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GENERAL MULTIVARIATE DEPENDENCE USING ASSOCIATED COPULAS

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Abstract:

- This paper studies the general multivariate dependence and tail dependence of a random vector. We analyse the dependence of variables going up or down, covering the 2^d orthants of dimension d and accounting for non-positive dependence. We extend definitions and results from positive to general dependence using the associated copulas. We study several properties of these copulas and present general versions of the tail dependence functions and tail dependence coefficients. We analyse the perfect dependence models, elliptical copulas and Archimedean copulas. We introduce the monotonic copulas and prove that the multivariate Student's t copula accounts for all types of tail dependence simultaneously while Archimedean copulas with strict generators can only account for positive tail dependence.

Key-Words:

- *non-positive dependence; tail dependence; copula theory; perfect dependence models; elliptical copulas; Archimedean copulas.*

AMS Subject Classification:

- 62H20, 60G70.

1. INTRODUCTION

A great deal of literature has been written on the analysis of the dependence structure between random variables. There is an increasing interest in the understanding of the dependencies between extreme values in what is known as tail dependence. However, the analysis of multivariate tail dependence in copula models has been exclusively focused on the positive case. Only the lower and upper tail dependence have been considered, leaving a void in the analysis of dependence structure implied by the use of these models. In this paper we tackle this issue by considering the dependence in the 2^d different orthants of dimension d for a random vector.

The use of the tail dependence coefficient (TDC) and the tail dependence function comes as a response to the inability of other measures when it comes to tail dependence (see [22, 13] and [20, Chapter 5]). This includes the Pearson's correlation coefficient and copula measures such as the Spearman's ρ, Kendall's τ and the Blomqvist's β.

The analysis of lower tail dependence has been derived using the copula, C, see e.g. [13, 22, 23]. In the context of nonparametric statistics, it is possible to measure upper tail dependence by using negative transformations or rotations. However, presenting a formal definition of upper tail dependence in the multivariate case and analysing it in copula models can not be achieved by the use of such methods. Also, trying to define it in terms of C becomes cumbersome in higher dimensions. By using the survival copula, the results and analysis of lower tail dependence have been generalised to upper tail dependence. For more on the analysis of the use of the survival copula for upper tail dependence, see [10, 23, 14, 15, 20, 27]. The study of non-positive tail dependence is also relevant when dealing with empirical data and in copula models analysis, see e.g. [32, 4]. In the case of copula models, the study of tail dependence helps in the understanding of the underlying assumptions implied by the use of these models. For example, the Student's t copula is often used to model data with only positive tail dependence. However, although this model accounts for the positive tail dependence, it also assumes the existence of negative tail dependence. Table 1 illustrates positive and negative tail dependence in the bivariate case which we generalise to the multivariate one.

Table 1: Tail dependence in the four different orthants of dimension two for variables X and Y.

	Lower Tail of X	Upper Tail of X
Lower Tail of Y	classical lower tail dependence	upper-lower tail dependence
Upper Tail of Y	lower-upper tail dependence	classical upper tail dependence

Although much has been written on the need to understand multivariate non-positive tail dependence, no formal definition has been presented. In this work we define the necessary concepts to study non-positive tail dependence in multivariate copula models. We use a copula approach and base our study on the associated copulas (see [13, p. 15]). If a copula is the distribution of $\mathbf{U}=(U_1,...,U_d)$, the associated copulas are the distribution functions of vectors of the form $(U_1,1-U_2,U_3,...,1-U_{d-1},1-U_d)$. The use of copulas of transformations for non-positive dependence is also suggested in [5, 30].

The reasoning behind the use of associated copulas is the same as for the use of the survival copula for upper tail dependence analysis. Similarly to that case, the definition and study of non-positive tail dependence is simplified by the use of these copulas. They enable us to present a unified definition of multivariate general tail dependence. This definition is consistent with generalisations from dimension 2 to d of positive tail dependence. The study of the associated copulas to analyse non-positive tail dependence is then a generalisation of the use of the copula and the survival copula for lower and upper tail dependence respectively.

The reminder of this work is divided in three sections: In the second section we present the concepts we use to study dependence in all the orthants. This includes general definitions of dependence and probability functions. We present a version of Sklar's theorem that proves that the copulas that link these general probability functions and its marginals are the associated copulas. We then present four propositions regarding these copulas. At the end of this section we present general definitions of the tail dependence functions and TDCs. In the third section we use the results obtained in Section 2 to study the perfect dependence models, elliptical copulas and Archimedean copulas. We present the copulas of the perfect dependence cases, which include non-positive perfect dependence. We call these copulas the monotonic copulas. We then characterise the associated elliptical copulas and obtain an expression for the associated tail dependence functions of the Student's t copula model. This model accounts for all 2^d types of tail dependence simultaneously. After that, we prove that, by construction, Archimedean copulas with strict generators can not account for non-positive tail dependence. We then present three examples with non-strict generators which account for negative tail dependence. At the end of this section we discuss a method for modelling arbitrary tail dependence using copula models. Finally, in the fourth section, we conclude and discuss future lines of research for general dependence.

Unless we specifically state it, all the definitions and results presented regarding general dependence are a contribution of this work.

2. ASSOCIATED COPULAS, TAIL DEPENDENCE FUNCTIONS AND TAIL DEPENDENCE COEFFICIENTS

In this section we analyse the dependence structure among random variables using copulas. Given a random vector $\mathbf{X} = (X_1, ..., X_d)$, we use the corresponding copula C and its associated copulas to analyse its dependence structure. For this we introduce a general type of dependence $\mathbf{D}$, one for each of the 2^d different orthants. This corresponds to the lower and upper movements of the different variables.

To analyse different dependencies, we introduce the $\mathbf{D}$-probability function and present a version of Sklar's theorem that states that an associated copula is the copula that links this function and its marginals. We present a formula to link all associated copulas and three results on monotone functions and associated copulas. We then introduce the associated tail dependence function and the associated tail dependence coefficient for the type of dependence $\mathbf{D}$. These functions generalise the positive (lower and upper) cases (extensively studied in [12, 13, 23]). With the concepts studied in this section, we aim to provide the tools to analyse the whole dependence structure among random variables, including non-positive dependence.

2.1. Copulas and dependence

The concept of copula was first introduced by [29], and is now a cornerstone topic in multivariate dependence analysis (see [13, 22, 20]). We now present the concepts of copula, general dependence and associated copulas that are fundamental for the rest of this work.

Definition 2.1. A multivariate copula $C(u_1, ..., u_d)$ is a distribution function on the d-dimensional-square $[0, 1]^d$ with standard uniform marginal distributions.

If C is the distribution function of $\mathbf{U} = (U_1, ..., U_d)$, we denote as $\widehat{C}$ the distribution function of $(1-U_1, ..., 1-U_d)$. C is used to link distribution functions with their corresponding marginals, accordingly we refer to C as the distributional copula. On the other hand, $\widehat{C}$ is used to link multivariate survival functions with their marginal survival functions, this copula is known as the survival copula.[1] Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with joint distribution function F, joint survival function $\overline{F}$, marginals F_i and marginal survival functions $\overline{F}_i$, for $i \in$

[1] We use the term distributional for C, to distinguish it from the other associated copulas. The notation for the survival copula corresponds to the one used in the seminal work of [13].

$\{1, ...d\}$. Two versions of Sklar's theorem guarantees the existence and uniqueness of a copulas C and $\widehat{C}$ which satisfy

$$F(x_1, ..., x_d) \;=\; C\big(F_1(x_1), ..., F_d(x_d)\big) \;, \tag{2.1}$$

$$\overline{F}(x_1, ..., x_d) \;=\; \widehat{C}\big(\overline{F}_1(x_1), ..., \overline{F}_d(x_d)\big) \;, \tag{2.2}$$

see [13, 22]. In the next section we generalise these equations using the concept of general dependence, which we now define.

Definition 2.2. In d dimensions, we call the vector $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence if each D_i is a boolean variable, whose value is either L (lower) or U (upper) for $i \in \{1, ...d\}$. We denote by Δ the set of all 2^d types of dependence.

Each type of dependence corresponds to the variables going up or down simultaneously. Tail dependence, which we define later, refers to the case when the variables go extremely up or down simultaneously. Two well known types of dependence are lower and upper dependence. Lower dependence refers to the case when all variables go down at the same time ($D_i = L$ for $i \in \{1, ..., d\}$) and upper dependence to the case when they all go up at the same time ($D_i = U$ for $i \in \{1, ..., d\}$). These two cases are examples of positive dependence and they have been extensively studied for tail dependence analysis, see e.g. [13, 22]. In the bivariate case the dependencies $\mathbf{D} = (L, U)$ and $\mathbf{D} = (U, L)$ correspond to one variable going up while the other one goes down. These are examples of negative dependence. Negative tail dependence is often present in financial time series, see [32, 4, 14]. Hence, in dimension 2 there are four types of dependence that correspond to the four quadrants. Note that, in dimension d, for each of the 2^d orthants we define a dependence $\mathbf{D}$.

Using the concept of dependence, we now present the associated copulas, see [13, Chapter 1, p. 15].

Definition 2.3. Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with corresponding copula C, which is the distribution function of the vector $(U_1, ..., U_d)$ with uniform marginals. Let Δ denote the set of all types of dependencies of Definition 2.2. For $\mathbf{D} = (D_1, ..., D_d) \in \Delta$, let $\mathbf{V_D} = (V_{D_1,1}, ..., V_{D_d,d})$ with

$$V_{D_i,i} = \begin{cases} U_i & \text{if } D_i = L \\ 1 - U_i & \text{if } D_i = U \end{cases} .$$

Note that $\mathbf{V_D}$ also has uniform marginals. We call the distribution function of $\mathbf{V_D}$, which is a copula, the associated $\mathbf{D}$-copula and denote it $C_{\mathbf{D}}$. We denote $\mathcal{A}_{\mathbf{X}} = \{C_{\mathbf{D}} |\ \mathbf{D} \in \Delta\}$, the set of 2^d associated copulas of the random vector $\mathbf{X}$. Also, for any $\emptyset \neq S \subseteq I$, let $\mathbf{D}(S)$ denote the corresponding $|S|$-dimensional marginal dependence of $\mathbf{D}$. Then the copula $C_{\mathbf{D}(S)}$, the distribution of the $|S|$-dimensional marginal vector $(V_{D_i,i} |\ i \in S)$, is known as a marginal copula of $C_{\mathbf{D}}$.

Note that the distributional and the survival copula are $C = C_{(L,...,L)}$ and $\widehat{C} = C_{(U,...,U)}$ respectively.

2.1.1. The **D**-probability function and its associated **D**-copula

The distributional copula C and the survival copula $\widehat{C}$ are used to explain the lower and upper dependence structure of a random vector respectively. We use the associated **D**-copula to explain the **D**-dependence structure of a random vector. For this, we first present the **D**-probability functions, which generalise the joint distribution and survival functions.

Definition 2.4. Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with marginal distributions F_i for $i \in \{1, ...d\}$ and $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence according to Definition 2.2. Define the event $\mathcal{B}_i(x_i)$ in the following way

$$\mathcal{B}_i(x_i) = \begin{cases} \{X_i \le x_i\} & \text{if } D_i = L \\ \{X_i > x_i\} & \text{if } D_i = U \end{cases}.$$

Then the corresponding **D**-probability function is

$$F_{\mathbf{D}}(x_1, ..., x_d) = P\left(\bigcap_{i=1}^{d} \mathcal{B}_i(x_i)\right).$$

We refer to

$$F_{D_i,i} = \begin{cases} F_i & \text{if } D_i = L \\ \overline{F}_i & \text{if } D_i = U \end{cases},$$

for $i \in \{1, ...d\}$ as the marginal functions of $F_{\mathbf{D}}$ (note that the marginals are either univariate distribution or survival functions).

In the bivariate case for example, there are four **D**-probability functions: $F(x_1, x_2)$, $\overline{F}(x_1, x_2)$, $F_{LU}(x_1, x_2) = P(X_1 \le x_1, X_2 > x_2)$ and $F_{UL}(x_1, x_2) = P(X_1 > x_1, X_2 \le x_2)$. In general, these functions complement the use of the joint distribution and survival functions in our analysis of dependence in the 2^d orthants.

The following theorem presents the associated copula $C_{\mathbf{D}}$ in terms of the $F_{\mathbf{D}}$ and its marginals. It is because of this theorem that we can use the associated copula $C_{\mathbf{D}}$ to analyse **D**-dependence. We restrict the proof to the continuous case (for Sklar's theorem for distribution functions see [20, 13, 22]).

Theorem 2.1. ***Sklar's theorem for* D*-probability functions and associated copulas.***

Let $\mathbf{X} = (X_1, ..., X_d)$ *be a random vector,* $\mathbf{D} = (D_1, ..., D_d)$ *a type of dependence,* $F_{\mathbf{D}}$ *its* $\mathbf{D}$*-probability function and* $F_{D_i,i}$ *for* $i \in \{1, ...d\}$ *the marginal functions of* $F_{\mathbf{D}}$ *as in Definition 2.4. Let the marginal functions of* $F_{\mathbf{D}}$ *be continuous and* $F^{\leftarrow}$ *denote the generalised inverse of* F*, defined as* $F^{\leftarrow}(u) := \inf\{x \in \mathbb{R} \,|\, F(x) \geq u\}$*. Then the associated copula* $C_{\mathbf{D}} : [0,1]^d \to [0,1]$*, satisfies, for all* $x_1, ..., x_2$ *in* $[-\infty, \infty]$*,*

$$F_{\mathbf{D}}(x_1, ..., x_d) \;=\; C_{\mathbf{D}}\big(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)\big) \,, \tag{2.3}$$

which is equivalent to

$$C_{\mathbf{D}}(u_1, ..., u_d) \;=\; F_{\mathbf{D}}\big(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d)\big) \,. \tag{2.4}$$

Conversely, let $\mathbf{D} = (D_1, ..., D_d)$ *be a dependence and* $F_{D_i,i}$ *a univariate distribution, if* $D_i = L$*, or a survival function, if* $D_i = U$*, for* $i \in \{1, ...d\}$*, then:*

- (**a**) *If* $C_{\mathbf{D}}$ *is a copula, then* $F_{\mathbf{D}}$ *in (2.3) defines a* $\mathbf{D}$*-probability function with marginals* $F_{D_i,i}$*,* $i \in \{1, ...d\}$*.*
- (**b**) *If* $F_{\mathbf{D}}$ *is any* $\mathbf{D}$*-probability function, then* $C_{\mathbf{D}}$ *in (2.4) is a copula.*

Proof: The proof of this theorem is analogous to the proof of Sklar's theorem for distribution functions. When two random variables have the same probability functions, we say they are equivalent in probability and denote it as $\overset{P}{\sim}$. In this general version of the theorem, we have that for the distribution function F_i, the events $\{X_i \leq x_i\} \overset{P}{\sim} \{F_i(X_i) \leq F_i(x_i)\}$ and $\{X_i > x_i\} \overset{P}{\sim} \{\overline{F}_i(X_i) \leq \overline{F}_i(x_i)\}$, for $i \in \{1, ..., d\}$ and $x_i \in [-\infty, \infty]$. This implies

$$P\big(\mathcal{B}_i(x_i)\big) \;=\; P\big(F_{D_i,i}(X_i) \leq F_{D_i,i}(x_i)\big) \,, \tag{2.5}$$

for $i \in \{1, ..., d\}$.

Considering equation (2.5) and Definition 2.4, we have that for any $x_1, ..., x_d$ in $[-\infty, \infty]$

$$F_{\mathbf{D}}(x_1, ..., x_d) \;=\; P\Big(F_{D_1,1}(X_1) \leq F_{D_1,1}(x_1), \; ..., \; F_{D_d,d}(X_d) \leq F_{D_d,d}(x_d)\Big) \,. \tag{2.6}$$

Using the continuity of F_i, we have that $F_i(X_i)$ is uniformly distributed (see [20, Proposition 5.2 (2)]). Hence, if we define $\mathbf{U} = (F_1(X_1), ..., F_d(X_d))$, its distribution function is a copula C. Note that in this case $\mathbf{V_D}$, defined as in Definition 2.3, is equal to $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$. It follows that the distribution function of $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$ is the associated copula $C_{\mathbf{D}}$, in which case equation (2.5) implies

$$C_{\mathbf{D}}\big(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)\big) = P\Big(F_{D_1,1}(X_1) \leq F_{D_1,1}(x_1), ..., F_{D_d,d}(X_d) \leq F_{D_d,d}(x_d)\Big) \,,$$

and equation (2.3) follows.

Now, one of the properties of the generalised inverse is that, when T is continuous, $T \circ T^{\leftarrow}(x) = x$ (see [20, Proposition A.3]). Hence, if we evaluate $F_{\mathbf{D}}$ in $(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d))$, using equation (2.3), we get equation (2.4). This equation explicitly represents $C_{\mathbf{D}}$ in terms of $F_{\mathbf{D}}$ and its marginals implying its uniqueness.

For the converse statement of the theorem, we have

(**a**) Let $\mathbf{U} = (U_1, ..., U_d)$ be the random vector with distribution function C. Define $\mathbf{X} = (X_1, ..., X_d) = (F^{\leftarrow}_{D_1,1}(U_1), ..., F^{\leftarrow}_{D_d,d}(U_d))$ and

$$\mathcal{B}_i(x_i) = \begin{cases} \{X_i \le x_i\} & \text{if } D_i = L \\ \{X_i > x_i\} & \text{if } D_i = U \end{cases} ,$$

for $i \in \{1, ...d\}$. Considering that $F(x) \le y \iff x \le F^{\leftarrow}(y)$, we have $\overline{F}^{\leftarrow}(x) \le y \iff x \ge \overline{F}(y)$. Using these properties, we get

$$\{U_i \le F_{D_i,i}(x_i)\} \overset{P}{\sim} \mathcal{B}_i(x_i) ,$$

for $i \in \{1, ...d\}$. Using this, the $\mathbf{D}$-probability function of $\mathbf{X}$ is

$$P\left(\bigcap_{i=1}^{d} \mathcal{B}_i(x_i)\right) = C\big(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)\big) .$$

This implies that $F_{\mathbf{D}}$ defined by (2.3) is the $\mathbf{D}$-probability function of $\mathbf{X}$ with marginals

$$P\big(\mathcal{B}_i(x_i)\big) = P\big(U_i \le F_{D_i,i}(x_i)\big) = F_{D_i,i}(x_i) ,$$

for $i \in \{1, ...d\}$.

(**b**) Similarly, let $(X_1, ..., X_d)$ be the random vector with $\mathbf{D}$-probability function $F_{\mathbf{D}}$. Define $\mathbf{U} = (U_1, ..., U_d) = (F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$ (note that the vector is uniformly distributed). Again, using the properties of the generalised inverse, we have

$$\{U_i \le u_i\} \overset{P}{\sim} \mathcal{B}_i\big(F^{\leftarrow}_{D_i,i}(u_i)\big) ,$$

for $i \in \{1, ...d\}$. Hence the distribution function of $\mathbf{U}$ is $F_{\mathbf{D}}\big(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d)\big)$, which implies that the function is a copula.

For the properties of the generalised inverse function used in this proof, see [20, Proposition A.3]. □

For this theorem we referred to generalised inverse functions as they are more general than inverse functions. However, whenever we are not proving a general property, we assume distribution functions have inverse functions.

Note that this theorem implies that in the continuous case $C_{\mathbf{D}}$ is the $\mathbf{D}$-probability function of $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$ characterised in (2.3). This theorem implies the importance of the associated copulas to analyse dependencies. It also implies the Fréchet bounds for the **D**-probability functions of Definition 2.4. The bounds can also be obtained similarly to [13, Theorems 3.1 and 3.5],

$$\begin{aligned}\max\Big\{0,\, F_{D_1,1}(x_1)+\cdots+F_{D_d,d}(x_d)-(d-1)\Big\} &\le F_{\mathbf{D}}(x_1,...,x_d)\\ &\le \min\Big\{F_{D_1,1}(x_1),...,F_{D_d,d}(x_d)\Big\}\,. \end{aligned} \tag{2.7}$$

2.1.2. Properties of the associated copulas

In the bivariate case, [13, Chapter 1], and [22, Chapter 2], presented the expressions to link the associated copulas with the distributional copula C. In the multivariate case [14, Equation 8.1] and [10, Theorem 3], presented the expression between the distributional and the survival copula and [5, Theorem 2.7] proved that is possible to express the associated copulas in terms of the distributional copula C. We now present a general equation for the relationship between any two associated copulas $C_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ in the multivariate case. The equation is based on all the subsets of the indices where the $\mathbf{D}^*$ and $\mathbf{D}^+$ are different.

Proposition 2.1. *Let $\mathbf{X}=(X_1,...,X_d)$ be a random vector with associated copulas $\mathcal{A}_{\mathbf{X}}$ and $\mathbf{D}^*=(D_1^*,...,D_d^*)$ and $\mathbf{D}^+=(D_1^+,...,D_d^+)$ any two types of dependence. Consider the following sets and notations: $I=\{1,...,d\}$; $I_1=\{i\in I|\ D_i^*=D_i^+\}$ and $I_2=\{i\in I|\ D_i^*\neq D_i^+\}$; $d_1=|I_1|$ and $d_2=|I_2|$; $S_j=\{$the subsets of size j of $I_2\}$ and $S_{j,k}=\{$The k-th element of $S_j\}$ for $j\in\{1,...,d_2\}$ and $k\in\left\{1,...,\binom{d_2}{j}\right\}$. We define $S_0=\emptyset$ and $S_{0,1}=\emptyset$; for each $S_{j,k}$ define $\mathbf{W}_{j,k}=(W_{j,k.1},...,W_{j,k,d})$ with*

$$W_{j,k,i}=\begin{cases}u_i & \text{if } i\in I_1\\ 1-u_i & \text{if } i\in S_{j,k}\\ 1 & \text{if } i\notin I_1\cup S_{j,k}\end{cases},$$

for $i\in\{1,...d\}$, $j\in\{0,...,d_2\}$ and $k\in\left\{1,...,\binom{d_2}{j}\right\}$.

Then the associated $\mathbf{D}^*$-copula $C_{\mathbf{D}^*}$ is expressed in terms of the $\mathbf{D}^+$-copula $C_{\mathbf{D}^+}$ according to the following equation

$$C_{\mathbf{D}^*}(u_1,...,u_d)=\sum_{j=0}^{d_2}(-1)^j\sum_{k=1}^{\binom{d_2}{j}}C_{\mathbf{D}^+}(\mathbf{W}_{j,k})\,. \tag{2.8}$$

Note that in the cases when at least a 1 appears in $\mathbf{W}_{j,k}$, $C_{\mathbf{D}^+}(\mathbf{W}_{j,k})$ becomes a marginal copula of $C_{\mathbf{D}^+}$.

Proof: Throughout this proof, it must be borne in mind that $C_{\mathbf{D}^*}$ is the distribution function of the random vector $\mathbf{V}_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ of $\mathbf{V}_{\mathbf{D}^+}$, defined according to Definition 2.3. Note that, for $i \in I_2$, $V_{D_i^*,i} = 1 - V_{D_i^+ i}$ and they are equal otherwise.

In the case $d_2 = 0$, we have $\mathbf{D}^* = \mathbf{D}^+$, $j \in \{0\}$ and $k \in \{1\}$ [2], hence (2.8) holds. We prove (2.8) by induction on d, the dimension; it can also be proven by induction on d_2, the number of elements in which $D_i^* \neq D_i^+$. Note that in dimension $d = 1$, a copula becomes the identity function. If $D_1^* \neq D_1^+$, the expression becomes $u_1 = 1 - (1 - u_1)$; the case $D_1^* = D_1^+$ has already been covered in $d_2 = 0$, and expression (2.8) holds.

Now, in dimension d, we prove the formula works if it works in dimension $d-1$. We obtain an expression for $C_{\mathbf{D}^*}(u_1, ..., u_d)$ using the induction hypothesis. Consider the dependencies, on the $(d-1)$-dimension, $\mathbf{F}^* = (D_1^*, ..., D_{d-1}^*)$ and $\mathbf{F}^+ = (D_1^+, ..., D_{d-1}^+)$. We use an apostrophe on the sets and notations of $\mathbf{F}^*$ and $\mathbf{F}^+$ to differentiate them from those of $\mathbf{D}^*$ and $\mathbf{D}^+$. It follows that $d' = d - 1$ and $I' = I - \{d\}$. By the induction hypothesis, equation (2.8) holds to express $C_{\mathbf{F}^*}$ in terms of $C_{\mathbf{F}^+}$. In terms of probabilities this is equivalent to

$$\begin{aligned} &P\Big(V_{D_i^*,1} \leq u_1,\ ...,\ V_{D_{d-1}^*,d-1} \leq u_{d-1}\Big) = \\ &= \sum_{j=0}^{d_2-1} (-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} P\Big(V_{D_1^+,1} \leq W'_{j,k,1},\ ...,\ V_{D_{d-1}^+,d-1} \leq W'_{j,k,d-1}\Big) . \end{aligned} \tag{2.9}$$

There are two cases to consider depending on whether D_d^* is equal to D_d^+ or not.

Case 1. $D_d^* = D_d^+$.

In this case, it follows that, $I_1' = I_1 - \{d\}$, $I_2' = I_2$, $d_2' = d_2$ *and* $V_{D_d^*,d} = V_{D_d^+,d}$. *If we intersect the events in equation (2.9) with the event* $\{V_{D_d^* d} \leq u_d\}$ *we get*

$$\begin{aligned} &P\Big(V_{D_i^*,1} \leq u_1,\ ...,\ V_{D_{d-1}^*,d-1} \leq u_{d-1},\ V_{D_d^* d} \leq u_d\Big) = \\ &= \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} P\Big(V_{D_1^+,1} \leq W'_{j,k,1},\ ...,\ V_{D_{d-1}^+,d-1} \leq W'_{j,k,d-1},\ V_{D_d^+,d} \leq u_d\Big) . \end{aligned} \tag{2.10}$$

Because $I_2' = I_2$, *in this case, for* $j \in \{1, ..., d_2\}$ *and* $k \in \left\{1, ..., \binom{d_2}{j}\right\}$, *the events* $S'_{j,k}$ *are equal to* $S_{j,k}$. *Considering this, and* $I_1' = I_1 - \{d\}$, *we have*

$$\left(\mathbf{W}'_{j,k}, u_d\right)_i = W_{j,k,i}$$

for $i \in \{1, ..., d\}$, *so* $(\mathbf{W}'_{j,k}, u_d) = \mathbf{W}_{j,k}$ *for* $j \in \{1, ..., d_2\}$ *and* $k \in \left\{1, ..., \binom{d_2}{j}\right\}$. *Equation (2.10) then implies:*

$$C_{D^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{D^+}(\mathbf{W}_{j,k}) .$$

[2] Note that we are using the convention $0! = 1$

Case 2. $D_d^* \neq D_d^+$.

In this case, it holds that, $I_1' = I_1$, $I_2' = I_2 - \{d\}$, $d_2' = d_2 - 1$. *To obtain an expression for* $C_{\mathbf{D}^*}(u_1, ..., u_d) = P(V_{D_i^*,1} \leq u_1, ..., V_{D_d^*,d} \leq u_d)$, *we use the induction hypothesis. Considering* $P(A) = P(A \cap B) + P(A \cap B^c)$, *we have*

$$\begin{aligned} P\Big(V_{D_i^*,1} \leq u_1,\ ...,\ V_{D_{d-1}^*,d-1} \leq u_{d-1}\Big) \ = \\ = \ P\Big(V_{D_i^*,1} \leq u_1,\ ...,\ V_{D_{d-1}^*,d-1} \leq u_{d-1},\ V_{D_d^*,d} \leq u_d\Big) \\ + \ P\Big(V_{D_i^*,1} \leq u_1,\ ...,\ V_{D_{d-1}^*,d-1} \leq u_{d-1},\ V_{D_d^*,d} \geq u_d\Big)\,, \end{aligned}$$

which implies

$$\begin{aligned} C_{\mathbf{D}^*}(u_1, ..., u_d) \ &= \ P\Big(V_{D_1^*,1} \leq u_1,\ ...,\ V_{d-1}^* \leq u_{d-1}\Big) \\ &\quad - P\Big(V_{D_1^*,1} \leq u_1,\ ...,\ V_{d-1}^* \leq u_{d-1},\ V_{D_d^*,d} \geq u_d\Big)\,. \end{aligned} \tag{2.11}$$

Note that, in this case $V_{D_d^*,d} = 1 - V_{D_d^+,d}$. *This implies that the event* $\{V_{D_d^*,d} \geq u_d\}$ *is equivalent to* $\{V_{D_d^+,d} \leq 1 - u_d\}$. *If we intersect the events involved in equation (2.9) with the event* $\{V_{D_d^*,d} \geq u_d\}$ *we get*

$$\begin{aligned} &P\Big(V_{D_1^*,1} \leq u_1,\ ...,\ V_{D_{d-1}^*,d-1} \leq u_{d-1},\ V_{D_d^*,d} \geq u_d\Big) \ = \\ &= \sum_{j=0}^{d_2-1} (-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} P\Big(V_{D_1^+,1} \leq W'_{j,k,1},\ ...,\ V_{D_{d-1}^+,d-1} \leq W'_{j,k,d-1},\ V_{D_d^+,d} \leq 1 - u_d\Big)\,. \end{aligned} \tag{2.12}$$

Combining equations (2.9), (2.11) and (2.12), we obtain

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2-1} (-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k}, 1) - \sum_{j=0}^{d_2-1} (-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k}, 1 - u_d)\,. \tag{2.13}$$

Note that, in this case, the sets I_2 *and* I_2' *satisfy* $I_2 = I_2' \cup \{d\}$.

The rest of the proof is based on the fact that for $j \in \{1, ..., d-1\}$ the elements of size j of I_2 are the elements of size j of I_2' plus the elements of size $j-1$ of I_2' attaching them $\{d\}$. Considering our notation, this means

$$S_j \ = \ S_j' \cup S_{j-1}''\,, \tag{2.14}$$

with $S_{j-1}'' = \big\{S_{j-1,k}'' = S_{j-1,k}' \cup \{d\} \;\big|\; k \in \big\{1, ..., \binom{d_2}{j}\big\}\big\}$ for $j \in \{1, ..., d-1\}$. Further to this, by definition of $\mathbf{W}_{j,k}$ we have the following three equalities:

$$\big(\mathbf{W}'_{j,k}, 1\big)_i = \begin{cases} u_i & \text{if } i \in I_1 \\ 1 - u_i & \text{if } i \in S'_{j,k} \\ 1 & \text{if } i \notin I_1 \cup S'_{j,k} \end{cases}, \qquad W_{j,k,i} = \begin{cases} u_i & \text{if } i \in I_1 \\ 1 - u_i & \text{if } i \in S_{j-1,k} \\ 1 & \text{if } i \notin I_1 \cup S_{j,k} \end{cases}$$

$$\text{and} \qquad \left(\mathbf{W}'_{j-1,k}, 1-u_d\right)_i = \begin{cases} u_i & \text{if } i \in I_1 \\ 1-u_i & \text{if } i \in S''_{j-1,k} \\ 1 & \text{if } i \notin I_1 \cup S''_{j-1,k} \end{cases},$$

for $i \in \{1, ...d\}$, $j \in \{1, ..., d-1\}$ and $k \in \left\{1, ..., \binom{d_2}{j}\right\}$. These three equalities and equation (2.14) imply that, for a fixed j, if we sum $C_{\mathbf{D}^+}$ evaluated in all of the $(\mathbf{W}'_{j,k}, 1)$ and $(\mathbf{W}'_{j,k}, 1-u_d)$ for different k, we get the sum of $C_{\mathbf{D}^+}$ evaluated on $\mathbf{W}_{j,k}$ for different k, that is:

$$\sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k}, 1) + \sum_{k=1}^{\binom{d_2-1}{j-1}} C_{\mathbf{D}^+}(\mathbf{W}'_{j-1,k}, 1-u_d) = \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}), \tag{2.15}$$

for $j \in \{1, ..., d-1\}$. Also, the equalities

$$\left(\mathbf{W}'_{0,1}, 1\right)_i = W_{0,1,i} \qquad \text{and} \qquad \left(\mathbf{W}'_{d-1,1}, 1-u_d\right)_i = W_{d,1,i}$$

hold for $i \in \{1, ...d\}$; the result is implied by these two equalities and equations (2.13) and (2.15). $\square$

Note that this expression is reflexible, meaning that it yields the same formula to express $C_{\mathbf{D}^+}$ in terms of $C_{\mathbf{D}^*}$. As a particular case, equation (2.8) can be used to express any associated copula in terms of the distributional copula C, which is the expression found in literature for copula models. A copula is said to be exchangeable if for every permutation $P\colon i \to p_i$ of $I = \{1, ..., d\}$, we have $C(u_1, ..., u_d) = C(u_{p_1}, ..., u_{p_d})$. In order to analyse the symmetry and exchangeability of copula models, we use the following definition.

Definition 2.5. Let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence, the complement dependence is defined as $\mathbf{D}^{\complement} = (D_1^{\complement}, ..., D_d^{\complement})$, with

$$D_i^{\complement} = \begin{cases} U & \text{if } D_i = L \\ L & \text{if } D_i = U \end{cases},$$

for $i \in \{1, ..., d\}$. We say that the random vector $\mathbf{X}$, with associated copulas $\mathcal{A}_{\mathbf{X}}$, is complement (reflection or radial) symmetric, if there exists $\mathbf{D}^* \in \Delta$, such that $C_{\mathbf{D}^*} = C_{\mathbf{D}^{*\complement}}$.

Note that $\mathbf{X}$ is symmetric if there exists one dependence which satisfies $C_{\mathbf{D}^*} = C_{\mathbf{D}^{*\complement}}$. Along with other important properties, in the following proposition we prove that, if it holds for one dependence, it holds for them all.

Proposition 2.2. *Let $\mathbf{X}$ be a vector with corresponding associated copulas $\mathcal{A}_{\mathbf{X}}$, and let $\mathbf{D}^*$, $\mathbf{D}^+$, $\mathbf{D}^\circ$ and $\mathbf{D}^\times$ be types of dependencies. Denote as $I_1(\mathbf{D}^1, \mathbf{D}^2)$ and $I_2(\mathbf{D}^1, \mathbf{D}^2)$ the elements where the corresponding dependencies are equal or different respectively. Then the following equivalences hold:*

(i) *If* $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^+}$ *and* $I_2(\mathbf{D}^*, \mathbf{D}^+) = I_2(\mathbf{D}^\times, \mathbf{D}^\circ)$ *then* $C_{\mathbf{D}^\times} \equiv C_{\mathbf{D}^\circ}$. *In particular,* $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^{*\complement}}$, *for some* $\mathbf{D}^*$, *implies* $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ *for all* $\mathbf{D} \in \Delta$.

(ii) *If* $C_{\mathbf{D}^\circ}$ *is exchangeable, then* $C_{\mathbf{D}^*}$ *is exchangeable over the elements of* $I_1(\mathbf{D}^*, \mathbf{D}^\circ)$ *and over the elements of* $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. *In particular, if* $C_{\mathbf{D}^\circ}$ *is exchangeable, then* $C_{\mathbf{D}^{\circ\complement}}$ *is exchangeable.*

Proof: **(i)** This follows from the fact $I_2(\mathbf{D}^*, \mathbf{D}^+) = I_2(\mathbf{D}^\times, \mathbf{D}^\circ) \implies I_2(\mathbf{D}^\times, \mathbf{D}^*) = I_2(\mathbf{D}^\circ, \mathbf{D}^+)$, which is easily verified considering the different cases. From Proposition 2.1, we have that the vectors $\mathbf{W}_{j,k}$ are the same in both cases, which implies

$$\begin{aligned} C_{\mathbf{D}^\times}(u_1, ..., u_d) &= \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^*}(\mathbf{W}_{j,k}) \\ &= \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}) \\ &= C_{\mathbf{D}^\circ}(u_1, ..., u_d) \,. \end{aligned}$$

In particular, note that $I_2(\mathbf{D}^*, \mathbf{D}^{*\complement}) = I_2(\mathbf{D}, \mathbf{D}^\complement) = \{1, ..., d\}$ for every $\mathbf{D} \in \Delta$. Then, $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^{*\complement}}$ implies $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for every $\mathbf{D} \in \Delta$.

(ii) From Proposition 2.1 we have

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k}) \,. \tag{2.16}$$

Consider $j \in \{0, ..., d_2\}$ and $k \in \left\{1, ..., \binom{d_2}{j}\right\}$, from the way it is defined, $W_{j,k,i} = u_i$ for every $i \in I_1(\mathbf{D}^*, \mathbf{D}^\circ)$. The exchangeability of $C_{\mathbf{D}^\circ}$ implies that $C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k})$ is exchangeable over $I_1(\mathbf{D}^*, \mathbf{D}^\circ)$. Hence, equation (2.16) implies that $C_{\mathbf{D}^*}$ is exchangeable over $I_1(\mathbf{D}^*, \mathbf{D}^\circ)$. Now, let $j \in \{0, ..., d_2\}$ be fixed, note that each $\mathbf{W}_{j,k}$, $k \in \left\{1, ..., \binom{d_2}{j}\right\}$, is based on a different subset of size j of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. Consider the sum $\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k})$ as a function, given that $C_{\mathbf{D}^\circ}$ is exchangeable and that the sum considers all the subsets of size j of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$, it follows that this function is exchangeable over $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. Equation (2.16) then implies that $C_{\mathbf{D}^*}$ is exchangeable over $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. In particular $C_{\mathbf{D}^{\circ\complement}}$ is exchangeable over $I_2(\mathbf{D}^\circ, \mathbf{D}^{\circ\complement}) = \{1, ..., d\}$. $\square$

It is well known that elliptical copulas satisfy $C = \widehat{C}$. Hence, it follows that in the bivariate case, $C_{LU} = C_{UL}$ and in three dimensions, for instance, $C_{ULU} = C_{LUL}$. Also, from (ii), it follows that the survival copulas of Archimedean families are exchangeable in all dimensions. These examples illustrate some of the applications of this proposition.

In the following proposition we prove that, same as the distributional copula, all associated copulas are invariant under strictly increasing transformations.

Proposition 2.3. *Let T_1,..., T_d be strictly increasing functions and $\mathbf{X} = (X_1, ..., X_d)$ a random vector with corresponding distribution function and marginals, $\mathbf{D}$ a type of dependence and $\mathbf{D}$-copula $C_{\mathbf{D}}$. Then, in the continuous case,*

$$\widetilde{\mathbf{X}} = \big(T_1(X_1), ..., T_d(X_d)\big)$$

also has the same corresponding $\mathbf{D}$-copula $C_{\mathbf{D}}$.

Proof: This result follows straightforwardly from the fact that the distributional copula is invariant under strictly increasing transformations (see [20, Proposition 5.6]) as all associated copulas are implied by this copula using Proposition 2.1. □

In the bivariate case, [22, Theorem 2.4.4] and [5, Theorem 2.7], characterised the copula after the use of strictly monotone functions on random variables. In the multivariate case, this can be done using the associated copulas as we show in the following proposition.

Proposition 2.4. *Let T_1,..., T_d be strictly monotone functions and $\mathbf{X} = (X_1, ..., X_d)$ a random vector with corresponding distributional copula C. Then the distributional copula of $\widetilde{\mathbf{X}} = (T_1(X_1), ..., T_d(X_d))$ is the associated $\mathbf{D}$-copula $C_{\mathbf{D}}$ of $\mathbf{X}$, with*

$$D_i = \begin{cases} L & \text{if } T_i \text{ is strictly increasing} \\ U & \text{if } T_i \text{ is strictly decreasing} \end{cases},$$

for $i \in \{1, .., d\}$, whose expression is given by Proposition 2.1.

Proof: By using the inverse functions of T_i and F_i, $i \in \{1, ..., d\}$ we have:

$$T_i(X_i) \leq \big(\widetilde{F}_i^{\leftarrow}(u_i)\big) \overset{P}{\sim} \mathcal{B}_i\big(F_{D_i,i}^{\leftarrow}(u_i)\big)\ ,$$

for $i \in \{1, ..., d\}$, with $\mathcal{B}_i$ as in Definition 2.4, which implies that the distributional copula of $\widetilde{\mathbf{X}}$ is $C_{\mathbf{D}}$. □

2.2. Associated tail dependence functions and tail dependence coefficients

Considering the results obtained so far, it is possible to introduce a general definition of tail dependence function and tail dependence coefficients considering the dependence $\mathbf{D}$. For the analysis of the conditions of the existence of the tail dependence function see [21]. The general expression of the tail dependence function is the following (for the positive case, see [23])

Definition 2.6. Let $I = \{1, ..., d\}$, $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with copula C, $\mathbf{D}$ a type of dependence and $C_{\mathbf{D}}$ the corresponding associated copula. For any $\emptyset \neq S \subseteq I$, let $C_{\mathbf{D}(S)}$ denote the corresponding marginal copula. Define the associated $\mathbf{D}(S)$-tail dependence functions $b_{\mathbf{D}(S)}$ of $C_{\mathbf{D}}$, $\emptyset \neq S \subseteq I$ as

$$b_{\mathbf{D}(S)}(w_i, i \in S) = \lim_{u \downarrow 0} \frac{C_{\mathbf{D}(S)}(uw_i,\, i \in S)}{u} \ , \qquad \forall w = (w_1, ..., w_d) \in \mathbb{R}_+^d \ .$$

Given that these functions come from the associated copulas, we call the set of all $\mathbf{D}$-tail dependence functions the associated tail dependence functions. When $S = \{1, ..., d\}$ we omit such subindex.

In particular, the corresponding TDCs are presented in the following definition (for the positive TDCs, see [23, 12]).

Definition 2.7. Consider the same conditions of Definition 2.6. Define the associated $\mathbf{D}(S)$-tail dependence coefficients $\lambda_{\mathbf{D}(S)}$ of $C_{\mathbf{D}}$, $\emptyset \neq S \subseteq I$ as

$$\lambda_{\mathbf{D}(S)} = \lim_{u \downarrow 0} \frac{C_{\mathbf{D}(S)}(u, ..., u)}{u} \ .$$

We say that $\mathbf{D}(S)$-tail dependence exists whenever $\lambda_{\mathbf{D}(S)} > 0$.

Note that

$$C_{\mathbf{D}(S)}(u, ..., u) = C_{\mathbf{D}}(u_1, ..., u_d) \geq C_{\mathbf{D}}(u, ..., u) \ ,$$

with $u_i = \begin{cases} u & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$, $i \in \{1, ..., d\}$. Because of this, $\lambda_{\mathbf{D}(S)} \geq \lambda_{\mathbf{D}}$, so $\mathbf{D}$-tail dependence implies $\mathbf{D}(S)$-tail dependence for all $\emptyset \neq S \subseteq I$.

3. MODELLING GENERAL DEPENDENCE

In this section we analyse general dependence and tail dependence in three examples of copula models. To this end we use the definitions and results obtained on the previous section. We first analyse the perfect dependence cases and obtain their corresponding copulas, this includes perfect non-positive dependence. We then study the elliptical copulas for which we characterise the associated copulas. Using this characterisation, we obtain an expression for the associated tail dependence functions of the Student's t copula, which accounts for all types of tail dependence simultaneously. After that we study the Archimedean copulas, we prove that they can only account for non-positive tail dependence when their generator is non-strict and present three examples when they do. At the end of the section we discuss a method for modelling general tail dependence using copula models. The analysis of general dependence presented in this section complements the analysis of positive tail dependence for these models.

3.1. Perfect dependence cases

We now analyse the most basic examples of copula models. They correspond to all the variables being either independent or perfectly dependent.

For the independence case, let $\mathbf{U} = (U_1, ..., U_d)$ be a random vector with $\{U_i\}_{i=1}^d$ independent uniform random variables. The distribution function of U is the copula $C(u_1, ..., u_d) = \prod_{i=1}^d u_i$, which is known as the independence copula. It follows that the associated copula are also equal to the independence copula. This is the copula of any random vector formed by independent variables.

Our analysis of perfect dependence corresponds to the distribution of vectors of the form $(W, -W, -W, ..., W, -W)$ with W a uniform random variable. From Definition 2.3 and Proposition 2.4 it follows that the distribution of a vector of this form is an associated copula of the vector $\mathbf{W} = (W, ..., W)$. The distributional copula of $\mathbf{W}$ is

$$C(u_1, ..., u_d) \;=\; \min\{u_i\}_{i=1}^d \,. \tag{3.1}$$

Given that $1 - W$ is also uniform it follows that this is also the survival copula, so the vector is symmetric. This copula is the comonotonic copula. Now, let $\mathbf{D}$ be a type of dependence and $I = \{1, ..., d\}$. Define $I_L = \{i \in I \mid D_i = L\}$ and $I_U = \{i \in I \mid D_i = L\}$. Let us assume that neither I_L nor I_U are empty. That is, we assume perfect non-positive dependence (the case of perfect positive dependence is covered in equation (3.1)). Then the associated $\mathbf{D}$-copula is

$$C_{\mathbf{D}}(u_1, ..., u_d) \;=\; P\Big(\big(W \le \min\{u_i\}_{i\in I_L}\big) \cap \big(W \ge \max\{1 - u_i\}_{i\in I_U}\big)\Big) \,.$$

It follows that, for $\min\{u_i\}_{i\in I_L} > \max\{1 - u_i\}_{i\in I_U}$, this probability is equal to zero; therefore, a general expression is

$$C_{\mathbf{D}}(u_1, ..., u_d) \;=\; \max\Big\{0,\, \min\{u_i\}_{i\in I_L} + \min\{u_i\}_{i\in I_U} - 1\Big\} \,. \tag{3.2}$$

In the bivariate case the associated (L, U)-copula C_{LU} is equal to the Fréchet lower bound for copulas, also known as the countermonotonic copula. Copulas of this form appear in perfect non-positive dependence, see [20, Example 5.22]. In the following proposition we prove that, in d dimensions, the copulas of (3.1) and (3.2) correspond not only to vectors of the form $(W, -W, W, ..., W, -W)$, but to the use of strictly monotone transformations on a random variable. Because of this, we call these copulas the monotonic copulas.

Proposition 3.1. *Let Z be a random variable, and let $\{T_i\}_{i=1}^d$ be strictly monotone functions, then the distributional copula of the vector $X =$*

$(T_1(Z),...,T_d(Z))$ is one of the monotonic copulas of equations (3.1) or (3.2) with $\mathbf{D}=(D_1,...,D_d)$,

$$D_i = \begin{cases} L & \text{if } T_i \text{ is strictly increasing} \\ U & \text{if } T_i \text{ is strictly decreasing} \end{cases} .$$

Conversely, consider a random vector $\mathbf{X}=(X_1,...,X_d)$ whose distributional copula is a monotonic copula of equation (3.1) or (3.2) for certain $\mathbf{D}$. Then there exist monotone functions $\{T_i\}_{i=1}^d$ and a random variable Z such that

$$(X_1,...,X_d) \stackrel{d}{=} \big(T_1(Z),...,T_d(Z)\big) , \tag{3.3}$$

the $\{T_i\}_{i=1}^d$ satisfy that T_i is strictly increasing if $D_i=L$ and strictly decreasing if $D_i=U$ for $i\in\{1,...,d\}$. In both cases the vector $\mathbf{X}$ is complement symmetric.

Proof: Let F be the distribution function of Z. Considering the uniform random variable $F(Z)$ it is clear that the copula of the d-dimensional vector $(Z,...,Z)$ is the Fréchet upper bound copula $\min\{u_i\}_{i=1}^d$ of equation (3.1). The result is then implied by Proposition 2.4.

The converse statement is a generalisation of [5, Theorem 3.1]. We have that the distributional copula of $\mathbf{X}$ is a monotonic copula for certain $\mathbf{D}$. Note that the associated $\mathbf{D}$-copula of $\mathbf{X}$ is the Fréchet upper bound copula. Let $\{\alpha_i\}_{i=1}^d$ be any invertible monotone functions that satisfy α_i is strictly increasing if $D_i=L$ and strictly decreasing if $D_i=U$ for $i\in\{1,...,d\}$. Proposition 2.4 implies that the copula of $\mathbf{A}=(\alpha_1(X_1),...,\alpha_d(X_d))$ is the Fréchet upper bound copula. According to [9, 6], there exists a random variable Z and strictly increasing $\{\beta_i\}_{i=1}^d$ such that

$$\big(\alpha_1(X_1),...,\alpha_d(X_d)\big) \stackrel{d}{=} \big(\beta_1(Z),...,\beta_d(Z)\big) .$$

By defining $T_i=\alpha_i^{-1}\circ\beta_i$ for $i\in\{1,...,d\}$ we get the result.

In both cases the associated copulas of $\mathbf{X}$ are the monotonic copulas implying that the vector is complement symmetric. □

Regarding tail dependence, suppose the vector $\mathbf{X}$ has distributional copula C^* equal to a monotonic copula $C_{\mathbf{D}}$ of equations (3.1) or (3.2) for certain $\mathbf{D}$. Considering Definition 2.3 of the associated copulas, this implies that $C^*_{\mathbf{D}}$ is the comonotonic copula. It follows that the $\mathbf{D}$ and $\mathbf{D}^{\mathsf{C}}$ tail dependence functions of the vector $\mathbf{X}$ are

$$b^*_{\mathbf{D}}(w_1,...,w_d) = b^*_{\mathbf{D}^{\mathsf{C}}}(w_1,...,w_d) = \min\{w_1,...,w_d\} .$$

The other associated copulas satisfy equation (3.2) for some $\mathbf{D}^0$. It follows that the corresponding tail dependence functions are equal to zero.

3.2. Elliptically contoured copulas

We now analyse the dependence structure of elliptically contoured copulas. We present the definition of this model, a result for its corresponding associated copulas and the associated tail dependence functions of the Student's t copula.

Elliptical distributions, were introduced by [17] and have been analysed by several authors (see e.g. [8, 11]). They have the following form.

Definition 3.1. The random vector $\mathbf{X} = (X_1, ..., X_d)$ has a multivariate elliptical distribution, denoted as $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$, if for $\mathbf{x} = (x_1, ..., x_d)'$ its characteristic function has the form

$$\varphi(\mathbf{x}; \mu, \Sigma) \;=\; \exp(i\,\mathbf{x}'\mu)\;\psi_d\!\left(\frac{1}{2}\,\mathbf{x}'\Sigma\,\mathbf{x}\right),$$

with μ a vector, $\Sigma = (\sigma_{ij})_{1 \le i,j \le d}$ a symmetric positive-definite matrix and $\psi_d(t)$ a function called the characteristic generator.

Elliptical contoured distributions include a large number of distributions (see [31, Appendix]). In the case when the joint density exists, several results have been obtained (see [11, 2, 19]). The corresponding copula is referred to as elliptical copula. This copula has also been subject to numerous analysis (see [7, 1, 5, 3]). Note that the process of standardising the marginal distributions of a vector uses strictly increasing transformations. From Proposition 2.3, we have that the copulas associated to $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$ are the same as the copulas associated to $\mathbf{X}^* \sim El_d(0, R, \psi)$. Here $R = \big(\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{iI}\sigma_{jJ}}}\big)_{1 \le i,j \le d}$ is the corresponding "correlation" matrix implied by $\Sigma = (\sigma_{ij})_{1 \le i,j \le d}$ (see [5, Theorem 5.2] or [7, 3]). Hence, we always assume $\mathbf{X} \sim El_d(R, \psi)$ with $R = (\rho_{ij})_{1 \le i,j \le d}$.

In general, there is no closed-form expression for elliptical copulas but they can be expressed as multivariate integrals of the joint density. In the following proposition we prove an identity for the associated copulas of the elliptical copula.

Proposition 3.2. *Let $\mathbf{X} \sim El_d(R, \psi)$ as in Definition 3.1, with correlation matrix $R = (\rho_{ij})_{1 \le i,j \le d}$, and let $\mathbf{D}$ be a type of dependence. Then the associated $\mathbf{D}$-copula of $\mathbf{X}$ is the same as the distributional copula of $\mathbf{X}^+ \sim El_d(\wp_{\mathbf{D}} R \wp'_{\mathbf{D}}, \psi)$, with $\wp_{\mathbf{D}}$ a diagonal matrix (all values in it are zero except for the values in its diagonal) $\wp_{\mathbf{D}} \in M_{d \times d}$, whose diagonal is $\mathbf{p} = (p_1, ..., p_d)$ with*

$$p_i = \begin{cases} 1 & \text{if } D_i = L \\ -1 & \text{if } D_i = U \end{cases},$$

for $i \in \{1, ..., d\}$.

Proof: The vector $\wp_{\mathbf{D}}\mathbf{X}$ is equal to $(T_1(X_1), ..., T_d(X_d))$ with $T_i(x) = p_i x$, $i \in \{1, ..., d\}$. Using Proposition 2.4, the distributional copula of $\wp_{\mathbf{D}}\mathbf{X}$ is the associated $\mathbf{D}$-copula of $\mathbf{X}$. From the stochastic representation of $\mathbf{X}$ (see [8]), it follows that $\wp_{\mathbf{D}}\mathbf{X} \sim El_d(\wp_{\mathbf{D}} R \wp'_{\mathbf{D}}, \psi)$ (see [5, Theorem 5.2]). □

Given that $C = \widehat{C}$ in elliptical copulas, we have that these copulas are symmetric. This can be easily verified considering that $\wp_{\mathbf{D}^{\complement}} = -\wp_{\mathbf{D}}$, for every dependence $\mathbf{D}$. This implies $\wp_{\mathbf{D}^{\complement}} \cdot R \cdot \wp'_{\mathbf{D}^{\complement}} = \wp_{\mathbf{D}} \cdot R \cdot \wp'_{\mathbf{D}}$. Hence, both $C_{\mathbf{D}}$ and $C_{\mathbf{D}^{\complement}}$ are equal to the distributional copula of $\mathbf{X}^+ \sim El_d(\wp_{\mathbf{D}} R \wp'_{\mathbf{D}}, \psi)$.

Proposition 3.2 makes it possible to use the results of elliptical copulas in associated copulas. This includes the analysis of tail dependence. In the bivariate case [18, 26] studied positive tail dependence in elliptical copulas under regular variation conditions. The Gaussian copula does not account for positive tail dependence, Proposition 3.2 implies that it does not account for tail dependence for any $\mathbf{D}$. In contrast the Student's t copula does account for tail dependence (see e.g. [14, 23, 3, 20]). The Student's t copula with ν degrees of freedom and correlation matrix R is expressed in terms of integrals and density $t_{\nu,R}$ as

$$C(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d\,|R|}} \left(1 + \frac{\mathbf{x}' R^{-1} \mathbf{x}}{\nu}\right)^{-\frac{\nu+d}{2}} d\mathbf{x}\,,$$

with $\mathbf{u} = (u_1, ..., u_d)$ and $\mathbf{x} = (x_1, ..., x_d)'$. [23] analysed in detail the extreme value properties of this copula and obtained an expression for the lower and upper tail dependence functions among other results. More recently, in the bivariate case, [14] obtained an expression for the $\mathbf{D} = (L, U)$ and the $\mathbf{D} = (U, L)$ tail dependence coefficients proving that this copula accounts for negative tail dependence. We now present the expression for the associated $\mathbf{D}$-tail dependence function of the multivariate Student's t copula. This result follows from [23, Theorem 2.3] and Proposition 3.2.

Proposition 3.3. *Let $\mathbf{X} = (X_1, ..., X_d)$ have multivariate t distribution with ν degrees of freedom, and correlation matrix $R = (\rho_{ij})_{1 \le i,j \le d}$, that is $\mathbf{X} \sim T_{d,\nu,R}$. Let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence. Then the associated $\mathbf{D}$-tail dependence function $b_{\mathbf{D}}$ is given by*

$$b_{\mathbf{D}}(w) = \sum_{j=1}^{d} w_j\, T_{d-1,\nu+1,R_j^*}\left(\sqrt{\frac{\nu+1}{1-\rho_{ij}^2}}\left[-\left(\frac{w_i}{w_j}\right)^{-\frac{1}{\nu}} + p_i p_j \rho_{ij}\right],\ i \in I_j\right),$$

with

$$R_j^* = \begin{pmatrix} 1 & \cdots & \rho^*_{1,j-1;j} & \rho^*_{1,j+1;j} & \cdots & \rho^*_{1,d;j} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho^*_{j-1,1;j} & \cdots & 1 & \rho^*_{j-1,j+1;j} & \cdots & \rho^*_{j-1,d;j} \\ \rho^*_{j+1,1;j} & \cdots & \rho^*_{j+1,j-1;j} & 1 & \cdots & \rho^*_{j+1,j-1;j} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^*_{d,1;j} & \cdots & \rho^*_{d,j-1;j} & \rho^*_{d,j+1;j} & \cdots & 1 \end{pmatrix};$$

$\rho^*_{i,k;j} = p_i p_k \frac{\rho_{ik} - \rho_{ij}\rho_{kj}}{\sqrt{1-\rho^2_{ij}}\sqrt{1-\rho^2_{kj}}}$, *the modified partial correlations;* $I_j = I - \{j\}$ *and*

$$p_j = \begin{cases} 1 & \text{if } D_j = L \\ -1 & \text{if } D_j = U \end{cases},$$

for $j \in \{1, ..., d\}$.

Proof: Proposition 3.2 implies that the associated $\mathbf{D}$-tail dependence function of the random vector $\mathbf{X} \sim T_{d,\nu,R}$ is the lower tail dependence function of the vector $\mathbf{X}^+ \sim T_{d,\nu,\wp_\mathbf{D} R \wp'_\mathbf{D}}$. The modified correlation matrix is $\wp_\mathbf{D} R \wp'_\mathbf{D} = R^* = (\rho^*_{ij})_{1\le i,j\le d}$, it follows that

$$(\rho^*_{ij})_{1\le i,j\le d} = (p_i p_j \rho_{ij})_{1\le i,j\le d}\,.$$

Hence $(\rho^*_{ij})^2 = p_i^2 p_j^2 \rho^2_{ij} = 1 \cdot 1 \cdot \rho^2_{ij} = \rho^2_{ij}$. Under this change, the partial correlations are modified as follows:

$$\rho^*_{i,k;j} = p_i p_k \frac{\rho_{ik} - \rho_{ij}\rho_{kj}}{\sqrt{1-\rho^2_{ij}}\sqrt{1-\rho^2_{kj}}}\,.$$

The result is then implied by [23, Theorem 2.3]. □

This proposition implies that the Student's t copula accounts for all 2^d dependencies simultaneously. It can happen that we have negative dependence and positive tail dependence. In that case, the variables might generally exhibit negative dependence but, when it comes to extreme values, they can also be positively dependent.

3.3. Archimedean copulas

Now we analyse the dependence structure of Archimedean copulas. We present the bivariate and multivariate definition of these copulas. We then prove that, when the generator is strict, they can only account for positive tail dependence. Finally, we present three examples with non-strict generators that account for negative tail dependence. For the analysis of positive tail dependence in these copulas we refer to [15, Propositions 2.5 and 3.3], [13, Theorems 4.12 and 4.15] and [22, Corollary 5.4.3]

Much has been written on Archimedean copulas and their applications to different areas of statistics. [28] provide an excellent monography of their history. For further references on their analysis we refer to the seminal works of [13, 22]. [13] analyses several examples with strict generators and [22] extends the analysis to non-strict generators. In order to consider both cases, we follow the notation used in [22].

A bivariate Archimedean copula is defined in terms of a generator, which we denote φ, in the following way:

$$C(u_1, u_2) \;=\; \varphi^{[-1]}\big(\varphi(u_1) + \varphi(u_2)\big)\;, \tag{3.4}$$

where $\varphi^{[-1]}(u) = \begin{cases} \varphi^{-1}(u) & \text{if } 0 \le u \le \varphi(0) \\ 0 & \text{if } \varphi(0) \le u \le \infty \end{cases}$, is the pseudo-inverse of φ. In order for this function to be a copula, the generator must satisfy the following properties:

i) $\varphi\colon [0,1] \to R^+ \cup \infty$,

ii) φ is continuous, strictly decreasing and convex,

iii) $\varphi(1) = 0$.

φ is called a strict generator when $\varphi(0) = \infty$. Note that, when φ is strict, $\varphi^{[-1]} = \varphi^{-1}$. ([3])

[16] proved that a strict generator gives a copula in any dimension d if and only if the generator inverse φ^{-1} is completely monotonic. In that case, the multivariate Archimedean copula is defined as

$$C(u_1, ..., u_d) \;=\; \varphi^{-1}\left(\sum_{i=1}^{d} \varphi(u_i)\right), \tag{3.5}$$

In the next proposition we prove that, by construction, Archimedean copulas with strict generators, do not account for any non-positive tail dependence.

Proposition 3.4. *Let C be an Archimedean copula with differentiable strict generator φ and let $\mathbf{D}$ be a non-positive type of dependence. Then, if the corresponding tail dependence function $b_{\mathbf{D}}$ exists, it is equal to zero.*

Proof: Let C be a bivariate Archimedean copula with strict generator φ. As we pointed out before, given that φ is strict, $\varphi^{[-1]} = \varphi^{-1}$. We begin this proof with the bivariate case and prove that $\lambda_{LU} = 0$.

Let $G(h) = \dfrac{\varphi^{-1}\big(\varphi(h) + \varphi(1-h)\big)}{h}$, by definition

$$\begin{aligned} \lambda_{LU} \;&=\; \lim_{h\to 0} \frac{C_{LU}(h,h)}{h} \\ &=\; \lim_{h\to 0} \frac{h - C(h, 1-h)}{h} \\ &=\; 1 - \lim_{h\to 0} G(h)\;. \end{aligned} \tag{3.6}$$

[3] In [13], the construction of Archimedean copulas covers the strict generator case when φ^{-1} is a Laplace transform, they denote such Laplace transform as ϕ.

Along with the three properties of the generator φ mentioned above, in this case it is strict and differentiable. This implies the following for φ^{-1}:

i) φ^{-1} is differentiable,

ii) φ^{-1} is strictly decreasing and convex,

iii) $\lim_{s\to\infty} \varphi^{-1}(s) = 0$.

Note that property iii) is only satisfied when the generator is strict, the behaviour of φ^{-1} around ∞ is fundamental in this proof. If we visualise the graphic of a function with such three features, it is intuitively straightforward that the slope of its tangent will tend to zero as $s \to \infty$, that is $\lim_{s\to\infty} (\varphi^{-1})'(s) = 0$. To prove this, note that, from ii), $(\varphi^{-1})'$ is always negative and increasing. This implies $(\varphi^{-1})'(s)$ converges, as $s \to \infty$, to $c \leq 0$. Suppose $c < 0$, this would imply that φ^{-1} crosses the x-axis. So it follows that $\lim_{s\to\infty} (\varphi^{-1})'(s) = 0$. Hence, we have

$$\begin{aligned} \lim_{s\to\infty} (\varphi^{-1})'(s) &= \lim_{x\to\infty} \lim_{y\to 0} \frac{\varphi^{-1}(x+y) - \varphi^{-1}(x)}{y} \\ &= 0\,. \end{aligned} \tag{3.7}$$

Also, φ is differentiable, strictly decreasing and $\varphi(1) = 0$, hence we have

$$-\infty < \varphi'(1) < 0\,. \tag{3.8}$$

If we take $x(h) = \varphi(h)$ and $y(h) = \varphi(1-h)$ in equation (3.7), we get:

$$\begin{aligned} 0 &= \lim_{h\to 0} \frac{\varphi^{-1}\big(\varphi(h) + \varphi(1-h)\big) - \varphi^{-1}\big(\varphi(h)\big)}{\varphi(1-h)} \\ &= \lim_{h\to 0} \frac{h\,G(h) - h}{\varphi(1-h) - \varphi(1)} \\ &= \frac{\lim_{h\to 0} 1 - G(h)}{\varphi'(1)}\,. \end{aligned}$$

From equation (3.6) and inequality (3.8), this this implies $\lambda_{LU} = 0$. Analogously, we get $\lambda_{UL} = 0$. The multivariate extension is straightforward: let C be a multivariate Archimedean copula and $\mathbf{D}$ a non-positive dependence. Then, there exist $i_1 < i_2$ such that $D_{i_1} \neq D_{i_2}$. Let $C_{(i_1,i_2)}$ be the bivariate marginal copula of C. Hence $\lambda_{(i_1,i_2),(D_{i_1},D_{i_2})} \geq \lambda_{\mathbf{D}}$ and, given that $C_{(i_1,i_2)}$ is also Archimedean, it satisfies $\lambda_{(i_1,i_2),(L,U)} = \lambda_{(i_1,i_2),(U,L)} = 0$. Then $\lambda_{\mathbf{D}} = 0$ follows. $\square$

The same holds for other multivariate constructions based on nesting of Archimedean copulas, such as the ones described in [13, Section 4.2].

When the generator is non-strict, Archimedean copulas can account for non-positive tail dependence. This is the case in the three bivariate examples presented in Table 2. These examples can be found in [22, Section 4.2]. The first two examples are the one-parameter copulas 4.2.7 and 4.2.8 in [22]. The third example is a two-parameter family of copulas known as the rational Archimedean copulas. The construction of these copulas can be found in [22, Subsection 4.5.2]. The expression is equation (4.5.9) and the generator is studied in p. 149 therein.

Table 2: Examples of Archimedean copulas with non-strict generators that account for negative tail dependence.

Generator $\varphi(s)$	Copula	b_{LU} and b_{UL}
$-\ln(\theta s+1-\theta)$, $0<\theta\leq 1$	$\max\left\{\theta u_1 u_2+(1-\theta)\,(u_1+u_2-1),\, 0\right\}$	$\min\left\{w_1,\,(1-\theta)w_2\right\}$, $\min\left\{(1-\theta)\,w_1,\,w_2\right\}$
$\dfrac{1-s}{1+(\theta-1)\,s}$, $\theta\geq 1$	$\max\left\{\dfrac{\theta^2 u_1 u_2-(1-u_1)\,(1-u_2)}{\theta^2-(\theta-1)^2\,(1-u_1)\,(1-u_2)},\, 0\right\}$	$\min\left\{w_1,\,\frac{w_2}{\theta^2}\right\}$, $\min\left\{\frac{w_1}{\theta^2},w_2\right\}$
see [22, p. 149], $0\leq\beta\leq 1-\lvert\alpha\rvert$	$\max\left\{\dfrac{u_1u_2-\beta(1-u_1)\,(1-u_2)}{1-\alpha(1-u_1)\,(1-u_2)},\, 0\right\}$	$\min\left\{w_1,\,\beta w_2\right\}$, $\min\left\{\beta w_1,\,w_2\right\}$

3.4. Use of rotations to model general tail dependence

We now discuss a method to model an arbitrary type of tail dependence using a copula model. The condition on the copula model is to account for, at least, one type of tail dependence. Similar procedures have been suggested in [25, Section 2.4] and [14, Example 8.1]. To illustrate how this procedure works, consider the bivariate Generalised Clayton copula, C^{GC} ([4]). This Archimedean copula accounts for upper tail dependence. Suppose that we are trying to model data that exhibits lower-upper tail dependence with a model C^* and want to use C^{GC} and the fact that it accounts for upper tail dependence. The use of this procedure implies defining $C^*_{LU}=\widehat{C}^{GC}$. And it holds that C^* accounts for lower-upper tail dependence. Using Proposition 2.1, $C^*(u_1,u_2)=u_1-C^{GC}(1-u_1,u_2)$. Note that the fact that C^{GC} also accounts for lower tail dependence implies that C^* accounts for upper-lower tail dependence. So, before using this technique, the whole dependence structure of the model and the data must be analysed.

We generalise this idea to model arbitrary $\mathbf{D}^\circ$-tail dependence using a copula model C that accounts for $\mathbf{D}^+$-tail dependence. Let $\mathcal{A}_{\mathbf{X}}=\{C_{\mathbf{D}}\,|\,\mathbf{D}\in\Delta\}$ be the associated copulas of model C, we know that $\lim\limits_{h\to 0}\frac{C_{\mathbf{D}^+}(h,\ldots,h)}{h}>0$. Now, define a $\mathbf{D}^\circ$-associated copula as $C^*_{\mathbf{D}^\circ}=C_{\mathbf{D}^+}$. By construction, as in the example, this

[4] $C^{GC}_{\theta,\delta}(u,v)=\left\{\left[(u^{-\theta}-1)^\delta+(v^{-\theta}-1)^\delta\right]^{\frac{1}{\delta}}+1\right\}^{-\frac{1}{\theta}}$.

copula model accounts for $\mathbf{D}^{\circ}$-tail dependence. The associated copulas, $\mathcal{A}^*_{\mathbf{X}} = \{C^*_{\mathbf{D}} \mid \mathbf{D} \in \Delta\}$, of this model can be obtained from $C^*_{\mathbf{D}^{\circ}}$, using Proposition 2.1. Note that the set $\mathcal{A}^*_{\mathbf{X}}$ is the same as $\mathcal{A}_{\mathbf{X}}$, but with rotated dependencies. The whole dependence structure of model C^* is implied by C.

4. CONCLUSIONS AND FUTURE WORK

In this section we discuss the main findings of this work and some future lines of research. In Section 2 we introduce the concepts to analyse, in the multivariate case, the whole dependence structure among random variables. We consider the 2^d different orthants of dimension d. We first introduce general dependence, the **D**-probability functions and the associated copulas. We then present a version of Sklar's theorem that proves that the associated copulas link the **D**-probability functions with their marginals. It is through this result that we are able to generalise the use of the distributional and survival copulas for positive dependence. In this generalisation we use the associated copulas to cover general dependence. We introduce an expression for the relationship among all associated copulas and present a proposition regarding symmetry and exchangeability. After that, we prove that they are invariant under strictly increasing transformations and characterise the copula of a vector after using monotone transformations. At the end of this section, we introduce the associated tail dependence functions and associated tail dependence coefficients of a random vector. With them we can analyse tail dependence in the different orthants.

In Section 3 we use the concepts and results obtained in Section 2 to analyse three examples of copula models. The first example corresponds to the perfect dependence models. We begin this analysis with the independence case and then consider perfect dependence, including perfect non-positive dependence. We find and expression for their copulas, which are a generalisation of the Fréchet copula bounds of the bivariate case. Given that they correspond to the use of strictly monotone transformations on a random variable, we call them the monotonic copulas. The second example corresponds to the elliptical copulas. In this case, we characterise the corresponding associated copulas. We then present an expression for the associated tail dependence function of the Student's t copula. This result proves that this copula model accounts for tail dependence in all orthants. The third example corresponds to Archimedean copulas. In this case, we prove that, if their generator is strict, they can only account for positive tail dependence. We then present three examples of Archimedean copulas with non-strict generators that account for negative tail dependence. After that we discuss a method for modelling arbitrary tail dependence using copula models.

There are several areas where future research regarding general dependence is worth being pursued. For instance, the use of **D**-probability functions is not restricted to copula theory. The analysis of probabilities in the multivariate case has sometimes been centered in distribution functions, but, just like survival functions, **D**-probability functions can serve different purposes in dependence analysis. Another possibility is the use of nonparametric estimators to measure non-positive tail dependence, as the use of these estimators has been restricted to the lower and upper cases. The results obtained in this work are useful in the understanding of the dependence structure implied by different copula models. As we have seen, without analysing general dependence, the analysis of these models is incomplete. Therefore, it is relevant to extend this analysis to models such as the hierarchical Archimedean copulas and vine copulas. The use of vine copulas has proven to provide a flexible approach to tail dependence and account for asymmetric positive tail dependence (see e.g. [24, 15]).

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