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PROPERTIES OF n-LAPLACE TRANSFORM RATIO ORDER AND $\mathcal{L}^{(n)}$-CLASS OF LIFE DISTRIBUTIONS

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Abstract:

- One notion of stochastic comparisons of non-negative random variables based on ratios of n^{th} derivative of Laplace transforms (n-Laplace transform order or shortly $\leq_{n\text{-Lt-r}}$ order) is introduced by Mulero *et al.* (2010). In addition, they studied some of its applications in frailty models. In this paper, we have focused on some further properties of this order. In particular, we have shown that $\leq_{n\text{-Lt-r}}$ order implies dual weak likelihood ratio order ($\leq_{\text{DWLR}}$ order). Moreover, $\leq_{n\text{-Lt-r}}$ order, under certain circumstances, implies likelihood ratio order ($\leq_{\text{lr}}$ order). Finally, the $\mathcal{L}^{(n)}$ ($\bar{\mathcal{L}}^{(n)}$)-class of life distribution is proposed and studied. This class reduces to $\mathcal{L}$ ($\bar{\mathcal{L}}$)-class if we take $n = 0$.

Key-Words:

- *likelihood ratio order; hazard rate order; shock models; dual weak likelihood ratio ordering; totally positive of order 2 (*TP_2*).*

AMS Subject Classification:

- 62E10, 60E05, 60E15.

1. INTRODUCTION, DEFINITIONS AND IMPLICATIONS

There are several stochastic orders that have been introduced in the literature based on Laplace transforms. For example, Laplace transform order ($\leq_{\text{Lt}}$ order) compares two random variables via their Laplace transforms. Moreover, Laplace transform ratio order ($\leq_{\text{Lt-r}}$ order) and reverse Laplace transform ratio order ($\leq_{\text{r-Lt-r}}$ order) which are presented based on ratios of Laplace transforms, studied by Shaked and Wong (1997). Recently, Li *et al.* (2009) introduced differentiated Laplace transform order ($\leq_{\text{d-Lt-r}}$ order) which is based on ratio of derivative of Laplace transforms, and then, Mulero *et al.* (2010) generalized differentiated Laplace transform order to n-Laplace transform order. In addition, one can see Rolski and Stoyan (1976), Alzaid *et al.* (1991) and Shaked and Shanthikumar (2007) for more details. The main purpose of this article is to study the n-Laplace transform ratio order. The $\mathcal{L}$ ($\bar{\mathcal{L}}$)-class of life distributions states that $\int_0^\infty e^{-st}\bar{F}(t)\,dt \geq (\leq) \int_0^\infty e^{-st}\bar{G}(t)\,dt$, where $\bar{G}(t) = e^{-t/\mu}$, $t \geq 0$ and $\mu = \int_0^\infty \bar{F}(t)\,dt$ which was introduced by Klefsjo (1983). He presented results concerning closure properties under some of this class reliability operations, under shock models and a certain cumulative damage model. Mitra *et al.* (1995), Sengupta (1995), Bhattacharjee and Sengupta (1996), Chaudhuri *et al.* (1996), Lin (1998), Lin and Hu (2000) and Klar (2002) have studied this topic.

Here, we give some preliminaries and definitions and study some new results that are used to present our main results. Various properties and its relationships to other stochastic orders, will be described in the next section.

Throughout the paper, we assume that X and Y are absolutely continuous and non-negative random variables and use the term *increasing* in place of *non-decreasing*.

For any absolutely continuous and non-negative random variable X with density function f and survival function $\bar{F}$, the Laplace transform of f is given by $L_X(s) = \int_0^\infty e^{-st} f(t)\,dt$, $s > 0$, and the Laplace transform of $\bar{F}$ is defined as

$$L_X^*(s) = \int_0^\infty e^{-st}\bar{F}(t)\,dt\,, \qquad s > 0\,.$$

It is easy to see that $L_X(s) = 1 - s L_X^*(s)$. For absolutely continuous and non-negative random variable Y with density function g and survival function $\bar{G}$, $L_Y(s)$ and $L_Y^*(s)$ can be defined similar to $L_X(s)$ and $L_X^*(s)$ respectively.

Think of X as representing the length of an interval. Let this interval be subject to a poissonian marking process with intensity s. Then the Laplace transform $L_X(s)$ is the probability that there are no marks in the interval.

$$\begin{aligned} P\{X \text{ has no marks}\} &= E\big(P\{X \text{ has no marks}\} \,|\, X\big) \\ &= E\big(P\{\text{the number of events in the interval } X \text{ is } 0 \,|\, X\}\big) \\ &= E\big(e^{-sX}\big) \\ &= L_X(s)\,. \end{aligned}$$

Note that

$$\begin{aligned} P\{\text{there are } n \text{ events in the interval } X \,|\, X\} &= \frac{(sX)^n}{n!}\, e^{-sX}\,, \\ P\{\text{the number of events in the interval } X \text{ is } 0 \,|\, X\} &= e^{-sX}\,, \end{aligned}$$

so,

$$E\big(P\{\text{there are } n \text{ events in the interval } X \,|\, X\}\big) \;=\; (-1)^n \frac{s^n}{n!}\, L_X^{(n)}(s)\,,$$

and

$$E\big(P\{\text{the number of events in the interval } X \text{ is } 0 \,|\, X\}\big) \;=\; L_X(s)\,.$$

Example 1.1 (Thinning of a Renewal Stream)**.** Assume that for a random point process the lengths of the time intervals between the points which are independent and equally distributed random variables with probability density f and Laplace transform $L_X(s)$. Such a point process is called a renewal stream. The process is subject to the following thinning operation. Each point is kept with probability $1-p$ and is removed with probability p and the removal of different points are independent. We will derive the Laplace transform $L_Y^{(n)}(s)$ for the time intervals in the new stream. Let Y be the length of the time interval from a point to the next in the thinned stream and let X be the length of the time interval from the same point to the next in the original stream of points. By conditioning with respect to whether the next point is kept or removed we get, if a catastrophe risk is added as described above,

$$\begin{aligned} L_Y^{(n)}(s) &= P(n \text{ catastrophe in } X) \\ &= (1-p)\,.\,P(n \text{ catastrophe in } X) \\ &\quad + \; p\,.\,P(n \text{ catastrophe in } X)\; P(n \text{ catastrophe in } Y)\,, \end{aligned}$$

we have used the fact that if the next point is removed the process starts from scratch again. Thus we have

$$L_Y^{(n)}(s) \;=\; (1-p)\, L_X^{(n)}(s) + p\, L_X^{(n)}(s)\, L_Y^{(n)}(s)\,,$$

which gives

$$L_Y^{(n)}(s) = \frac{(1-p)\, L_X^{(n)}(s)}{1 - p\, L_X^{(n)}(s)}\,.$$

If, for instance, X has an exponential distribution with parameter λ, which is the case if the stream is a Poisson process,

$$L_X^{(n)}(s) \;=\; (-1)^n \int_0^\infty \lambda\, x^n\, e^{-x(s+\lambda)} \;=\; (-1)^n \frac{\lambda\, n!}{(s+\lambda)^{n+1}}$$

and

$$L_Y^{(n)}(s) \;=\; \frac{(-1)^n\,(1-p)\,\lambda\, n!}{(s+\lambda)^{n+1} - (-1)^n\, p\,\lambda\, n!}\,.$$

For instance, if $n = 0$ then

$$L_Y(s) \;=\; \frac{(1-p)\,\lambda}{s + \lambda(1-p)}\,,$$

thus the lengths of the time intervals in the new stream have exponential distributions with parameter $\lambda(1-p)$.

Thinning of streams of points appears in many applications in operations research, in technology and in biology. For instance, consider the stream of customers arriving at a supermarket, and make this stream thinner by considering only those customers which buy a certain item.

Example 1.2 (Waiting Time for the M/G/1 System)**.** In this system, the customers arrive according to a Poisson process with parameter λ. There is one service station and we assume the queue discipline is “first come-first served”. Let X_n be the waiting time of customer number n which the density function of X_n is denoted by f_n. We now assume that the customers on arrival at the system are marked with probability $1-s$. If the waiting time X_n of customer number n is t, the conditional probability that m marked customer arrive during this time is

$$\sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\cdot \binom{k}{m}\,(1-s)^m\, s^{k-m}\,,$$

so,

$$\begin{aligned} P(m \text{ marked customer during } X_n) \;&=\\ &= \left(\frac{1-s}{s}\right)^m \sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!}\binom{k}{m}\int_0^\infty e^{-\lambda t}\, t^k f_n(t)\, dt\\ &= \left(\frac{1-s}{s}\right)^m \sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!}\binom{k}{m}\,(-1)^k\, L_{X_n}^{(k)}(s)\,. \end{aligned}$$

Recall that X is said to be smaller than Y in the Laplace transform order (denoted by $X \le_{\mathrm{Lt}} Y$), if $L_X(s) \ge L_Y(s)$, $\forall\, s > 0$. Shaked and Wong (1997) established and extensively investigated stochastic orderings based on ratios of

Laplace transform. They said that X is smaller than Y in the Laplace transform ratio order (and denoted by $X \leq_{\text{Lt-r}} Y$) if $\frac{L_X(s)}{L_Y(s)}\left(\frac{1-sL_X^*(s)}{1-sL_Y^*(s)}\right)$ is increasing in $s > 0$. Also, X is smaller than Y in the reverse Laplace transform ratio order (denoted by $X \leq_{\text{r-Lt-r}} Y$) if $\frac{1-L_X(s)}{1-L_Y(s)}\left(\frac{L_X^*(s)}{L_Y^*(s)}\right)$ is increasing in $s > 0$. It is evident that $X \leq_{\text{Lt-r}} (\leq_{\text{r-Lt-r}})\, Y$ implies $X \leq_{\text{Lt}} Y$.

Li *et al.* (2009) introduced a new stochastic order upon Laplace transform with applications. They said that X is smaller than Y in the differentiated Laplace transform ratio order (denoted by $X \leq_{\text{d-Lt-r}} Y$) if $\frac{L'_X(s)}{L'_Y(s)}$ is increasing in $s > 0$. They demonstrated that

$$X \leq_{\text{d-Lt-r}} Y \implies X \leq_{\text{Lt-r}} (\leq_{\text{r-Lt-r}})\, Y\,.$$

For two random variables X and Y with densities f and g and survival functions $\bar{F}$ and $\bar{G}$ respectively, we say that X is smaller than Y in the likelihood ratio order ($X \leq_{\text{lr}} Y$) if $\frac{g(t)}{f(t)}$ is increasing in t and say that X is smaller than Y in the hazard rate order ($X \leq_{\text{hr}} Y$) if $\frac{\bar{G}(t)}{\bar{F}(t)}$ is increasing in t. For more details of other stochastic orders one can see Shaked and Shanthikumar (2007).

Indeed, their new order has been constructed using the first derivative of the Laplace transform of density functions rather than the own Laplace transform. In order to clarify and further to determine how does the comparison affect, Mulero *et al.* (2010) considered, in general, the n^{th} derivative of the Laplace transform. As a useful observation, for example, the order based on ratios of the Laplace transform as it increases or decreases, may be important to present much information about comparison of two random variables. Moreover, as shown in the continue, for a special shock model, it is highly motivated to be considered in the case of comparison of number of shocks according to $\leq_{\text{lr}}$ order. Thus, they introduced a new partial orderings as below:

Definition 1.1. We say that X is smaller than Y in n-Laplace transform ratio (denoted by $X \leq_{n\text{-Lt-r}} Y$) if

$$\frac{L_X^{(n)}(s)}{L_Y^{(n)}(s)} \text{ is increasing in } s > 0\,, \tag{1.1}$$

in which $n \geq 0$ is an integer and $L_X^{(n)}(s)$ denotes n^{th} derivative of $L_X(s)$ and similarly for Y.

We can define $\leq_{n\text{-Lt}^*\text{-r}}$ order for n^{th} derivative of $L_X^*(s)$ in a same manner.

Example 1.3. As pointed out in Mulero (2010), when $X_i \sim Gamma(\alpha_i, \beta_i)$, $i = 1, 2$, then $X_1 \leq_{n\text{-Lt-r}} X_2$ holds if for every $n \geq 0$, $\beta_1 \geq \beta_2$ and $\alpha_2 \geq \alpha_1$. It can be seen in this case that if $\alpha_1 = \alpha_2 = 1$ and $\beta_1 \geq \beta_2$ then $X_1 \leq_{n\text{-Lt}^*\text{-r}} X_2$.

2. MAIN RESULTS

In this section, we present some results for $\leq_{n\text{-Lt-r}}$ order and then we discuss $\leq_{n\text{-Lt-r}}$ order for shock models. The same results can be obtained for $\leq_{n\text{-Lt}^*\text{-r}}$.

2.1. Basic properties

First of all, stochastic orders which have connections to $\leq_{n\text{-Lt-r}}$ order have been described.

Theorem 2.1. *Let X_1 and X_2 be absolutely continuous and non-negative iid random variables with density functions $f_1(\cdot)$ and $f_2(\cdot)$ respectively, and n be a non-negative integer. Then for any n, we have:*

(**a**) *If $X_1 \leq_{\mathrm{lr}} X_2$ then $X_1 \leq_{n\text{-Lt-r}} X_2$.*

(**b**) *If f_1 and f_2 are both bounded on $[0,\infty)$ then for all n, $X_1 \leq_{n\text{-Lt-r}} X_2$ implies that $X_1 \leq_{\mathrm{lr}} X_2$.*

(**c**) *If $X_1 \leq_{n+1\text{-Lt-r}} X_2$ then $X_1 \leq_{n\text{-Lt-r}} X_2$.*

Proof: As we know that a non-negative function $h(x,y)$ is said to be TP_2 (RR_2) if

$$\begin{vmatrix} h(x_1,y_1) & h(x_1,y_2) \\ h(x_2,y_1) & h(x_2,y_2) \end{vmatrix} \geq (\leq)\ 0$$

for every $x_1 \leq x_2$ and $y_1 \leq y_2$.

(**a**) It is easy to verify, that $t^n e^{-st}$ is RR_2 in $s>0$ and in $t>0$. So, by Karlin (1968, Lemma 1.1 on p. 99) it follows that

$$\int_0^\infty t^n e^{-st} f_j(t)\,dt\,,$$

is RR_2 in $j \in \{1,2\}$ and in $s>0$, that is,

$$\frac{\int_0^\infty t^n e^{-st} f_2(t)\,dt}{\int_0^\infty t^n e^{-st} f_1(t)\,dt}\,,$$

is decreasing in $s>0$. Hence we have the result.

(**b**) Let $X_1 \leq_{n\text{-Lt-r}} X_2$, so, by Widder (1946), we have

$$\lim_{n\to\infty} L^{(n)}_{X_1}(s)\big|_{s=\frac{n+1}{t}} = f_1(t)\,,$$

and similarly we have for Y. So, $\frac{f_2(t)}{f_1(t)}$ is increasing in $t>0$.

(**c**) Since

$$\begin{aligned}
-L^{(n)}_{X_i}(s) &= \int_s^\infty L^{(n+1)}_{X_i}(t)\,dt \\
&= \int_0^\infty \Big[L^{(n+1)}_{X_i}(t)\,1_{(s,\infty)}(t)\Big]\,dt\,,
\end{aligned}$$

and $L^{(n+1)}_{X_i}(s)$ is $\mathrm{TP}_2(i,t)$ and $1_{(s,\infty)}(t)$ is $\mathrm{TP}_2(t,s)$, so, by "basic composition theorem" in Karlin (1968), $L^{(n)}_{X_i}(s)$ is $\mathrm{TP}_2(i,s)$, and thus $\frac{L^{(n)}_{X_2}(s^{-1})}{L^{(n)}_{X_1}(s^{-1})}$ is increasing in $s>0$. □

Note that the inverse of the above theorem necessarily does not establish, for this case, see the following example:

Example 2.1. Let $P(X=0)=P(X=1)=\frac{1}{2}$ and $P(Y=0)=\frac{2}{4}$, $P(Y=1)=\frac{1}{4}$, $P(Y=2)=\frac{1}{4}$. It is clear that $X\le_{\mathrm{lr}}Y$ is invalid, but $X\le_{n\text{-Lt-r}}Y$ and $X\le_{n\text{-Lt}^*\text{-r}}Y$ are true.

There is no relationship between $\le_{n\text{-Lt-r}}$ and $\le_{n\text{-Lt}^*\text{-r}}$ orderings which is mentioned by Shaked and Wong (1997).

Example 2.2. Let $P(X=1)=P(X=2)=P(X=3)=\frac{1}{3}$ and $P(Y=0)=P(Y=1)=\frac{1}{4}$ and $P(Y=2)=\frac{1}{2}$. Then, $X\le_{n\text{-Lt-r}}Y$ and $X\le_{n\text{-Lt}^*\text{-r}}Y$ are not hold.

The inverse of part (c) of Theorem 2.1 is not necessarily established. This review is the following example:

Example 2.3. Let $P(X=1)=\frac{3}{4}$, $P(X=2)=P(X=3)=\frac{1}{8}$, $P(Y=1)=\frac{1}{4}$ and $P(Y=2)=\frac{3}{4}$. Then,

- (**i**) for $n=1$, Li *et al.* (2009) showed that $X\le_{1\text{-Lt-r}}Y$,
- (**ii**) for $n=2$, we have, $\frac{d}{ds}\frac{L^{(n)}_X(s)}{L^{(n)}_Y(s)}=\frac{1}{2}\,\frac{68e^{-s}-18e^{-2s}-108e^{-3s}}{(1+12e^{-s})}$, which for $s=0.1$ is equal to -1.4, for $s=1$ is equal to 1.59 and for $s=10$ gives 0.0015, so $X\not\le_{2\text{-Lt-r}}Y$.

The next result can be easily built and thus is presented only with proof of part (c).

Theorem 2.2.

(**a**) *Let $\{X_j\}$ and $\{Y_j\}$ be two sequences of random variables such that $X_j \to X$ and $Y_j \to Y$ in distribution. If $X_j \le_{n\text{-Lt-r}} Y_j$, $j = 1, 2, \ldots$, then $X \le_{n\text{-Lt-r}} Y$.*

(**b**) *Let X and Y be non-negative random variables with moments μ_i and ν_i, $i = 1, 2, \ldots$, respectively, $(\mu_0 = \nu_0 = 1)$. Then, $X \le_{n\text{-Lt-r}} Y$ if and only if*

$$\frac{\sum_{i=0}^{\infty} \frac{(-s)^i}{i!}\, \mu_{n+i}}{\sum_{i=0}^{\infty} \frac{(-s)^i}{i!}\, \nu_{n+i}} \quad \text{is increasing in } s > 0\,.$$

(**c**) *Let X, Y and Θ be random variables such that $[X \,|\, \Theta = \theta] \le_{n\text{-Lt-r}} [Y \,|\, \Theta = \theta']$ for all θ and θ' in the support of Θ. Then $X \le_{n\text{-Lt-r}} Y$.*

Proof: We only present the proof of part (**c**). The proof of parts (**a**) and (**b**) is clear.

With similar arguments to Theorem 5.B.8 of Shaked and Shanthikumar (2007) we have

$$\frac{L_X^{(n)}(s)}{L_Y^{(n)}(s)} = \frac{E_\Theta\Big(L_{[X|\Theta]}^{(n)}(s)\Big)}{E_\Theta\Big(L_{[Y|\Theta]}^{(n)}(s)\Big)}\,.$$

On the other hand $\frac{d}{ds}\frac{L_{[X|\theta]}^{(n)}(s)}{L_{[Y|\theta']}^{(n)}(s)} \ge 0$, if and only if

$$L_{[X|\theta]}^{(n+1)}(s)\, L_{[Y|\theta']}^{(n)}(s) - L_{[X|\theta]}^{(n)}(s)\, L_{[Y|\theta']}^{(n+1)}(s) \ge 0\,,$$

for all θ and θ' in the support of Θ. Consequently,

$$E_\Omega\Big(L_{[X|\theta]}^{(n+1)}(s)\, L_{[Y|\theta']}^{(n)}(s) - L_{[X|\theta]}^{(n)}(s)\, L_{[Y|\theta']}^{(n+1)}(s)\Big) \ge 0\,,$$

where $\Omega = (\theta, \theta')$, and the proof is complete. □

Theorem 2.3. *Let $f(t)$ and $g(t)$ be both bounded on $[0, \infty)$. If $X \le_{n\text{-Lt-r}} Y$, then $X \le_{\mathrm{DWLR}} Y$.*

Proof: If $X \le_{n\text{-Lt-r}} Y$, then $\frac{L_X^{(n)}(s)}{L_Y^{(n)}(s)} \ge \frac{E(X^n)}{E(Y^n)}$ so,

$$\lim_{n\to\infty} \frac{L_X^{(n)}(s)}{L_Y^{(n)}(s)}\bigg|_{s=\frac{n+1}{t}} \ge \lim_{n\to\infty} \frac{E(X^n)}{E(Y^n)} = c$$

where $0 < c \le 1$. So by Widder (1946), if $f(t)$ and $g(t)$ are both bounded on $[0, \infty)$, then we have $f(t) \ge c g(t)$, from which we conclude $X \le_{\mathrm{DWLR}} Y$. (Note that if $c > 1$ then $\int f(t)\, dt \ge 1$.) □

2.2. Shock models

A device is subjected to shocks arriving according to a Poisson process with parameter λ. Then the lifetime T_1 of the system is given by $T_1 = \sum_{j=1}^{N_1} X_j$, where N_1 denote the number of shocks survived by the system and X_j is the random interval time between the $j-1$ and j^{th} shocks. Suppose further that the device has probability $\bar{P}_k = P(N_1 > k)$ for all $k \in N$ of surviving the first k shocks, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \dots$. Also, let $p_{k+1} = \bar{P}_k - \bar{P}_{k+1}$, $k = 0, 1, 2, ...$, then, the probability function of the device is given by

$$f(t_1) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t_1}(\lambda t_1)^k}{k!} \lambda p_{k+1} \,.$$

The survival function of this device is given by

$$\bar{F}(t_1) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t_1}(\lambda t_1)^k}{k!} \bar{P}_k \,.$$

Consider another device which is also subjected to shocks arriving according to a Poisson process with the same parameter λ. Then the lifetime T_2 of the system is given by $T_2 = \sum_{j=1}^{N_2} Y_j$, where N_2 denote the number of shocks survived by the system and Y_j is the random interval time between the $j-1$ and j^{th} shocks. The device has probability $\bar{Q}_k = P(N_2 > k)$ for all $k \in N$ of surviving the first k shocks, where $1 = \bar{Q}_0 \geq \bar{Q}_1 \geq \dots$. Also, $q_{k+1} = \bar{Q}_k - \bar{Q}_{k+1}$, $k = 0, 1, 2, ...$, then, the probability function of the device is given by

$$g(t_2) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t_2}(\lambda t_2)^k}{k!} \lambda q_{k+1} \,.$$

The corresponding survival function of this device is given by

$$\bar{G}(t_2) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t_2}(\lambda t_2)^k}{k!} \bar{Q}_k \,.$$

Theorem 2.4. *Let N_1, N_2, T_1 and T_2 be random variables as above. If $N_1 \leq_{\text{lr}} N_2$ then $T_1 \leq_{n\text{-Lt-r}} T_2$.*

Proof: Let us denote $L_{T_i}^{(n)}(s) = L_i^{(n)}(s),\ i = 1, 2$. We have

$$L_i^{(n)}(s) = \sum_{k=0}^{\infty} (-1)^n \frac{(n+k)!}{k!} \frac{\lambda^{k+1}}{(\lambda+s)^{n+k+1}} \, p_{k+1} \,,$$

in which $(-1)^n \frac{(n+k)!}{k!} \frac{\lambda^{k+1}}{(\lambda+s)^{n+k+1}}$ is $\text{RR}_2(s,k)$ and p_{k+1} is $\text{TP}_2(k,i)$, so, by Karlin (1968, Lemma 1.1 on p. 99) it follows that $L_i^{(n)}(s)$ is $\text{RR}_2(s,i)$, therefore $\frac{L_2^{(n)}(s)}{L_1^{(n)}(s)}$ is decreasing in $s > 0$, or equivalently, $T_1 \leq_{n\text{-Lt-r}} T_2$. $\square$

3. $\mathcal{L}^{(n)}$-CLASS

The $\mathcal{L}$ ($\bar{\mathcal{L}}$)-class of life distributions for which $\int_0^\infty e^{-st}\bar{F}(t)\,dt \geq (\leq) \int_0^\infty e^{-st}\bar{G}(t)\,dt$, where $\bar{G}(t) = e^{-t/\mu}$, $t \leq 0$ and $\mu = \int_0^\infty \bar{F}(t)\,dt$ has been introduced by Klefsjo (1983). He presented results concerning closure properties under some usual reliability operations and studied some shock models and a certain cumulative damage model. The $\mathcal{L}$-class is strictly larger than the well known HNBUE class (the harmonic new better than used in expectation class of life distributions) in which a life distribution F is said to be HNBUE if $\int_t^\infty \bar{F}(x)\,dx \leq \mu\exp(-\frac{t}{\mu})$ for all $t \geq 0$. The $\mathcal{L}$-class of life distributions has attracted a great deal of attention (for more details see Lin 1998).

3.1. Basic properties of $\mathcal{L}^{(n)}$-class

We will define the class $\mathcal{L}^{(n)}$ ($\bar{\mathcal{L}}^{(n)}$)-class of life distributions based on n^{th} derivative of Laplace transform in the same manner of $\mathcal{L}$ ($\bar{\mathcal{L}}$)-class.

Definition 3.1. Let X be a non-negative random variable with life distribution F, survival function $\bar{F} = 1-F$ and finite mean $\mu = \int_0^\infty \bar{F}(t)\,dt$. We say that the life distributions F belongs to the $\mathcal{L}^{(n)}$-class if

$$\int_0^\infty t^n e^{-st}\bar{F}(t)\,dt \;\geq\; n!\left(\frac{\mu}{1+s\,\mu}\right)^{n+1}, \qquad \text{for} \quad s \geq 0\,. \tag{3.1}$$

If the reversed inequality holds we shall say that F belongs to the $\bar{\mathcal{L}}^{(n)}$-class.

Theorem 3.1.

(**a**) *If $X \in \mathcal{L}^{(n)}$-class and $X \in \bar{\mathcal{L}}^{(n)}$-class, then X has exponential distribution with mean μ.*

(**b**) *$\mathcal{L}^{(n)} \subset \mathcal{L}^{(n-1)}$ for all $n = 1, 2, \ldots$.*

(**c**) *$\mathcal{L}^{(n)} = \mathcal{L}_0^{(n)} \cup \mathcal{L}_+^{(n)}$, in which $\mathcal{L}_+^{(n)}$ denote the class of all distributions F having support $S_F \subset (0,\infty)$, mean $\mu < \infty$, and satisfying the relation (3.1). Also, denote by $\mathcal{L}_0^{(n)} = \{F_0\}$, where F_0 is the degenerate at 0.*

(**d**) *$F \in \mathcal{L}^{(n)}$ if and only if $\int_0^\infty t^n e^{-st} f(t)\,dt \leq n!\,\frac{\mu^n}{(1+s\mu)^{n+1}}$.*

Proof: (**a**) By assumptions

$$\int_0^\infty t^n e^{-st}\big(\bar{F}(t) - e^{-t/\mu}\big)\,dt \;=\; 0\,. \tag{3.2}$$

Due to statistical completeness property of the exponential distribution, it follows that $\bar{F}(t) = e^{-t/\mu}$, $\forall\, t \geq 0$.

(**b**) By equation (3.1), the random variable X belongs to $\mathcal{L}^{(n)}$-class if

$$\int_0^\infty t^n e^{-st}\big(\bar{F}(t) - e^{-t/\mu}\big)\, dt \,\geq\, 0\,,$$

that gives

$$\int_x^\infty \int_0^\infty t^n e^{-st}\big(\bar{F}(t) - e^{-t/\mu}\big)\, dt\, ds \,\geq\, 0\,,$$

so

$$\int_0^\infty t^{n-1}\big(\bar{F}(t) - e^{-t/\mu}\big) \int_x^\infty t\, e^{-st}\, ds\, dt \,\geq\, 0\,,$$

therefore

$$\int_0^\infty t^{n-1} e^{-xt}\big(\bar{F}(t) - e^{-t/\mu}\big)\, dt \,\geq\, 0\,,$$

which means $X \in \mathcal{L}^{(n-1)}$-class.

(**c**) Using (3.1) we conclude that for all $s \geq 0$

$$n!\left(\frac{\mu}{1+s\,\mu}\right)^{n+1} \leq \big(1 - F(0)\big) \int_0^\infty t^n e^{-st}\, dt\,,$$

from which we get

$$\left(\frac{\mu}{1+s\,\mu}\right)^{n+1} \leq \frac{1}{s^{n+1}}\big(1 - F(0)\big)\,,$$

hence, $\mu^{n+1} \leq \big(1 - F(0)\big)\big(\mu + \frac{1}{s}\big)^{n+1}$. Letting $s \to \infty$ yields $\mu^{n+1} F(0) \leq 0$, and with similar discuss to Lin (1998) obtain the result.

(**d**) Using $L_X^{(n)}(s) = n\, {L_X^*}^{(n-1)}(s) - s\, {L_X^*}^{(n)}(s)$ for all $n = 1, 2, 3, ...$, the desired result follows. □

Theorem 3.2. *If X has distribution function F and Y has exponential distribution with mean μ, such that $E(X^n) = E(Y^n)$, then, $Y \leq_{n\text{-Lt}^*\text{-r}} X$ implies that $X \in \mathcal{L}^{(n)}$.*

Proof: Note that $Y \leq_{n\text{-Lt}^*\text{-r}} X$ so, $\frac{{L_Y^*}^{(n)}(s)}{{L_X^*}^{(n)}(s)}$ is increasing in $s \geq 0$. Hence,

$$\frac{\displaystyle\int_0^\infty t^n e^{-t(s+\frac{1}{\mu})}\, dt}{\displaystyle\int_0^\infty t^n e^{-st} \bar{F}(t)\, dt} \geq \lim_{s \to 0} \frac{\displaystyle\int_0^\infty t^n e^{-t(s+\frac{1}{\mu})}\, dt}{\displaystyle\int_0^\infty t^n e^{-st} \bar{F}(t)\, dt} = 1\,,$$

which means X is in $\mathcal{L}^{(n)}$. □

We are going to give another interesting characterization of the $\mathcal{L}_+^{(n)}$-class through the equilibrium transformation, which will be used to estimate the moments of $F \in \mathcal{L}_+^{(n)}$ and to characterize the exponential distribution. Let X be a non-negative random variable with distribution function F and finite mean $\mu > 0$. Then the equilibrium transformation F_e of F is defined by

$$F_e(x) = \frac{1}{\mu}\int_0^x \bar{F}(t)\,dt \qquad \text{for} \quad x \geq 0\,.$$

The distribution F_e is known by the names equilibrium distribution and let the random variable with distribution function F_e is denoted by X_e.

Theorem 3.3. *Let X be a positive random variable with distribution function F and finite mean $\mu > 0$. If $X \in \mathcal{L}_+^{(n)}$ and $E(X^n e^{-sX}) \geq E(X_e^n e^{-sX_e})$ then $X \in \mathcal{L}_+^{(n-1)}$.*

Proof: Note that

$$\begin{aligned}
E(X^n e^{-sX}) \;&\geq\; E(X_e^n e^{-sX_e}) \iff \\
\iff\;& \frac{1}{\mu}\int_0^\infty t^n e^{-st}\bar{F}(t)\,dt \;\leq\; \int_0^\infty t^n e^{-st} f(t)\,dt \\
\iff\;& \frac{1}{\mu}\int_0^\infty t^n e^{-st}\bar{F}(t)\,dt \;\leq\; n\int_0^\infty t^{n-1}e^{-st}\bar{F}(t)\,dt - s\int_0^\infty t^n e^{-st}\bar{F}(t)\,dt \\
\iff\;& \frac{1+s\mu}{\mu}\int_0^\infty t^n e^{-st}\bar{F}(t)\,dt \;\leq\; n\int_0^\infty t^{n-1}e^{-st}\bar{F}(t)\,dt\,,
\end{aligned}$$

since $X \in \mathcal{L}_+^{(n)}$ then $\int_0^\infty t^{n-1}e^{-st}\bar{F}(t) \geq (n-1)!\,(\frac{\mu}{1+s\mu})^n$ that means $X \in \mathcal{L}_+^{(n-1)}$. $\square$

Block and Savits (1980) considered

$$a_n(s) = \frac{(-1)^n}{(n)!}\,L_X^{*\,(n)}(s)\,, \qquad n = 0, 1, 2, \ldots\,, \quad s > 0\,,$$

and set $\alpha_{n+1}(s) = s^{n+1}a_n(s)$ for $n = 0, 1, 2, \ldots$, $s > 0$. So, $X \in \mathcal{L}^{(n)}$ if and only if

$$\alpha_{n+1}(s) \;\geq\; \left(\frac{s\mu}{1+s\mu}\right)^{n+1}.$$

Block and Savits (1980) supposed that $\{N_s(t),\ t \geq 0\}$ be a Poisson process with rate $s > 0$. They showed, if X is a random variable with survival function $\bar{F}(u)$, then

$$\begin{aligned}
\alpha_{n+1}(s) \;&=\; s\int_0^\infty \frac{e^{-su}(su)^n}{n!}\,\bar{F}(u)\,du \\
&=\; s\int_0^\infty P\{N_s(u) = n\}\,\bar{F}(u)\,du \\
&=\; s\int_0^\infty P\{N_s(u) > n\}\,dF(u) \\
&=\; P\{N_s(X) > n\}\,.
\end{aligned}$$

Furthermore, if $Y_1, Y_2, ...$ are the (exponential) arrival times for the process, then

$$\alpha_{n+1}(s) = P\left(\sum_{i=1}^{n+1} Y_i \le X\right) = \int_0^\infty G^{(n+1)}(u)\, dF(u)\,, \tag{3.3}$$

where $\bar{G}(u) = \exp(-su)$, $u \ge 0$. Thus (3.3) shows that the $\{\alpha_n(s),\ n \ge 1\}$ are the discrete survival probabilities for a special case of the random threshold cumulative damage model of Esary *et al.* (1973).

3.2. $\mathcal{L}^{(n)}$ ($\bar{\mathcal{L}}^{(n)}$) class for discrete life distributions

Let ξ be a strictly positive integer valued random variable and denote $\bar{P}_k = P(\xi > k)$, $k = 0, 1, 2, ...$, the corresponding survival probabilities. Also, suppose that $1 = \bar{Q}_0 \ge \bar{Q}_1 \ge \bar{Q}_2 \ge ...$ denote the corresponding survival probabilities of a geometric distribution with mean

$$\mu = \sum_{k=0}^{\infty} \bar{Q}_k = \sum_{k=0}^{\infty} \bar{P}_k\,,$$

that is,

$$\bar{Q}_k = (1 - 1/\mu)^k\,, \qquad k = 0, 1, 2, ...$$

Since the discrete counterpart to Laplace transform is the probability generating function, we consider the following natural definition:

Definition 3.2. A discrete life distribution and its survival probabilities $\bar{P}_k$, $k = 0, 1, 2, ...$, with finite mean $\sum\limits_{k=0}^{\infty} \bar{P}_k = \mu$ are in $\mathcal{L}^{(n)}$ ($\bar{\mathcal{L}}^{(n)}$) class if

$$\sum_{k=n}^{\infty} \frac{k!}{(k-n)!}\, \bar{P}_k\, p^{k-n} \ge (\le)\ \frac{n!\,\mu(\mu-1)^n}{\big(p + (1-p)\,\mu\big)^{n+1}}\,, \qquad \text{for} \quad 0 \le p \le 1\,.$$

Example 3.1. Let $\bar{P}_0 = 1$, $\bar{P}_1 = \frac{1}{2}$, $\bar{P}_2 = \frac{1}{4}$, $\bar{P}_3 = \frac{1}{5}$ and $\bar{P}_k = 0$ for $k = 4, 5, 6, ...$ Then $\bar{P}_k$ is belong to $\mathcal{L}$-class but is not belong to $\mathcal{L}^{(1)}$.

Preservation of $\mathcal{L}^{(n)}$ and $\bar{\mathcal{L}}^{(n)}$ classes under mixture and convolutions are studied as follow:

Mixtures:

Let $\{F_\Theta\}$ be a family of life distribution, where Θ is random variable with distribution $H(\theta)$, the mixture F of F_Θ according to H is $F(t) = \int F_\Theta(t)\, dH(\theta)$. If each F_Θ is an exponential distribution and therefore DFR, then F is DFR that follows F not in $\mathcal{L}^{(n)}$.

Consider $\bar{G}(t) = \int \bar{G}_\Theta(t)\, dH(\theta)$, $t \geq 0$, where $\bar{G}_\Theta(t) = \exp(-t/\mu_\Theta)$ in which $\mu_\Theta = \int_0^\infty \bar{F}_\Theta(t)\, dt$ and $\mu = \int_0^\infty \bar{F}(t)\, dt$. If every F_Θ is in $\bar{\mathcal{L}}^{(n)}$ so,

$$\begin{aligned}\int_0^\infty t^n e^{-st}\big(\bar{G}(t) - \bar{F}(t)\big)\, dt &= \int_0^\infty t^n e^{-st} \int_\theta \big(\bar{G}_\Theta(t) - \bar{F}_\Theta(t)\big)\, dH(\theta)\, dt \\ &= \int_\theta \int_0^\infty t^n e^{-st}\big(\bar{G}_\Theta(t) - \bar{F}_\Theta(t)\big)\, dt\, dH(\theta) \;\geq\; 0\,.\end{aligned}$$

Convolutions:

Let θ_1 and θ_2 be two independent random variables with life distributions F_1 and F_2 with means μ_1 and μ_2, respectively, belonging to $\mathcal{L}^{(n)}$ class. If $\bar{G}_1(t) = \exp(-t/\mu_1)$, $t \geq 0$, and $\bar{G}_2(t) = \exp(-t/\mu_2)$, $t \geq 0$, then $\theta_1 + \theta_2$, that has life distribution $F_1 * F_2$, belongs to $\mathcal{L}^{(n)}$, too. With a similar argument to Klefsjo (1983), by using properties of the Laplace transform of convolutions we get

$$\int_0^\infty t^n e^{-st}\, \overline{F_1 * F_2}(t)\, dt \;\geq\; \int_0^\infty t^n e^{-st}\, \overline{G_1 * G_2}(t)\, dt\,,$$

due to the fact $G_1 * G_2$ is IFR, it follows that $\theta_1 + \theta_2$ is in $\mathcal{L}^{(n)}$ class. Note that since $G_1 * G_2$ is IFR then it also follows that $\bar{\mathcal{L}}^{(n)}$ class is not closed under convolutions.

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