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OPEN MARKOV CHAIN SCHEME MODELS FED BY SECOND ORDER STATIONARY AND NON STATIONARY PROCESSES

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Abstract:

- We introduce a schematic formalism for the time evolution of a random open population divided into classes.

  With a Markov chain model, allowing for population entrances, we consider the flow of incoming members modeled by a time series - either ARIMA for the number of new incomings or SARMA for the residuals of a deterministic sigmoid type trend - and we detail the time series structure of the elements in each class.

  A practical application to real data from a credit portfolio is presented.

Key-Words:

- *Markov chains, Open Markov chain models, Second order processes, ARIMA, SARMA, Credit Risk.*

AMS Subject Classification:

- 60J10, 91D99, 62M10, 91B8.

1. INTRODUCTION

An usual application of a Markov chain model considers a *closed* population with each member being assigned a certain class at each date; the *random* transition of each element among the classes is governed by the transition probabilities. In the homogeneous case - the transition probabilities do not depend on the date at which the transitions occur - and, in the case where there are both transient and recurrent states in the Markov chain, the main emphasis is on the asymptotic behavior. Under that perspective, the transient type events do not matter on long run distributions. In a more realistic model, the population under scrutiny may be changing by the persistent arrival of new members and the events related to the so called "transient" states acquire new significance, as they may persist in time, as the inflow of new elements in the population continues indefinitely.

The consideration of Markov models with a population inflow, the so called *open* Markov models, may be set to start , according to [2], with a work by Gani (see [9]) and was much developed in subsequent years, as perfectly shown in the references mentioned in Bartholomew's work [2, p. 80]. Previously, [24] have obtained a mathematical model to predict distributions of staff and analyse long term impacts on patterns of recruitment and promotions. The case of Poisson recruitment in discrete time open Markov chain model was first dealt in [15], where expressions for the first and second moments of the classes probability distributions were obtained.

There has been remarkable work on the extension of discrete to continuous time Markov and semi-Markov models, such as the developments obtained in [18], [11], [12] and [13]. An important set of contributions to this theme has been detailed in [22] and, in particular, we would like to highlight the works [23], [21], [20], [14] and [16].

The motivation for the present work lays mainly in extending the previous results in the characterization of stable populations lead by discrete time Markov chain transitions: for instance, already in [7] the asymptotic behavior of the classes subpopulation averages is obtained in the case of an exponential input process and a detailed study of stability in terms of relative proportions among classes is also presented; in [14] the asymptotic behavior is described when the Poisson input parameter satisfies general regularity conditions. Having in mind the study not only of the expected values but also of the laws of the subpopulations in the classes, in [4] we considered a sequence of Poisson inflows and studied the probability distributions of the subpopulation classes relying on the fact that, by the *randomized sampling principle* (see [8, p. 268]), the subpopulations are Poisson distributed and independent of each other.

In this work, we consider that the inflow of new population elements is modeled by a time series - to wit, a second order stationary process or stationary with deterministic trend - and we study possible descriptions of the subpopulations, in particular in the transient states, as time flows.

Section 2 introduces the model and some preliminary results and notations that we be the basis of our developments. Section 3 contains the main results obtained in this paper which allowed us to perform an application to credit consumption in Section 4.

2. OPEN MARKOV CHAIN SCHEME MODEL

2.1. The model

Consider a population model driven by a Markov chain defined by a sequence of initial distributions given, for $n \geq 1$, by $(\mathbf{q}^n)^t = (q_1^n, q_2^n, \ldots, q_{r_\star}^n)$ and a transition matrix $\mathbf{P} = [p_{ij}], 1 \leq i, j \leq r_\star$. After the first transition, supposing that the initial distribution is performed according to $(\mathbf{q}^1)^t$ at date $n = 1$, the new value of the proportion of the population, for instance, in state 1, is the proportion of those which stay in state 1 plus the proportion of those who come to state 1 from states 2 to $r_\star$. That is:

$$p_{11}q_1^1 + p_{21}q_2^1 + \cdots + p_{r1}q_{r_\star}^1 = \sum_{i=1}^{r_\star} p_{i1}q_i^1$$

and so, the new values of the proportions in all states, after one transition, can be recovered from $\mathbf{P}^\intercal\mathbf{q} = (\mathbf{q}^\intercal\mathbf{P})^\intercal$ and, after n transitions, by $(\mathbf{P}^{(n)})^\intercal\mathbf{q} = (\mathbf{q}^\intercal\mathbf{P}^{(n)})^\intercal$, with $\mathbf{P}^{(0)} = \mathbf{I}$, $\mathbf{P}^{(1)} = \mathbf{P}$ and, by induction, $\mathbf{P}^{(n+1)} = \mathbf{P} \circ \mathbf{P}^{(n)}$. Let us stress, as a notation convention, that all vectors are column vectors.

Now suppose that we want to account for the evolution of the **expected** number of elements in each class supposing that, at each date $k \in \{1, \ldots, n\}$, a random number X_k of new elements enters the population. Just after the second cohort enters the population, a first transition occurs in the first cohort driven by the Markov chain law and so on and so forth. Table 1 summarizes this accounting process. Remark that at each step k we distribute multinomially the new random arrivals X_k according to the probability vector $\mathbf{q}^k$ and the elements in each class are redistributed according to the Markov chain transition matrix $\mathbf{P}$.

Table 1: Accounting of n Markov cohorts each with an initial distribution

Date	1	2	...	$n-1$	n
1	$\mathbb{E}[X_1](\mathbf{q}^1)^\intercal$	$\mathbb{E}[X_1](\mathbf{q}^1)^\intercal\mathbf{P}$	...	$\mathbb{E}[X_1](\mathbf{q}^1)^\intercal\mathbf{P}^{(n-2)}$	$\mathbb{E}[X_1](\mathbf{q}^1)^\intercal\mathbf{P}^{(n-1)}$
2	–	$\mathbb{E}[X_2](\mathbf{q}^2)^\intercal$	...	$\mathbb{E}[X_2](\mathbf{q}^2)^\intercal\mathbf{P}^{(n-3)}$	$\mathbb{E}[X_2](\mathbf{q}^2)^\intercal\mathbf{P}^{(n-2)}$
...	...	...	...	...	...
n	–	–	–	–	$\mathbb{E}[X_n](\mathbf{q}^n)^\intercal$

At date n, if we suppose that each new set of customers, a cohort, evolves independently from any one of the already existing sets of customers but, accordingly to the same Markov chain model, we may recover the total **expected** number of elements in each class at date n by computing the sum:

$$\overline{\mathbf{Y}_n} = \sum_{k=1}^{n} \mathbb{E}[X_k](\mathbf{q}^k)^\intercal \, \mathbf{P}^{(n-k)} \ . \tag{2.1}$$

Each vector component corresponds precisely to the **expected** number of elements in each class. This formula - for a constant initial distribution, i.e., $\mathbf{q}^k \equiv \mathbf{q}$ - is well known; see, for a deduction using conditional expectations, [2, p. 52: (3.2)]. In this paper, in order to further study the properties of $(\overline{\mathbf{Y}_n})_{n\geq 1}$, given the properties of a stochastic process $\mathbb{X} = (X_k)_{k\geq 1}$, we will randomize formula (2.1) by considering, instead, for $n \geq 1$:

$$\mathbf{Y}_n = \sum_{k=1}^{n} X_k(\mathbf{q}^k)^\intercal \, \mathbf{P}^{(n-k)} \ . \tag{2.2}$$

Despite the fact that the expressions in (2.1) and (2.2) share the same expected value, i.e, $\overline{\mathbf{Y}_n} = \mathbb{E}[Y_n]$, there is no obvious way to study the probability distribution of the number of elements in each of the population classes, except in the case where the $(X_k)_{k\geq 1}$ new elements are Poisson distributed or independent (see [4]). However, this is not the case for a typical ARMA time series.

Ideally, the most fruitful approach comes from knowing the joint distribution of the entrances $(X_k)_{k\geq 1}$ and of the Markov chain. As this is not the case here, we will call the stochastic process $(\mathbf{Y}_n)_{n\geq 1}$ an **open Markov chain scheme model** for the time evolution of the number of elements in each class [1].

For the case of a non-homogeneous Markov chain, the denomination non-homogeneous Markov system was used, in the context of this work, for the first time in [20], according to [19].

We note that some preliminary results on this problem have already been developed in [6].

2.2. Preliminary results and notations

We will introduce now the notions and main results, allowing to give meaning to the Cramer spectral representation theorem (see [3], [17] or [1]).

In the following, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is identified with $[-\pi, +\pi[$ by the map $\lambda \mapsto e^{i\lambda}$.

[1] Observe that $\sum_{j=1}^{r_\star} \sum_{i=1}^{r_\star} p_{ij} q_i^j = \sum_{i=1}^{r_\star} \left(\sum_{j=1}^{r_\star} p_{ij} \right) q_i = 1$, if, for instance, the initial distribution does not depend on j; the same being true for the powers of the transition matrix.

Definition 2.1. A **centered uncorrelated random field (CURF)** Z on $\mathbb{T}$ is a map from $\mathcal{B}(\mathbb{T})$, the Borel subsets of $\mathbb{T}$, into $L^2((\Omega,\mathcal{A},\mathbb{P}),(\mathbb{C},\mathcal{B}(\mathbb{C}))$, the Lebesgue space of (classes) of square integrable random variables taking values in the complex numbers $\mathbb{C}$, such that:

1. Z is centered: $\forall A \in \mathcal{B}(\mathbb{T})\ ,\ \mathbb{E}\left[Z(A)\right] = 0\ .$
2. The images of disjoint Borel sets are uncorrelated, i.e.: $\forall A, B \in \mathcal{B}(\mathbb{T}) :$ $A \cap B = \emptyset \Rightarrow \left(\mathbb{E}\left[Z(A)\cdot\overline{Z(B)}\right] = 0 \text{ and } Z(A\cup B) = Z(A) + Z(B)\right)$
3. Z is mean-square upper continuous:
$$\forall (A_n)_{n\geq 1}\ :\ A_n \downarrow \emptyset \Rightarrow \lim_{n\to+\infty} Z(A_n) =_{L^2} 0\ .$$

The following result characterizes the structure of a CURF by means of bounded positive measure defined over the Borel sets of the torus.

Theorem 2.1. *A map from $\mathcal{B}(\mathbb{T})$ into the centered random variables of the Lebesgue space $L^2((\Omega,\mathcal{A},\mathbb{P}),(\mathbb{C},\mathcal{B}(\mathbb{C}))$ is a centered uncorrelated random field (CURF) if and only if there exists a bounded positive measure μ, named the **basis** of Z such that:*
$$\forall A, B \in \mathcal{B}(\mathbb{T})\ ,\ \mathbb{E}\left[Z(A)\cdot\overline{Z(B)}\right] = \mu(A\cap B)\ .$$

The next result gives sense to the stochastic integral naturally associated to a CURF by means of an isometry between Hilbert spaces of square integrable functions.

Theorem 2.2 (CURF stochastic integral)**.** *Let Z be a CURF on $\mathbb{T}$ with basis μ. There exists an unique isometry $\widetilde{Z}$ from $L^2((\mathbb{T},\mathcal{B}(\mathbb{T}),\mu))$ into $L^2(((\Omega,\mathcal{A},\mathbb{P}),(\mathbb{C},\mathcal{B}(\mathbb{C})))$ such that for all $A \in \mathcal{B}(\mathbb{T})$, $\widetilde{Z}(\mathbb{1}_A) = Z(A)$. We have that:*

1. *$\widetilde{Z}$ is a centered isometry*
$$\forall f \in L^2((\mathbb{T},\mathcal{B}(\mathbb{T}),\mu))\ ,\ \mathbb{E}\left[\widetilde{Z}(f)\right] = 0\ .$$
2. *The image of $\widetilde{Z}$ is the closure of the vector space spanned by the random variables obtained from Z, that is:*
$$\widetilde{Z}\left(L^2((\mathbb{T},\mathcal{B}(\mathbb{T}),\mu))\right) = \overline{\mathcal{V}(\{Z(A) : A \in \mathcal{B}(\mathbb{T})\}}\ .$$

Remark 2.1. For each $f \in L^2((\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu))$ we denote the isometry $\widetilde{Z}$ as a **stochastic integral** as follows:

$$\widetilde{Z}(f) = \int_{[-\pi,+\pi[} f(\lambda) dZ(\lambda).$$

Remark 2.2. Moreover, we stress the important result that all the centered isometries between the L^2 spaces mentioned above are generated by a CURF.

Covariances of stochastic processes are nonnegative-definite functions and these, in turn, are represented by positive bounded measures on the torus.

Definition 2.2. A function γ from $\mathbb{Z}$ into $\mathbb{C}$ is **nonnegative-definite** if and only if $\gamma(n) = \overline{\gamma(-n)}$, for $n \in \mathbb{Z}$ and

$$\forall r \geq 1,\ \forall z_1, \ldots, z_r \in \mathbb{C},\ \forall n_1, \ldots, n_r \in \mathbb{Z},\ \sum_{i,j=1}^{r} z_i \overline{z}_j \gamma(n_i - n_j) \geq 0\ .$$

Theorem 2.3 (Bochner-Herglotz)**.** *A necessary and sufficient condition for a function γ to be nonnegative-definite is that there exists a positive bounded measure on $\mathbb{T}$, which is unique, such that:*

$$\forall n \in \mathbb{Z},\ \gamma(n) = \int_{[-\pi,+\pi[} e^{i\lambda n} d\mu(\lambda)\ .$$

Definition 2.3. A stochastic process $\mathbb{X} = (X_n)_{n \in \mathbb{Z}}$ is **second order stationary** if and only if:

1. All random variables are square integrable, that is:

$$\forall n \in \mathbb{Z} \quad \mathbb{E}\left[|X_n|^2\right] < +\infty\ .$$

2. Both the mean and the covariance functions (sequences) of the process, given, for all $n, m \in \mathbb{Z}$, respectively, by $M(n) := \mathbb{E}[X_n]$ and $\Gamma(m,n) := \mathbb{E}\left[(X_m - \mathbb{E}[X_m])\overline{(X_n - \mathbb{E}[X_n])}\right]$, are invariant by time translations, and so:

$$\forall m, m \in \mathbb{Z}\ ,\ M(n) = M \in \mathbb{R} \text{ and } \Gamma(n,m) = \gamma(m-n)\ ,$$

for some function γ defined on $\mathbb{Z}$.

Remark 2.3. We may verify that γ is a nonnegative-definite function as defined in Definition 2.2, thus justifying the application of the Bochner-Herglotz theorem to obtain a representation of a second order stationary process.

Example 2.1 (White noise). A process $\mathbb{W} = (W_n)_{n\in\mathbb{Z}}$ is a **white noise** if the random variables are centered, square integrable and, moreover, uncorrelated, that is, if:

$$\forall n, m \in \mathbb{Z} \ : \ n \neq m \Rightarrow \Gamma(n,m) = 0 \ .$$

An example of white noise is given by a sequence of independent centered random variables with common variance.

Example 2.2 (ARMA process). A process $\mathbb{X} = (X_n)_{n\in\mathbb{Z}}$ is an ARMA(p,q) process if there exists a white noise $\mathbb{W} = (W_n)_{n\in\mathbb{Z}}$ and two complex sequences $a_1, a_2, \dots a_p$ and $b_1, b_2, \dots b_q$ such that

$$\sum_{k=0}^{p} a_k X_{n-k} = \sum_{l=0}^{q} b_l W_{n-l}. \tag{2.3}$$

Formula (2.3) is called a **canonical ARMA relation** (see [1, p. 80]) if the polynomials $P(z) = \sum_{k=0}^{p} a_k z^k$ and $Q(z) = \sum_{l=0}^{q} b_l z^l$ have no common factor, P has all his roots with modulus *strictly greater* than 1, Q has all his roots with modulus *greater or equal* than 1 and $P(0) = Q(0) = 1$. It is a remarkable result (see [1, p. 81]), that will prove useful in the following, that, if a stochastic process $\mathbb{X}$ satisfies a canonical ARMA relation with a white noise $\mathbb{W}$ then, this white noise is unique and it is named the **innovation** of $\mathbb{X}$.

We now obtain the representation of a second order stationary stochastic process by the positive bounded measure associated to its covariance.

Definition 2.4 (Spectral measure). Let $\mathbb{X} = (X_n)_{n\in\mathbb{Z}}$ be a second order stationary process. The **spectral measure** of $\mathbb{X}$ is the unique positive bounded measure $\mu_{\mathbb{X}}$ on $\mathbb{T}$ representing the covariance of the process that is, such that,

$$\forall m, n \in \mathbb{Z}, \ \Gamma(m,n) = \gamma(m-n) = \int_{[-\pi,+\pi[} e^{i\lambda(m-n)} d\mu_{\mathbb{X}}(\lambda) \ .$$

In vue of future application the particular case of real valued processes deserves special mention.

Remark 2.4. In the case that $\mathbb{X}$ is real valued then the spectral measure $\mu_{\mathbb{X}}$ on $\mathbb{T}$ is invariant by the symmetry ϕ defined on $\mathbb{T}$ by $\phi(z) = \overline{z}$ for all $z \in \mathbb{T}$.

Remark 2.5. If the spectral measure $\mu_{\mathbb{X}}$ is absolutely continuous with respect to the Lebesgue measure on the torus then, by Radon-Nikodym theorem, $\mu_{\mathbb{X}}$ admits a density $f_{\mathbb{X}}$ with respect to the Lebesgue measure and we call this density the **spectral density** of $\mathbb{X}$.

Example 2.3 (White noise)**.** A white noise $\mathbb{W} = (W_n)_{n\in\mathbb{Z}}$ with the random variables having common variance σ^2 has a spectral density given by:

$$f_{\mathbb{W}}(\lambda) = \frac{\sigma^2}{2\pi} \ .$$

Example 2.4 (ARMA process)**.** The spectral density $f_{\mathbb{X}}$ corresponding to the canonical ARMA relation in Example 2.2 is given, using the same notations, by:

$$f_{\mathbb{X}}(\lambda) = \frac{\sigma^2}{2\pi}\left|\frac{Q(e^{-i\lambda})}{P(e^{-i\lambda})}\right|^2 \ .$$

We now state the theorem allowing to represent second order stationary stochastic process as a CURF.

Theorem 2.4 (Cramer theorem)**.** *Let $\mathbb{X} = (X_n)_{n\in\mathbb{Z}}$ be a second order stationary process with spectral measure $\mu_{\mathbb{X}}$. Then, there exists an unique CURF $Z_{\mathbb{X}}$ on $\mathbb{T}$ with basis $\mu_{\mathbb{X}}$ such that:*

$$\forall n \in \mathbb{Z},\ X_n = \int_{[-\pi,+\pi[} e^{i\lambda n} dZ_{\mathbb{X}}(\lambda) \ .$$

We will need the following observation clarifying the structure of the Cramer representation of a time inverted process.

Remark 2.6 (Time inversion)**.** Let μ^ϕ be the image of μ by the symmetry ϕ. As the map $f \mapsto f\circ\phi$ is an isometry from $L^2(\mu^\phi)$ onto $L^2(\mu)$, then the map

$$f \mapsto \int_{[-\pi,+\pi[} f\circ\phi(\lambda) dZ_{\mathbb{X}}(\lambda)$$

is also an isometry from $L^2(\mu^\phi)$ into $L^2(\Omega)$. Now, by Remark 2.2 above, there exists an unique CURF $Z^\phi_{\mathbb{X}}$ with basis μ^ϕ, the **symmetric** CURF of $Z_{\mathbb{X}}$, such that for all $f \in L^2(\mu^\phi)$:

$$\int_{[-\pi,+\pi[} f(\lambda) dZ^\phi_{\mathbb{X}}(\lambda) = \int_{[-\pi,+\pi[} f\circ\phi(\lambda) dZ_{\mathbb{X}}(\lambda) \ . \tag{2.4}$$

As a consequence, $\mathbb{X}^{\leftarrow} = (X_{-n})_{n\in\mathbb{Z}}$, the time inversion of $\mathbb{X}$, has a spectral representation given by:

$$X_{-n} = \int_{[-\pi,+\pi[} e^{-i\lambda n} dZ_{\mathbb{X}}(\lambda) = \int_{[-\pi,+\pi[} e^{i\lambda n} dZ^\phi_{\mathbb{X}}(\lambda) \ .$$

We will now introduce a special class of processes that will prove useful in the following results.

Definition 2.5 (Evanescent process)**.** A centered stochastic $\mathbb{X} = (X_n)_{n\in\mathbb{Z}}$ process is called an **evanescent process** at time $+\infty$ iff:

$$\lim_{n\to+\infty} \mathbb{E}\left[|X_n|^2\right] = 0 \ .$$

Remark 2.7. Any linear combination of centered evanescent processes at time $+\infty$ is a centered evanescent process at time $+\infty$.

3. SECOND ORDER FEDS OF A MARKOV CHAIN SCHEME

In this section we considere a Markov chain scheme fed by a stochastic process. Let $\mathbf{P}$ be the transition matrix of the Markov chain. We will suppose that the transition matrix may be written in the following form:

$$\mathbf{P} = \begin{bmatrix} \mathbf{T} & \mathbf{S}_1 \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \ , \tag{3.1}$$

where $\mathbf{T}$ is the $t_\star \times t_\star$ transition matrix between transient states, $\mathbf{S}_1$ the $t_\star \times (r_\star - t_\star)$ matrix of one step transitions between the transient and the recurrent states and $\mathbf{R}$ the $(r_\star - t_\star) \times (r_\star - t_\star)$ transition matrix between the recurrent states. A straitghforward computation shows that:

$$\mathbf{P}^{(n)} = \begin{bmatrix} \mathbf{T}^{(n)} & \mathbf{S}_n \\ \mathbf{0} & \mathbf{R}^{(n)} \end{bmatrix} \ , \ n > 1$$

with $\mathbf{S}_n = \mathbf{S}_{n-1}\mathbf{R} + \mathbf{T}^{(n-1)}\mathbf{S}_1 = \sum_{i=0}^{n-1} \mathbf{T}^{(i)}\mathbf{S}_1\mathbf{R}^{(n-1-i)}$.

We now write the successive cohorts vectors of classifications for new arriving elements, at time period k, as

$$(\mathbf{q}^k)^\intercal = \left[(\mathbf{t}^k)^\intercal \middle| (\mathbf{r}^k)^\intercal\right] \ , \tag{3.2}$$

with $(\mathbf{t}^k)^\intercal$ the vector of the initial classification probabilities for the transient states and $(\mathbf{r}^k)^\intercal$ the vector of the initial classification probabilities for the recurrent states. Using (3.1) and (3.2), formula (2.2) may be written as

$$\mathbf{Y}_n = \left[\mathbf{Y}_n^1 \mid \mathbf{Y}_n^2\right] = \left[\sum_{k=1}^{n} X_k (\mathbf{t}^k)^\intercal \mathbf{T}^{(n-k)} \,\middle|\, \sum_{k=1}^{n} X_k \left((\mathbf{t}^k)^\intercal \mathbf{S}_{n-k} + (\mathbf{r}^k)^\intercal \mathbf{R}^{(n-k)} \right) \right] . \tag{3.3}$$

Formula (3.3) allow us to estimate the number of elements in each subpopulation (transient or recurrent). However, for the reasons pointed in the introduction and for technical reasons that will become apparent in the following, we will consider only the transient states part of the transition matrix.

At first, we will suppose that the feeding process is stationary. The main result will be the following:

Theorem 3.1. *Consider an open Markov chain scheme model, with a diagonizable matrix* $\mathbf{P}$, *written as in (3.1), and a constant vector of initial classification probabilities* $(\mathbf{q}^k)^\intercal \equiv (\mathbf{q})^\intercal$, *defined as in (3.2).*

If the open Markov chain scheme model is fed by a real valued ARMA process then the population in each of the transient states may be described as a sum of an ARMA processes with an evanescent process.

Proof: Suppose that $\mathbb{X} = (X_k)_{k\in\mathbb{Z}}$ is a second order stationary time series. Recall that by the Cramer representation theorem (see [1, p. 51]) also stated above, we have that, for all $k \in \mathbb{Z}$,

$$X_k = \int_{[-\pi,+\pi[} e^{i\lambda k} dZ_{\mathbb{X}}(\lambda) ,$$

with $Z_{\mathbb{X}}$ the spectral field of $\mathbb{X}$, the unique CURF associated to $\mathbb{X}$ (see [1, p. 38] for a definition). Reporting this representation in the Markov chain scheme given by formula (2.2) we get that

$$\mathbf{Y}_n = \sum_{k=1}^{n} \left(\int_{[-\pi,+\pi[} e^{i\lambda k} dZ_{\mathbb{X}}(\lambda) \right) (\mathbf{q}^k)^\intercal\ \mathbf{P}^{(n-k)} . \tag{3.4}$$

Considering that the transition matrix of the transient states $\mathbf{T}$ is diagonalizable, it may be written as:

$$\mathbf{T} = \sum_{j=1}^{t_\star} \eta_j \boldsymbol{\alpha}_j \boldsymbol{\beta}_j^\intercal ,$$

with $(\eta_j)_{j\in\{1,\ldots,t_\star\}}$ the eigenvalues, $(\boldsymbol{\alpha}_j)_{j\in\{1,\ldots,t_\star\}}$ the left eigenvectors and with $(\boldsymbol{\beta}_j)_{j\in\{1,\ldots,t_\star\}}$ the right eigenvectors of $\mathbf{T}$ (see [8] or [10]).We observe that $j \in \{1,\ldots,t_\star\}$ corresponds to a transient state if and only if $\mid \eta_j \mid< 1$. Considering also that, for $k \geq 1$, we have $\mathbf{t}^k \equiv \mathbf{t}$, we will have, for $n \geq 1$,

$$\begin{aligned} \mathbf{Y}_n^1 &= \sum_{j=1}^{t_\star} \left(\int_{[-\pi,+\pi[} \left(\sum_{k=1}^{n} e^{i\lambda k}\ \eta_j^{(n-k)} \right) dZ_{\mathbb{X}}(\lambda) \right) \mathbf{t}^\intercal\ \boldsymbol{\alpha}_j \boldsymbol{\beta}_j^\intercal = \\ &= \sum_{j=1}^{t_\star} \left(\int_{[-\pi,+\pi[} e^{-i\lambda n} \left[\frac{1-(e^{i\lambda}\ \eta_j)^{n+1}}{1-e^{i\lambda}\ \eta_j} \right] dZ_{\mathbb{X}}(\lambda) \right) \mathbf{t}^\intercal\ \boldsymbol{\alpha}_j \boldsymbol{\beta}_j^\intercal . \end{aligned} \tag{3.5}$$

We now define, for each $j \in \{1,\ldots,s\}$ and $n \geq 1$, $W_n^j = W_n^{1,j} - W_n^{2,j}$ with

$$W_n^{1,j} := \int_{[-\pi,+\pi[} e^{-i\lambda n} \left[\frac{1}{1-e^{i\lambda}\eta_j} \right] dZ_{\mathbb{X}}(\lambda) \ , W_n^{2,j} := \int_{[-\pi,+\pi[} \left[\frac{e^{i\lambda}\ \eta_j^{n+1}}{1-e^{i\lambda}\eta_j} \right] dZ_{\mathbb{X}}(\lambda)$$

and we observe that $h^{1,j}(\lambda) := \frac{1}{1-e^{i\lambda}\eta_j}$ and $h^{2,j}(\lambda) := \frac{e^{i\lambda}\ \eta_j^{n+1}}{1-e^{i\lambda}\eta_j}$ are both $L^2([-\pi,+\pi])$ functions due to $|\eta_j| < 1$. We will deal separately with these two components.

Firstly, we show that $\mathbb{W}^{2,j} = (W_n^{2,j})_{n\geq 1}$ is an *evanescent process* at $+\infty$, according to Definition 2.5. In fact, with $\mu_{\mathbb{X}}$ the spectral measure of $\mathbb{X}$, we have that:

$$\mathbb{E}\left[\left|W_n^{2,j}\right|^2\right] = \int_{[-\pi,+\pi[} \left|\frac{e^{i\lambda}\ \eta_j^{n+1}}{1-e^{i\lambda}\eta_j}\right|^2 d\mu_{\mathbb{X}}(\lambda) \leq \frac{|\eta_j|^{2n+2}}{|1-|\eta_j||^2}\ \mu_{\mathbb{X}}\left([-\pi,+\pi[\right)\ ,$$

and so, as $\mu_{\mathbb{X}}$ is bounded and $|\eta_j| < 1$, we have, with exponential rate given by $|\eta_j|^{2n+2}$,

$$\lim_{n\to+\infty} \mathbb{E}\left[\left|W_n^{2,j}\right|^2\right] = 0\,.$$

In fact, due to the exponential convergence to zero of the second order moments, the convergence of the process $\mathbb{W}^{2,j}$ to zero is in the almost sure sense. Let $0 < \epsilon < 1$, then, as,

$$\mathbb{P}\left[\left|W_n^{2,j}\right| > |\eta_j|^{\epsilon n}\right] \leq \frac{\mathbb{E}\left[\left|W_n^{2,j}\right|^2\right]}{|\eta_j|^{2\epsilon n}} \leq \frac{|\eta_j|^{2n+2}}{|\eta_j|^{2\epsilon n}\,|1-|\eta_j||^2}\mu_{\mathbb{X}}\left([-\pi,\pi[\right) =$$
$$= |\eta_j|^{2n(1-\epsilon)}\ \frac{|\eta_j|^2}{|1-|\eta_j||^2}\mu_{\mathbb{X}}\left([-\pi,\pi[\right)\ ,$$

we have that, for some constant c,

$$\sum_{n=1}^{+\infty} \mathbb{P}\left[\left|W_n^{2,j}\right| > |\eta_j|^{\epsilon n}\right] < \sum_{n=1}^{+\infty} |\eta_j|^{2n(1-\epsilon)} < +\infty\ ,$$

with $\lim_{n\to+\infty} |\eta_j|^{2n(1-\epsilon)} = 0$, thus showing (see [3, p. 370]) the almost sure convergence to zero at $+\infty$ of the process $\mathbb{W}^{2,j}$.

Secondly, we have that $\mathbb{W}^{1,j} = (W_n^{1,j})_{n\geq 1}$ defines a second-order stationary stochastic process obtained from $\mathbb{X}$ from time inversion (see Remark 2.6 above) and via the filter given by $\overline{h^{1,j}}$ see [1, p. 58]). In fact, by formula (2.4) we have

$$W_n^{1,j} = \int_{[-\pi,+\pi[} e^{-i\lambda n} h^{1,j}(\lambda) dZ_{\mathbb{X}}(\lambda) = \int_{[-\pi,+\pi[} e^{i\lambda n} h^{1,j}(-\lambda) dZ_{\mathbb{X}}^{\phi}(\lambda) =$$
$$= \int_{[-\pi,+\pi[} e^{i\lambda n} \overline{h^{1,j}}(\lambda) dZ_{\mathbb{X}}^{\phi}(\lambda)$$

and so, considering the filter defined by $\overline{h^{1,j}}$ and the second order stationary process defined by the CURF $Z_{\mathbb{X}}^{\phi}$, we prove the stated result.

We will now suppose that $\mathbb{X}$ is a real valued *ARMA* process. For real processes, as stated in Remark 2.4, the spectral measure is invariant under ϕ

and then, the spectral density is also invariant under ϕ. As the CURF $Z^{\phi}_{\mathbb{X}}$ has basis $\mu^{\phi}_{\mathbb{X}} \equiv \mu_{\mathbb{X}}$ and $f^{\phi}_{\mathbb{X}} \equiv f_{\mathbb{X}}$ [2] then, we see that each $\mathbb{W}^{1,j}$ the filtered process, is also *ARMA* process, due to the fact that by multiplying the spectral density by $\left|h^{1,j}\right|^2$ we only introduce roots strictly larger than 1 in the denominator and so, by resorting to the form of the spectral density in the canonical ARMA relation as stated in Examples 2.2 and 2.4, we still have an ARMA process. In fact, let $f_{\mathbb{W}^{1,j}}$ be the spectral density of the process $\mathbb{W}^{1,j}$, obtained from $\mathbb{X}$ by filtering by the square integrable function $h^{1,j}$. We have that, with the notations being used, (3.6)

$$f_{\mathbb{W}^{1,j}}(\lambda) = f^{\phi}_{\mathbb{X}}(\lambda)\left|h^{1,j}(\lambda)\right|^2 = f_{\mathbb{X}}(\lambda)\left|h^{1,j}(\lambda)\right|^2 = \frac{\sigma^2}{2\pi}\left|\frac{Q(e^{-i\lambda})}{P(e^{-i\lambda})(1-e^{-i\lambda}\eta_j)}\right|^2 .$$

The polynomial $R(z) := P(z)(1-z\eta_j)$ still has all its roots with modulus strictly greater than one and still verifies $R(0) = 1$. If $1/\eta_j$ is not a root of Q then, the representation of $f_{\mathbb{W}^{1,j}}$ still is a canonical ARMA relation. If Q admits $(1-z\eta_j)$ as a factor then, writing $Q(z) = S(z)(1-z\eta_j)$ we have the represention

$$f_{\mathbb{W}^{1,j}}(\lambda) = \frac{\sigma^2}{2\pi}\left|\frac{S(e^{-i\lambda})}{P(e^{-i\lambda})}\right|^2 ,$$

with S having all its roots with modulus strictly greater than one, still verifying $S(0) = 1$ and with P and S still not having any common roots. So, this representation is still a canonical ARMA relation. We may so observe that $\mathbb{Y}$ is asymptotically - due to the evanescent process - a linear combination of *ARMA* processes.

We are now going to show that any linear combination of the $\mathbb{W}^{i,j}$ still is an ARMA process. That results from the fact that the innovation noise of each $\mathbb{W}^{i,j}$ coincides with the innovation noise of $\mathbb{X}$ (see [3, p. 210], for the general idea).

Let the spectral density of the process $\mathbb{X}$ be written according to the canonical ARMA representation as,

$$f_{\mathbb{X}}(\lambda) = \frac{\sigma^2}{2\pi}\left|\frac{Q(e^{-i\lambda})}{P(e^{-i\lambda})}\right|^2 .$$

As the process $\mathbb{W}^{1,j} = (W^{1,j}_n)_{n\geq 1}$ admits the spectral representation given by

$$W^{1,j}_n = \int_{[-\pi,+\pi[} e^{i\lambda n}\overline{h^{1,j}(\lambda)}dZ^{\phi}_{\mathbb{X}}(\lambda) ,$$

this process has a density that may be written according the canonical ARMA representation as

$$f_{\mathbb{W}^{1,j}}(\lambda) = \frac{\sigma^2}{2\pi}\left|\frac{Q^{1,j}(e^{-i\lambda})}{P^{1,j}(e^{-i\lambda})}\right|^2$$

[2] If the spectral density of an ARMA process is a rational function with real functions the the AR and MA polynomials may be chosen with real coefficients (see [1, p. 77]).

with $Q^{1,j}$ and $P^{1,j}$ such that, either

$$Q(e^{-i\lambda}) = Q^{1,j}(e^{-i\lambda})\left(1 - e^{-i\lambda}\eta_j\right) \text{ and } P^{1,j}(e^{-i\lambda}) = P(e^{-i\lambda})$$

or

$$Q^{1,j}(e^{-i\lambda}) = Q(e^{-i\lambda}) \text{ and } P^{1,j}(e^{-i\lambda}) = P(e^{-i\lambda})(1 - e^{-i\lambda}) .$$

We observe that as $d\mu_{\mathbb{X}}(\lambda) = f_{\mathbb{X}}(\lambda)d\mathrm{Leb}(\lambda)$, then we have that

$$\mu_{\mathbb{X}}\left(\left\{Q^{1,j}(e^{-i\lambda}) = 0\right\}\right) = \mu_{\mathbb{X}}\left(\left\{Q(e^{-i\lambda}) = 0\right\}\right) = 0 .$$

Now, let ε^j be the innovation noise of $\mathbb{W}^{1,j}$. We then have:

$$\begin{aligned} d\mu_{\varepsilon^j}(\lambda) &= \frac{1}{f_{\mathbb{W}^{1,j}}(\lambda)}\mathbb{I}_{\left\{Q^{1,j}(e^{-i\lambda})\neq 0\right\}}(\lambda)d\mu_{\mathbb{W}^{1,j}}(\lambda) = \\ &= \frac{1}{f_{\mathbb{W}^{1,j}}(\lambda)}\mathbb{I}_{\left\{Q^{1,j}(e^{-i\lambda})\neq 0\right\}}(\lambda)\left|h^{1,j}(\lambda)\right|^2 d\mu_{\mathbb{X}}(\lambda) = \\ &= \mathbb{I}_{\left\{Q(e^{-i\lambda})\neq 0\right\}}(\lambda)d\mathrm{Leb}(\lambda) = \frac{1}{f_{\mathbb{X}}(\lambda)}\mathbb{I}_{\left\{Q(e^{-i\lambda})\neq 0\right\}}(\lambda)d\mu_{\mathbb{X}}(\lambda) . \end{aligned}$$

We are now going to show that ε^j is the innovation noise of $\mathbb{X}$, that is, for $m < n$, $\mathbb{E}\left[X_m\varepsilon^j_n\right] = 0$, and so the innovation noise of $\mathbb{W}^{1,j}$ does not depend on j. For that we use the spectral representation of both processes, and so, considering any function ψ such that

$$|\psi(\lambda)|^2 = \frac{1}{f_{\mathbb{X}}(\lambda)}\mathbb{I}_{\left\{Q(e^{-i\lambda})\neq 0\right\}}(\lambda) ,$$

we have, using the isometry property of the stochastic integral and the Cauchy theorem,

$$\begin{aligned} &\mathbb{E}\left[\left(\int_{[-\pi,+\pi[} e^{i\lambda m}dZ_{\mathbb{X}}(\lambda)\right)\cdot\overline{\left(\int_{[-\pi,+\pi[} e^{i\lambda n}dZ_{\varepsilon^j}(\lambda)\right)}\right] \\ &= \mathbb{E}\left[\left(\int_{[-\pi,+\pi[} e^{i\lambda m}dZ_{\mathbb{X}}(\lambda)\right)\cdot\overline{\left(\int_{[-\pi,+\pi[} e^{i\lambda n}\psi(\lambda)dZ_{\mathbb{X}}(\lambda)\right)}\right] \\ &= \int_{[-\pi,+\pi[} e^{i\lambda(m-n)}\overline{\psi(\lambda)}d\mu_{\mathbb{X}}(\lambda) \\ &= \int_{[-\pi,+\pi[} e^{i\lambda(m-n)}\frac{\sqrt{2\pi}}{\sigma}\overline{\left(\frac{P(e^{-i\lambda})}{Q(e^{-i\lambda})}\right)}\frac{\sigma^2}{2\pi}\frac{Q(e^{-i\lambda})}{P(e^{-i\lambda})}\overline{\left(\frac{Q(e^{-i\lambda})}{P(e^{-i\lambda})}\right)}\mathbb{I}_{\left\{Q(e^{-i\lambda})\neq 0\right\}}d\mathrm{Leb}(\lambda) \\ &= \frac{\sigma}{\sqrt{2\pi}}\int_{[-\pi,+\pi[} e^{-i\lambda(n-m)}\frac{Q(e^{-i\lambda})}{P(e^{-i\lambda})}d\mathrm{Leb}(\lambda) \\ &= \frac{\sigma i}{\sqrt{2\pi}}\int_{\mathbb{T}} z^{n-(m+1)}\frac{Q(z)}{P(z)}dz = 0 . \end{aligned}$$

Let $\varepsilon = (\varepsilon_k)_{k\in\mathbb{Z}}$ be the common innovation noise of all the processes $\mathbb{W}^{1,j}$. We then have (see [1, p. 81]) that for each j and some (square) integrable sequence $(c^j_k)_{k\geq 0}$ we may write,

$$\mathbb{W}^{1,j}_n = \sum_{k\geq 0} c^j_k\varepsilon_{n-k} ,$$

and so for $\alpha_j, \alpha_l \in \mathbb{C}$, as $\left(\alpha_j c_k^j + \alpha_l c_k^l\right)_{k\geq 0}$ is a (square) integrable sequence,

$$\alpha_j \mathbb{W}_n^{1,j} + \alpha_l \mathbb{W}_n^{1,l} = \sum_{k\geq 0} \left(\alpha_j c_k^j + \alpha_l c_k^l\right) \varepsilon_{n-k} \ ,$$

and $\alpha_j \mathbb{W}^{1,j} + \alpha_l \mathbb{W}^{1,l}$ is an ARMA process. □

We now deal with the case of some non stationary processes which are relevant for the applications.

Theorem 3.2. *Under the same conditions of Theorem 3.1, if the open Markov chain scheme model is fed by a real valued ARIMA or SARMA process then the population in each of the transient states may be described as a sum of a deterministic trend plus a linear combination of ARMA processes plus an evanescent process.*

Proof: Let $\mathbb{X} = (X_n)_{n\in\mathbb{Z}}$ be an ARMA process. Let $s, d \geq 1$ be integers and consider the following functions defined, for $i, j \in \{0, 1, \ldots, s-1\}$ and $\alpha, \beta \in \{0, 1, \ldots, d-1\}$, by:

$$U_{i,\alpha}(x) = x^\alpha \cos\left(\frac{2\pi i}{s} x\right) \qquad \text{and} \qquad V_{j,\beta}(x) = x^\beta \sin\left(\frac{2\pi j}{s} x\right) \ .$$

Consider now the function given by linear combinations with complex coefficients of the functions $U_{i,\alpha}$ and $V_{j,\beta}$ as

$$P_{(s,d)}(x) = \sum_{0\leq i\leq s-1, 0\leq \alpha\leq d-1} a_{i,\alpha} U_{i,\alpha}(x) + \sum_{0\leq j\leq s-1, 0\leq \beta\leq d-1} b_{j,\beta} V_{j,\beta}(x) \ .$$

Then the process $\mathbb{T} = (T_n)_{n\in\mathbb{Z}}$ represented as

$$T_n = P_{(s,d)}(n) + \sum_{j=0}^{n} \gamma_j X_{n-j} \ ,$$

is an identifiable ARIMA or SARMA process for an appropriate choice of s, d and the complex coefficients $a_{i,\alpha}$, $b_{j,\beta}$ and γ_j (see [1, p. 87, 89]). Moreover, every identifiable ARIMA or SARMA process can be represented in that form. Consider now an open Markov chain scheme fed by $\mathbb{T}$. We have the obvious decomposition:

$$\begin{aligned}
\mathbf{Y}_n &= \sum_{k=1}^{n} T_k (\mathbf{q}^k)^\intercal \, \mathbf{P}^{(n-k)} = \\
&= \sum_{k=1}^{n} P_{(s,d)}(k) (\mathbf{q}^k)^\intercal \, \mathbf{P}^{(n-k)} + \sum_{k=1}^{n} \left(\sum_{j=0}^{k} \gamma_j X_{k-j}\right) (\mathbf{q}^k)^\intercal \, \mathbf{P}^{(n-k)} = \\
&= \sum_{k=1}^{n} P_{(s,d)}(k) (\mathbf{q}^k)^\intercal \, \mathbf{P}^{(n-k)} + \sum_{j=0}^{n} \gamma_j \left(\sum_{k=j}^{n} X_{k-j} (\mathbf{q}^k)^\intercal \, \mathbf{P}^{(n-k)}\right)
\end{aligned}$$

and so, as the right-hand term of the last sum is a linear combination of ARMA processes the result follows. □

Remark 3.1. Using the results in [4], we note that, at least on average, the asymptotic behavior of the subpopulations can be described.

4. AN APPLICATION TO COMSUMPTION CREDIT

4.1. Real data

In this section we will present fittings of a second order processes to real data of consumption credit portfolio from a Cape Verdean bank.

In this portfolio, we defined five risk classes, according to the number of days in delay of the monthly reimbursements, as shown in Table 2, and an extra class for the clients leaving the portfolio.

Table 2: Portfolio risk classes

Risk Class	**Number of days in delay**
RC1	0 - 30
RC2	31-60
RC3	61-90
RC4	91-120
RC5	> 120
RC6	Leaving

In each month, each client is classified into the risk class that refers to his number of days in delay of reimbursments. Only fully paid contracts are allowed to move to risk class 6.

The transition matrix, estimated from portfolio data, is given by:

$$\mathbf{P} = \begin{bmatrix} 0.934735 & 0.026566 & 0 & 0 & 0 & 0.038698 \\ 0.518363 & 0.285733 & 0.195903 & 0 & 0 & 0 \\ 0.009076 & 0.372018 & 0.248963 & 0.369943 & 0 & 0 \\ 0 & 0.007835 & 0.335464 & 0.205361 & 0.450928 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1}$$

Naturally, each new client is initially placed in risk class 1, and so,

$$(\mathbf{q}^k)^\intercal \equiv (\mathbf{q})^\intercal = [\ 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\ ]$$

In previous works (see [4] and [5]), using this data and the related client information in the consumption credit database, we provided models for the spread to be applied to each client and related this to the global spread of the portfolio, estimated using an open Markov chain model with the number of entrances modeled by a sequence of Poisson laws. We recall that one of the motivations for the present work is to develop a model with the entrances of new clients modeled by a time series.

The data on the number of new monthly clients arriving to the portfolio corresponds to a monthly sequence of 106 observations. The fitting was performed using Wolfram Mathematica and, for illustration purposes, we adopted two different approaches. In the first one, we fitted a time series directly to data. In the second, we firstly fitted a sigmoid type function to data, as in [4], and then, a time series to the residuals of the sigmoid fitting.

The results obtained for the first approach are illustrated in Table 3.

As shown in Table 3 the best model for the entrance data, under both the AIC and the BIC criterias, is an ARIMA$[0, 1, 1]$ model.

Table 3: Fitting the entrance data directly

	Candidate	BIC
1	**ARIMAProcess[0, 1, 1]**	**909.593**
2	ARIMAProcess[1, 1, 0]	912.706
3	ARIMAProcess[0, 1, 2]	913.637
4	ARIMAProcess[1, 1, 1]	918.713
5	ARIMAProcess[1, 1, 2]	919.631
6	ARIMAProcess[0, 1, 0]	925.378
7	ARIMAProcess[1, 2, 1]	929.179
8	ARProcess[2]	932.164
9	ARProcess[3]	935.583
10	ARIMAProcess[2, 2, 2]	935.643

	Candidate	AIC
1	**ARIMAProcess[0, 1, 1]**	**897.294**
2	ARIMAProcess[0, 1, 2]	898.366
3	ARIMAProcess[1, 1, 0]	900.518
4	ARIMAProcess[1, 1, 2]	901.762
5	ARIMAProcess[1, 1, 1]	903.686
6	ARIMAProcess[1, 2, 1]	914.666
7	ARIMAProcess[2, 2, 2]	916.189
8	ARIMAProcess[0, 1, 0]	916.863
9	ARProcess[2]	917.8
10	ARProcess[3]	918.702

For the second approach, we show, in Figure 1, on the left side graphic, both the data and the fitted sigmoid type function and, in the right side graphic, the correspondent residuals.

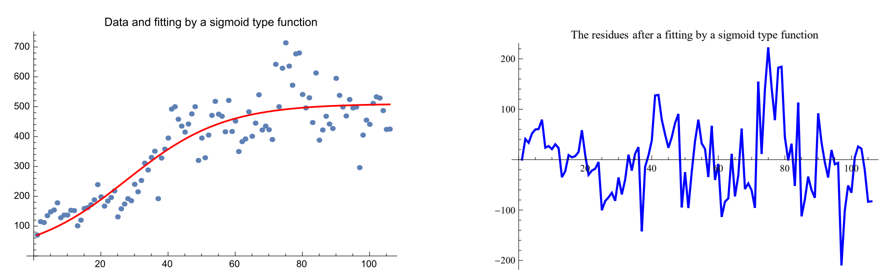

Figure 1: Fitting a sigmoid type function to data and to the residuals.

In Table 4 it is shown that the best model for the residuals of the fitting of the entrance data by a sigmoid type function, under both the AIC and the BIC criterias, is the SARMA$[(1, 0), (1, 0)_{34}]$ model.

Table 4: Fitting the residues of a fitting by a sigmoid function

	Candidate	AIC
1	**SARMAProcess[{1, 0}, {1, 0}$_{34}$]**	**876.091**
2	SARMAProcess[{1, 0}, {2, 0}$_{34}$]	876.237
3	SARMAProcess[{1, 0}, {1, 1}$_{34}$]	878.53
4	SARMAProcess[{2, 0}, {1, 0}$_{34}$]	879.641
5	SARMAProcess[{1, 0}, {2, 1}$_{34}$]	880.596
6	SARMAProcess[{1, 1}, {1, 0}$_{34}$]	883.077
7	ARProcess[2]	884.778
8	SARMAProcess[{2, 1}, {1, 0}$_{34}$]	885.059
9	SARMAProcess[{1, 0}, {0, 1}$_{34}$]	886.115
10	ARProcess[1]	886.819

	Candidate	BIC
1	**SARMAProcess[{1, 0}, {1, 0}$_{34}$]**	**882.283**
2	SARMAProcess[{1, 0}, {2, 0}$_{34}$]	883.098
3	SARMAProcess[{1, 0}, {1, 1}$_{34}$]	885.067
4	SARMAProcess[{2, 0}, {1, 0}$_{34}$]	886.016
5	SARMAProcess[{1, 0}, {2, 1}$_{34}$]	887.306
6	SARMAProcess[{1, 1}, {1, 0}$_{34}$]	888.927
7	ARProcess[2]	889.879
8	SARMAProcess[{2, 1}, {1, 0}$_{34}$]	890.959
9	SARMAProcess[{1, 0}, {0, 1}$_{34}$]	891.028
10	ARProcess[1]	891.08

In Table 5 we present the results on the parameter estimation for both the ARIMA and the SARMA processes.

Table 5: The ARIMA and SARMA parameter tables

ARIMA model

	Estimate	Standard Error	t-Statistic	P-Value
b_1	−0.398645	0.0890771	−4.47528	9.63596×10^{-6}

SARMA model

	Estimate	Standard Error	t-Statistic	P-Value
a_1	0.438344	0.0949723	4.6155	5.522×10^{-6}
α_1	0.478033	0.0931484	5.13196	6.53092×10^{-7}

4.2. A simulation study

In this section we will compare, by means of a simulation study, two ways of obtaining the distribution of the number of elements in each risk class. First, we simulate 300 paths of the Markov chain model and compute the observed proportions of elements in each one of the six classes. We will also simulate the number of elements in each class according to the two models fitted in the previous section. The results are presented in Tables 6 and 7. The results in the first table show that the sub-population in class 5 is slightly larger in both ARIMA and SARMA models, when compared with the direct simulation of the Markov chain. As the class 5 population measures the most part of the risk of the portfolio, both the ARIMA and SARMA models are conservative, but not excessively.

Table 6: Proportions in each class by simulation of the Markov chain and of the Markov chain scheme models ARIMA and SARMA

	Class 1	Class 2	Class 3	Class 4	Class 5	Class 6
MARKOV	0.0366667	0.00333333	0.00333333	0.00333333	0.0266667	0.926667
ARIMA	0.106427	0.00594949	0.00328688	0.00362677	0.0305018	0.850208
SARMA	0.130596	0.00705139	0.00362903	0.00378776	0.0307309	0.824204

We also computed the relative proportions of elements in the population in each one of the five transient classes. The results show a remarkable difference between the Markov chain simulation and the Markov chain scheme fed with the sum of a sigmoid trend plus a SARMA process, thus showing the advantage of an open model.

Table 7: Conditional proportions for the 5 transient classes - simulation

	Class 1	Class 2	Class 3	Class 4	Class 5
MARKOV	0.5	0.0454545	0.0454545	0.0454545	0.363637
ARIMA	0.710498	0.0397184	0.021943	0.0242121	0.203628
SARMA	0.742888	0.0401113	0.0206435	0.0215464	0.17481

We simulated 300 paths for each of the two models fitted in section 4, to wit, the ARIMA$[0,1,1]$ and the SARMA$[(1,0),(1,0)_{34}]$. We computed the mean and the standard deviation for each classe and the correspondent one standard deviation confidence interval. Despite the paths in the ARIMA$[0,1,1]$ possibly taking negative values we computed the corresponding number of elements in each class. The results in Table 8 clearly show that the SARMA$[(1,0),(1,0)_{34}]$ model, for the residuals of a sigmoid type function fitting, is much more adequate to describe the evolution of the entrance of new clients in the credit portfolio.

Table 8: Data at date 106 and confidence intervals from models

	Class 1	Class 2	Class 3	Class 4	Class 5	Class 6
Data	5307	284	144	149	1203	32256
ARIMA	[-1614,5904]	[-81,321]	[-36,168]	[-33,179]	[-248,1478]	[-6164,40436]
SARMA	[4669,5593]	[255,299]	[134,151]	[142,156]	[1158,1257]	[31112,33654]

In Figure 2 we observe that the results given by the SARMA$[(1,0),(1,0)_{34}]$ model are more meaningful. In fact, in the empirical distribution of the simulated number of elements, negative numbers occur in both classes 1 and 5. [3]

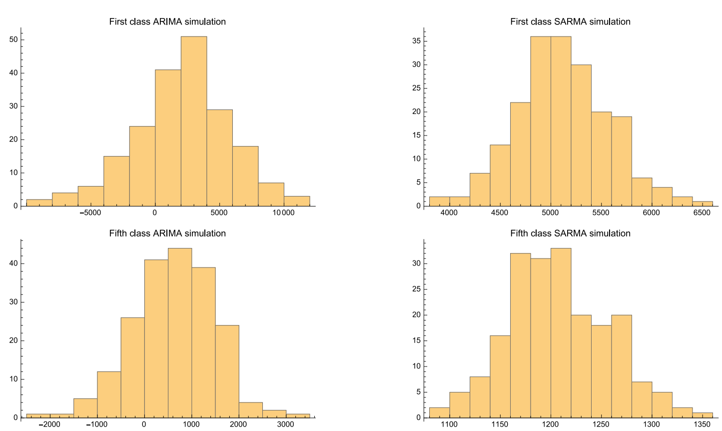

Figure 2: Simulated empirical distributions in classes 1 and 5.

[3] All Wolfram Mathematica 10 computational files used in this work are available at http://ferrari.dmat.fct.unl.pt/personal/mle/pps/pm-mle2009a.html.

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