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EXPANSIONS FOR QUANTILES AND MULTIVARIATE MOMENTS OF EXTREMES FOR HEAVY TAILED DISTRIBUTIONS

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Abstract:

- Let $X_{n,r}$ be the r-th largest of a random sample of size n from a distribution function $F(x) = 1 - \sum_{i=0}^{\infty} c_i\, x^{-\alpha - i\beta}$ for $\alpha > 0$ and $\beta > 0$. An inversion theorem is proved and used to derive an expansion for the quantile $F^{-1}(u)$ and powers of it. From this an expansion in powers of $\left(n^{-1}, n^{-\beta/\alpha}\right)$ is given for the multivariate moments of the extremes $\{X_{n,n-s_i},\, 1 \le i \le k\}/n^{1/\alpha}$ for fixed $\mathbf{s} = (s_1, ..., s_k)$, where $k \ge 1$. Examples include the Cauchy, Student's t, F, second extreme distributions and stable laws of index $\alpha < 1$.

Key-Words:

- *Bell polynomials; extremes; inversion theorem; moments; quantiles.*

AMS Subject Classification:

- 62E15, 62E17.

1. INTRODUCTION

For $1 \leq r \leq n$, let $X_{n,r}$ be the r-th largest of a random sample of size n from a continuous distribution function F on $\mathbb{R}$, the real numbers. Let f denote the density function of F when it exists.

The study of the asymptotes of the moments of $X_{n,r}$ has been of considerable interest. McCord [12] gave a first approximation to the moments of $X_{n,1}$ for three classes. This showed that a moment of $X_{n,1}$ can behave like any positive power of n or $n_1 = \log n$. (Here, log is to the base e.) Pickands [15] explored the conditions under which various moments of $(X_{n,1} - b_n)/a_n$ converge to the corresponding moments of the extreme value distribution. It was proved that this is indeed true for all F in the domain of attraction of an extreme value distribution provided that the moments are finite for sufficiently large n. Nair [13] investigated the limiting behavior of the distribution and the moments of $X_{n,1}$ for large n when F is the standard normal distribution function. The results provided rates of convergence of the distribution and the moments of $X_{n,1}$. Downey [4] derived explicit bounds for $\mathbb{E}[X_{n,1}]$ in terms of the moments associated with F. The bounds were given up to the order $o\left(n^{1/\rho}\right)$, where $\int_{-\infty}^{\infty} |x|^\rho \, dF(x)$ is defined, so $\mathbb{E}[X_{n,1}]$ grows slowly with the sample size. For other work, we refer the readers to Ramachandran [16], Hill and Spruill [9] and Hüsler *et al.* [10].

The main aim of this paper is to study multivariate moments of $\{X_{n,n-s_i}, 1 \leq i \leq k\}$ for fixed $\mathbf{s} = (s_1, ..., s_k)$, where $k \geq 1$. We suppose F is heavy tailed, *i.e.*,

$$1 - F(x) \sim Cx^{-\alpha} \tag{1.1}$$

as $x \to \infty$ for some $C > 0$ and $\alpha > 0$. For a nonparametric estimate of α, see Novak and Utev [14].

There are many practical examples giving rise to $\{X_{n,n-s_i}, 1 \leq i \leq k\}$ for heavy tailed F. Perhaps the most prominent example is the Hill's estimator (Hill [8]) for the *extremal index* given by

$$-\log X_{n,n-k} + k^{-1} \sum_{i=1}^{k} \log X_{n,n-i+1} \,.$$

Clearly, this is a function of $X_{n,n-s_i}$, $1 \leq i \leq k$. Real life applications of the Hill's estimator are far too many to list.

Since Hill [8], many other estimators have been proposed for the extremal index, see Gomes and Guillou [6] for an excellent review of such estimators. Each of these estimators is a function of $X_{n,n-s_i}$, $1 \leq i \leq k$. No doubt that many more estimators taking the form of a function of $X_{n,n-s_i}$, $1 \le i \le k$ will be proposed in the future.

A possible application of the results in this paper is to assess optimality of these estimators. Suppose we can write the general form of the estimators as

$$\omega = \omega\big(X_{n,n-s_1}, X_{n,n-s_2}, ..., X_{n,n-s_k}; \boldsymbol{\mu}\big) , \tag{1.2}$$

where $\boldsymbol{\mu}$ contains some parameters, which include k itself. The optimum values of $\boldsymbol{\mu}$ can be based on criteria like bias and mean squared error. For example, $\boldsymbol{\mu}$ could be chosen as the value minimizing the bias of ω or the value minimizing the mean squared error of ω. If (1.2) can be expanded as

$$\omega = \sum_{\theta_1,\theta_2,...,\theta_k} a\big(\theta_1, \theta_2, ..., \theta_k; \boldsymbol{\mu}\big) \prod_{i=1}^{k} X_{n,n-s_i}^{\theta_i}$$

then the bias and mean squared error of ω can be expressed as

$$\mathrm{Bias}(\omega) = \sum_{\theta_1,\theta_2,...,\theta_k} a\big(\theta_1, \theta_2, ..., \theta_k; \boldsymbol{\mu}\big) \, \mathbb{E}\left[\prod_{i=1}^{k} X_{n,n-s_i}^{\theta_i}\right] - \omega$$

and

$$\begin{aligned} \mathrm{MSE}(\omega) = & \sum_{\theta_1,\theta_2,...,\theta_k} \sum_{\vartheta_1,\vartheta_2,...,\vartheta_k} a\big(\theta_1, \theta_2, ..., \theta_k; \boldsymbol{\mu}\big)\, a\big(\vartheta_1, \vartheta_2, ..., \vartheta_k; \boldsymbol{\mu}\big) \, \mathbb{E}\left[\prod_{i=1}^{k} X_{n,n-s_i}^{\theta_i+\vartheta_i}\right] \\ & - \left\{ \sum_{\theta_1,\theta_2,...,\theta_k} a\big(\theta_1, \theta_2, ..., \theta_k; \boldsymbol{\mu}\big) \, \mathbb{E}\left[\prod_{i=1}^{k} X_{n,n-s_i}^{\theta_i}\right] \right\}^2 + \big[\mathrm{Bias}(\omega)\big]^2 , \end{aligned}$$

respectively. Both involve multivariate moments of $X_{n,n-s_i}$, $1 \le i \le k$. Expressions for the latter are given in Section 2, in particular, Theorem 2.2. Hence, general estimators can be developed for $\boldsymbol{\mu}$ which minimize bias, mean squared error, etc. Such developments could apply to any future estimator (also to any past estimator) of the extremal index taking the form of (1.2).

Note that $U_{n,r} = F\left(X_{n,r}\right)$ is the r-th order statistics from $U(0,1)$. For $1 \le r_1 < r_2 < \cdots < r_k \le n$ set $U_{n,\mathbf{r}} = \{U_{n,r_i}, 1 \le i \le k\}$. By Section 14.2 of Stuart and Ord [17], $U_{n,\mathbf{r}}$ has the multivariate beta density function

$$U_{n,\mathbf{r}} \sim B\left(\mathbf{u} : \mathbf{r}\right) = \prod_{i=0}^{k} \left(u_{i+1} - u_i\right)^{r_{i+1}-r_i-1} / B_n(\mathbf{r}) \tag{1.3}$$

on $0 < u_1 < \cdots < u_k < 1$, where $u_0 = 0$, $u_{k+1} = 1$, $r_0 = 0$, $r_{k+1} = n+1$ and

$$B_n(\mathbf{r}) = \prod_{i=1}^{k} B\left(r_i, r_{i+1} - r_i\right) . \tag{1.4}$$

David and Johnson [3] expanded $X_{n,r_i} = F^{-1}(U_{n,r_i})$ about $u_{n,i} = \mathbb{E}[U_{n,r_i}] = r_i/(n+1)$: $X_{n,r_i} = \sum_{j=0}^{\infty} G^{(j)}(u_{n,i})(U_{n,i} - u_{n,i})^j/j!$, where $G(u) = F^{-1}(u)$ and $G^{(j)}(u) = d^jG(u)/du^j$, and using the properties of (1.3) showed that if $\mathbf{r}$ depends on n in such a ways that $\mathbf{r}/n \to \mathbf{p} \in (\mathbf{0}, \mathbf{1})$ as $n \to \infty$ then the m-th order cumulants of $X_{n,\mathbf{r}} = \{X_{n,r_i}, 1 \le i \le k\}$ have magnitude $O\left(n^{1-m}\right)$ — at least for $n \le 4$, so that the distribution function of $X_{n,\mathbf{r}}$ has a multivariate Edgeworth expansion in powers of $n^{-1/2}$. (Alternatively one can use James and Mayne [11] to derive the cumulants of $X_{n,\mathbf{r}}$ from those of $U_{n,\mathbf{r}}$.) The method requires the derivatives of F at $\left\{F^{-1}(p_i), 1 \le i \le k\right\}$ so breaks down if $p_i = 0$ or $p_k = 1$ — which is the situation we study here.

In Withers and Nadarajah [18], we showed that for fixed $\mathbf{r}$ when (1.1) holds the distribution of $X_{n,n\mathbf{1}-\mathbf{r}}$ (where $\mathbf{1}$ is the vector of ones in $\mathbb{R}^k$), suitably normalized tends to a certain multivariate extreme value distribution as $n \to \infty$, and so obtained the leading terms of the expansions of its moments in inverse powers of n. Here, we show how to extend those expansions when

$$F^{-1}(u) = \sum_{i=0}^{\infty} b_i (1-u)^{\alpha_i} \tag{1.5}$$

with $\alpha_0 < \alpha_1 < \cdots$, that is, $\{1 - F(x)\}\, x^{-1/\alpha_0}$ has a power series in $\{x^{-\delta_i} \colon \delta_i = (\alpha_i - \alpha_0)/\alpha_0\}$. Hall [7] considered (1.5) with $\alpha_i = i - 1/\alpha$, but did not give the corresponding expansion for $F(x)$ or expansions in inverse powers of n. He applied it to the Cauchy. In Section 2, we demonstrate the method when

$$1 - F(x) = x^{-\alpha} \sum_{i=0}^{\infty} c_i\, x^{-i\beta}\,, \tag{1.6}$$

where $\alpha > 0$ and $\beta > 0$. In this case, (1.5) holds with $\alpha_i = (i\beta - 1)/\alpha$. In Section 3, we apply it to the Student's t, F and second extreme value distributions and to stable laws of exponent $\alpha < 1$. The appendix gives the inverse theorem needed to pass from (1.6) to (1.5), and expansions for powers and logs of series.

We use the following notation and terminology. Let $(x)_i = \Gamma(x+i)/\Gamma(x)$ and $\langle x\rangle_i = \Gamma(x+1)/\Gamma(x-i+1)$. An inequality in $\mathbb{R}^k$ consists of k inequalities. For example, for $\mathbf{x}$ in $\mathbb{C}^k$, where $\mathbb{C}$ is the set of complex numbers, $\mathrm{Re}(\mathbf{x}) < \mathbf{0}$ means that $\mathrm{Re}(x_i) < 0$ for $1 \le i \le k$. Also let $I(A) = 1$ if A is true and $I(A) = 0$ if A is false. For $\boldsymbol{\theta} \in \mathbb{C}^k$ let $\overline{\boldsymbol{\theta}}$ denote the vector with $\overline{\theta}_i = \sum_{j=1}^{i} \theta_j$.

2. MAIN RESULTS

For $1 \le r_1 < \cdots < r_k \le n$ set $s_i = n - r_i$. Here, we show how to obtain expansions in inverse powers of n for the moments of the $X_{n,\mathbf{s}}$ for fixed $\mathbf{r}$ when (1.5) holds, and in particular when the upper tail of F satisfies (1.6).

Theorem 2.1. *Suppose (1.6) holds with c_0, α, $\beta > 0$. Then $F^{-1}(u)$ is given by (1.5) with $\alpha_i = ia - 1/\alpha$, $a = \beta/\alpha$ and $b_i = C_{i,1/\alpha}$, where $C_{i,\psi} = c_0^{\psi} \widehat{C}_i\left(-\psi, c_0, \mathbf{x}^*\right)$ of (A.3) and $x_i^* = x_i^*\left(a, 1, \mathbf{c}\right)$ of (A.4). In particular,*

$$\begin{aligned}
C_{0,\psi} &= c_0^{\psi} ,\\
C_{1,\psi} &= \psi\, c_0^{\psi-a-1} c_1 ,\\
C_{2,\psi} &= \psi\, c_0^{\psi-2a-2} \left\{c_0 c_2 + (\psi - 2a - 1)\, c_1^2/2\right\} ,\\
C_{3,\psi} &= \psi\, c_0^{\psi-3a-3} \left[c_0^2 c_2 + (\psi-3a-1)\, c_0 c_1 c_2 + \left\{(\psi+1)_2/6(\psi+3a/2)(a+1)\right\} c_1^3\right] ,
\end{aligned}$$

and so on. Also for any θ in $\mathbb{R}$,

$$\left\{F^{-1}(u)\right\}^{\theta} = \sum_{i=0}^{\infty} (1-u)^{ia-\psi} C_{i,\psi} \tag{2.1}$$

at $\psi = \theta/\alpha$.

On those rate occasions, where the coefficients $d_i = C_{i,1/\alpha}$ in $F^{-1}(u) = \sum_{i=0}^{\infty} (1-u)^{ia-1/\alpha} d_i$ are known from some alternative formula then one can use $C_{i,\psi} = d_0^{\theta} \widehat{C}_i\left(\theta, 1/d_0, \mathbf{d}\right)$ of (A.3).

Proof of Theorem 2.1: By Theorem A.1 with $k = 1$, we have $x^{-\alpha} = \sum_{i=0}^{\infty} x_i^* (1-u)^{1+ia}$ at $u = F(x)$, where

$$\begin{aligned}
x_0^* &= c_0^{-1} ,\\
x_1^* &= c_0^{-a-2} c_1 ,\\
x_2^* &= c_0^{-2a-3} \left\{-c_0 c_2 + (a+1) c_1^2\right\} ,\\
x_3^* &= c_0^{-3a-4} \left\{-c_0^2 c_3 + (2+3a)\, c_0 c_1 c_2 - (2+3a)\,(1+a)\, c_1^2/2\right\} ,
\end{aligned}$$

and so on. So, for S of (A.1), $x^{-\alpha} = c_0^{-1} v\left[1 + c_0 S\left(v^a, \mathbf{x}^*\right)\right]$ at $v = 1-u$. Now apply (A.2). □

Lemma 2.1. *For $\boldsymbol{\theta}$ in $\mathbb{C}^k$,*

$$\mathbb{E}\left[\prod_{i=1}^{k}(1-U_{n,r_i})^{\theta_i}\right] = b_n\left(\mathbf{r}:\overline{\boldsymbol{\theta}}\right) , \tag{2.2}$$

where

$$b_n\left(\mathbf{r}:\overline{\boldsymbol{\theta}}\right) = \prod_{i=1}^{k} b\left(r_i-r_{i-1}, n-r_i+1:\overline{\theta}_i\right) \tag{2.3}$$

and $b\,(\alpha,\beta:\theta) = B(\alpha,\beta+\theta)/B(\alpha,\beta)$. *Also in (1.4),*

$$B_n\left(\mathbf{r}\right) = \prod_{i=1}^{k} B\left(r_i-r_{i-1}, n-r_i+1\right) . \tag{2.4}$$

Since $B(\alpha,\beta)=\infty$ for $\mathrm{Re}\beta\le 0$, for (2.2) to be finite we need $n-r_i+1+\mathrm{Re}\,\overline{\theta}_i>0$ for $1\le i\le k$.

Proof of Lemma 2.1: Let I_k denote the left hand side of (2.2). Then $I_k = \displaystyle\int B_n(\mathbf{u}:\mathbf{r})\prod_{i=1}^{k}(1-u_i)^{\theta_i}\,du_1\cdots du_k$ integrated over $0<u_1<\cdots<u_k<1$ by (1.3). So, (2.2), (2.4) hold for $k=1$. Set $s_i=(u_i-u_{i-1})\,/\,(1-u_{i-1})$. Then

$$I_2 = \int_0^1 u_1^{r_1-1}(1-u_1)^{\theta_1}\int_{u_1}^1 (u_2-u_1)^{r_2-r_1-1}(1-u_2)^{r_3-r_2-1+\theta_2}\,du_2\big/B_n(\mathbf{r}) ,$$

which is the the right hand side of (2.2) with denominator replaced by the right hand side of (2.3). Putting $\boldsymbol{\theta}=\mathbf{0}$ gives (2.2), (2.4) for $k=2$. Now use induction. □

Lemma 2.2. *In Lemma 2.1, the restriction*

$$1\le r_1<\cdots<r_k\le n \quad \textit{may be relaxed to} \quad 1\le r_1\le\cdots\le r_k\le n . \tag{2.5}$$

Proof: For $k=2$, the second factor in the right hand side of (2.3) is $b(r_2-r_1,\ n-r_2+1:\overline{\theta}_2) = f\left(\overline{\theta}_2\right)/f(0)$, where $f\left(\overline{\theta}_2\right)=\Gamma\left(n-r_2+1+\overline{\theta}_2\right)/\Gamma\left(n-r_1+1+\overline{\theta}_2\right)=1$ if $r_2=r_1$ and the first factor is $b\left(r_1,n-r_1+1:\overline{\theta}_1\right)=\mathbb{E}\left[(1-U_{n,r_1})^{\overline{\theta}_1}\right]$. Similarly, if $r_i=r_{i-1}$, the i-th factor is 1 and the product of the others is $\mathbb{E}\left[\displaystyle\prod_{j=1,j\neq i}^{k}\left(1-U_{n,r_j}\right)^{\theta_j^*}\right]$, where $\theta_j^*=\theta_j$ for $j\neq i-1$ and $\theta_j^*=\theta_{i-1}+\theta_i$ for $j=i-1$. □

Corollary 2.1. *In any formulas for $\mathbb{E}\left[g\left(X_{n,\mathbf{r}}\right)\right]$ for some function g, (2.5) holds. In particular it holds for the moments and cumulants of $X_{n,\mathbf{r}}$.*

This result is very important as it means we can dispense with treating the 2^{k-1} cases $r_i < r_{i+1}$ or $r_i = r_{i+1}$, $1 \le i \le k-1$ separately. For example, Hall [7] treats the two cases for $\cos(X_{n,\mathbf{r}}, X_{n,\mathbf{s}})$ separately and David and Johnson [3] treat the 2^{k-1} cases for the k-th order cumulants of $X_{n,\mathbf{r}}$ separately for $k \le 4$.

Theorem 2.2. *Under the conditions of Theorem 2.1,*

$$\mathbb{E}\left[\prod_{i=1}^{k} X_{n,r_i}^{\theta_i}\right] = \sum_{i_1,\ldots,i_k=0}^{\infty} C_{i_1,\psi_1} \cdots C_{i_k,\psi_k} b_n\left(\mathbf{r} : \bar{\mathbf{i}}a - \overline{\boldsymbol{\theta}}/\alpha\right) \tag{2.6}$$

with b_n as in (2.3), where $\boldsymbol{\psi} = \boldsymbol{\theta}/\alpha$. All terms are finite if $\mathrm{Re}\,\overline{\boldsymbol{\theta}} < (\mathbf{s}+1)\,\alpha$, where $s_i = n - r_i$.

Lemma 2.3. *For α, β positive integers and θ in $\mathbb{C}$,*

$$b(\alpha, \beta : \theta) = \prod_{j=\beta}^{\alpha+\beta-1} (1 + \theta/j)^{-1} . \tag{2.7}$$

So, for $\boldsymbol{\theta}$ in $\mathbb{C}^k$,

$$b_n\left(\mathbf{r} : \overline{\boldsymbol{\theta}}\right) = \prod_{i=1}^{k} \prod_{j=s_i+1}^{s_{i-1}} \left(1 + \overline{\theta}_i/j\right)^{-1} , \tag{2.8}$$

where $s_i = n - r_i$ and $r_0 = 0$.

Proof: The left hand side of (2.7) is equal to $\Gamma(\beta+\theta)\Gamma(\alpha+\beta)\,/ \{\Gamma(\beta+\theta+\alpha)\Gamma(\beta)\}$. But $\Gamma(\alpha+x)/\Gamma(x) = (x)_\alpha$, so (2.7) holds, and hence (2.8). $\square$

From (2.3) we have, interpreting $\prod_{i=2}^{k-1} b_i$ as 1,

Lemma 2.4. *For $s_i = n - r_i$,*

$$b_n\left(\mathbf{r} : \overline{\boldsymbol{\theta}}\right) = B\left(\mathbf{s} : \overline{\boldsymbol{\theta}}\right) n! \,/\, \Gamma\left(n + 1 + \overline{\theta}_1\right) , \tag{2.9}$$

where

$$B\left(\mathbf{s} : \overline{\boldsymbol{\theta}}\right) = \Gamma\left(s_1 + 1 + \overline{\theta}_1\right) (s_1!)^{-1} \prod_{i=2}^{k} b\left(s_{i-1} - s_i, s_i + 1 : \overline{\theta}_i\right)$$

does not depend on n for fixed $\mathbf{s}$.

Lemma 2.5. *We have*

$$n!/\Gamma(n+1+\theta) \;=\; n^{-\theta}\sum_{i=0}^{\infty} e_i(\theta)n^{-i} \;,$$

where

$$\begin{aligned}
&e_0(\theta) = 1\;, \quad e_1(\theta) = -(\theta)_2/2\;, \quad e_2(\theta) = (\theta)_3(3\theta+1)/24\;,\\
&e_3(\theta) = -(\theta)_4(\theta)_2/(4!\cdot 2)\;, \quad e_4(\theta) = (\theta)_5\left(15\theta^3+30\theta^2+5\theta-2\right)/(5!\cdot 48)\;,\\
&e_5(\theta) = -(\theta)_6(\theta)_2\left(3\theta^2+7\theta-2\right)/(6!\cdot 16)\;,\\
&e_6(\theta) = (\theta)_7\left(63\theta^5+315\theta^4+315\theta^3-91\theta^2-42\theta+16\right)/(7!\cdot 576)\;,\\
&e_7(\theta) = -(\theta)_8(\theta)_2\left(9\theta^4+54\theta^3+51\theta^2-58\theta+16\right)/(8!\cdot 144)\;.
\end{aligned}$$

Proof: Apply equation (6.1.47) of Abramowitz and Stegun [1]. □

So, (2.6), (2.9) yield the joint moments of $X_{n,\mathbf{r}}n^{-1/\alpha}$ for fixed $\mathbf{s}$ as a power series in $(1/n, n^{-\alpha})$:

Corollary 2.2. *Under the conditions of Theorem 2.1,*

$$\mathbb{E}\left[\prod_{i=1}^{k} X_{n,n-s_i}^{\theta_i}\right] = \sum_{j=0}^{\infty} n!\,\Gamma\left(n+1+ja-\overline{\psi}_1\right)^{-1} C_j\left(\mathbf{s}:\boldsymbol{\psi}\right)\;, \tag{2.10}$$

where $\boldsymbol{\psi} = \boldsymbol{\theta}/\alpha$ *and*

$$C_j\left(\mathbf{s}:\boldsymbol{\psi}\right) \;=\; \sum\left\{C_{i_1,\psi_1}\cdots C_{i_k,\psi_k} B\left(\mathbf{s}:\bar{\mathbf{i}}a-\overline{\boldsymbol{\psi}}\right):\; i_1+\cdots+i_k = j\right\}.$$

So, if $\mathbf{s}$, $\boldsymbol{\theta}$ are fixed as $n\to\infty$ and $\mathrm{Re}\left(\overline{\boldsymbol{\theta}}\right) < (\mathbf{s}+\mathbf{1})\,\alpha$, then the left hand side of (2.10) is equal to

$$n^{\psi_1}\sum_{i,j=0}^{\infty} n^{-i-ja}\, e_i\left(ja-\overline{\psi}_1\right) C_j\left(\mathbf{s}:\boldsymbol{\psi}\right)\;. \tag{2.11}$$

If a is rational, say $a = M/N$ then the left hand side of (2.10) is equal to

$$n^{\overline{\psi}_1}\sum_{m=0}^{\infty} n^{-m/N} d_m\left(\mathbf{s}:\boldsymbol{\psi}\right)\;, \tag{2.12}$$

where

$$\begin{aligned}
d_m\left(\mathbf{s}:\boldsymbol{\psi}\right) \;&=\; \sum\left\{e_i\left(ja-\overline{\psi}_1\right) C_j\left(\mathbf{s}:\boldsymbol{\psi}\right):\; iN+jM = m\right\}\\
&=\; \sum\left\{e_{m-ja}\left(ja-\overline{\psi}_1\right) C_j\left(\mathbf{s}:\boldsymbol{\psi}\right):\; 0\le j\le m/a\right\}
\end{aligned}$$

if $N=1$; so for d_m to depend on c_1 and not just c_0 we need $m\le M$.

The leading term in (2.11) does not involve c_1 so may be deduced from the multivariate extreme value distribution that the law of $X_{n,n-s_i}$, suitably normalized, tends to. The same is true of the leading terms of its cumulants. See Withers and Nadarajah [18] for details.

The leading terms in (2.11) are

$$n^{\overline{\psi}_1}\left[\left\{1-n^{-1}\langle\overline{\psi}_1\rangle_2/2\right\}C_0\left(\mathbf{s}:\boldsymbol{\psi}\right)+n^{-a}C_1\left(\mathbf{s}:\boldsymbol{\psi}\right)+O\left(n^{-2a_0}\right)\right],$$

where

$$\begin{aligned}
a_0 &= \min(a,1)\,,\\
C_0\left(\mathbf{s}:\boldsymbol{\psi}\right) &= c_0 B\left(\mathbf{s}:-\overline{\boldsymbol{\psi}}\right)\,,\\
C_1\left(\mathbf{s}:\boldsymbol{\psi}\right) &= c_0^{\overline{\psi}_1-a-2}c_1\sum_{j=1}^{k}\psi_j B\left(\mathbf{s}:a\mathbf{I}_j-\overline{\boldsymbol{\psi}}\right)
\end{aligned}$$

and $I_{j,m}=I(m\le j)$. For $k=1$,

$$\begin{aligned}
C_0(s:\psi) &= c_0^{\psi}(s+1)_{-\psi} = c_0^{\psi}/\langle s\rangle_{\psi}\,,\\
C_1(s:\psi) &= \psi c_0^{\psi-a-1}c_1(s+1)_{a-\psi} = \psi c_0^{\psi-a-1}c_1/\langle s\rangle_{\psi-a}\,.
\end{aligned}$$

Set

$$\pi_{\mathbf{s}}(\lambda) = b\left(s_1-s_2,s_2+1:\lambda\right) = \prod_{j=s_2+1}^{s_1}1/\left(1+\lambda/j\right)$$

for λ an integer. For example, $\pi_{\mathbf{s}}(1)=(s_2+1)/(s_1+1)$ and $\pi_{\mathbf{s}}(-1)=s_1/s_2$. Then for $k=2$,

$$\begin{aligned}
C_0\left(\mathbf{s}:\lambda\mathbf{1}\right) &= c_0^{2\lambda}\langle s_1\rangle_{2\lambda}^{-1}\,\pi_{\mathbf{s}}(-\lambda)\\
&= c_0^{2}\left(s_1-1\right)^{-1}s_2 \quad \text{for } \lambda=1\\
&= c_0^{2}\langle s_2-2\rangle_2^{-1}\langle s_2\rangle_2^{-1} \quad \text{for } \lambda=2
\end{aligned}$$

and

$$\begin{aligned}
C_1\left(\mathbf{s}:\lambda\mathbf{1}\right) &= \lambda\,c_0^{2\lambda-a-1}c_1\langle s_1\rangle_{2\lambda-a}^{-1}\left\{\pi_{\mathbf{s}}(-\lambda)+\pi_{\mathbf{s}}(a-\lambda)\right\}\\
&= \lambda\,c_0^{1-a}c_1\langle s_1\rangle_{2-a}^{-1}\left\{s_1/s_2+\pi_{\mathbf{s}}(a-1)\right\} \quad \text{for } \lambda=1\\
&= \lambda\,c_0^{3-a}c_1\langle s_1\rangle_{4-a}^{-1}\left\{\langle s_1\rangle_2\langle s_2\rangle_2^{-1}+\pi_{\mathbf{s}}(a-2)\right\} \quad \text{for } \lambda=2\,.
\end{aligned}$$

Set $\lambda=1/\alpha$, $Y_{n,s}=X_{n,n-s}/\left(nc_0\right)^{\lambda}$ and $E_{\mathbf{c}}=\lambda\,c_0^{-a-1}c_1$. Then for $s>\lambda-1$

$$\mathbb{E}\left[Y_{n,s}\right] = \left\{1-n^{-1}\langle\lambda\rangle_2/2\right\}\langle s\rangle_{\lambda}^{-1}+n^{-a}E_{\mathbf{c}}\langle s\rangle_{\lambda-a}^{-1}+O\left(n^{-2a_0}\right) \tag{2.13}$$

and for $s_1>2\lambda-1$, $s_2>\lambda-1$, $s_1\ge s_2$,

$$\mathbb{E}\left[Y_{n,s_1}Y_{n,s_2}\right] = \left\{1-n^{-1}\langle 2\lambda\rangle_2/2\right\}B_{2,0}+n^{-a}E_{\mathbf{c}}D_a+O\left(n^{-2a_0}\right)\,, \tag{2.14}$$

where $B_{2,0} = \langle s_1 \rangle_{2\lambda}^{-1} \pi_{\mathbf{s}}(-\lambda)$, $D_a = \langle s_1 \rangle_{2\lambda-a}^{-1} \{\pi_{\mathbf{s}}(-\lambda) + \pi_{\mathbf{s}}(a-\lambda)\}$ and

$$\text{(2.15)} \qquad \operatorname{Cov}(Y_{n,s_1}, Y_{n,s_2}) \;=\; F_0 + F_1/n + E_{\mathbf{c}} F_2/n + O\left(n^{-2a_0}\right) ,$$

where $F_0 = B_{2,0} - \langle s_1 \rangle_\lambda^{-1} \langle s_2 \rangle_\lambda^{-1}$, $F_1 = \langle \lambda \rangle_2 \langle s_1 \rangle_\lambda^{-1} \langle s_2 \rangle_\lambda^{-1} - \langle 2\lambda \rangle_2 B_{2,0}/2$ and $F_2 = D_a - \langle s_1 \rangle_\lambda^{-1} \langle s_2 \rangle_{\lambda-a}^{-1} - \langle s_1 \rangle_{\lambda-a}^{-1} \langle s_2 \rangle_\lambda^{-1}$. Similarly, we may use (2.11) to approximate higher order cumulants. If $a = 1$ this gives $\mathbb{E}[Y_{n,\mathbf{s}}]$ and $\operatorname{Cov}(Y_{n,s_1}, Y_{n,s_2})$ to $O\left(n^{-2}\right)$.

Example 2.1. Suppose $\alpha = 1$. Then $Y_{n,s} = X_{n,n-s}/(nc_0)$, $E_{\mathbf{c}} = c_0^{-a-1} c_1$, $B_{2,0} = -F_1 = (s_1-1)^{-1} s_2^{-1}$, $F_0 = \langle s_1 \rangle_2^{-1} s_2^{-1}$, $D_a = \langle s_1 \rangle_{2-a}^{-1} G_a$, where $G_a = s_1 s_2^{-1} + \pi_{\mathbf{s}}(a-1)$ for $s_1 \geq s_2$, $G_a = 2$ for $s_1 = s_2$ and $F_2 = D_a - s_1^{-1} \langle s_2 \rangle_{1-a}^{-1} - s_2^{-1} \langle s_1 \rangle_{1-a}^{-1}$. So,

$$\text{(2.16)} \qquad \mathbb{E}[Y_{n,s}] \;=\; s^{-1} + n^{-a} E_{\mathbf{c}} \langle s \rangle_{1-a}^{-1} + O\left(n^{-2a_0}\right)$$

for $s > 0$ and (2.14)–(2.15) hold if

$$\text{(2.17)} \qquad s_1 > 1 \,, \quad s_2 > 0 \,, \quad s_1 \geq s_2 \,.$$

A little calculation shows that $C_0(\mathbf{s} : \mathbf{1}) = c_0^k B_{k,0}$, $C_1(\mathbf{s} : \mathbf{1}) = c_0^{k-a-1} c_1 B_{k,\cdot}$, and

$$\begin{aligned} \mathbb{E}\left[\prod_{i=1}^k Y_{n,s_i}\right] &= \left\{1 + n^{-1} \langle k \rangle_2 / 2\right\} B_{k,0} + n^{-a} E_{\mathbf{c}} B_{k,\cdot} + O\left(n^{-2a_0}\right) \\ &= m_0(\mathbf{s}) + n^{-1} m_1(\mathbf{s}) + n^{-a} m_a(\mathbf{s}) + O\left(n^{-2a_0}\right) \end{aligned}$$

say for $s_i > k-i$, $1 \leq i \leq k$ and $s_1 \geq \cdots \geq s_k$, where

$$\begin{aligned} B_{k,\cdot} &= \sum_{j=1}^k B_{k,j} \,, \\ B_{k,0} &= \prod_{i=1}^k 1/(s_1 - k + i) \,, \\ B_{k,j} &= \prod_{i=1}^{j-1} (s_i - k + a + i)^{-1} \langle s_j - k + j + 1 \rangle_{a-1} \prod_{i=j+1}^k (s_i - k + i)^{-1} \,, \\ B_{k,k} &= \prod_{i=1}^{k-1} (s_i - k + a + i)^{-1} \langle s_k \rangle_{1-a}^{-1} \end{aligned}$$

for $s_i > k - i$ and $1 \leq j < k$. For example, $B_{1,0} = s_1$, $B_{2m,0} = (s_1-1)^{-1} s_2^{-1}$ and $B_{3,0} = (s_1-2)^{-1} (s_2-1)^{-1} s_3^{-1}$. So, $\kappa_n(\mathbf{s}) = \kappa(Y_{n,s_1}, ..., Y_{n,s_k})$, the joint cumulant of $(Y_{n,s_1}, ..., Y_{n,s_k})$, is given by $\kappa_n(\mathbf{s}) = \kappa_0(\mathbf{s}) + n^{-1}\kappa_1(\mathbf{s}) + n^{-a}\kappa_a(\mathbf{s}) +$

$O\left(n^{-2a_0}\right)$, where, for example,

$$\begin{aligned}
\kappa_0\left(s_1,s_2,s_3\right) &= m_0\left(s_1,s_2,s_3\right)-m_0\left(s_1\right)m_0\left(s_2,s_3\right)-m_0\left(s_2\right)m_0\left(s_1,s_3\right)\\
&\quad -m_0\left(s_3\right)m_0\left(s_1,s_2\right)+2\prod_{i=1}^{3}m_0\left(s_i\right)\\
&= 2\left(s_1+s_2-2\right)D\left(s_1,s_2,s_3\right)\ ,
\end{aligned}$$

$$\begin{aligned}
\kappa_1\left(s_1,s_2,s_3\right) &= m_1\left(s_1,s_2,s_3\right)-m_0\left(s_1\right)m_1\left(s_2,s_3\right)-m_0\left(s_2\right)m_1\left(s_1,s_3\right)\\
&\quad -m_0\left(s_3\right)m_1\left(s_1,s_2\right)\\
&= 2\left\{s_2\left(1-2s_1\right)+s_1-s_1^2\right\}/D\left(s_1,s_2,s_3\right) \qquad \text{since} \quad m_1\left(s_1\right)=0\ ,
\end{aligned}$$

$$\begin{aligned}
\kappa_a\left(s_1,s_2,s_3\right) &= m_a\left(s_1,s_2,s_3\right)-m_0\left(s_1\right)m_a\left(s_2,s_3\right)-m_a\left(s_1\right)m_0\left(s_2,s_3\right)\\
&\quad -m_0\left(s_2\right)m_a\left(s_1,s_3\right)-m_a\left(s_2\right)m_0\left(s_1,s_3\right)-\ m_0\left(s_3\right)m_a\left(s_1,s_2\right)\\
&\quad -m_a\left(s_3\right)m_0\left(s_1,s_2\right)+2\,m_0\left(s_1\right)m_0\left(s_2\right)m_a\left(s_3\right)\\
&\quad +2\,m_0\left(s_3\right)m_0\left(s_1\right)m_a\left(s_2\right)+\ 2\,m_0\left(s_2\right)m_0\left(s_3\right)m_a\left(s_1\right)\ ,
\end{aligned}$$

where $D\left(s_1,s_2,s_3\right)=\langle s_1\rangle_3\,\langle s_2\rangle_2\,s_3$.

Consider the case $a=1$. Then $\kappa_a\left(s_1,s_2,s_3\right)=0$ so

$$\begin{aligned}
\kappa_n\left(s_1,s_2,s_3\right) &= 2\Big\{s_1+s_2-2+n^{-1}\left(s_2\left(1-2s_1\right)+s_1-s_1^2\right)\Big\}/D\left(s_1,s_2,s_3\right)\\
&\quad +O\left(n^{-2}\right)\ . \qquad (2.18)
\end{aligned}$$

Set $s. = \sum_{j=1}^{k} s_j$. Then

$$\begin{aligned}
&B_{1,\cdot}=B_{1,1}-1\ , \qquad B_{2,2}=1/s_2\ , \qquad B_{2,2}=1/s_2\ , \qquad B_{2,2}=s_1\ ,\\
&B_{2,\cdot}=s_1^{-1}+s_2^{-1}=\left(s_1+s_2\right)/\left(s_1s_2\right)\ ,\\
&B_{3,1}=\left(s_2-1\right)^{-1}s_3^{-1}\ , \qquad B_{3,2}=\left(s_1-1\right)^{-1}s_3^{-1}\ , \qquad B_{3,3}=\left(s_1-1\right)^{-1}s_2^{-1}\ ,\\
&B_{3,\cdot}=\left\{s_2\left(s.-2\right)-s_3\right\}\left(s_1-1\right)^{-1}\langle s_2\rangle_2^{-1}s_3^{-1}\ ,\\
&B_{4,1}=\left(s_2-2\right)^{-1}\left(s_3-1\right)^{-1}s_4^{-1}\ , \qquad B_{4,2}=\left(s_1-2\right)^{-1}\left(s_3-1\right)^{-1}s_4^{-1}\ ,\\
&B_{4,3}=\left(s_1-2\right)^{-1}\left(s_2-1\right)^{-1}s_4^{-1}\ , \qquad B_{4,4}=\left(s_1-2\right)^{-1}\left(s_2-1\right)^{-1}s_3^{-1}\ ,\\
&B_{4,\cdot}=\left\{s.s_3\left(s_2-2\right)+s_3\left(s_2-4s_2+4\right)-s_2s_4\right\}\left\{\left(s_1-2\right)\langle s_2-2\rangle_2\,\langle s_3\rangle_2\,s_4\right\}^{-1}\ .
\end{aligned}$$

Also $E_{\mathbf{c}}=c_0^{-2}c_1$, $D_a=s_1^{-1}+s_2^{-1}$, $F_2=0$, and

$$\mathbb{E}\left[Y_{n,s}\right]=s^{-1}+n^{-1}E_{\mathbf{c}}+O\left(n^{-2}\right) \qquad \text{for} \quad s>0\ , \tag{2.19}$$

$$\mathbb{E}\left[Y_{n,s_1}Y_{n,s_2}\right]=\left(1-n^{-1}\right)B_{2,\cdot}+n^{-1}E_{\mathbf{c}}D_a+O\left(n^{-2}\right) \qquad \text{if} \quad (2.17)\ \text{holds}\ , \tag{2.20}$$

$$\operatorname{Cov}\left(Y_{n,s_1},Y_{n,s_2}\right)=\langle s_1\rangle_2^{-1}s_2^{-1}\left(s_2-n^{-1}s_1\right)+O\left(n^{-2}\right) \qquad \text{if} \quad (2.17)\ \text{holds}\ . \tag{2.21}$$

In the case $a\geq 2$, (2.19)–(2.21) hold with $E_{\mathbf{c}}$ replaced by 0. In the case $a\leq 1$, (2.14)–(2.16) with $a_0=a$ give terms $O\left(n^{-2a}\right)$ with the n^{-1} terms disposable if $a\leq 1/2$.

We now investigate what extra terms are needed to make (2.19)–(2.21) depend on c when $a = 1$ or 2.

Example 2.2. $\alpha = \beta = 1$. Here, we find the coefficients of n^{-2}. By (2.12),

$$\begin{aligned} d_2(\mathbf{s}:\boldsymbol{\psi}) &= \sum_{j=0}^{2} e_{2-j}\left(j-\overline{\psi}_1\right) C_j(\mathbf{s}:\boldsymbol{\psi}) \\ &= e_2\left(-\overline{\psi}_1\right) C_0(\mathbf{s}:\boldsymbol{\psi}) + e_1\left(1-\overline{\psi}_1\right) C_1(\mathbf{s}:\boldsymbol{\psi}) + C_2(\mathbf{s}:\boldsymbol{\psi}) \\ &= C_2(\mathbf{s}:\boldsymbol{\psi}) \qquad \text{if } \ \overline{\psi}_1 = 1 \ \text{ or } \ 2\,. \end{aligned}$$

For $k = 1$, $C_2(s:\psi) = C_{2,\psi}(s+1)_{2-\psi}$, where $C_{2,\psi} = \psi c_0^{\psi-4}\left\{c_0c_2 + (\psi-3)c_1^2/2\right\}$, so $d_2(s:1) = (s+1)F_{\mathbf{c}}$, where $F_{\mathbf{c}} = c_0^{-3}\left(c_0c_2 - c_1^2\right)$, so in (2.19) we may replace $O\left(n^{-2}\right)$ by $n^{-2}(s+1)F_{\mathbf{c}}c_0^{-1} + O\left(n^{-3}\right)$. For $k = 2$,

$$\begin{aligned} C_2(\mathbf{s}:\mathbf{1}) &= \sum\Big\{C_{i,1}C_{j,1}B(\mathbf{s}:0,j-1):\ i+j=2\Big\} \\ &= C_{0,1}C_{2,1}\big\{B(\mathbf{s}:0,1) + B(\mathbf{s}:0,-1)\big\} + C_{1,1}^2 B(\mathbf{s}:0,0)\ , \end{aligned}$$

where $B(\mathbf{s}:0,\lambda) = b(s_1-s_2, s_2+1:\lambda) = \pi_{\mathbf{s}}(\lambda)$, so $d_2(\mathbf{s}:\mathbf{1}) = C_2(\mathbf{s}:\mathbf{1}) - D_{2,\mathbf{s}}H_{\mathbf{c}} + c_0^{-2}c_1^2$, where $D_{2,\mathbf{s}} = (s_2+1)(s_1+1)^{-1} + s_1s_2^{-1}$, $H_{\mathbf{c}} = c_0^{-2}\left(c_0c_2 - c_1^2\right)$ and in (2.20) we may replace $O\left(n^{-2}\right)$ by $n^{-2}d_2(\mathbf{s}:\mathbf{1})\,c_0^{-2} + O\left(n^{-3}\right)$. Upon simplifying this gives

$$\operatorname{Cov}\left(Y_{n,s_1}, Y_{n,s_2}\right) = \langle s_1\rangle_2^{-1}s_2^{-1}\left(1-n^{-1}s_1\right) - c_0^{-2}H_{\mathbf{c}}F_{3,\mathbf{s}}\,n^{-2} + O\left(n^{-2}\right)\ ,$$

where $F_{3,\mathbf{s}} = (s_2+1)/\langle s_1\rangle_2 + s_2^{-1}$.

Example 2.3. $\alpha = 1$, $\beta = 2$. So, $a = 2$, $\lambda = 1$, $\boldsymbol{\psi} = \boldsymbol{\theta}$. By (2.12),

$$\begin{aligned} d_2(\mathbf{s}:\boldsymbol{\psi}) &= \sum_{j=0}^{1} e_{2-2j}\left(2j-\overline{\psi}_1\right) C_j(\mathbf{s}:\boldsymbol{\psi}) \\ &= e_2\left(-\overline{\psi}_1\right) C_0(\mathbf{s}:\boldsymbol{\psi}) + C_1(\mathbf{s}:\boldsymbol{\psi}) \\ &= C_1(\mathbf{s}:\boldsymbol{\psi}) \qquad \text{if } \ \overline{\psi}_1 = 0,\ 1 \ \text{ or } \ 2\,. \end{aligned}$$

For $k = 1$,

$$C_1(s:\psi) = \psi\, c_0^{\psi-3}c_1\langle s\rangle_{\psi-2}^{-1} = \begin{cases} c_0^{-2}c_1(s+1), & \text{if } \psi = 1\,, \\ 2\,c_0^{-1}c_1\,, & \text{if } \psi = 2\,, \end{cases}$$

so $\mathbb{E}\left[Y_{n,s}\right] = s^{-1} + c_0^{-3}c_1(s+1)n^{-2} + O\left(n^{-3}\right)$ for $s > 0$. For $k = 2$, $C_1(\mathbf{s}:\mathbf{1}) = c_0^{-1}c_1D_{2,\mathbf{s}}$ for $D_{2,\mathbf{s}}$ above, so

$$\mathbb{E}\left[Y_{n,s_1}Y_{n,s_2}\right] = \left(1-n^{-1}\right)(s_1-1)^{-1}s_2^{-1} + n^{-2}c_0^{-3}c_1\,D_{2,\mathbf{s}} + O\left(n^{-3}\right)$$

and

$$\operatorname{Cov}\left(Y_{n,s_1}, Y_{n,s_2}\right) = \langle s_1\rangle_2^{-1}s_2^{-1}\left(1-n^{-1}s_1\right) - n^{-2}c_0^{-3}c_1\,F_{3,\mathbf{s}} + O\left(n^{-3}\right)\ .$$

3. EXAMPLES

Example 3.1. For Student's t distribution, $X = t_N$ has density function

$$\left(1 + x^2/N\right)^{-\gamma} g_N \;=\; \sum_{i=0}^{\infty} d_i\, x^{-2\gamma-2i} \;,$$

where $\gamma = (N+1)/2$, $g_N = \Gamma(\gamma) / \left\{\sqrt{N\pi}\;\Gamma(N/2)\right\}$ and $d_i = \binom{-\gamma}{i} N^{\gamma+i} g_N$. So, (1.6) holds with $\alpha = N$, $\beta = 2$ and $c_i = d_i/(N+2i)$. In particular,

$$\begin{aligned}
c_0 &= N^{\gamma-1} g_N \;,\\
c_1 &= -\gamma N^{\gamma+1} (N+2)^{-1} g_N \;=\; -N^{\gamma+1} (N+1)\,(N+2)^{-1}\, g_N/2 \;,\\
c_2 &= (\gamma)_2\, N^{\gamma+2} (N+4)^{-1} g_N/2 \;,\\
c_3 &= -(\gamma)_3\, N^{\gamma+3} g_N (N+6)^{-1}/6 \;,
\end{aligned}$$

and so on. So, $a = 2/N$ and (2.12) gives an expression in powers of $n^{-a/2}$ if N is odd or n^{-a} if N is even. The first term in (2.12) to involve c_1, not just c_0, is the coefficient of n^{-a}.

Putting $N = 1$ we obtain

Example 3.2. For the Cauchy distribution, (1.6) holds with $\alpha = 1$, $\beta = 2$ and $c_i = (-1)^i (2i+1)^{-1} \pi^{-1}$. So, $a = 2$, $\psi = \theta$, $C_{0,\psi} = \pi^{-\psi}$, $C_{1,\psi} = -\psi\, \pi^{2-\psi}/3$, $C_{2,\psi} = \psi\, \pi^{4-\psi} \{1/5 + (\psi - 5)/a\}$ and $C_{3,\psi} = -\psi\, \pi^{6-\psi} \{1/105 - 2\psi/15 + (\psi+1)_2/162\}$. By Example 2.3, $Y_{n,s} = (\pi/n)\, X_{n,n-s}$ satisfies

$$\mathbb{E}\left[Y_{n,s}\right] \;=\; s^{-1} - n^{-2} \pi^2 (s+1) + O\left(n^{-3}\right) \tag{3.1}$$

for $s > 0$ and when (2.17) holds

$$\mathbb{E}\left[Y_{n,s_1} Y_{n,s_2}\right] \;=\; \left(1 - n^{-1}\right) (s_1 - 1)^{-1}\, s_2^{-1} - n^{-2} \pi^2\, D_{2,\mathbf{s}}/3 + O\left(n^{-3}\right) \tag{3.2}$$

for $D_{2,\mathbf{s}} = (s_2+1)\,/\,(s_1+1) + s_1/s_2$ and

$$\mathrm{Cov}\left(Y_{n,s_1}, Y_{n,s_2}\right) \;=\; \langle s_1\rangle_2^{-1} s_2^{-1} \left(1 - n^{-1} s_1\right) + n^{-2} \pi^2\, F_{3,\mathbf{s}}/3 + O\left(n^{-3}\right)$$

for $F_{3,\mathbf{s}} = (s_2+1)\,/\langle s_1\rangle_2 + s_2^{-1}$. Page 274 of Hall [7] gave the first term in (3.1) and (3.2) when $s_1 = s_2$ but his version of (3.2) for $s_1 > s_2$ replaces $(s_1-1)^{-1}\, s_2^{-1}$ and $D_{2,\mathbf{s}}$ by complicated expressions each with $s_1 - s_2$ terms. The joint order of order three for $\{Y_{n,s_i}, 1 \le i \le 3\}$ is given by (2.18). Hall points out that $F^{-1}(u) = \cot\left(\pi - \pi u\right)$, so $F^{-1}(u) = \sum_{i=0}^{\infty} (1-u)^{2i-1} C_{i,1}$, where $C_{i,1} = \left(-4\pi^2\right)^i \pi^{-1} B_{2,i}/(2i)!$.

Example 3.3. Consider the F distribution. For $N, M \geq 1$, set $\nu = M/N$, $\gamma = (M+N)/2$ and $g_{M,N} = \nu^{M/2}/B\,(M/2, N/2)$. Then $X = F_{M,N}$ has density function

$$x^{M/2}\,(1+\nu x)^{-\gamma}\, g_{M,N} \;=\; \nu^{-\gamma}\, x^{-N/2}\left(1+\nu^{-1}x^{-1}\right)^{-\gamma} g_{M,N} \;=\; \sum_{i=0}^{\infty} d_i\, x^{-N/2-i}\ ,$$

where $d_i = h_{M,N}\binom{-\gamma}{i}\nu^i$ and $h_{M,N} = g_{M,N}\,\nu^{-\gamma} = \nu^{-N/2}/B(M/2, N/2)$. So, for $N > 2$, (2.1) holds with $\alpha = N/2 - 1$, $\beta = 1$ and $c_i = d_i/(N/2 + i - 1)$. If $N = 4$ then $\alpha = 1$ and Examples 2.1–2.2 apply. Otherwise (2.13)–(2.15) give $\mathbb{E}\,[Y_{n,\mathrm{s}}]$, $\mathbb{E}\,[Y_{n,s_1} Y_{n,s_2}]$ and $\mathrm{Cov}\,(Y_{n,s_1}, Y_{n,s_2})$ to $O\left(n^{-2a_0}\right)$, where $Y_{n,s} = X_{n,n-s}/\,(nc_0)\,\lambda$, $\lambda = 1/\alpha$, $a = 2/(N-2)$, $a_0 = \min(a,1) = a$ if $N \geq 4$ and $a_0 = \min(a,1) = 1$ if $N < 4$.

Example 3.4. Consider the stable laws. Page 549 of Feller [5] proves that the general stable law of index $\alpha \in (0,1)$ has density function

$$\sum_{k=1}^{\infty} |x|^{-1-\alpha k} a_k(\alpha,\gamma)\ ,$$

where $a_k(\alpha,\gamma) = (1/\pi)\,\Gamma(k\alpha+1)\left\{(-1)^k/k!\right\}\sin\{k\pi(\gamma-\alpha)/2\}$ and $|\gamma| \leq \alpha$. So, for $x > 0$ its distribution function F satisfies (2.1) with $\beta = \alpha$ and $c_i = a_{i+1}(\alpha,\gamma)\,\gamma^{-1}(i+1)^{-1}$. Since $a = 1$ the first two moments of $Y_{n,s} = X_{n,n-s}/\,(nc_0)^{\lambda}$, where $\lambda = 1/\alpha$ are $O\left(n^{-2}\right)$ by (2.13)–(2.15).

Example 3.5. Finally, consider the second extreme value distribution. Suppose $F(x) = \exp\left(-x^{-\alpha}\right)$ for $x > 0$, where $\alpha > 0$. Then (1.6) holds with $\beta = \alpha$ and $c_i = (-1)^i/(i+1)!$. Since $a = 1$ the first two moments of $Y_{n,s} = X_{n,n-s}/n^{1/\alpha}$ are given to $O\left(n^{-2}\right)$ by (2.13)–(2.15).

APPENDIX: AN INVERSION THEOREM

Given $x_j = y_j/j!$ for $j \geq 1$ set

$$S = \widehat{S}(t, \mathbf{x}) = \sum_{j=1}^{\infty} x_j \, t^j = S(t, \mathbf{y}) = \sum_{j=1}^{\infty} y_j \, t^j/j! \, . \tag{A.1}$$

The partial ordinary and exponential Bell polynomials $\widehat{B}_{r,i}(\mathbf{x})$ and $B_{r,i}(\mathbf{y})$ are defined for $r = 0, 1, ...$ by

$$S^i = \sum_{r=i}^{\infty} t^r \widehat{B}_{r,i}(\mathbf{x}) = i! \sum_{r=i}^{\infty} t^r B_{r,i}(\mathbf{y})/r! \, .$$

So, $\widehat{B}_{r,0}(\mathbf{x}) = B_{r,0}(\mathbf{y}) = I(r = 0)$, $\widehat{B}_{r,i}(\lambda \mathbf{x}) = \lambda^i \widehat{B}_{r,i}(\mathbf{x})$ and $B_{r,i}(\lambda \mathbf{y}) = \lambda^i B_{r,i}(\mathbf{y})$. They are tabled on pages 307–309 of Comtet [2] for $r \leq 10$ and 12. Note that

$$(1 + \lambda S)^{\alpha} = \sum_{r=0}^{\infty} t^r \widehat{C}_r = \sum_{r=0}^{\infty} t^r C_r/r! \, , \tag{A.2}$$

where

$$\widehat{C}_r = \widehat{C}_r \left(\alpha, \lambda, \mathbf{x}\right) = \sum_{i=0}^{r} \widehat{B}_{r,i}(\mathbf{x}) \binom{\alpha}{i} \lambda^i \tag{A.3}$$

and

$$C_r = C_r \left(\alpha, \lambda, \mathbf{y}\right) = \sum_{i=0}^{r} B_{r,i}(\mathbf{y}) \, \langle \alpha \rangle_i \, \lambda^i \, .$$

So, $\widehat{C}_0 = 1$, $\widehat{C}_1 = \alpha \lambda x_1$, $\widehat{C}_2 = \alpha \lambda x_2 + \langle \alpha \rangle_2 \lambda^2 x_1^2/2$, $\widehat{C}_3 = \alpha \lambda x_3 + \langle \alpha \rangle_2 \lambda^2 x_1 x_2 + \langle \alpha \rangle_3 \lambda^3 x_1^3/6$ and $C_0 = 1$, $C_1 = \alpha \lambda y_1$, $C_2 = \alpha \lambda y_2 + \langle \alpha \rangle_2 \lambda^2 y_1^2$. Similarly,

$$\log(1 + \lambda S) = \sum_{r=1}^{\infty} t^r \widehat{D}_r = \sum_{r=1}^{\infty} t^r D_r/r!$$

and

$$\exp(\lambda S) = 1 + \sum_{r=1}^{\infty} t^r \widehat{B}_r = 1 + \sum_{r=1}^{\infty} t^r B_r/r! \, ,$$

where

$$\widehat{D}_r = \widehat{D}_r(\lambda, \mathbf{x}) = -\sum_{i=1}^{r} \widehat{B}_{r,i}(\mathbf{x}) \, (-\lambda)^i/i! \, ,$$

$$D_r = D_r(\lambda, \mathbf{y}) = -\sum_{i=1}^{r} B_{r,i}(\mathbf{y}) \, (-\lambda)^i/(i-1)! \, ,$$

$$\widehat{B}_r = \widehat{B}_r(\lambda, \mathbf{x}) = \sum_{i=1}^{r} \widehat{B}_{r,i}(\mathbf{x})\, \lambda^i / i!$$

and

$$B_r = B_r(\lambda, \mathbf{y}) = \sum_{i=1}^{r} B_{r,i}(\mathbf{y})\, \lambda^i \ .$$

Here, $\widehat{B}_r(1, \mathbf{x})$ and $B_r(1, \mathbf{y})$ are known as the *complete* ordinary and exponential Bell polynomials. If $x_j = y_j = 0$ for j even, then $S = t^{-1} \sum_{j=1}^{\infty} X_j t^{2j}$, where $X_j = x_{2j-1}$, so

$$S^i = t^{-i} \sum_{r=i}^{\infty} t^{2r} \widehat{B}_{r,i}(\mathbf{X}) \qquad \text{and} \qquad \exp(\lambda S) = 1 + \sum_{k=1}^{\infty} t^k \widehat{B}_k \ ,$$

where

$$\widehat{B}_k = \sum \left\{ \widehat{B}_{r,i}(\mathbf{X}) \lambda^i / i! \colon\ i = 2r - k,\ k/2 < r \le k \right\} .$$

The following derives from Lagrange's inversion formula.

Theorem A.1. *Let k be a positive integer and a any real number. Suppose*

$$v/u = \sum_{i=0}^{\infty} x_i\, u^{ia} = \sum_{i=0}^{\infty} y_i\, v^{ia} / i!$$

with $x_0 \neq 0$. Then

$$(u/v)^k = \sum_{i=0}^{\infty} x_i^*\, v^{ia} = \sum_{i=0}^{\infty} y_i^* v^{ia} / (ia)! \ ,$$

where $x_i^* = x_i^*\,(a, k, \mathbf{x})$ and $y_i^* = y_i^*\,(a, k, \mathbf{y})$ are given by

$$\text{(A.4)} \qquad x_i^* = k\, n^{-1} \widehat{C}_i\,(-n, 1/x_0, \mathbf{x}) = k\, x_0^{-n} \sum_{j=0}^{i} (n+1)_{j-1}\, \widehat{B}_{i,j}\,(\mathbf{x})\,(-x_0)^{-j} / j!$$

and

$$\text{(A.5)} \qquad y_i^* = k\, n^{-1} C_i\,(-n, 1/y_0, \mathbf{y}) = k\, y_0^{-n} \sum_{j=0}^{i} (n+1)_{j-1}\, B_{i,j}(\mathbf{y})\, (-y_0)^{-j} \ ,$$

respectively, where $n = k + a\,i$.

Proof: u/v has a power series in v^a so that $(u/v)^k$ does also. A little work shows that (A.4)–(A.5) are correct for $i = 0, 1, 2, 3$ and so by induction that $x_i^* x_0^{ia}$ and $y_i^* y_0^{ia}$ are polynomials in a of degree $i - 1$. Hence, (A.4)–(A.5) will hold true for all a if they hold true for all positive integers a. Suppose then a is a positive integer. Since $v/u = x_0\left(1 + x_0^{-1}S\right)$ for $S = \widehat{S}\left(u^a, \mathbf{x}\right) = S\left(u^a, \mathbf{y}\right)$, the coefficient of u^{ai} in $(v/u)^{-n}$ is $x_0^{-n}\widehat{C}_i\left(-n, 1/x_0, \mathbf{x}\right) = y_0^{-n}C_i\left(-n, 1/y_0, \mathbf{y}\right)/(n-k)!$. Now set $n = k + ai$ and apply Theorem A in page 148 of Comtet [2] to $v = f(u) = \sum_{i=0}^{\infty} x_i u^{1+ai}$. □

Theorem F in page 15 of Comtet [2] proves (A.4) for the case $k = 1$ and a a positive integer.

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