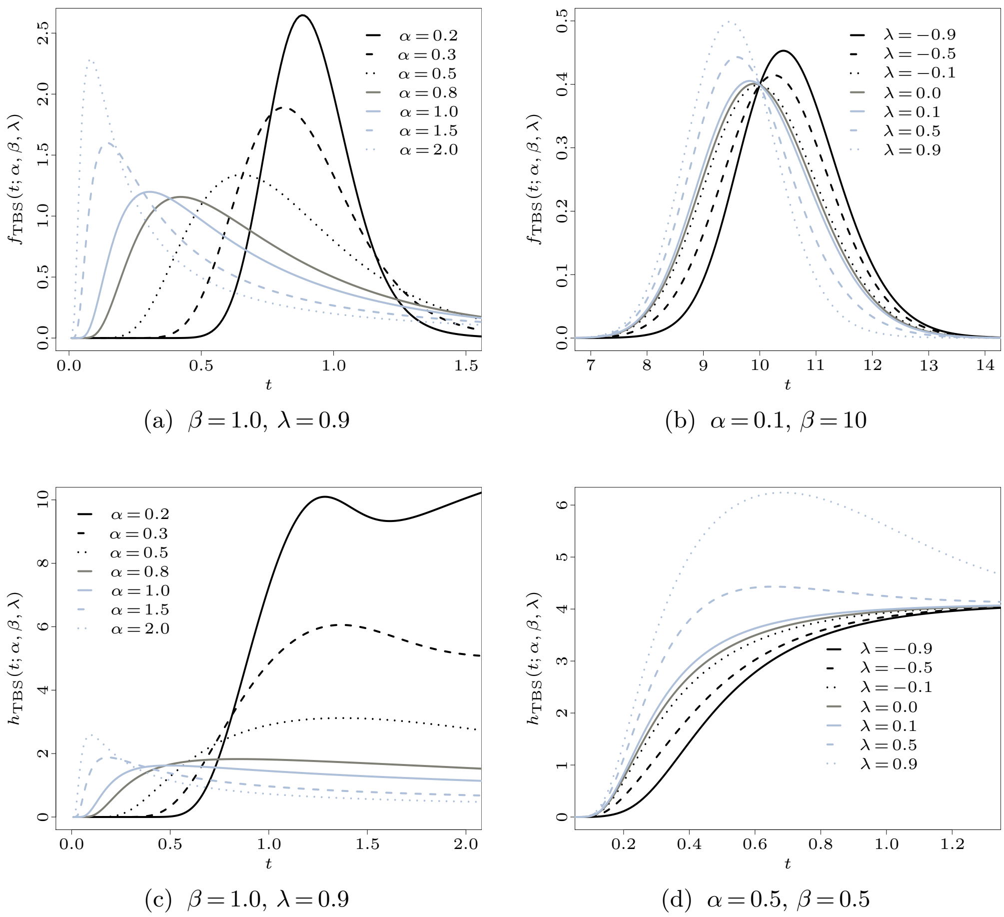


Figure 1: Plots of the TBS PDF (first panel/row) and FR (second panel/row) for the indicated value of its parameters.

We can verify that the TBS FR is upside-down. From (2.2), note that:

- **(i)** $h_{\mathrm{TBS}}(t;\alpha,\beta,\lambda)/h_{\mathrm{BS}}(t;\alpha,\beta)$ is decreasing in t for $\lambda \geq 0$;
- **(ii)** $h_{\mathrm{TBS}}(t;\alpha,\beta,\lambda)/h_{\mathrm{BS}}(t;\alpha,\beta)$ is increasing in t for $\lambda \leq 0$;
- **(iii)** $h_{\mathrm{BS}}(t;\alpha,\beta) \leq h_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) \leq (1+\lambda)\,h_{\mathrm{TBS}}(t;\alpha,\beta,\lambda)$ for $\lambda \geq 0$;
- **(iv)** $(1+\lambda)\,h_{\mathrm{BS}}(t;\alpha,\beta) \leq h_{\mathrm{TBS}}(t;\alpha,\beta,\lambda) \leq h_{\mathrm{BS}}(t;\alpha,\beta)$ for $\lambda \leq 0$; and

(**v**) $\lim_{t\to 0} h_{\rm TBS}(t;\alpha,\beta,\lambda) = 0$ and $\lim_{t\to\infty} h_{\rm TBS}(t;\alpha,\beta,\lambda) = 1/(2\alpha^2\beta)$, that is, the limiting behaviors of the FRs of the TBS and BS distributions are the same.

Observe that the expression given in (2.2) may also be written as

$$h_{\rm TBS}(t;\alpha,\beta,\lambda) \;=\; p(t)\, h_{\rm BS}(t;\alpha,\beta) + \big(1-p(t)\big)\, h_{\text{skew-BS}}(t;\alpha,\beta,1)\,,$$

where $p(t) = \big((1+\lambda)\,\big(1-\Phi\big((1/\alpha)\,\rho(t/\beta)\big)\big)\big)/\big(1-(1+\lambda)\,\Phi\big((1/\alpha)\,\rho(t/\beta)\big) + \lambda\big(\Phi\big((1/\alpha)\,\rho(t/\beta)\big)\big)^2\big)$, whereas $h_{\rm BS}$ and $h_{\text{skew-BS}}$ are the FRs of the BS and skew-BS distributions, respectively.

Many important features of a distribution can be obtained through its moments. Let $T_1 \sim {\rm BS}(\alpha,\beta)$ and $T_2 \sim \text{skew-BS}(\alpha,\beta,1)$. Then, their rth moments are

$$\text{(2.3)} \qquad {\rm E}(T_1^r) \;=\; \frac{\beta^r\alpha^{2\,r}}{2^{3r-1}} \sum_{k=0}^{r}\sum_{i=0}^{k} \binom{2\,r}{2\,k}\binom{k}{i}\left(\frac{\alpha^2}{4}\right)^{i-k},$$

$$\text{(2.4)} \qquad {\rm E}(T_2^r) \;=\; \beta^r \sum_{k=0}^{r}\sum_{i=0}^{k} \binom{r}{k}\binom{k}{i}\, 2^i \left(\frac{\alpha}{2}\right)^{k+1} w_{k+1;k-i}\,, \qquad r = 1,2,...,$$

where $w_{a,b} = {\rm E}\big(Z^a(\sqrt{\alpha^2Z^2+4})^b\big)$ and $Z \sim \text{skew-normal}(0,1,1)$; see Azzalini (1985). The rth moment of $T \sim {\rm TBS}(\alpha,\beta,\lambda)$ can be written as ${\rm E}(T^r) = (1+\lambda)\,{\rm E}(T_1^r) - \lambda{\rm E}(T_2^r)$. Then, using the results presented in (2.3) and (2.4), we obtain

$${\rm E}(T^r) \;=\; \beta^r \left(\sum_{k=0}^{r}\sum_{j=0}^{k}\left((1+\lambda)\binom{2\,r}{2\,k}\binom{k}{j}\frac{\alpha^{2(r-k+j)}}{2^{3(r-k+j)-1}} - \lambda\binom{r}{k}\binom{k}{j}\, 2^j\left(\frac{\alpha}{2}\right)^{k+1} w_{k+1;k-j}\right)\right).$$

Therefore, the first four moments of $T \sim {\rm TBS}(\alpha,\beta,\lambda)$ are

$$\begin{aligned}
{\rm E}(T) \;&=\; \mu \;=\; \beta\left(1+\frac{\alpha^2}{2}\right)\left(1+\lambda-\lambda\left(1+\frac{\alpha\, w_{1,1}}{(2+\alpha^2)}\right)\right),\\
{\rm E}(T^2) \;&=\; \beta^2\left(1+2\alpha^2+\frac{3}{2}\,\alpha^4\right)\left(1+\lambda-\lambda\left(1+\frac{2\,\alpha\, w_{1,1}+\alpha^3 w_{3,1}}{2+4\alpha^2+3\alpha^4}\right)\right),\\
{\rm E}(T^3) \;&=\; \beta^3\left(1+\frac{9}{2}\,\alpha^2+9\alpha^4+\frac{15}{2}\,\alpha^6\right)\\
&\quad\times\left(1+\lambda-\lambda\left(1+\frac{3\,\alpha\, w_{1,1}+4\,\alpha^3 w_{3,1}+\alpha^5 w_{5,1}}{2+9\,\alpha^2+18\,\alpha^4+15\,\alpha^6}\right)\right),\\
{\rm E}(T^4) \;&=\; \beta^4\left(1+8\,\alpha^2+30\,\alpha^4+60\,\alpha^6+\frac{105}{2}\,\alpha^8\right)\\
&\quad\times\left(1+\lambda-\lambda\left(1+\frac{4\,\alpha\, w_{1,1}+10\,\alpha^3 w_{3,1}+6\,\alpha^5 w_{5,1}+\alpha^7 w_{7,1}}{2+16\,\alpha^2+60\,\alpha^4+120\,\alpha^6+105\,\alpha^8}\right)\right).
\end{aligned}$$

Thus, the rth moment of $T \sim \mathrm{TBS}(\alpha, \beta, \lambda)$ about its mean is

$$\begin{aligned}\mathrm{E}\big((T-\mu)^r\big) \;=\; & (1+\lambda)\sum_{j=0}^{r}\binom{r}{j}(\mu_1-\mu)^{r-j}\,\mathrm{E}\big((T_1-\mu_1)^j\big)\\ & -\lambda\sum_{j=0}^{r}\binom{r}{j}(\mu_2-\mu)^{r-j}\,\mathrm{E}\big((T_2-\mu_2)^j\big),\end{aligned}$$

where $\mu_1 = \beta\left(1+\alpha^2/2\right)$ and $\mu_2 = \beta\left(1+\alpha\, w_{1,1}+\alpha^2/2\right)$. Hence, the corresponding second, third and fourth moments about the mean are

$$\begin{aligned}\mathrm{E}\big((T-\mu)^2\big) \;&=\; \mathrm{Var}(T) \;=\; (1+\lambda)\left((\mu_1-\mu)^2+\sigma_1^2\right)-\lambda\big((\mu_2-\mu)^2+\sigma_2^2\big),\\ \mathrm{E}\big((T-\mu)^3\big) \;&=\; (1+\lambda)\left((\mu_1-\mu)^3+3\sigma_1^2+\mu_1^{(3)}\right)-\lambda\left((\mu_2-\mu)^3+3\sigma_2^2+\mu_2^{(3)}\right),\\ \mathrm{E}\big((T-\mu)^4\big) \;&=\; (1+\lambda)\left((\mu_1-\mu)^4+6\,(\mu_1-\mu)^2\sigma_1^2+6\,(\mu_1-\mu)\,\mu_1^{(3)}+\mu_1^{(4)}\right)\\ &\quad -\lambda\left((\mu_2-\mu)^4+6\,(\mu_2-\mu)^2\sigma_2^2+6\,(\mu_1-\mu)\,\mu_2^{(3)}+\mu_2^{(4)}\right),\end{aligned}$$

where

$$\begin{aligned}\sigma_1^2 \;&=\; \mathrm{Var}(T_1) \;=\; \alpha^2\beta^2\big(1+(5/4)\,\alpha^2\big),\\ \sigma_2^2 \;&=\; \mathrm{Var}(T_2) \;=\; (\beta^2/4)\left(4\,\alpha^2-\alpha^2\,w_{1,1}^2+2\,\alpha^3\,w_{3,1}-2\,\alpha^3\,w_{1,1}+5\,\alpha^4\right),\\ \mu_1^{(3)} \;&=\; \beta^3\alpha^4\big(3+(11/2)\,\alpha^2\big),\\ \mu_1^{(4)} \;&=\; \beta^4\alpha^4\big(3+(45/2)\,\alpha^2+(633/16)\,\alpha^4\big),\\ \mu_2^{(3)} \;&=\; \mu_1^{(3)}+(\alpha^3\beta^3/4)\Big(2\,\alpha^2\,w_{5,1}+2\,w_{3,1}-3\,\alpha^2\,w_{3,1}-3\,\alpha\,w_{1,1}\,w_{3,1}\\ &\quad +w_{1,1}^3+3\,\alpha\,w_{1,1}^2-6\,\alpha^2\,w_{1,1}-6\,w_{1,1}\Big),\end{aligned}$$

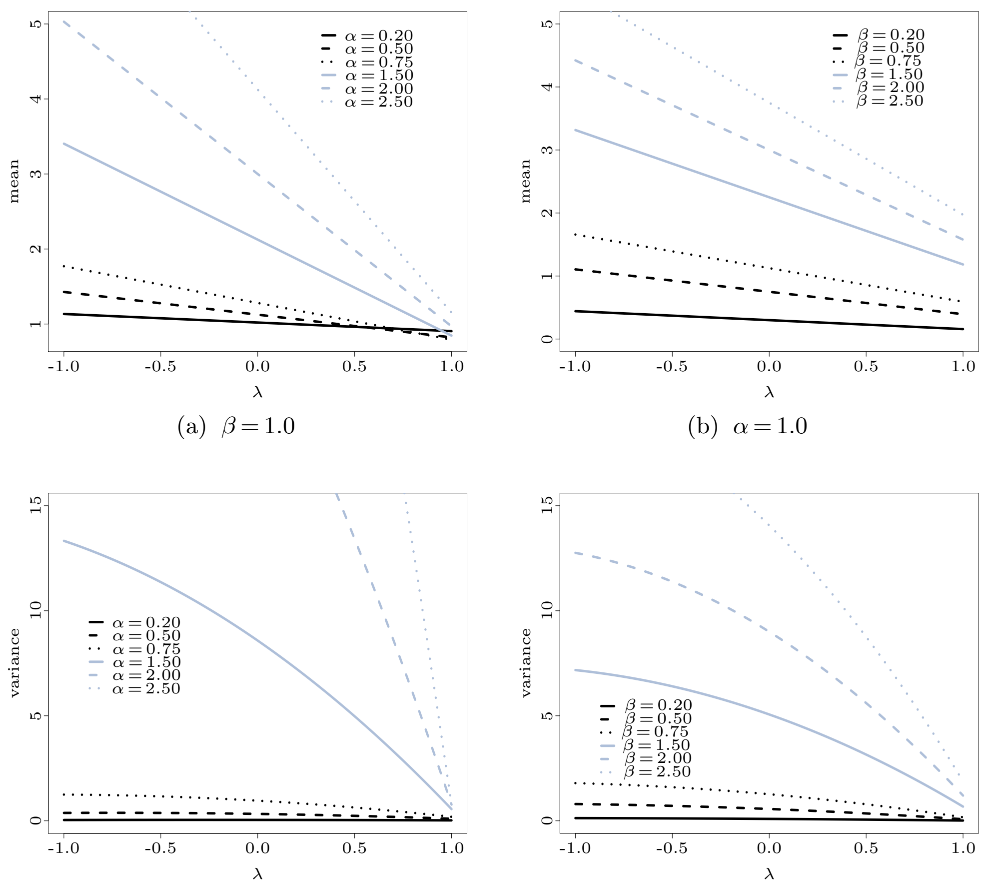
and

$$\begin{aligned}\mu_2^{(4)} \;&=\; \mu_1^{(4)}+(\alpha^4\beta^4/16)\Big(24\,\alpha^2\,w_{1,1}\,w_{2,1}+12\,w_{1,1}\,w_{3,1}^2-16\,\alpha^2\,w_{1,1}\,w_{5,1}\\ &\quad +18\,\alpha^2\,w_{1,1}^2-96\,\alpha\,w_{1,1}+16\,\alpha\,w_{5,1}-12\,\alpha\,w_{1,1}^3+8\,\alpha^3\,w_{7,1}\\ &\quad -3\,w_{1,1}^4+24\,w_{1,1}^2-16\,\alpha^3\,w_{5,1}+12\,\alpha^3\,w_{3,1}-180\,\alpha^3\,w_{1,1}\\ &\quad +16\,\alpha\,w_{3,1}-16\,w_{1,1}\,w_{3,1}\Big).\end{aligned}$$

Figure 2 presents graphical plots of the mean (first panel/row) and variance (second panel/row) of the TBS distribution for different values of α, β and λ. Note that the mean and variance decrease as λ increases, but the mean and variance, generally, increases as α and β increase. The QF of $T \sim \mathrm{TBS}(\alpha, \beta, \lambda)$ is

$$t_{\mathrm{TBS}}(q;\alpha,\beta,\lambda) \;=\; \begin{cases}\beta\left(\frac{\alpha}{2}\,\Phi^{-1}(q^*)+\left(1+\frac{\alpha^2}{4}\,\Phi^{-1}(q^*)^2\right)^{1/2}\right)^2, & \lambda\neq 0;\\[2ex] \beta\left(\frac{\alpha}{2}\,\Phi^{-1}(q)+\left(1+\frac{\alpha^2}{4}\,\Phi^{-1}(q)^2\right)^{1/2}\right)^2, & \lambda=0;\end{cases}$$

where $q^* = \left(1+\lambda-\sqrt{(1+\lambda)^2-4\,\lambda\,q}\right)/2\,\lambda$, for $q \in [0,1]$. Random numbers for the TBS distribution can be generated from the TBS QF, which is detailed by Algorithm 1.



(a) $\beta = 1.0$ (b) $\alpha = 1.0$

(c) $\beta = 1.0$ (d) $\alpha = 1.0$

Figure 2: Plots of the mean (first panel/row) and variance (second panel/row) of the TBS distribution for the indicated value of its parameters.

Algorithm 1 – Random number generator from the TBS distribution

1: Generate a random number u from $U \sim \mathrm{U}(0,1)$;

2: Set values for α, β and λ of $T \sim \mathrm{TBS}(\alpha,\beta,\lambda)$;

3: If $\lambda \neq 0$, then compute a random number

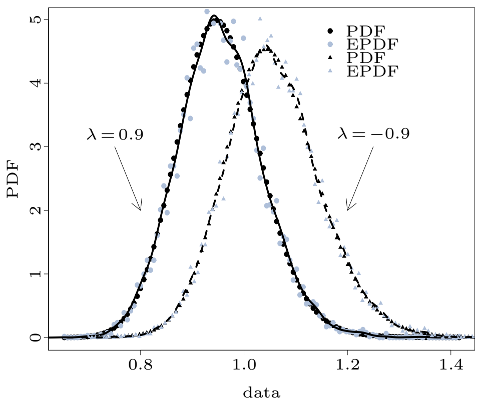
$$t \;=\; \beta\left(\frac{\alpha}{2}\,\Phi^{-1}(u^*) + \left(1+\frac{\alpha^2}{4}\,\Phi^{-1}(u^*)^2\right)^{1/2}\right)^2$$

from $T \sim \mathrm{TBS}(\alpha,\beta,\lambda)$, with $u^* = \bigl(1+\lambda-\sqrt{(1+\lambda)^2-4\,\lambda\,u}\,\bigr)/(2\lambda)$; otherwise

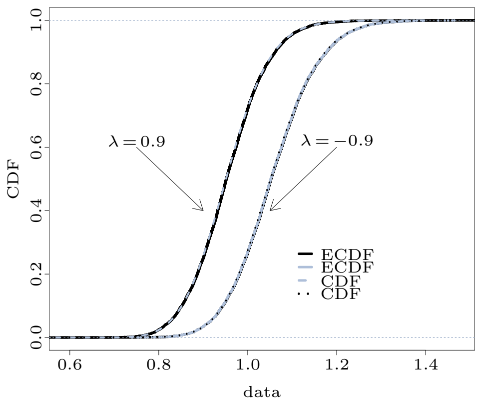
$$t \;=\; \beta\left(\frac{\alpha}{2}\,\Phi^{-1}(u) + \left(1+\frac{\alpha^2}{4}\,\Phi^{-1}(u)^2\right)^{1/2}\right)^2 ;$$

4: Repeat steps 1 to 3 until the required amount of random numbers to be completed.

From Figure 3, note that the generator of random numbers proposed in Algorithm 1 seems to be appropriate for simulating data from a TBS distribution. We implement this algorithm in the `R` statistical software (R Core Team, 2016) and generate 10000 random numbers, considering the following values of the parameters: $\alpha = 0.1$, $\beta = 1.0$ and $\lambda \in \{-0.9, 0.9\}$. The empirical PDF (EPDF), the empirical CDF (ECDF) and the kernel density estimate (KDE) are obtained using these random numbers. Figure 3 (a) shows that the midpoints are consistent with the values obtained through the TBS PDF. Figure 3 (b) allows us to compare the ECDF and TBS CDF, which are detected to be similar.



(a)



(b)

Figure 3: EPDF and TBS PDF with its KDE in solid and dashed lines (a) and ECDF and TBS CDF (b) for simulated data.

3. PARAMETER ESTIMATION, ITS PERFORMANCE AND DIAGNOSTICS

In this section, we use the ML method to estimate the TBS distribution parameters. In addition, by means of MC simulations, we study the performance of the ML estimators. Furthermore, we provide diagnostic tools to detect influential data.

3.1. ML estimation

Let $T_1, ..., T_n$ be a random sample from the TBS distribution with vector of parameters $\boldsymbol{\theta} = (\alpha, \beta, \lambda)^\top$ and $t_1, ..., t_n$ be their observations (data). The log-likelihood function for $\boldsymbol{\theta}$ is

$$\begin{aligned}\ell(\boldsymbol{\theta}) \;=\; & n\log\big(\kappa(\alpha,\beta)\big) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(t_i+\beta) \\ & -\frac{1}{2\,\alpha^2}\sum_{i=1}^{n}\tau(t_i/\beta) + \sum_{i=1}^{n}\log\Big(1+\lambda\big(1-2\,\Phi(v_i)\big)\Big),\end{aligned} \tag{3.1}$$

where $v_i = (1/\alpha)\,\rho(t_i/\beta)$. The ML estimate $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha},\widehat{\beta},\widehat{\lambda})^\top$ is obtained by solving the likelihood equations $U_\alpha = U_\beta = U_\lambda = 0$ simultaneously, where U_α, U_β and U_λ are the components of the score vector $\boldsymbol{U}(\boldsymbol{\theta}) = (U_\alpha, U_\beta, U_\lambda)^\top$ given by

$$\begin{aligned}U_\alpha \;&=\; -\frac{n}{\alpha}\left(1+\frac{2}{\alpha^2}\right) + \frac{1}{\alpha^3}\sum_{i=1}^{n}\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}\right) + \frac{2\,\lambda}{\alpha}\sum_{i=1}^{n}\frac{v_i\,\phi(v_i)}{1+\lambda-2\,\lambda\,\Phi(v_i)}\,, \\ U_\beta \;&=\; -\frac{n}{2\,\beta} + \sum_{i=1}^{n}\frac{1}{t_i+\beta} + \frac{1}{2\,\alpha^2\,\beta}\sum_{i=1}^{n}\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}\right) - \frac{2\,\lambda}{\alpha\,\beta}\sum_{i=1}^{n}\left(\frac{\tau\big(\sqrt{t_i/\beta}\big)\,\phi(v_i)}{1+\lambda-2\,\lambda\,\Phi(v_i)}\right), \\ U_\lambda \;&=\; \sum_{i=1}^{n}\frac{1-2\,\Phi(v_i)}{1+\lambda\big(1-2\,\Phi(v_i)\big)}\,,\end{aligned}$$

with ϕ being the standard normal PDF. The equations $U_\alpha = U_\beta = U_\lambda = 0$ cannot be solved analytically, so that iterative techniques, such as bisection, Newton–Raphson and secant methods, may be used; see Lange (2001) and McNamee and Pa (2013). To obtain the ML estimates of the model parameters, we employ the subroutine `MaxBFGS` of the `Ox` software; see Doornik (2006). This subroutine uses the analytical derivatives to maximize $\ell(\boldsymbol{\theta})$; see Nocedal and Wright (1999) and Press *et al.* (2007). As starting values for the numerical procedure, we suggest to consider

$$\widetilde{\alpha} \;=\; \big(s/\widetilde{\beta}+\widetilde{\beta}/r-2\big)^{1/2}, \qquad \widetilde{\beta} = (s\,r)^{1/2}, \qquad \widetilde{\lambda} = 0\,,$$

where $s = (1/n)\sum_{i=1}^{n} t_i$ and $r = 1/\big((1/n)\sum_{i=1}^{n}(1/t_i)\big)$; see Birnbaum and Saunders (1969a) and Leiva (2016, pp. 40–42).

To construct approximate confidence intervals and hypothesis tests for the parameters, we use the normal approximation of the distribution of the ML estimator of $\boldsymbol{\theta} = (\alpha,\beta,\lambda)^\top$. Specifically, assume that regularity conditions are fulfilled in the interior of the parameter space but not on the boundary; see Cox and Hinkley (1974). Then, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta})$ is $\mathrm{N}_3(\mathbf{0},\boldsymbol{\Sigma}_{\boldsymbol{\theta}})$, where $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is the asymptotic variance–covariance matrix of $\widehat{\boldsymbol{\theta}}$, which can be approximated from the observed information matrix $\boldsymbol{K}(\boldsymbol{\theta}) = -\boldsymbol{J}(\boldsymbol{\theta})$, where $\boldsymbol{J}(\boldsymbol{\theta})$ is the Hessian matrix $\boldsymbol{J}(\boldsymbol{\theta}) = \partial^2\ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\,\partial\boldsymbol{\theta}^\top$, whose elements are

$$
\begin{aligned}
J_{\alpha\alpha} &= \frac{n}{\alpha^2} + \frac{6n}{\alpha^4} - \frac{4\lambda}{\alpha^2}\sum_{i=1}^{n} \frac{v_i\,\phi(v_i)}{1+\lambda-2\lambda\,\Phi(v_i)}\left(v_i^2 + \frac{v_i\,\phi(v_i)}{1+\lambda-2\lambda\,\Phi(v_i)} - 2\right)\\
&\quad - \frac{3}{4}\sum_{i=1}^{n}\left(\frac{t_i}{\beta} + \frac{\beta}{t_i}\right),\\
J_{\alpha\beta} &= \frac{2\lambda}{\alpha^2\beta}\sum_{i=1}^{n} \frac{\tau\big(\sqrt{t_i/\beta}\big)\,\phi(v_i)}{1+\lambda\big(1-2\Phi(v_i)\big)}\left(\frac{2\lambda\,v_i\,\phi(v_i)}{1+\lambda\big(1-2\Phi(v_i)\big)} + (v_i)^2 - 1\right)\\
&\quad - \frac{1}{\alpha^3\beta}\sum_{i=1}^{n}\left(\frac{t_i}{\beta} - \frac{\beta}{x_i}\right),\\
J_{\alpha\lambda} &= -\frac{2}{\alpha}\sum_{i=1}^{n}\left(\frac{v_i\,\phi(v_i)}{\big(1+\lambda-2\lambda\,\Phi(v_i)\big)^2}\right),\\
J_{\beta\lambda} &= \frac{2}{\alpha\beta}\sum_{i=1}^{n}\left(\frac{\tau\big(\sqrt{t_i/\beta}\big)\,\phi(v_i)}{\big(1+\lambda-2\lambda\,\Phi(v_i)\big)^2}\right),\\
J_{\beta\beta} &= \frac{\lambda}{\alpha\beta^2}\sum_{i=1}^{n}\frac{\phi(v_i)}{1+\lambda-2\lambda\,\Phi(v_i)}\\
&\quad \times\left(3\sqrt{\frac{t_i}{\beta}} - \sqrt{\frac{\beta}{t_i}} - \frac{\big(\tau(t_i/\beta)\big)^2}{\alpha}\left(v_i - \frac{2\lambda\,\phi(v_i)}{1+\lambda-2\lambda\,\Phi(v_i)}\right)\right)\\
&\quad + \frac{n}{2\beta^2} - \sum_{i=1}^{n}\frac{1}{(t_i+\beta)^2} - \frac{1}{\alpha^2\beta^3}\sum_{i=1}^{n} t_i,\\
J_{\lambda\lambda} &= -\sum_{i=1}^{n}\left(\frac{1-2\Phi(v_i)}{1+\lambda\big(1-2\Phi(v_i)\big)}\right)^2.
\end{aligned}
$$

Thus, this trivariate normal distribution can be used to construct approximate confidence intervals and regions for the model parameters. Note that asymptotic $100(1-\gamma/2)\%$ confidence intervals for α, β and λ are, respectively, established as

$$
\widehat{\alpha} \pm z_{1-\gamma/2}\big(\widehat{\mathrm{Var}}(\widehat{\alpha})\big)^{1/2}, \qquad \widehat{\beta} \pm z_{1-\gamma/2}\big(\widehat{\mathrm{Var}}(\widehat{\beta})\big)^{1/2}, \qquad \widehat{\lambda} \pm z_{1-\gamma/2}\big(\widehat{\mathrm{Var}}(\widehat{\lambda})\big)^{1/2},
$$

where $\widehat{\mathrm{Var}}(\widehat{\theta}_j)$ is the jth diagonal element of $\boldsymbol{K}^{-1}(\widehat{\boldsymbol{\theta}})$ related to each parameter θ_j, for $j = 1, 2, 3$, with $\theta_1 = \alpha$, $\theta_2 = \beta$, $\theta_3 = \lambda$, and $z_{\gamma/2}$ is the $100(1-\gamma/2)$th quantile of the standard normal distribution. Note that the estimated asymptotic standard errors (SEs) of the each estimator can be obtained from the square root of the diagonal element of $\boldsymbol{K}^{-1}(\widehat{\boldsymbol{\theta}})$.

3.2. Simulation study

We present a numerical experiment to evaluate the performance of the ML estimators $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\lambda}$. The simulation was performed using the `Ox` software. A number of 10000 MC replications were considered, sample sizes $n \in \{25, 50, 75, 100, 200, 400, 800\}$, the combination of the parameters $(\alpha, \beta) \in \{(0.10, 1.00), (0.50, 1.00), (1.50, 1.00), (2.00, 1.00)\}$ and $\lambda \in \{-0.80, -0.50, -0.20, 0.20, 0.50, 0.80\}$. Without loss of generality, we fix β at 1.00 in all experiments, because this is a scale parameter. Table 1 presents the empirical bias and square root of mean squared error of the estimators of the TBS distribution parameters. From this table, note that, generally, the bias decreases as n increases, evidencing that the ML estimators $\widehat{\alpha}$ and $\widehat{\beta}$ are asymptotically unbiased. Observe that, when varying the values of λ, the distributions of the estimators of α and β show, in general, symmetrical behaviors. In addition, when the parameter α increases, the bias of $\widehat{\beta}$ increases. Note also that the estimator $\widehat{\lambda}$ is more biased than $\widehat{\alpha}$ and $\widehat{\beta}$, considering all scenarios. Also in all of the cases, the square root of the mean square error decreases as n increases, proving that the ML estimators of the TBS distribution parameters have good precision, as known. It is important to mention that some iterations did not converge during the simulations, due possibly to the complexity of the function to be maximized or because of the difficulty to provide a good initial value from λ.

3.3. Influence diagnostics

Local influence is based on the curvature of the plane of the log-likelihood function; see Leiva *et al.* (2014b, 2016b). In the case of the TBS model given in (2.1), let $\boldsymbol{\theta} = (\alpha, \beta, \lambda)^\top$ and $\ell(\boldsymbol{\theta}|\boldsymbol{\omega})$ be the parameter vector and the log-likelihood function related to this model perturbed by $\boldsymbol{\omega}$, respectively. The perturbation vector $\boldsymbol{\omega}$ belongs to a subset $\Omega \in \mathbb{R}^n$ and $\boldsymbol{\omega}_0$ is an $n \times 1$ non-perturbation vector, such that $\ell(\boldsymbol{\theta}|\boldsymbol{\omega}_0) = \ell(\boldsymbol{\theta})$, for all $\boldsymbol{\theta}$. The corresponding likelihood distance (LD) is

$$\mathrm{LD}(\boldsymbol{\omega}) = 2\big(\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widehat{\boldsymbol{\theta}}_\omega)\big), \tag{3.2}$$

where $\widehat{\boldsymbol{\theta}}_\omega$ denotes the ML estimate of $\boldsymbol{\theta}$ upon the perturbed TBS model used to assess the influence of the perturbation on the ML estimate, whereas $\ell(\widehat{\boldsymbol{\theta}})$ is the usual likelihood function given in (3.1). Cook (1987) showed that the normal curvature for $\boldsymbol{\theta}$ in the direction of the vector $\boldsymbol{d}$, with $\|\boldsymbol{d}\| = 1$, is expressed as $C_d(\boldsymbol{\theta}) = 2|\boldsymbol{d}^\top \boldsymbol{\Delta}^\top \boldsymbol{J}(\boldsymbol{\theta})^{-1} \boldsymbol{\Delta}\, \boldsymbol{d}|$, where $\boldsymbol{\Delta}$ is a $3 \times n$ perturbation matrix with elements $\Delta_{ji} = \partial^2 \ell(\boldsymbol{\theta}|\boldsymbol{\omega})/\partial\theta_j\, \partial\omega_i$ evaluated at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, for $j = 1, 2, 3$, $i = 1, ..., n$, and $\boldsymbol{J}(\boldsymbol{\theta})$ is the corresponding Hessian matrix.

Table 1: Empirical bias and square root of mean squared error (in parentheses) of the parameter estimator for the indicated value of α, β, λ, n.

n	λ	$\alpha=0.10,\ \beta=1.00$			$\alpha=0.50,\ \beta=1.00$			$\alpha=1.50,\ \beta=1.00$			$\alpha=2.00,\ \beta=1.00$		
		$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$
25	−0.80	−0.0096 (0.0172)	0.0209 (0.0305)	0.3329 (0.3892)	−0.0509 (0.0871)	0.1198 (0.1722)	0.3472 (0.4095)	−0.1741 (0.2818)	0.3332 (0.5057)	0.3022 (0.3853)	−0.2717 (0.4122)	0.4616 (0.6787)	0.3296 (0.4115)
	−0.50	−0.0038 (0.0152)	0.0064 (0.0243)	0.0926 (0.2381)	−0.0177 (0.0763)	0.0331 (0.1222)	0.0760 (0.2328)	−0.0670 (0.2359)	0.0970 (0.3309)	0.0502 (0.2467)	−0.1000 (0.3250)	0.1457 (0.4034)	0.0898 (0.2494)
	−0.20	−0.0013 (0.0147)	−0.0087 (0.0248)	−0.1734 (0.2738)	−0.0048 (0.0745)	−0.0430 (0.1179)	−0.2053 (0.3139)	−0.0221 (0.2232)	−0.0581 (0.2662)	−0.1827 (0.2988)	−0.0306 (0.3040)	−0.0315 (0.3069)	−0.1447 (0.2613)
	0.20	−0.0012 (0.0145)	0.0093 (0.0253)	0.1731 (0.2738)	−0.0064 (0.0738)	0.0500 (0.1292)	0.1952 (0.3005)	−0.0138 (0.2236)	0.1562 (0.3613)	0.1920 (0.3042)	−0.0247 (0.3055)	0.1369 (0.3875)	0.1453 (0.2637)
	0.50	−0.0042 (0.0152)	−0.0061 (0.0236)	−0.1010 (0.2436)	−0.0196 (0.0765)	−0.0181 (0.1128)	−0.0736 (0.2416)	−0.0673 (0.2380)	−0.0104 (0.2817)	−0.0511 (0.2469)	−0.1038 (0.3267)	−0.0357 (0.3175)	−0.0896 (0.2508)
	0.80	−0.0099 (0.0175)	−0.0198 (0.0292)	−0.3239 (0.3822)	−0.0484 (0.0869)	−0.0897 (0.1338)	−0.3350 (0.3960)	−0.1726 (0.2788)	−0.1858 (0.2976)	−0.3058 (0.3889)	−0.2662 (0.4045)	−0.2336 (0.3510)	−0.3278 (0.4107)
50	−0.80	−0.0070 (0.0129)	0.0165 (0.0251)	0.2627 (0.3466)	−0.0386 (0.0667)	0.0962 (0.1406)	0.2907 (0.3771)	−0.1295 (0.2199)	0.2662 (0.4066)	0.2680 (0.3640)	−0.1916 (0.3083)	0.3345 (0.5025)	0.2754 (0.3675)
	−0.50	−0.0015 (0.0110)	0.0030 (0.0200)	0.0506 (0.2407)	−0.0062 (0.0560)	0.0160 (0.0988)	0.0420 (0.2400)	−0.0244 (0.1733)	0.0541 (0.2515)	0.0360 (0.2483)	−0.0395 (0.2321)	0.0862 (0.2943)	0.0650 (0.2415)
	−0.20	0.0008 (0.0106)	−0.0108 (0.0220)	−0.1897 (0.2940)	0.0067 (0.0549)	−0.0527 (0.1037)	−0.2051 (0.3136)	0.0179 (0.1645)	−0.0743 (0.2149)	−0.1669 (0.2879)	0.0055 (0.2139)	−0.0476 (0.2298)	−0.1282 (0.2489)
	0.20	0.0008 (0.0106)	0.0108 (0.0223)	0.1893 (0.2928)	0.0059 (0.0544)	0.0596 (0.1177)	0.1942 (0.3017)	0.0173 (0.1646)	0.1312 (0.2860)	0.1668 (0.2870)	0.0104 (0.2168)	0.1156 (0.2930)	0.1283 (0.2492)
	0.50	−0.0018 (0.0108)	−0.0033 (0.0195)	−0.0618 (0.2423)	−0.0066 (0.0550)	−0.0070 (0.0960)	−0.0432 (0.2405)	−0.0203 (0.1732)	0.0066 (0.2374)	−0.0315 (0.2462)	−0.0427 (0.2356)	−0.0235 (0.2572)	−0.0696 (0.2437)
	0.80	−0.0068 (0.0128)	−0.0159 (0.0244)	−0.2663 (0.3514)	−0.0374 (0.0652)	−0.0777 (0.1149)	−0.2848 (0.3678)	−0.1275 (0.2167)	−0.1605 (0.2584)	−0.2659 (0.3639)	−0.1932 (0.3104)	−0.1912 (0.2992)	−0.2727 (0.3694)
75	−0.80	−0.0057 (0.0110)	0.0144 (0.0230)	0.2320 (0.3297)	−0.0293 (0.0560)	0.0764 (0.1202)	0.2270 (0.3430)	−0.1118 (0.1889)	0.2359 (0.3620)	0.2490 (0.3488)	−0.1584 (0.2681)	0.2803 (0.4342)	0.2371 (0.3370)
	−0.50	−0.0005 (0.0095)	0.0018 (0.0185)	0.0342 (0.2449)	−0.0023 (0.0469)	0.0107 (0.0909)	0.0250 (0.2409)	−0.0075 (0.1480)	0.0419 (0.2244)	0.0307 (0.2446)	−0.0213 (0.1989)	0.0661 (0.2504)	0.0566 (0.2322)
	−0.20	0.0017 (0.0093)	−0.0114 (0.0211)	−0.1991 (0.3046)	0.0101 (0.0473)	−0.0504 (0.0999)	−0.2012 (0.3082)	0.0200 (0.1370)	−0.0656 (0.1905)	−0.1422 (0.2664)	0.0134 (0.1761)	−0.0500 (0.1988)	−0.1083 (0.2278)
	0.20	0.0014 (0.0091)	0.0112 (0.0213)	0.1909 (0.2965)	0.0089 (0.0467)	0.0603 (0.1115)	0.1785 (0.3028)	0.0221 (0.1384)	0.1120 (0.2508)	0.1414 (0.2652)	0.0164 (0.1781)	0.0940 (0.2485)	0.1104 (0.2307)
	0.50	−0.0009 (0.0093)	−0.0024 (0.0181)	−0.0471 (0.2400)	−0.0027 (0.0469)	−0.0046 (0.0872)	−0.0318 (0.2384)	−0.0084 (0.1491)	0.0038 (0.2200)	−0.0311 (0.2458)	−0.0207 (0.1960)	−0.0140 (0.2333)	−0.0570 (0.2332)
	0.80	−0.0059 (0.0112)	−0.0144 (0.0226)	−0.2426 (0.3408)	−0.0307 (0.0570)	−0.0681 (0.1063)	−0.2469 (0.3442)	−0.1069 (0.1887)	−0.1469 (0.2402)	−0.2408 (0.3450)	−0.1571 (0.2657)	−0.1661 (0.2691)	−0.2337 (0.3359)
100	−0.80	−0.0048 (0.0099)	0.0130 (0.0217)	0.2072 (0.3183)	−0.0286 (0.0528)	0.0758 (0.1195)	0.2349 (0.3233)	−0.1012 (0.1730)	0.2152 (0.3348)	0.2272 (0.3296)	−0.1346 (0.2385)	0.2401 (0.3852)	0.2091 (0.3107)
	−0.50	0.0000 (0.0084)	0.0008 (0.0180)	0.0204 (0.2489)	0.0005 (0.0430)	0.0072 (0.0869)	0.0176 (0.2447)	−0.0053 (0.1353)	0.0418 (0.2114)	0.0369 (0.2440)	−0.0147 (0.1790)	0.0539 (0.2296)	0.0490 (0.2259)
	−0.20	0.0020 (0.0082)	−0.0111 (0.0203)	−0.1950 (0.3011)	0.0120 (0.0430)	−0.0527 (0.0967)	−0.1909 (0.2905)	0.0217 (0.1219)	−0.0596 (0.1740)	−0.1236 (0.2492)	0.0157 (0.1562)	−0.0396 (0.1763)	−0.0885 (0.2105)
	0.20	0.0022 (0.0081)	0.0116 (0.0208)	0.1910 (0.2955)	0.0116 (0.0424)	0.0598 (0.1087)	0.1895 (0.2931)	0.0218 (0.1231)	0.1005 (0.2302)	0.1266 (0.2516)	0.0131 (0.1557)	0.0750 (0.2141)	0.0883 (0.2091)
	0.50	−0.0004 (0.0084)	−0.0015 (0.0176)	−0.0330 (0.2399)	−0.0010 (0.0418)	−0.0020 (0.0853)	−0.0257 (0.2366)	−0.0026 (0.1342)	0.0052 (0.2048)	−0.0308 (0.2406)	−0.0140 (0.1788)	−0.0028 (0.2171)	−0.0481 (0.2262)
	0.80	−0.0049 (0.0098)	−0.0126 (0.0212)	−0.2129 (0.3250)	−0.0285 (0.0510)	−0.0659 (0.1025)	−0.2401 (0.3448)	−0.0965 (0.1712)	−0.1363 (0.2235)	−0.2255 (0.3300)	−0.1342 (0.2384)	−0.1455 (0.2500)	−0.2093 (0.3116)
200	−0.80	−0.0031 (0.0079)	0.0091 (0.0187)	0.1530 (0.2852)	−0.0209 (0.0434)	0.0591 (0.1046)	0.1863 (0.3146)	−0.0744 (0.1414)	0.1624 (0.2709)	0.1780 (0.2829)	−0.0935 (0.1885)	0.1619 (0.2863)	0.1455 (0.2440)
	−0.50	0.0009 (0.0071)	−0.0007 (0.0168)	−0.0011 (0.2470)	0.0040 (0.0357)	0.0011 (0.0810)	0.0035 (0.2458)	0.0070 (0.1115)	0.0187 (0.1767)	0.0160 (0.2201)	−0.0003 (0.1376)	0.0301 (0.1785)	0.0300 (0.1906)
	−0.20	0.0024 (0.0067)	−0.0107 (0.0192)	−0.1820 (0.2963)	0.0121 (0.0352)	−0.0463 (0.0884)	−0.1712 (0.2969)	0.0138 (0.0870)	−0.0353 (0.1346)	−0.0717 (0.1941)	0.0106 (0.1097)	−0.0261 (0.1331)	−0.0525 (0.1657)
	0.20	0.0022 (0.0066)	0.0108 (0.0193)	0.1767 (0.2896)	0.0119 (0.0344)	0.0544 (0.1005)	0.1704 (0.2758)	0.0157 (0.0887)	0.0578 (0.1692)	0.0739 (0.1976)	0.0092 (0.1081)	0.0427 (0.1522)	0.0503 (0.1638)
	0.50	0.0003 (0.0068)	−0.0005 (0.0165)	−0.0202 (0.2386)	0.0024 (0.0345)	0.0005 (0.0802)	−0.0187 (0.2351)	0.0057 (0.1102)	0.0121 (0.1813)	−0.0167 (0.2213)	−0.0002 (0.1395)	0.0027 (0.1812)	−0.0276 (0.1928)
	0.80	−0.0033 (0.0080)	−0.0096 (0.0189)	−0.1635 (0.2992)	−0.0229 (0.0424)	−0.0565 (0.0943)	−0.2092 (0.3319)	−0.0729 (0.1410)	−0.1092 (0.1947)	−0.1794 (0.2867)	−0.0871 (0.1857)	−0.0977 (0.2032)	−0.1414 (0.2419)
400	−0.80	−0.0019 (0.0066)	0.0061 (0.0164)	0.1043 (0.2538)	−0.0139 (0.0369)	0.0436 (0.0923)	0.1391 (0.2816)	−0.0518 (0.1143)	0.1117 (0.2098)	0.1254 (0.2238)	−0.0547 (0.1498)	0.0962 (0.2075)	0.0886 (0.1817)
	−0.50	0.0011 (0.0060)	−0.0009 (0.0160)	−0.0048 (0.2436)	0.0053 (0.0308)	−0.0006 (0.0759)	0.0005 (0.2387)	0.0106 (0.0902)	0.0048 (0.1430)	0.0047 (0.1874)	0.0061 (0.1072)	0.0097 (0.1365)	0.0068 (0.1525)
	−0.20	0.0022 (0.0055)	−0.0097 (0.0176)	−0.1595 (0.2790)	0.0116 (0.2770)	−0.0387 (0.0813)	−0.1410 (0.0298)	0.0072 (0.0592)	−0.0142 (0.0994)	−0.0323 (0.1424)	0.0040 (0.0760)	−0.0064 (0.0980)	−0.0165 (0.1215)
	0.20	0.0019 (0.0054)	0.0097 (0.0178)	0.1567 (0.2751)	0.0109 (0.0287)	0.0439 (0.0902)	0.1348 (0.2673)	0.0070 (0.0593)	0.0269 (0.1108)	0.0317 (0.1415)	0.0043 (0.0764)	0.0178 (0.1041)	0.0188 (0.1220)
	0.50	0.0006 (0.0057)	−0.0002 (0.0153)	−0.0164 (0.2352)	0.0042 (0.0298)	0.0028 (0.0757)	−0.0135 (0.2301)	0.0098 (0.0905)	0.0177 (0.1572)	−0.0041 (0.1877)	0.0056 (0.1061)	0.0068 (0.1431)	−0.0108 (0.1529)
	0.80	−0.0021 (0.0066)	−0.0066 (0.0166)	−0.1165 (0.2671)	−0.0186 (0.0371)	−0.0497 (0.0903)	−0.1870 (0.3240)	−0.0502 (0.1138)	−0.0739 (0.1588)	−0.1213 (0.2242)	−0.0503 (0.1477)	−0.0577 (0.1659)	−0.0854 (0.1807)
800	−0.80	−0.0011 (0.0055)	0.0041 (0.0141)	0.0728 (0.2216)	−0.0095 (0.0320)	0.0322 (0.0808)	0.1045 (0.2482)	−0.0348 (0.0951)	0.0733 (0.1612)	0.0821 (0.1731)	−0.0244 (0.1233)	0.0498 (0.1568)	0.0474 (0.1400)
	−0.50	0.0010 (0.0051)	−0.0006 (0.0147)	0.0010 (0.2286)	0.0050 (0.0266)	−0.0001 (0.0707)	0.0043 (0.2266)	0.0091 (0.0713)	−0.0015 (0.1112)	−0.0037 (0.1466)	0.0061 (0.0765)	0.0008 (0.0969)	−0.0012 (0.1116)
	−0.20	0.0018 (0.0045)	−0.0074 (0.0155)	−0.1224 (0.2494)	0.0086 (0.0239)	−0.0266 (0.0681)	−0.0960 (0.2350)	0.0030 (0.0407)	−0.0037 (0.0730)	−0.0079 (0.1063)	0.0017 (0.0523)	0.0007 (0.0723)	−0.0028 (0.0918)
	0.20	0.0018 (0.0045)	0.0076 (0.0160)	0.1218 (0.2506)	0.0079 (0.0233)	0.0307 (0.0769)	0.0925 (0.2305)	0.0027 (0.0410)	0.0093 (0.0760)	0.0097 (0.1069)	0.0005 (0.0525)	0.0040 (0.0736)	0.0012 (0.0910)
	0.50	0.0004 (0.0047)	−0.0011 (0.0141)	−0.0290 (0.2227)	0.0040 (0.0261)	0.0028 (0.0708)	−0.0108 (0.2230)	0.0078 (0.0707)	0.0124 (0.1238)	−0.0005 (0.1465)	0.0033 (0.0749)	0.0057 (0.1015)	−0.0036 (0.1113)
	0.80	−0.0013 (0.0055)	−0.0043 (0.0141)	−0.0777 (0.2273)	−0.0157 (0.0331)	−0.0424 (0.0827)	−0.1607 (0.2990)	−0.0293 (0.0925)	−0.0437 (0.1311)	−0.0743 (0.1723)	−0.0229 (0.1234)	−0.0258 (0.1416)	−0.0449 (0.1384)

A local influence diagnostic is generally based on index plots. For example, the index graph of the eigenvector $\boldsymbol{d}_{\max}$ related to the maximum eigenvalue of $\boldsymbol{B}(\boldsymbol{\theta}) = -\boldsymbol{\Delta}^\top \boldsymbol{J}(\boldsymbol{\theta})^{-1}\boldsymbol{\Delta}$, $C_{d_{\max}}(\boldsymbol{\theta})$ say, evaluated at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, can detect those cases that, under small perturbations, exercise a high influence on $\mathrm{LD}(\boldsymbol{\omega})$ given in (3.2). In addition to the direction vector of maximum normal curvature, $\boldsymbol{d}_{\max}$ say, another direction of interest is $\boldsymbol{d}_i = \boldsymbol{e}_{in}$, which corresponds to the direction of the case i, where $\boldsymbol{e}_{in}$ is an $n \times 1$ vector of zeros with a value equal to one at the ith position, that is, $\{\boldsymbol{e}_{in}, 1 \leq i \leq n\}$ is the canonical basis of $\mathbb{R}^n$. Thus, the normal curvature is $C_i(\boldsymbol{\theta}) = 2|b_{ii}|$, where b_{ii} is the ith diagonal element of $\boldsymbol{B}(\boldsymbol{\theta})$, for $i = 1, ..., n$, evaluated at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$. The case i is considered as potentially influential if $C_i(\widehat{\boldsymbol{\theta}}) > 2\overline{C}(\widehat{\boldsymbol{\theta}})$, where $\overline{C}(\widehat{\boldsymbol{\theta}}) = \sum_{i=1}^n C_i(\widehat{\boldsymbol{\theta}})/n$. This procedure is called total local influence of the case i; see Liu *et al.* (2016).

Consider the log-likelihood function given in (3.1). We obtain the respective perturbation matrix $\boldsymbol{\Delta}$, which is already evaluated at the non-perturbation vector $\boldsymbol{\omega}_0$, under the scheme of case-weight perturbation. Then, we want to evaluate whether cases with different weights in the log-likelihood function affect the ML estimate of $\boldsymbol{\theta}$. This scheme is the most used to assess local influence in a model. The log-likelihood function of the TBS model perturbed by the case-weight scheme is

$$\ell(\boldsymbol{\theta}|\boldsymbol{\omega}) = \sum_{i=1}^n \ell_i(\boldsymbol{\theta}|\omega_i) = \sum_{i=1}^n \omega_i \, \ell_i(\boldsymbol{\theta}).$$

Then, taking its derivative with respect to $\boldsymbol{\omega}^\top$, we obtain $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\beta, \boldsymbol{\Delta}_\alpha, \boldsymbol{\Delta}_\lambda)^\top$. After evaluating at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, the elements of $\boldsymbol{\Delta}_\alpha$, $\boldsymbol{\Delta}_\beta$ and $\boldsymbol{\Delta}_\lambda$ are

$$\begin{aligned}
\Delta_\alpha^{(i)} &= -\frac{1}{\alpha}\left(1 + \frac{2}{\alpha^2}\right) + \frac{1}{\alpha^3}\left(\frac{t_i}{\beta} + \frac{\beta}{t_i}\right) + \frac{2\lambda\, v_i\, \phi(v_i)}{\alpha\big(1 + \lambda - 2\lambda\,\Phi(v_i)\big)}, \qquad i = 1, ..., n, \\
\Delta_\lambda^{(i)} &= \frac{1 - 2\Phi(v_i)}{1 + \lambda\big(1 - 2\Phi(v_i)\big)}, \\
\Delta_\beta^{(i)} &= -\frac{1}{2\beta} + \frac{1}{t_i + \beta} + \frac{1}{2\alpha^2\beta}\left(\frac{t_i}{\beta} + \frac{\beta}{t_i}\right) - \frac{2\lambda}{\alpha\beta}\left(\frac{\tau\big(\sqrt{t_i/\beta}\big)\,\phi(v_i)}{1 + \lambda - 2\lambda\,\Phi(v_i)}\right).
\end{aligned}$$

4. APPLICATIONS TO REAL-WORLD DATA

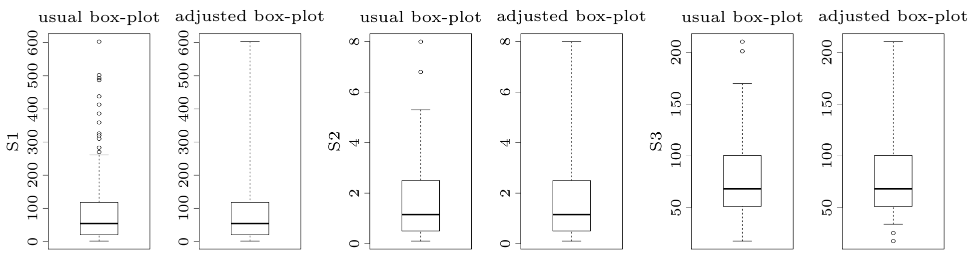
In this section, we apply the obtained results for the new model to three data sets, illustrating its potential applications. The results are compared to other competing BS distributions. All the computations were done using the `Ox` software. For each data set, we estimate the unknown parameters of the associated distribution by the ML method and evaluate its goodness of fit with suitable methods.

4.1. Exploratory analysis

The first data set (S1) corresponds to the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes ($n = 188$); S1 can be obtained from Proschan (1963). The second data set (S2) is related to vinyl chloride concentration (in mg/L) obtained from clean upgradient monitoring wells ($n = 34$); S2 can be obtained from Bhaumik *et al.* (2009). The third data set (S3) corresponds to protein amount (in g) in the restricted diet for adult patients in a Chilean hospital ($n = 61$); S3 and more details about these data can be obtained from Leiva *et al.* (2014a). Table 2 provides some descriptive measures for the three data sets, which include central tendency statistics, the standard deviation (SD) and the coefficients of variation (CV), skewness (CS) and kurtosis (CK), among others. From these exploratory analyses, we detect asymmetrical distributions with positive skewness in all of the cases and different kurtosis levels. Figure 5 (first panel/row) shows the histograms of S1, S2 and S3, from which it is possible to observe these features.

Table 2: Descriptives statistics for the indicated data set.

Statistic	Data set		
	S1	S2	S3
n	188	34	61
Minimum	1.0	0.10	17.8
Median	54.0	1.15	68.2
Mean	92.7	1.88	80.4
Maximum	603.0	8.00	210.3
SD	107.9	1.95	42.3
CS	2.1	1.53	1.2
CK	4.9	1.72	4.0



(a) S1 (b) S2 (c) S3

Figure 4: Box-plots for the indicated data set.

Figure 4 displays the usual and adjusted box-plots, where the latter is useful in cases when the data follow a skew distribution; see Rousseeuw *et al.* (2016). From Figure 4, note that potential outliers considered by the usual box-plot are not outliers in the adjusted box-plot. This is an indication that no outliers are present at the right tail in all of the studied data sets. Figure 5 (second panel/row) confirms these facts by means of the influence index plots, which do not detect atypical cases. Therefore, the TBS distribution can be a good candidate for modeling these data sets. We compare the TBS distribution to other generalizations of the BS distribution, such as the three-parameter MOBS, exponentiated BS (EBS) and two-parameter BS distributions with the EBS and MOBS PDFs being: $f_{\mathrm{EBS}}(x;\alpha,\beta,a) = a f_{\mathrm{BS}}(x;\alpha,\beta)\, F_{\mathrm{BS}}(x;\alpha,\beta)^{a-1}$, for $x>0$, $a>0$, and $f_{\mathrm{MOBS}}(x;\eta,\alpha,\beta) = \eta f_{\mathrm{BS}}(x;\alpha,\beta)/\big(1-(1-\eta)(1-F_{\mathrm{BS}}(x;\alpha,\beta))\big)^2$, for $x>0$, $\eta>0$.

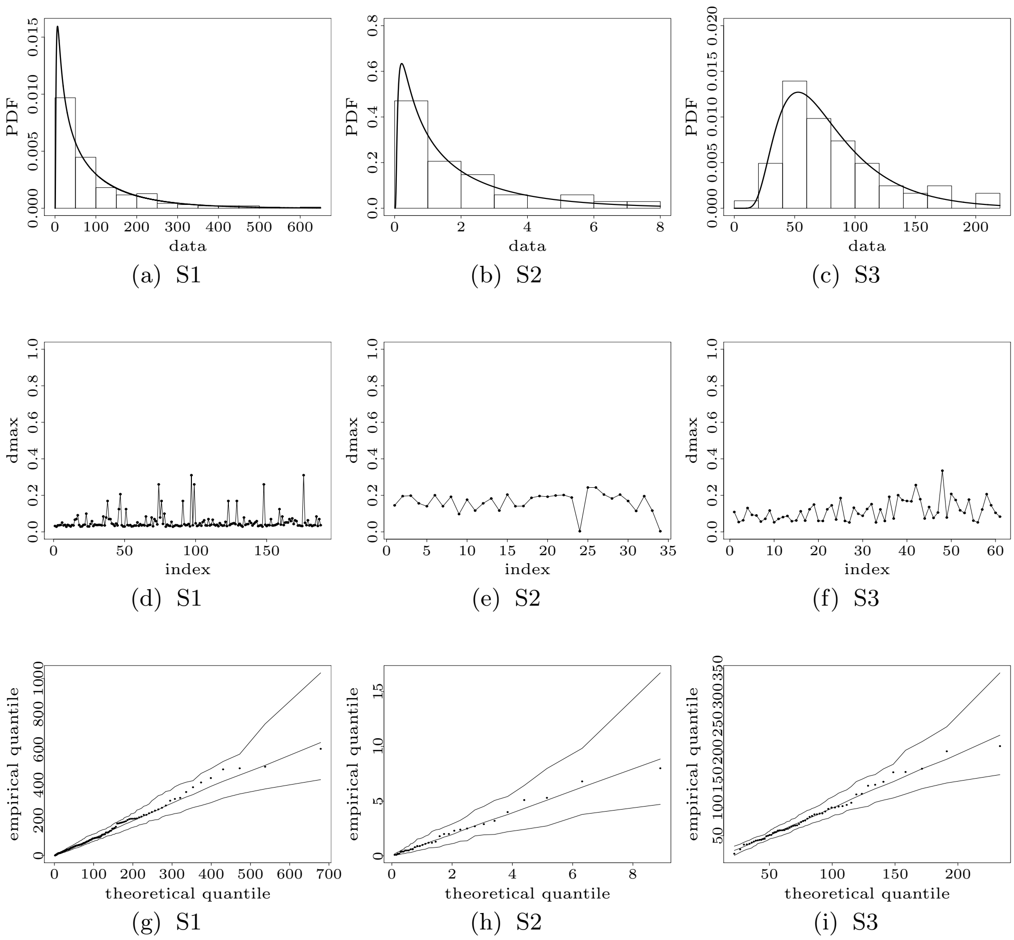


Figure 5: Histograms with estimated PDF (first panel/row), influence index (second panel/row) and plots QQ plots with envelope (third panel/row) for the TBS distribution based on the indicated data set.

4.2. Confirmatory analysis

Table 3 lists the ML estimates of the parameters and the estimated asymptotic SEs in parentheses of the corresponding estimators for the four distributions fitted to S1, S2 and S3. From this table and using the asymptotic distributions of the ML estimators proved by the simulation study of Section 3.2, we evaluate whether the additional parameters of the EBS, MOBS and TBS distributions are significatively different from zero or not for each data set. Note that, for S1, the additional parameter is always significatively different from zero at 5% for all EBS, MOBS and TBS distributions, indicating that the BS distribution should model S1 poorly. This is not the case of S2 and S3, where only the MOBS parameter is significatively different from zero at 5% in both cases.

Table 3: ML estimates (with estimated SE in parenthesis) for the indicated parameter, distribution and data set.

Distribution	S1			S2			S3		
	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$
TBS(α, β, λ)	1.7432	23.407	-0.8649	1.3435	0.7525	-0.5496	0.5207	73.138	0.1133
	(0.1347)	(3.4801)	(0.1179)	(0.2526)	(0.3250)	(0.5869)	(0.0495)	(17.662)	(0.8308)
MOBS(η, α, β)	2.1975	1.5556	26.2253	1.8193	0.7451	1.2899	0.9127	0.5197	72.6927
	(0.5016)	(0.0923)	(4.3453)	(1.0968)	(0.1691)	(0.2793)	(0.7345)	(0.0471)	(17.025)
EBS(α, β, a)	2.1790	13.5871	2.5381	1.6597	0.4705	2.0973	0.4290	92.414	0.5295
	(0.2755)	(4.4206)	(0.5224)	(0.5221)	(0.3889)	(1.3836)	(0.7842)	(209.16)	(3.1384)
BS(α, β)	1.5147	41.3240	—	1.2745	1.0203	—	0.5199	70.857	—
	(0.0783)	(3.4959)	—	(0.1546)	(0.1826)	—	(0.0471)	(4.5565)	—

To confirm these facts, we apply goodness-of-fit tests detecting what distribution adjusts better each data set. We consider the Anderson–Darling (AD), Cramér–von Mises (CM) and Kolmogorov–Smirnov (KS) statistics; see Barros *et al.* (2014). Table 4 provides the p-values of the corresponding tests for S1, S2 and S3.

Table 4: p-value of the indicated statistic, model and data set.

Distribution	S1			S2			S3		
	KS	CM	AD	KS	CM	AD	KS	CM	AD
TBS	0.7919	0.5819	0.4715	0.9820	0.9332	0.9215	0.9996	0.9459	0.9202
MOBS	0.4789	0.1337	0.0903	0.9765	0.9167	0.9126	0.9995	0.9409	0.9160
EBS	0.7369	0.2015	0.1568	0.9828	0.9257	0.9212	0.9967	0.8925	0.8810
BS	0.1064	0.0501	0.0166	0.8441	0.8253	0.7129	0.9985	0.9409	0.9001

Thus, according to these tests, the TBS distribution fits the three data sets better than the other distributions, that is, such p-values indicate that all of the null hypotheses are strongly not rejected for the TBS distribution. Also, we compare the four distributions using the Akaike (AIC) and Bayesian (BIC) information criteria, as well as the Bayes factor (BF) to evaluate the magnitude of the difference between two BIC values; see Kass and Raftery (1995). Note that the BF coincides with the likelihood ratio test for nested models. We compute the AIC and BIC for the four distributions, whereas the BF is obtained to compare the distribution having a smaller BIC to the others. Decision about the best fit is made according to the interpretation of the BF presented in Table 6 of Leiva *et al.* (2015b). Table 5 provides the values of AIC, BIC and BF, indicating that the TBS distribution provides the best fit for S1 and a very competitive performance for S2 and S3.

Table 5: Value of the information criteria and BF for indicated model and data set.

Distribution	S1			S2			S3		
	AIC	BIC	BF	AIC	BIC	BF	AIC	BIC	BF
TBS	1467.1000	1476.8000	—	5.0275	9.6065	2.5891	418.7600	425.1000	4.0939
MOBS	1471.6000	1481.3000	4.4000	4.9694	9.5485	2.5311	418.7700	425.1000	4.1100
EBS	1469.0000	1478.7000	1.8000	5.0111	9.5902	2.5728	418.7800	425.1100	4.1000
BS	1481.5000	1488.0000	11.2000	3.9647	7.0174	—	416.7800	421.0000	—

These good results of the TBS distribution can be supported graphically in Figure 5, which displays the histograms with the estimated TBS PDFs (first panel/row) and the quantile versus quantile (QQ) plot with envelope (third panel/row). The QQ plot allows us to compare the empirical CDF and the estimated TBS CDF. All of these results of goodness of fit allow us to conclude the superiority of the TBS distribution in relation to the BS, EBS and MOBS distributions to model S1, S2 and S3. This shows the potential of the TBS distribution and the importance of the additional parameter. In addition, because the TBS distribution presents the best fit to the studied data sets, we analyze the influence of small perturbations in the ML estimates of its parameters. We use the scheme of case-weight perturbation. Figure 5 (second panel/row) sketches the influence index plot based on the TBS distribution for each data set. An inspection of these plots reveals that, as mentioned, none case appears with outstanding influence on the ML estimates of the TBS distribution parameters.

5. CONCLUSIONS AND FUTURE RESEARCH

We have used the transmutation method to define a new distribution that generalizes the Birnbaum–Saunders model, named the transmuted Birnbaum–Saunders distribution. Some relevant characteristics of the new distribution have been derived, such as the probabilistic functions, as well moments and a generator of random numbers. We have estimated the model parameters with the maximum likelihood method and its good performance has been evaluated by means of Monte Carlo simulations. Score vector and Hessian matrix were derived to infer about the model parameters. Diagnostic tools have been obtained to detect locally influential data in the maximum likelihood estimates. Potential applications of the new distribution have been considered by using three real-world data sets. Goodness-of-fit methods have demonstrated the suitable performance of the transmuted Birnbaum–Saunders distribution to these data in comparison to other versions of the Birnbaum–Saunders distribution. We hope that the new proposed distribution may attract wider applications in statistics. Modeling based on fixed, random and mixed effects, including semi-parametric formulations and non-parametric estimation of kernel, can be conducted with this new distribution. Multivariate versions, as well as copula methods, could also be addressed by the new transmuted Birnbaum–Saunders distribution.

ACKNOWLEDGMENTS

The authors thank the editors and reviewers for their constructive comments on an earlier version of this manuscript which resulted in this improved version. This research was supported by Capes and CNPq from Brazil, and FONDECYT 1160868 grant from Chile.

REFERENCES

[1] Ahmed, S.; Castro-Kuriss, C.; Leiva, V.; Flores, E. and Sanhueza, A. (2010). Truncated version of the Birnbaum–Saunders distribution with an application in financial risk, *Pakistan Journal of Statistics*, **26**, 293–311.

[2] Aryal, G. (2013). Transmuted log-logistic distribution, *Journal of Statistics Applications and Probability*, **2**, 11–20.

[3] Aryal, G. and Tsokos, C. (2009). On the transmuted extreme value distribution with application, *Nonlinear Analysis: Theory, Methods and Applications*, **71**, 1401–1407.

[4] Aryal, G. and Tsokos, C. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution, *European Journal of Pure and Applied Mathematics*, **4**, 89–102.

[5] Ashour, S. and Eltehiwy, M. (2013). Transmuted Lomax distribution, *American Journal of Applied Mathematics and Statistics*, **1**, 121–127.

[6] Athayde, E.; Azevedo, C.; Leiva, V. and Sanhueza, A. (2012). About Birnbaum–Saunders distributions based on the Johnson system, *Communications in Statistics: Theory and Methods*, **41**, 2061–2079.

[7] Azevedo, C.; Leiva, V.; Athayde, E. and Balakrishnan, N. (2012). Shape and change point analyses of the Birnbaum–Saunders-t hazard rate and associated estimation, *Computational Statistics and Data Analysis*, **56**, 3887–3897.

[8] Azzalini, A. (1985). A class of distributions which includes the normal ones, *Scandinavian Journal of Statistics*, **11**, 171–178.

[9] Balakrishnan, N.; Gupta, R.; Kundu, D.; Leiva, V. and Sanhueza, A. (2011). On some mixture models based on the Birnbaum–Saunders distribution and associated inference, *Journal of Statistical Planning and Inference*, **141**, 2175–2190.

[10] Barros, M.; Leiva, V.; Ospina, R. and Tsuyuguchi, A. (2014). Goodness-of-fit tests for the Birnbaum–Saunders distribution with censored reliability data, *IEEE Transactions on Reliability*, **63**, 543–554.

[11] Bhaumik, D.; Kapur, K. and Gibbons, R. (2009). Testing parameters of a gamma distribution for small samples, *Technometrics*, **51**, 326–334.

[12] Birnbaum, Z.W. and Saunders, S.C. (1969a). Estimation for a family of life distributions with applications to fatigue, *Journal of Applied Probability*, **6**, 328–347.

[13] Birnbaum, Z.W. and Saunders, S.C. (1969b). A new family of life distributions, *Journal of Applied Probability*, **6**, 319–327.

[14] Bourguignon, M.; Silva, R. and Cordeiro, G. (2014). A new class of fatigue life distributions, *Journal of Statistical Computation and Simulation*, **84**, 2619–2635.

[15] Castillo, N.; Gómez, H. and Bolfarine, H. (2011). Epsilon Birnbaum–Saunders distribution family: Properties and inference, *Statistical Papers*, **52**, 871–883.

[16] Cook, R.D. (1987). Influence assessment, *Journal of Applied Statistics*, **14**, 117–131.

[17] Cordeiro, M. and Lemonte, J. (2011). The β-Birnbaum–Saunders distribution: An improved distribution for fatigue life modeling, *Computational Statistics and Data Analysis*, **55**, 1445–1461.

[18] Cornish, E.A. and Fisher, R.A. (1937). Moments and cumulants in the specification of distributions, *Review of the International Statistical Institute*, **5**, 307–320.

[19] Cox, D.R. and Hinkley, D.V. (1974). *Theoretical Statistics*, Chapman and Hall, London, UK.

[20] Desousa, M.F.; Saulo, H.; Leiva, V. and Scalco, P. (2017). On a tobit-Birnbaum–Saunders model with an application to antibody response to vaccine, *Journal of Applied Statistics*, page in press available at `http://dx.doi.org/10.1080/02664763.2017.1322559`.

[21] Díaz-García, J. and Leiva, V. (2005). A new family of life distributions based on elliptically contoured distributions, *Journal of Statistical Planning and Inference*, **128**, 445–457.

[22] Doornik, J. (2006). *An Object-Oriented Matrix Language*, Timberlake Consultants Press, London, UK.

[23] Edgeworth, F.Y. (1917). On the mathematical representation of statistical data, *Journal of the Royal Statistical Society A*, **80**, 65–83; 266–288; 411–437.

[24] Ferreira, M.; Gomes, M.I. and Leiva, V. (2012). On an extreme value version of the Birnbaum–Saunders distribution, *REVSTAT Statistical Journal*, **10**, 181–210.

[25] Fierro, R.; Leiva, V.; Ruggeri, F. and Sanhueza, A. (2013). On a Birnbaum–Saunders distribution arising from a non-homogeneous Poisson process, *Statistical and Probability Letters*, **83**, 1233–1239.

[26] Garcia-Papani, F.; Uribe-Opazo, M.A.; Leiva, V. and Aykroyd, R.G. (2017). Birnbaum–Saunders spatial modelling and diagnostics applied to agricultural engineering data, *Stochastic Environmental Research and Risk Assessment*, **31**, 105–124.

[27] Gómez, H.; Olivares-Pacheco, J.F. and Bolfarine, H. (2009). An extension of the generalized Birnbaum–Saunders distribution, *Statistical and Probability Letters*, **79**, 331–338.

[28] Gradshteyn, I. and Randzhik, I. (2000). *Table of Integrals, Series, and Products*, Academic Press, New York, US.

[29] Guiraud, P.; Leiva, V. and Fierro, R. (2009). A non-central version of the Birnbaum–Saunders distribution for reliability analysis, *IEEE Transactions on Reliability*, **58**, 152–160.

[30] Johnson, N.; Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, volume 1, Wiley, New York, US.

[31] Johnson, N.; Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions*, volume 2, Wiley, New York, US.

[32] Johnson, N.L. (1949). Systems of frequency curves generated by methods of translation, *Biometrika*, **36**, 149–176.

[33] Kass, R. and Raftery, A. (1995). Bayes factors, *Journal of the American Statistical Association*, **90**, 773–795.

[34] Khan, M. and King, R. (2013). Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution, *European Journal of Pure and Applied Mathematics*, **6**, 66–88.

[35] Khan, M. and King, R. (2014). A new class of transmuted inverse Weibull distribution for reliability analysis, *American Journal of Mathematical and Management*, **33**, 261–286.

[36] Kotz, S.; Leiva, V. and Sanhueza, A. (2010). Two new mixture models related to the inverse Gaussian distribution, *Methodology and Computing in Applied Probability*, **12**, 199–212.

[37] Lange, K. (2001). *Numerical Analysis for Statisticians*, Springer, New York, US.

[38] Leão, J.; Leiva, V.; Saulo, H. and Tomazella, V. (2017a). A survival model with Birnbaum–Saunders frailty for uncensored and censored cancer data, *Brazilian Journal of Probability and Statistics*, page in press.

[39] Leão, J.; Leiva, V.; Saulo, H. and Tomazella, V. (2017b). Birnbaum–Saunders frailty regression models: Diagnostics and application to medical data, *Biometrical Journal*, **59**, 291–314.

[40] Leiva, V. (2016). *The Birnbaum–Saunders Distribution*, Academic Press, New York, US.

[41] Leiva, V.; Athayde, E.; Azevedo, C. and Marchant, C. (2011). Modeling wind energy flux by a Birnbaum–Saunders distribution with unknown shift parameter, *Journal of Applied Statistics*, **38**, 2819–2838.

[42] Leiva, V.; Ferreira, M.; Gomes, M.I. and Lillo, C. (2016a). Extreme value Birnbaum–Saunders regression models applied to environmental data, *Stochastic Environmental Research and Risk Assessment*, **30**, 1045–1058.

[43] Leiva, V.; Liu, S.; Shi, L. and Cysneiros, F. (2016b). Diagnostics in elliptical regression models with stochastic restrictions applied to econometrics, *Journal of Applied Statistics*, **43**, 627–642.

[44] Leiva, V.; Marchant, C.; Ruggeri, F. and Saulo, H. (2015a). A criterion for environmental assessment using Birnbaum–Saunders attribute control charts, *Environmetrics*, **26**, 463–476.

[45] Leiva, V.; Marchant, C.; Saulo, H.; Aslam, M. and Rojas, F. (2014a). Capability indices for Birnbaum–Saunders processes applied to electronic and food industries, *Journal of Applied Statistics*, **41**, 1881–1902.

[46] Leiva, V.; Rojas, E.; Galea, M. and Sanhueza, A. (2014b). Diagnostics in Birnbaum–Saunders accelerated life models with an application to fatigue data, *Applied Stochastic Models in Business and Industry*, **30**, 115–131.

[47] Leiva, V.; Ruggeri, F.; Saulo, H. and Vivanco, J.F. (2017). A methodology based on the Birnbaum–Saunders distribution for reliability analysis applied to nano-materials, *Reliability Engineering and System Safety*, **157**, 192–201.

[48] Leiva, V.; Sanhueza, A. and Angulo, J.M. (2009). A length-biased version of the Birnbaum–Saunders distribution with application in water quality, *Stochastic Environmental Research and Risk Assessment*, **23**, 299–307.

[49] Leiva, V.; Santos-Neto, M.; Cysneiros, F.J.A. and Barros, M. (2014c). Birnbaum–Saunders statistical modelling: A new approach, *Statistical Modelling*, **14**, 21–48.

[50] Leiva, V.; Santos-Neto, M.; Cysneiros, F.J.A. and Barros, M. (2016c). A methodology for stochastic inventory models based on a zero-adjusted Birnbaum–Saunders distribution, *Applied Stochastic Models in Business and Industry*, **32**, 74–89.

[51] Leiva, V.; Saulo, H.; Leão, J. and Marchant, C. (2014d). A family of autoregressive conditional duration models applied to financial data, *Computational Statistics and Data Analysis*, **79**, 175–191.

[52] Leiva, V.; Tejo, M.; Guiraud, P.; Schmachtenberg, O.; Orio, P. and Marmolejo, F. (2015b). Modeling neural activity with cumulative damage distributions, *Biological Cybernetics*, **109**, 421–433.

[53] Lemonte, A. (2013). A new extension of the Birnbaum–Saunders distribution, *Brazilian Journal of Probability and Statistics*, **27**, 133–149.

[54] Lillo, C.; Leiva, V.; Nicolis, O. and Aykroyd, R.G. (2016). L-moments of the Birnbaum–Saunders distribution and its extreme value version: Estimation, goodness of fit and application to earthquake data, *Journal of Applied Statistics*, page in press available at `http://dx.doi.org/10.1080/02664763.2016.1269729`.

[55] Liu, S.; Leiva, V.; Ma, T. and Welsh, A. (2016). Influence diagnostic analysis in the possibly heteroskedastic linear model with exact restrictions, *Statistical Methods and Applications*, **5**, 227–249.

[56] Louzada, F. and Granzotto, D. (2016). The transmuted log-logistic regression model: A new model for time up to first calving of cows, *Statistical Papers*, **57**, 623–640.

[57] Marchant, C.; Bertin, K.; Leiva, V. and Saulo, H. (2013). Generalized Birnbaum–Saunders kernel density estimators and an analysis of financial data, *Computational Statistics and Data Analysis*, **63**, 1–15.

[58] Marchant, C.; Leiva, V. and Cysneiros, F.J.A. (2016a). A multivariate log-linear model for Birnbaum–Saunders distributions, *IEEE Transactions on Reliability*, **65**, 816–827.

[59] Marchant, C.; Leiva, V.; Cysneiros, F.J.A. and Vivanco, J.F. (2016b). Diagnostics in multivariate generalized Birnbaum–Saunders regression models, *Journal of Applied Statistics*, **43**, 2829–2849.

[60] Martinez, G.; Bolfarine, H. and Gómez, H. (2014). An alpha-power extension for the Birnbaum–Saunders distribution, *Statistics*, **48**, 896–912.

[61] McNamee, J. and Pa, V. (2013). *Numerical Methods for Roots of Polynomials, Part II*, Academic Press, New York, US.

[62] Merovci, F. and Puka, L. (2014). Transmuted Pareto distribution, *ProbStat Forum*, **7**, 1–11.

[63] Mohammadi, K.; Alavi, O. and McGowan, J.G. (2017). Use of Birnbaum–Saunders distribution for estimating wind speed and wind power probability distributions: A review, *Energy Conversion and Management*, **143**, 109–122.

[64] Mroz, T. (2013a). Transmuted Lindley distribution, *International Journal of Open Problems in Computer Science*, **6**, 63–72.

[65] Mroz, T. (2013b). Transmuted Rayleigh distribution, *Austrian Journal of Statistics*, **42**, 21–31.

[66] Nocedal, J. and Wright, S. (1999). *Numerical Optimization*, Springer, New York, US.

[67] Owen, W. (2006). A new three-parameter extension to the Birnbaum–Saunders distribution, *IEEE Transactions on Reliability*, **55**, 475–479.

[68] Pearson, K. (1895). Mathematical contribution to the theory of evolution. II. Skew variation in homogeneous material, *Philosophical Transactions of the Royal Society of London A*, **186**, 343–414.

[69] Press, W.; Teulosky, S.; Vetterling, W. and Flannery, B. (2007). *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press, Cambridge, UK.

[70] Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate, *Technometrics*, **15**, 375–383.

[71] R Core Team (2016). *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria.

[72] Rojas, F.; Leiva, V.; Wanke, P. and Marchant, C. (2015). Optimization of contribution margins in food services by modeling independent component demand, *Revista Colombiana de Estadística*, **38**, 1–30.

[73] Rousseeuw, P.; Croux, C.; Todorov, V.; Ruckstuhl, A.; Salibian-Barrera, M.; Verbeke, T.; Koller, M. and Maechler, M. (2016). *robustbase: Basic Robust Statistics*, R package version 0.92-6.

[74] Saboor, A.; Kamal, M. and Ahmad, M. (2015). The transmuted exponential-Weibull distribution with applications, *Pakistan Journal of Statistics*, **13**, 229–250.

[75] Santos-Neto, M.; Cysneiros, F.J.A.; Leiva, V. and Ahmed, S. (2012). On new parameterizations of the Birnbaum–Saunders distribution, *Pakistan Journal of Statistics*, **28**, 1–26.

[76] Santos-Neto, M.; Cysneiros, F.J.A.; Leiva, V. and Barros, M. (2014). On new parameterizations of the Birnbaum–Saunders distribution and its moments, estimation and application, *REVSTAT Statistical Journal*, **12**, 247–272.

[77] Santos-Neto, M.; Cysneiros, F.J.A.; Leiva, V. and Barros, M. (2016). Reparameterized Birnbaum–Saunders regression models with varying precision, *Electronic Journal of Statistics*, **10**, 2825–2855.

[78] Saulo, H.; Leão, J.; Leiva, V. and Aykroyd, R.G. (2017). Birnbaum–Saunders autoregressive conditional duration models applied to high-frequency financial data, *Statistical Papers*, pages in press available at `http://dx.doi.org/10.1007/s00362-017-0888-6`.

[79] Saulo, H.; Leão, V. and Bourguignon, M. (2012). The Kumaraswamy Birnbaum–Saunders distribution, *Journal of Statistical Theory and Practice*, **1**, 1–13.

[80] Saulo, H.; Leiva, V.; Ziegelmann, F.A. and Marchant, C. (2013). A nonparametric method for estimating asymmetric densities based on skewed Birnbaum–Saunders distributions applied to environmental data, *Stochastic Environmental Research and Risk Assessment*, **27**, 1479–1491.

[81] Sharma, V.; Singh, S. and Singh, U. (2014). A new upside-down bathtub shaped hazard rate model for survival data analysis, *Applied Mathematics and Computation*, **239**, 242–253.

[82] Shaw, W. and Buckley, I. (2009). The alchemy of probability distributions: Beyond Gram–Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, *Working Paper*, available at `http://arxiv.org/abs/0901.0434v1`:1–8.

[83] Tiana, Y.; Tiana, M. and Zhu, Q. (2014). Transmuted linear exponential distribution: A new generalization of the linear exponential distribution, *Communications in Statistics: Simulation and Computation*, **43**, 2661–2677.

[84] Vilca, F. and Leiva, V. (2006). A new fatigue life model based on the family of skew-elliptical distributions, *Communications in Statistics: Theory and Methods*, **35**, 229–244.

[85] Vilca, F.; Sanhueza, A.; Leiva, V. and Christakos, G. (2010). An extended Birnbaum–Saunders model and its application in the study of environmental quality in Santiago, Chile, *Stochastic Environmental Research and Risk Assessment*, **24**, 771–782.

[86] Villegas, C.; Paula, G. and Leiva, V. (2011). Birnbaum–Saunders mixed models for censored reliability data analysis, *IEEE Transactions on Reliability*, **60**, 748–758.

[87] Volodin, I. and Dzhungurova, O. (2000). On limit distribution emerging in the generalized Birnbaum–Saunders model, *Journal of Mathematical Science*, **99**, 1348–1366.

[88] Wanke, P. and Leiva, V. (2015). Exploring the potential use of the Birnbaum–Saunders distribution in inventory management, *Mathematical Problems in Engineering*, Article ID 827246:1–9.