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GENERALIZED LEAST SQUARES AND WEIGHTED LEAST SQUARES ESTIMATION METHODS FOR DISTRIBUTIONAL PARAMETERS

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Abstract:

- Regression procedures are often used for estimating distributional parameters because of their computational simplicity and useful graphical presentation. However, the resulting regression model may have heteroscedasticity and/or correction problems and thus, weighted least squares estimation or alternative estimation methods should be used. In this study, we consider generalized least squares and weighted least squares estimation methods, based on an easily calculated approximation of the covariance matrix, for distributional parameters. The considered estimation methods are then applied to the estimation of parameters of different distributions, such as Weibull, log-logistic and Pareto. The results of the Monte Carlo simulation show that the generalized least squares method for the shape parameter of the considered distributions provides for most cases better performance than the maximum likelihood, least-squares and some alternative estimation methods. Certain real life examples are provided to further demonstrate the performance of the considered generalized least squares estimation method.

Key-Words:

- *probability plot; heteroscedasticity; autocorrelation; generalized least squares; weighted least squares; shape parameter.*

1. INTRODUCTION

Regression procedures are often used for estimating distributional parameters. In this procedure, the distribution function is transformed to a linear regression model. Thus, least squares (LS) estimation and other regression estimation methods can be employed to estimate parameters of a specified distribution. In the literature, the parameters of the Pareto and Weibull distributions, in particular, have been estimated by such methods, since these distributions have been commonly used in reliability and survival analysis as well as engineering (Abernethy, 1996; Boudt *et al.* 2011; Kantar and Usta, 2008; Genschel and Meeker, 2010; Hung, 2001; Lu *et al.* 2004; Baxter 1980). In addition to the Pareto and Weibull distributions, Burr-type, Gumbel, logistic and log-logistic distributions have been studied by regression estimation methods (Bergman, 1986; Hossain and Howlader, 1996; Wang and Cheng, 2010; Zhang *et al.* 2007; Zhang *et al.* 2008; Zyl, 2012; Zyl and Schall, 2012; Usta, 2013; Kantar and Arik, 2014; Kantar and Yildirim, 2015). One of the main advantages of using regression procedures for estimating parameters is that their implementation is simple in the case of complete data, censoring data or data with outliers. Nevertheless, as is well known, the resulting regression model may have unequal variance (heteroscedastic) (Engeman and Keefe, 1982; Boudt *et al.* 2011; Zyl, 2012; Zyl and Schall, 2012) and/or correction problems (Engeman and Keefe, 1982) and thus, the weighted least squares (WLS) estimation or alternative methods should be used (Engeman and Keefe, 1982; Lu and Tao, 2007; Zyl, 2012; Zyl and Schall, 2012). For example, Engeman and Keefe (1982) consider generalized least squares estimation of the Weibull distribution by means of a linear regression model. Hung (2001), Lu *et al.* (2004), Zyl and Schall (2012) emphasize that a weight function should be used when performing regression methods, and propose different weights using large sample properties of the empirical distribution function or order statistics, to stabilize the variance in order to perform the WLS estimation method for the Weibull distribution. Zhang *et al.* (2008) discuss alternative WLS estimation methods for the Weibull distribution. On the other hand, Malik (1970) studied the LS method, ridge regression and maximum product of spacing methods to estimate parameters for the Pareto distribution, while Zyl (2012) considered the Laplace distributed errors (LAD) (Koenker and Bassett, 1978) and Box–Cox regression to stabilize variance. It can be seen that Zyl's (2012) WLS and Lu and Tao' (2007) WLS perform almost as well as the maximum likelihood (ML) estimation method for the Pareto distribution.

In this article, we consider generalized least squares (GLS) and WLS estimation methods for distributional parameters by easily calculating an approximation of the variance-covariance matrix. GLS and WLS are then applied to the estimation of the parameters of the Weibull, Pareto and log-logistic distributions. The simulation results show that the proposed estimation methods, particularly

GLS for the shape parameter of the considered distributions, provide better performance than the ML, LS and some alternative WLS estimation methods, for most of the considered sample sizes.

The rest of this paper is organized as follows: Section 2 provides the process of estimation of distributional parameters via regression models for the Weibull, Pareto and log-logistic distributions. Section 3 introduces the GLS and WLS estimation methods and their application to each of these distributions. Alternative estimation methods for the Weibull, Pareto or log-logistic distributions are briefly discussed in Section 4. To show the performance of the considered GLS and WLS methods, a simulation study is presented in Section 5. A number of real-data examples are discussed in Section 6 and, finally, the last section summarizes the conclusions of the study.

2. ESTIMATION OF DISTRIBUTIONAL PARAMETERS VIA REGRESSION MODELS

Probability plots that use the quantile function of the random variable are often used for different objectives, such as, (i) to draw conclusions from data, (ii) to estimate the parameters of the considered distribution, (iii) to apply to both complete and censored data and (iv) to show graphical presentation. (For other advantages, see Nelson, 2004)

By taking into account the objective (ii) of probability plots, the distribution function is transformed into a linear regression model, so that various regression estimation methods can then be used to estimate the parameters of the specified distribution.

The probability density function (pdf) and cumulative distribution function (cdf) of the Weibull random variable are respectively given in the following equations:

$$f(x,\lambda,\alpha) \,=\, \frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-(\frac{x}{\lambda})^{\alpha}}, \qquad \text{for} \quad x>0\ , \tag{2.1}$$

$$F(x,\lambda,\alpha) \,=\, 1-e^{-(\frac{x}{\lambda})^{\alpha}}, \qquad \text{for} \quad x>0\ , \tag{2.2}$$

where λ is the scale parameter and α is the shape parameter. The Weibull distribution is a reversed J-shaped, bell shaped and exponential distribution for $\alpha<1$, $\alpha>1$ and $\alpha=1$, respectively. The Weibull distribution appears similar to a normal distribution for $\alpha=3.4$ (Kantar and Senoglu, 2008).

After some algebraic manipulation, equation (2.2) can be expressed as follows:

$$\ln\big[-\ln(1-F(x))\big] = \alpha \ln x - \alpha \ln \lambda \ , \tag{2.3}$$

$$\ln x = \ln \lambda + \frac{1}{\alpha} \ln\big[-\ln(1-F(x))\big] \ . \tag{2.4}$$

For a sample size of n and $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ the regression model is rewritten as:

$$\ln x_{(i)} = \ln \lambda + \frac{1}{\alpha} \ln\big[-\ln(1-F(x_{(i)}))\big] \ , \tag{2.5}$$

where $\ln x_{(i)}$ is the ith order statistics of the logarithm of the sample from the Weibull distribution. $\frac{i-a}{(n+b)}, (0 \leq a \leq 0.5, 0 \leq b \leq 1)$ is used as estimate of $F(x_{(i)})$ where i is the rank of the data point in the sample in ascending order. For complete samples, $\frac{i}{(n+1)}$ and $\frac{i-0.3}{(n+0.4)}$ are generally used (Tiryakioglu and Hudak, 2007; Zyl, 2012, Kantar and Yildirim, 2015).

If we replace $\ln x_{(i)}$ with Y_i, $\ln \lambda$ with β_0, $\frac{1}{\alpha}$ with β_1 and $\ln[-\ln(1-F(x_{(i)}))]$ with X_i, the regression model (2.5) occurs as:

$$Y_i = \beta_0 + \beta_1 X_i \ . \tag{2.6}$$

By using the regression model given in (2.5), the LS and other regression estimation methods can be easily employed to estimate the parameters of the Weibull distribution.

The Pareto cdf is given as follows:

$$F(x, k, \eta) = 1 - \left(\frac{k}{x}\right)^{\eta} , \tag{2.7}$$

where k is the scale parameter and η is the shape parameter. The Pareto distribution, which is generally used to model extreme values, is skewed and heavy-tailed.

After algebraic manipulation, equation (2.7) can be expressed as follows:

$$\ln x = \ln k - \frac{1}{\eta} \ln(1-F(x)) \ . \tag{2.8}$$

For the ordered sample, the regression model for the Pareto distribution is rewritten as:

$$\ln x_{(i)} = \ln k - \frac{1}{\eta} \ln(1-F(x_{(i)})) \ . \tag{2.9}$$

If we replace $\ln x_{(i)}$ with Y_i, $\ln k$ with β_0, $-\frac{1}{\eta}$ with β_1 and $\ln(1-F(x_{(i)}))$ with X_i, the linear regression model is obtained for the Pareto distribution.

The cdf of the log-logistic random variable is given as follows:

$$F(x) = 1 - \left(1 + \left(\frac{x}{\gamma}\right)^{\delta}\right)^{-1}, \qquad x \geq 0\,, \quad \delta, \gamma > 0\,, \tag{2.10}$$

where γ is the scale parameter and δ is the shape parameter. For $\delta > 1$, the log-logistic distribution is unimodal and its variance decreases as δ increases. The log-logistic distribution has been widely-used in hydrology to model stream flow (Ahmad *et al.* 1988; Ashkar and Mahdi, 2006; Chen, 2006).

Similar to the Weibull and Pareto distributions, the obtained regression model for the log-logistic distribution is presented as follows:

$$\ln(x) \;=\; \frac{1}{\delta} \ln\!\big((1 - F(x))^{-1} - 1\big) + \ln(\gamma)\,, \tag{2.11}$$

which may be written as:

$$\ln(x_{(i)}) \;=\; \frac{1}{\delta} \ln\!\big((1 - F(x_{(i)}))^{-1} - 1\big) + \ln(\gamma)\,. \tag{2.12}$$

In conclusion, the parameters of the Weibull, Pareto and log-logistic distributions can be estimated respectively using ordinary LS estimation for equations (2.5), (2.9) and (2.12).

It should be noted that the common LS and other regression estimation procedures applied in the literature for the Weibull and Pareto distributions use the least squares regression of X on Y, $(X_i = \beta_0 + \beta_1 Y_i)$ (Genschel and Meeker, 2010; Hossain and Howlader, 1996; Hung, 2001; Kantar and Arik, 2014; Kantar and Yildirim, 2015; Lu *et al.* 2004; Wang and Cheng, 2010; Zhang *et al.* 2008; Zyl, 2012; Zyl and Schall, 2012). To the best of our knowledge, only Zhang *et al.* (2007) compare these two LS estimation methods for the Weibull using intensive Monte Carlo simulations, finding that LS of Y on X provides better estimators than LS of X on Y.

3. GENERALIZED LEAST SQUARES ESTIMATION AND WEIGHTED LEAST SQUARES ESTIMATION METHODS FOR DISTRIBUTIONAL PARAMETERS: THE CASES OF WEIBULL, PARETO AND LOG-LOGISTIC DISTRIBUTIONS

The most obvious point to be noticed is that since the sample is ordered in the models (2.5), (2.9) and (2.12), $\ln x_{(i)}$ is also ordered. For this reason, the covariance matrices of the dependent variable of these models are not in the form $\sigma^2\mathbf{I}$, but of $\sigma^2\mathbf{V} = \sum$, where σ^2 is unknown and $\mathbf{V}$ is known (White, 1969; Engeman and Keefe, 1982). In this case, the LS estimates of the coefficients may not have minimum variance. In such cases, alternative estimation approaches to stabilize variances can be used.

Generalized least squares (GLS) estimation is an efficient method for estimating the unknown coefficients of a linear regression model when the observations have unequal variance and there is a certain degree of correlation between the observations. In the linear regression model given in (2.6), if the form of the variance of $\mathbf{Y} = (Y_1, ..., Y_n)$ is $\sigma^2\mathbf{V} = \sum$, GLS minimizes

$$(\mathbf{Y} - \mathbf{X}\beta)'\,\mathbf{V^{-1}}(\mathbf{Y} - \mathbf{X}\beta)\ , \tag{3.1}$$

which is solved by

$$\hat{\beta}_{GLS} = (\mathbf{X'V^{-1}X})\,\mathbf{X'}\,\mathbf{V^{-1}Y}\ , \tag{3.2}$$

where $\hat{\beta}_{\mathbf{GLS}}$ is the vector of the GLS estimates of $\beta = (\beta_1, \beta_2)$ and $\mathbf{X}$ is the matrix of ones and x_i. In addition, the GLS estimates are equivalent to applying ordinary LS to a linearly transformed form of the data. That is, we can write $\mathbf{V} = \mathbf{SS'}$, where $\mathbf{S}$ is a triangular matrix, using Cholesky decomposition. The LS estimates obtained by regressing $\mathbf{S^{-1}Y}$ on $\mathbf{S^{-1}X}$ are equal to the GLS estimates. Thus, $\mathbf{Var}(\mathbf{S^{-1}Y}) = \mathbf{S^{-1}Var(Y)(S^{-1})'} = \sigma^\mathbf{2}\mathbf{S^{-1}V(S^{-1})'} = \sigma^\mathbf{2}\mathbf{I}$. The transformed form of the data is uncorrelated, with constant variance.

On occasion, the observations are uncorrelated or have a small enough correlation to be ignored, but have unequal variance. That is, the covariance-matrix is diagonal, say $\mathbf{W}$, but does not have equal diagonal elements. WLS estimation can be used in this situation. WLS estimate is obtained as follows:

$$\hat{\beta}_{WLS} = (\mathbf{X'W^{-1}X})\,\mathbf{X'}\,\mathbf{W^{-1}Y}\ . \tag{3.3}$$

Now, the problem is to estimate the $\mathbf{V}$ matrix for the considered distributions. Taking into account equation (2.5) for the Weibull distribution, it is noted that the cumulative function F is transformed into $\ln[-\ln(1 - F(x))]$ and the random variable X is transformed into $\ln X$. If these transformations are respectively denoted by TF and TX, the regression model for the Weibull distribution given in (2.5) can be expressed as follows:

$$TX(X_i) = \beta_0 + \beta_1 TF(F_i)\ , \tag{3.4}$$

where $\beta_0 = \ln\lambda$ and $\beta_1 = \frac{1}{\alpha}$. Taking the expectation, variance and covariance of both sides yields:

$$E(TX(X_i)) = \beta_0 + \beta_1 E(TF(F_i))\ , \tag{3.5}$$

$$Var(TX(X_i)) = \beta_1^2 Var(TF(F_i))\ , \tag{3.6}$$

$$Cov(TX(X_i), TX(X_j)) = \beta_1^2 Cov(TF(F_i), TF(F_j))\ . \tag{3.7}$$

By using Taylor expansion, it is possible to approximate the expectation of the observation and also variance and covariance between the observations. Taylor series expansion of $TF(F_i)$ about the value $F0_i = \frac{i}{n+1}$, $i = 1, 2, ..., n$, is given by:

$$TF(F_i) \approx TF(F0_i) + TF'(F0_i)(F - F0_i) + \frac{1}{2}TF''(F0_i)(F - F0_i)^2 \; . \tag{3.8}$$

Taking the expectation of both sides yields:

$$E(TF(F_i)) \approx E(TF(F0_i)) + \frac{1}{2}TF''(F0_i)E((F - F0_i)^2) \; . \tag{3.9}$$

Similarly, taking the variance of both sides yields:

$$Var(TF(F_i)) \approx (TF'(F0_i))^2 Var(F) = (TF'(F0_i))^2 \frac{F0_i(1 - F0_i)}{n+2} \tag{3.10}$$

A similar closed-form approximate formula for the covariance between $TF(F_i)$ and $TF(F_j)$ is

$$Cov(TF(F_i), TF(F_j)) \approx TF'(F0_i)TF'(F0_j)\frac{F0_i(1 - F0_j)}{n+2}, \quad i < j \; . \tag{3.11}$$

(See Blom, 1962; White, 1969; Engeman and Keefe, 1982). Thus,

$$E(TX(X_i)) \;\approx\; \beta_0 + \beta_1(TF(F0_i)) + \frac{1}{2}TF''(F0_i)\frac{F0_i(1 - F0_j)}{n+2} \; , \tag{3.12}$$

$$Var(TX(X_i)) \;\approx\; \beta_1^2(TF'(F0_i))^2 \frac{F0_i(1 - F0_i)}{n+2} \; , \tag{3.13}$$

$$Cov(TX(X_i), TX(X_j)) \;\approx\; \beta_1^2 TF'(F0_i)TF'(F0_j)\frac{F0_i(1 - F0_j)}{n+2}, \quad i < j \; , \tag{3.14}$$

where $TF(F) = \ln(-\ln(1 - F))$, $TF'(F) = \frac{-1}{(1-F)\ln(1-F)}$ and $TF''(F) = \frac{\ln(1-F)+1}{((1-F)\ln(1-F))^2}$

$$Var(TX(X_i)) = Cov(TX(X_i), TX(X_j)) \approx \frac{\beta_1^2}{n+2}\frac{i}{(n+1-i)}\frac{1}{\ln(\frac{n+1-i}{n+1})}\frac{1}{\ln(\frac{n+1-j}{n+1})} \; . \tag{3.15}$$

Thus, considering $\frac{\beta_1^2}{n+2}$ as $\sigma^2\mathbf{V}$ as in Engeman and Keefe (1982), the approximate formula for the $\mathbf{V}$ matrix can be expressed as follows:

$$v_{ij} = \frac{i}{(n+1-i)}\frac{1}{\ln(\frac{n+1-i}{n+1})}\frac{1}{\ln(\frac{n+1-j}{n+1})} \; , \qquad i \leq j \; , \tag{3.16}$$

where v_{ij} is an element of the $\mathbf{V}$ matrix. Thus, in order to apply GLS and WLS estimation methods for the Weibull distribution, the covariance matrix and the matrix of the diagonal elements of the covariance matrix are expressed respectively as follows:

$$\mathbf{V} = \begin{pmatrix} v_{11} & ... & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & ... & v_{nn} \end{pmatrix} , \tag{3.17}$$

$$\mathbf{W} = \begin{pmatrix} v_{11} & ... & 0 \\ \vdots & \ddots & \vdots \\ 0 & ... & v_{nn} \end{pmatrix} . \tag{3.18}$$

For the Weibull distribution:

$$\hat{\beta}_{GLS1} = (\mathbf{X'V^{-1}X})^{-1}\mathbf{X'V^{-1}Y} \ ,$$
$$\hat{\beta}_{WLS} = (\mathbf{X'W^{-1}X})^{-1}\mathbf{X'W^{-1}Y} \ ,$$

where $\mathbf{Y} = [\ln(x_{(1)}), ..., \ln(x_{(n)})]$, $\mathbf{X} = \begin{pmatrix} 1 & \ln(-\ln(1-\hat{F}_1)) \\ \vdots & \vdots \\ 1 & \ln(-\ln(1-\hat{F}_n)) \end{pmatrix}$, $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1)$, where $\hat{\beta}_0 = \ln\hat{\lambda}$ and $\hat{\beta}_1 = \frac{1}{\hat{\alpha}}$, $\hat{\lambda} = \exp(\hat{\beta}_0)$ and $\hat{\alpha} = \frac{1}{\hat{\beta}_1}$.

It should be noted that independent variables in equations (2.5), (2.9) and (2.12) are unobserved, but are estimated differently from classical regression analysis. In order to estimate regression coefficients, we replaced the independent variable with its estimate.

Considering the expected value of $E(TF(F_i))$ given in (3.9) and (3.12), we can choose the independent variable as $TF(F0_i) + \frac{1}{2}TF''(F0_i)Var(F)$ to reduce bias in the GLS procedure. Nevertheless, unbiasedness of the resulting estimator is not claimed, since $TX(X_i)$, as the dependent variable, has an estimate of mean. Consequently, the considered generalized estimation procedure here is denoted by GLS2 for the purpose of distinction.

Thereby, the design matrix of the regression model for the Weibull distribution is given as follows:

$$\mathbf{Z} = \begin{pmatrix} 1 & \ln(-\ln(1-\hat{F}_1)) - 0.5 - \frac{\ln(1-\hat{F}_1)+1}{((1-\hat{F}_1)\ln(1-\hat{F}_1))^2} \\ \vdots & \vdots \\ 1 & \ln(-\ln(1-\hat{F}_n)) - 0.5 - \frac{\ln(1-\hat{F}_n)+1}{((1-\hat{F}_n)\ln(1-\hat{F}_n))^2} \end{pmatrix} . \tag{3.19}$$

Consequently,
$$\hat{\beta}_{GLS2} = (\mathbf{Z'V^{-1}Z})^{-1}\mathbf{Z'V^{-1}Y} \ .$$

Taken into account the model given in (2.9) for the Pareto distribution, $TF(F) = \ln(1-F)$, $TF'(F) = \frac{-1}{1-F}$ and $TF''(F) = \frac{-1}{(1-F)^2}$

The approximate formula for the $\mathbf{V}$ matrix for the Pareto distribution can be expressed as follows:

$$v_{ij} = \frac{1}{(n+1-i)} \ , \qquad i \leq j \ . \tag{3.20}$$

Similarly, the $\mathbf{V}$ matrix for the log-logistic distribution is:

$$v_{ij} = \frac{(n+1)^2}{i(n+1-i)} \ , \qquad i \leq j \ . \tag{3.21}$$

where $TF(F) = \ln(\frac{F}{1-F})$, $TF'(F) = \frac{1}{1-F}$ and $TF''(F) = \frac{2F-1}{(F(1-F))^2}$.

Similar to the Weibull distribution, WSL, GLS1 and GLS2 estimation methods can be applied to estimate the parameters of the Pareto and log-logistic distributions using the approximate covariance matrices. It should also be highlighted that the obtained covariance matrix and the proposed GLS1 estimation method coincide with GLS for the Weibull studied in (Engeman and Keefe, 1982). However, Engeman and Keefe (1982) compare GLS estimation with ML for sample size $n = 25$ and find that GLS for the shape parameter of the Weibull distribution performs better than ML estimation. In this study, we compare GLS1 for the Weibull distribution with existing alternative WLS estimation methods for different sample sizes and shape parameter cases.

In conclusion, the considered WLS, GLS1 and GLS2 estimation methods are based on explicit functions of the sample observations and are therefore easy to compute, without the typical computational complexity of ML (Kantar and Senoglu, 2008; Gebizlioglu *et al.* 2011). Also, the standard error of the WLS, GLS1 and GLS2 estimates are easily calculated taking the square roots of the diagonal elements of $(\mathbf{X'W^{-1}X})^{-1}\sigma^2$, $(\mathbf{X'V^{-1}X})^{-1}\sigma^2$ and $(\mathbf{Z'V^{-1}Z})^{-1}\sigma^2$ respectively.

4. DISCUSSION OF OTHER ALTERNATIVE ESTIMATION METHODS

Many estimators have been proposed in the literature for the parameters of the Weibull distribution, and these estimators have been compared according to different criteria (Bergman, 1986; Gebizlioglu *et al.* 2011; Hassanein 1971; Hossain and Zimmer 2003; Hossain and Howlader, 1996; Hung, 2001; Marks, 2005; Kantar and Senoglu, 2008; Kantar and Usta, 2008; Zhang *et al.* 2008; Prakash and Singh, 2009; Zyl and Schall, 2012). The ML estimator, generally preferred due to its good theoretical properties for large sample sizes ($n > 100$), (Kantar and Senoglu, 2008), may have poor small sample performance (Kantar and Senoglu, 2008; Marks, 2005; Teimouri and S. Nadarajah, 2012). Moreover, ML requires an iterative numerical method for most distributions, such as Newton–Raphson. Among other alternative estimators, the most popular is LS estimation because of its computational simplicity in the case of complete data, censoring data and data with outliers (Genschel and Meeker, 2010; Hossain and Zimmer, 2003; Hung, 2001; Lu *et al.* 2004; Zhang *et al.* 2007). However, it is known that LS estimates for distributional parameters may give misleading inferences since there is a heteroscedasticity or correlation problem. With this in mind, GLS (Engeman and Keefe, 1982) and WLS are proposed in (Hung, 2001; Lu *et al.* 2004; Zyl and Schall, 2012), demonstrating that WLS totally outperforms the LS method. Also, Zyl and Schall's (2012) WLS performs almost as well as the ML estimation.

The Pareto distribution has been widely-studied in the literature (Quandt, 1966; Saksena and Johnson, 1984; Likes, 1969; Baxter, 1980). Since ML of its shape parameter is biased (Baxter, 1980; Saksena and Johnson, 1984), Malik (1970) studied the LS, ridge regression and maximum product of spacing methods to estimate the parameters for the Pareto distribution. Hossain and Zimmer (2000) consider LS for the Pareto distribution, showing the superiority of LS estimation over ML estimation. In addition, Lu and Tao (2007) provide a new weighting function for WLS, in order to estimate the parameters of the Pareto distribution. They show that their WLS method demonstrates better performance than classical LS estimation for the Pareto distribution.

On the other hand, the ML, LS, moment, generalized moment and probability weighted moment estimators are considered as estimation methods for log-logistic distribution in the literature (Ashkar and Mahdi, 2006; Chen, 2006; Kantar and Arik, 2014; Rao and Kantam, 2012). The moment estimator for the log-logistic distribution is not widely-used due to constraints $\delta > 1$ and $\delta > 2$. Among those mentioned, ML is the most-preferred estimation method.

5. MONTE CARLO SIMULATION

This section presents Monte Carlo simulation carried out to compare the performance of the proposed GLS1, GLS2 and WLS in comparison with ML, LS estimation methods, and also certain existing WLS estimation methods for the parameters of Weibull, Pareto and Log-logistic distributions.

Bias and RMSE for parameters are calculated using $20,000$ simulated samples. All computations for the simulation are performed using MATLAB 10.1. We consider sample sizes $n = 10, 20, 30, 50, 100$ and 250. While shape parameters are taken as $0.5, 1, 2, 4$ for the Pareto distribution, for log-logistic and Weibull distributions, shape parameters are taken as $1, 2, 3$ and 6 in common with previous studies. Also, without any loss of generality, the scale parameter is taken to be equal to 1.

Table 1 shows the RMSE and bias values for ML, LS estimation for regression of X on Y (LS1), known as classical LS in the literature, LS estimation for regression of Y on X (LS2), which is considered in this study, WLS (Zyl&Schall), WLS (Hung) and WLS (Lu *et al.*) and the considered WLS, GLS1 and GLS2 in this study, for the shape parameter of the Weibull distribution. From the simulation results presented in Table 1, the following conclusions may be summarized:

According to the RMSE criterion:

- (**a**) GLS1 and GLS2 apparently show better performance than others for most considered sample sizes.
- (**b**) While GLS1 provides less RMSE than others for $n = 10$, GLS1 and GLS2 show similar and best performance for $n = 20, 30, 50, 100$.
- (**c**) WLS estimation performs better than LS1 for all sample sizes and shape parameter cases.
- (**d**) LS2 performs better than LS1 for all considered sample sizes except $n = 10$. The same result is observed in the study of Zhang *et al.*, 2007.

According to bias criterion:

- (**a**) GLS2 is clearly the best estimator in terms of bias. Particularly, a superior performance of GLS2 is observed for $n \geq 20$.
- (**b**) Next to GLS2, LS2, GLS1 and WLS provide similar good performance for $n = 10, 20$ with ML being best for other $n = 50, 100, 250$.

From the simulation results presented in Table 2 for the shape parameter of the Pareto distribution, the following conclusions may be summarized:

According to RMSE:

- (**a**) While the proposed GLS1 estimation shows better performance than ML, LS1, LS2, WLS (Lu&Tao) and ELS (Lu&Tao) for most of the considered sample sizes, GLS2 is the best for $n = 250$. Also, the considered WLS is the best performer next to GLS1 for $n = 10, 20, 30, 100$ with GLS2 presenting the second best performance after GLS1 for $n = 50$.
- (**b**) The considered WLS estimation methods and WLS (Lu and Tao) show similar performance.
- (**c**) ML shows the worst performance compared with others, in terms of RMSE for $n = 10$. This result matches the study of Lu and Tao (2007).
- (**d**) LS2 has smaller RMSE than LS1 for all the considered shape parameter cases and sample sizes except $n = 10$.

Table 1: RMSE and Bias of the estimated shape parameters of the Weibull distribution.

Methods	Shape 1		Shape 2		Shape 3		Shape 6	
	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias
	n=10							
ML	0.38863	−0.16511	0.76759	−0.33705	1.13419	−0.51566	2.31548	−0.98691
LS1	0.31833	0.13005	0.63634	0.26033	0.95289	0.39420	1.90491	0.78612
LS2	0.31990	0.05931	0.63967	0.11783	0.95616	0.18046	1.91233	0.36114
WLS (Z&S)	0.29821	0.13978	0.58720	0.27390	0.88842	0.40626	1.78828	0.84593
WLS (Hung)	0.30320	0.13897	0.60026	0.27159	0.90529	0.40397	1.81904	0.83832
WLS (Lu)	0.29903	0.14035	0.58920	0.27497	0.89115	0.40837	1.79431	0.84926
WLS	0.31800	0.06117	0.63271	0.11755	0.95065	0.17788	1.92669	0.37458
GLS1	0.29000	0.06362	0.56862	0.12068	0.86533	0.17573	1.73207	0.38308
GLS2	0.35385	−0.06119	0.56109	−0.04698	0.89027	−0.16520	1.96762	−0.38306
	n=20							
ML	0.22154	−0.07498	0.43758	−0.14731	0.64809	−0.21834	1.33219	−0.44944
LS1	0.23171	0.10663	0.46301	0.21645	0.69459	0.31827	1.38990	0.63916
LS2	0.21958	0.05772	0.43834	0.11909	0.65791	0.17143	1.31574	0.34487
WLS (Z&S)	0.20379	0.09617	0.40523	0.19618	0.60474	0.29597	1.22168	0.57716
WLS (Hung)	0.21225	0.08519	0.41520	0.17664	0.63069	0.26699	1.26565	0.50690
WLS (Lu)	0.20472	0.09610	0.40707	0.19616	0.60777	0.29605	1.22675	0.57687
WLS	0.21970	0.05956	0.43600	0.12022	0.65285	0.18496	1.31841	0.36573
GLS1	0.19204	0.05495	0.37992	0.11169	0.56485	0.17069	1.15498	0.33209
GLS2	0.19931	−0.02895	0.41816	−0.07534	0.55577	−0.05771	1.13999	−0.07700
	n=30							
ML	0.16694	−0.04893	0.33061	−0.09807	0.50327	−0.14007	0.99289	−0.27909
LS1	0.19433	0.09079	0.38810	0.18201	0.58324	0.27222	1.16341	0.53894
LS2	0.18047	0.05201	0.36069	0.10431	0.54226	0.15547	1.08346	0.30588
WLS (Z&S)	0.16227	0.07152	0.32184	0.14273	0.49135	0.21621	0.97494	0.44393
WLS (Hung)	0.17208	0.05960	0.34207	0.11918	0.52139	0.18016	1.03151	0.37621
WLS (Lu)	0.16303	0.07120	0.32345	0.14211	0.49372	0.21527	0.97947	0.44253
WLS	0.18075	0.05161	0.35854	0.10327	0.54220	0.15931	1.07667	0.33157
GLS1	0.15281	0.04580	0.30270	0.09133	0.46125	0.14013	0.91585	0.28804
GLS2	0.15024	−0.02217	0.30412	−0.01153	0.44822	−0.04302	0.90238	−0.02257
	n=50							
ML	0.11983	−0.02859	0.24544	−0.06272	0.36191	−0.08748	0.72335	−0.16474
LS1	0.15472	0.07106	0.30869	0.14283	0.46362	0.21366	0.92958	0.42794
LS2	0.14211	0.04228	0.28304	0.08543	0.42558	0.12742	0.85239	0.25580
WLS (Z&S)	0.12252	0.04702	0.24491	0.08928	0.36714	0.14028	0.74707	0.28503
WLS (Hung)	0.13334	0.03707	0.26475	0.07007	0.39962	0.10809	0.81009	0.22344
WLS (Lu)	0.12317	0.04661	0.24597	0.08849	0.36893	0.13909	0.75062	0.28238
WLS	0.14114	0.04233	0.28107	0.07777	0.42417	0.12536	0.85600	0.25945
GLS1	0.11502	0.03513	0.23203	0.06517	0.34612	0.10363	0.70692	0.21860
GLS2	0.11894	−0.01230	0.23650	−0.02100	0.34015	−0.03437	0.68911	−0.06076
	n=100							
ML	0.08173	−0.01503	0.16438	−0.02775	0.24102	−0.04263	0.47159	−0.07239
LS1	0.11206	0.04903	0.22417	0.09773	0.33142	0.14372	0.66981	0.29164
LS2	0.10253	0.03039	0.20509	0.06024	0.30433	0.08797	0.61356	0.18014
WLS (Z&S)	0.08532	0.02365	0.17234	0.05307	0.25084	0.07527	0.50474	0.16409
WLS (Hung)	0.09497	0.01702	0.18984	0.04145	0.27751	0.05683	0.56174	0.12774
WLS (Lu)	0.08567	0.02331	0.17300	0.05260	0.25179	0.07440	0.50692	0.16229
WLS	0.10173	0.02789	0.20430	0.06213	0.30194	0.09223	0.60435	0.18702
GLS1	0.08068	0.02223	0.16310	0.04667	0.23956	0.06941	0.47439	0.14998
GLS2	0.08112	−0.00479	0.15633	−0.01075	0.23748	−0.01722	0.47863	−0.03175
	n=250							
ML	0.05067	−0.00450	0.10093	−0.01063	0.15032	−0.01562	0.30313	−0.02655
LS1	0.07090	0.02789	0.14099	0.05622	0.21292	0.08332	0.42111	0.16382
LS2	0.06573	0.01797	0.13096	0.03675	0.19744	0.05341	0.39137	0.10486
WLS (Z&S)	0.05505	0.01139	0.10954	0.02045	0.16313	0.03184	0.32897	0.07036
WLS (Hung)	0.06107	0.00848	0.12296	0.01354	0.18259	0.02266	0.36949	0.05389
WLS (Lu)	0.05514	0.01126	0.10982	0.02018	0.16352	0.03146	0.32979	0.06964
WLS	0.06745	0.01893	0.13191	0.03565	0.19723	0.05478	0.39537	0.11392
GLS1	0.05235	0.01343	0.10281	0.02536	0.15375	0.03858	0.30788	0.08173
GLS2	0.04909	−0.00189	0.09938	−0.00330	0.14755	−0.00684	0.30200	−0.00780

Table 2: RMSE and Bias of the estimated shape parameters of the Pareto distribution.

Methods	Shape 0.5 RMSE	Shape 0.5 Bias	Shape 1 RMSE	Shape 1 Bias	Shape 2 RMSE	Shape 2 Bias	Shape 4 RMSE	Shape 4 Bias
				n=10				
ML	0.26682	−0.12505	0.53541	−0.25050	1.06415	−0.49718	2.13514	−1.00263
LS1	0.20968	0.04864	0.41600	0.09824	0.83662	0.19479	1.66619	0.39608
LS2	0.22352	0.01044	0.44241	0.02185	0.89118	0.04120	1.77565	0.08861
ELS	0.24558	−0.04463	0.49464	−0.09034	0.98164	−0.17592	1.96201	−0.36301
WLS (L&T)	0.19586	−0.00777	0.39277	−0.01558	0.78208	−0.02760	1.56403	−0.06245
WLS	0.19118	−0.00860	0.38300	−0.01950	0.76350	−0.00124	1.52641	−0.00795
GLS1	0.18815	0.00937	0.37398	0.01637	0.74324	0.03592	1.48892	0.06459
GLS2	0.21636	−0.04981	0.43160	−0.09986	0.86333	−0.20072	1.72788	−0.39918
				n=20				
ML	0.14642	−0.05530	0.29084	−0.11156	0.58265	−0.22108	1.16502	−0.44376
LS1	0.14772	0.05070	0.29604	0.09971	0.59634	0.19960	1.18038	0.39813
LS2	0.14621	0.02396	0.29318	0.04620	0.59092	0.09233	1.16898	0.18253
ELS	0.15748	−0.01357	0.31413	−0.02760	0.63213	−0.05491	1.25870	−0.11178
WLS (L&T)	0.12506	0.00255	0.24789	0.00402	0.49643	0.01074	0.99388	0.01982
WLS	0.12426	0.00388	0.24623	0.00693	0.49396	0.01578	0.98857	0.03018
GLS1	0.11975	0.01462	0.23707	0.02849	0.47619	0.05864	0.95094	0.11581
GLS2	0.12905	−0.02108	0.25999	−0.04308	0.51875	−0.09169	1.03973	−0.18273
				n=30				
ML	0.10913	−0.03567	0.21762	−0.07012	0.43559	−0.14240	0.87266	−0.28254
LS1	0.12375	0.04583	0.24788	0.09080	0.49645	0.18579	0.99177	0.36783
LS2	0.11925	0.02400	0.23871	0.04717	0.47713	0.09810	0.95442	0.19228
ELS	0.12570	−0.00582	0.25086	−0.00965	0.50424	−0.02141	1.01027	−0.04358
WLS (L&T)	0.09935	0.00262	0.19872	0.00624	0.39603	0.01096	0.79497	0.02404
WLS	0.09900	0.00322	0.19817	0.00742	0.39477	0.01342	0.79265	0.02855
GLS1	0.09482	0.01333	0.18955	0.0278	0.37870	0.05361	0.75963	0.10923
GLS2	0.10082	−0.01303	0.20118	−0.02577	0.40632	−0.05303	0.80194	−0.11129
				n=50				
ML	0.07906	−0.02046	0.15777	−0.04141	0.31351	−0.08160	0.62915	−0.16547
LS1	0.09967	0.03844	0.19854	0.07558	0.39837	0.15476	0.79066	0.30625
LS2	0.09372	0.02144	0.18723	0.04185	0.37493	0.08732	0.74424	0.17248
ELS	0.09798	0.00172	0.19465	−0.00282	0.38897	−0.00561	0.77529	−0.00025
WLS (L&T)	0.07629	0.00218	0.15245	0.00409	0.30287	0.00911	0.60790	0.01549
WLS	0.07616	0.00236	0.15212	0.00452	0.30241	0.00988	0.60706	0.01720
GLS1	0.07273	0.01106	0.14481	0.02164	0.28832	0.04439	0.57757	0.08671
GLS2	0.07446	−0.00845	0.15212	−0.01880	0.30202	−0.03505	0.60540	−0.06028
				n=100				
ML	0.05310	−0.01009	0.10578	−0.01982	0.21128	−0.04089	0.42800	−0.08124
LS1	0.07279	0.02822	0.14520	0.05607	0.29094	0.11272	0.57808	0.22687
LS2	0.06765	0.01673	0.13485	0.03315	0.26997	0.06692	0.53589	0.13466
ELS	0.06993	0.00239	0.13950	0.00553	0.27776	0.00878	0.55709	0.02359
WLS (L&T)	0.05376	0.00107	0.10691	0.00273	0.21361	0.00432	0.43429	0.00801
WLS	0.05367	0.00108	0.10686	0.00270	0.21335	0.00457	0.43377	0.00810
GLS1	0.05089	0.00726	0.10159	0.01491	0.20246	0.02852	0.41018	0.05789
GLS2	0.05186	−0.00413	0.10262	−0.00746	0.20545	−0.01435	0.41371	−0.03020
				n=250				
ML	0.03225	−0.00393	0.06527	−0.00813	0.13035	−0.01499	0.26125	−0.03125
LS1	0.04690	0.01701	0.09390	0.03438	0.18550	0.06628	0.37319	0.13153
LS2	0.04376	0.01060	0.08753	0.02151	0.17273	0.04044	0.34929	0.08020
ELS	0.04416	0.00226	0.08871	0.00508	0.17827	0.00943	0.35890	0.01919
WLS (L&T)	0.03376	0.00068	0.06812	0.00093	0.13531	0.00327	0.27164	0.00474
WLS	0.03376	0.00066	0.06804	0.00087	0.13535	0.00327	0.27137	0.00457
GLS1	0.03178	0.00390	0.06419	0.00758	0.12853	0.01631	0.25743	0.03136
GLS2	0.03107	−0.00171	0.06313	−0.00292	0.12684	−0.00505	0.25707	−0.01208

According to bias criterion:

(**a**) GLS2 estimation outperforms GLS1 for $n = 50, 100, 250$.
(**b**) WLS and GLS1 estimation show better performance than LS1 and LS2.
(**c**) WLS (Lu and Tao) and the considered WLS present similar bias and display the best performance of the considered other methods
(**d**) LS2 performs better than LS1 for all considered sample sizes and shape parameter cases.

Table 3: RMSE and Bias of the estimated shape parameters of the log-logistic distribution.

Methods	Shape							
	1		2		3		6	
	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias
	n=10							
ML	0.38796	−0.15471	0.78149	−0.3109	1.16347	−0.45895	2.32594	−0.91451
LS1	0.30902	0.15929	0.62230	0.31840	0.92998	0.47446	1.85716	0.95062
LS2	0.29362	0.08694	0.59277	0.17352	0.88646	0.25647	1.76698	0.51585
WLS	0.29648	0.07146	0.59574	0.14163	0.89171	0.21789	1.78314	0.43795
GLS1	0.29472	0.08268	0.59331	0.16425	0.88798	0.25136	1.77551	0.50485
GLS2	0.33087	−0.05343	0.65867	−0.10590	0.97803	−0.14401	1.97124	−0.31417
	n=20							
ML	0.22390	−0.06861	0.45489	−0.13848	0.67829	−0.20789	1.34570	−0.41794
LS1	0.22448	0.12767	0.44931	0.25450	0.67293	0.38268	1.34322	0.76474
LS2	0.20281	0.07589	0.40629	0.15064	0.60819	0.22676	1.21250	0.45194
WLS	0.19860	0.05444	0.40275	0.10805	0.60008	0.16147	1.18873	0.32162
GLS1	0.19875	0.07033	0.40239	0.13916	0.60100	0.20922	1.19166	0.41605
GLS2	0.20748	−0.02236	0.41765	−0.04474	0.62353	−0.06622	1.24679	−0.13392
	n=30							
ML	0.17141	−0.04370	0.34395	−0.08501	0.52345	−0.14420	1.03422	−0.27163
LS1	0.18629	0.10648	0.36970	0.21193	0.55713	0.32123	1.11029	0.63844
LS2	0.16603	0.06451	0.32959	0.12839	0.49574	0.19560	0.98802	0.38663
WLS	0.16011	0.04180	0.32274	0.08621	0.48336	0.11330	0.96052	0.24106
GLS1	0.16092	0.05816	0.32357	0.11810	0.48363	0.16259	0.96633	0.34148
GLS2	0.16449	−0.01475	0.32823	−0.02932	0.48997	−0.03890	0.98450	−0.08392
	n=50							
ML	0.12621	−0.02668	0.25227	−0.04722	0.37653	−0.07609	0.76238	−0.16317
LS1	0.14506	0.08190	0.29183	0.16550	0.43646	0.24620	0.86976	0.48919
LS2	0.12857	0.05090	0.25852	0.10325	0.38689	0.15309	0.77050	0.30204
WLS	0.12240	0.02660	0.24763	0.05935	0.36726	0.08315	0.73900	0.15699
GLS1	0.12289	0.04194	0.24840	0.08941	0.36860	0.12961	0.73981	0.24918
GLS2	0.12421	−0.00941	0.24635	−0.01618	0.36919	−0.02661	0.73388	−0.04230
	n=100							
ML	0.08709	−0.01411	0.17256	−0.02669	0.25856	−0.04118	0.52055	−0.08304
LS1	0.10268	0.05523	0.20460	0.10987	0.30874	0.16597	0.61404	0.32834
LS2	0.09175	0.03532	0.18259	0.06989	0.27599	0.10623	0.54827	0.20855
WLS	0.08723	0.01382	0.17290	0.02787	0.25850	0.04125	0.52305	0.08135
GLS1	0.08647	0.02528	0.17223	0.05236	0.25759	0.07728	0.51743	0.15365
GLS2	0.08571	−0.00277	0.16993	−0.00706	0.25560	−0.01324	0.50743	−0.01613
	n=250							
ML	0.05291	−0.00510	0.10644	−0.01131	0.15879	−0.01616	0.32245	−0.03197
LS1	0.06344	0.03021	0.12690	0.06063	0.19083	0.09199	0.38095	0.18491
LS2	0.05809	0.01975	0.11620	0.03977	0.17459	0.06069	0.34850	0.12221
WLS	0.05426	0.00616	0.10873	0.01105	0.16357	0.01762	0.33027	0.03516
GLS1	0.05344	0.01334	0.10712	0.02560	0.15994	0.03929	0.32485	0.07890
GLS2	0.05431	−0.00109	0.10664	−0.00448	0.16027	−0.00499	0.32287	−0.00997

The results for the shape parameter of the log-logistic distribution are presented in Table 3.

According to RMSE:

(**a**) GLS1, GLS2 and WLS show better performance than ML and LS1 in terms of RMSE for all sample sizes except $n = 250$.

According to bias criterion:

(**a**) GLS2 apparently shows the best performance compared with others in terms of bias.

(**b**) WLS and GLS1 are the best performers next to GLS2 for all sample sizes.

In summary, it may be concluded that while the proposed WLS, GLS1 and GLS2 for the shape parameter of Pareto and log-logistic distributions are good alternatives to ML and LS1, the proposed GLS1 and GLS2 for the shape parameter of the Weibull distribution can be preferable estimators in terms of RMSE. If we only consider bias criterion, GLS2 is apparently the best alternative estimator for the shape parameter of the Weibull distribution. In other words, the bias reduction is achieved by GLS2.

Moreover, we found that the LS2 estimation for the shape parameter of Pareto and log-logistic distributions performs better than LS1 in terms of RMSE, similar to the result of the Weibull distribution (Zhang *et al.*, 2007).

Additionally, it can be deduced from all the simulation studies for scale parameters of the considered distributions that the considered GLS1, GLS2 and WLS are in competition with existing estimation methods. Simulation results for the scale parameter are available from the author upon request.

6. REAL LIFE EXAMPLES

In this section, we aim to show the performance of GLS by considering certain real applications.

Example 1

This example was studied with Pareto distribution in Clark, 2013. The sample consists of U.S. Weather/Climate Disasters, taken from the National Climatic Data Center and represents total economic damage from weather events in the U.S. for 1980–2011, adjusted to 2012 dollars. The sample size is 36. Using Q–Q plots, Kolmogorov–Smirnov and Chi-square tests, we show that the Pareto distribution can be used to model this data.

When we use LS1, LS2, GLS1 and GLS2 to estimate the parameters of the Pareto distribution, the descriptive statistics concerning their residuals are given in Table 4. Also, Jarque–Bera (JB) and Durbin–Watson (DW) tests are provided to test normality and autocorrelation, respectively.

The p-values, 0.075 and 0.080, for the Durbin–Watson (DW) test of null hypothesis that errors of the linear regression model are uncorrelated, show that there may be autocorrelation between residuals obtained from LS1 and LS2. On the other hand, the

p-values, 0.2855 and 0.2805 of the DW for the residuals of GLS1 and GLS2, respectively show that the null hypothesis cannot be rejected, that is, the residuals of GLS1 and GLS2 are not autocorrelated.

Table 4: Descriptive statistics and normality test results of regression residuals of the LS1, LS2, GLS1 and GLS2 for the Pareto distribution. (**Note:** Regression residuals' maximum (max), minimum (min), mean, variance (var), skewness (skew.), kurtosis (kurt.) values, also p-value and test value of Jarque Bera (JB) and Durbin Watson (DW) tests for the residuals are presented.)

Methods	Descriptive statistics				JB Test		DW Test	
	min	max	skew.	kurt.	p-val.	Test val.	p-val.	Test val.
LS1	−0.2638	0.1153	−1.6353	8.1862	0.0010	56.3904	0.0075	1.2658
LS2	−0.2430	0.1090	−1.5279	7.8415	0.0010	49.1681	0.0080	1.2723
GLS1	−0.9217	2.2652	1.1770	3.9269	0.0149	9.6002	0.2855	2.4067
GLS2	−1.1162	2.7486	1.2189	3.9606	0.0130	10.2977	0.2805	2.3987

We now calculate the estimates of the scale and shape parameters of the Pareto distribution using the estimation methods mentioned in this study. (See Table 7).

Table 5: Parameter estimates for the Pareto distribution.

Parameters	ML	LS1	LS2	ELS	WLS (Zyl&Schall)	WLS	GLS1	GLS2
Scale	5.3000	5.0354	5.0623	5.2445	5.0907	5.0891	5.1696	5.1744
Shape	1.1902	1.0680	1.0744	1.1754	1.1069	1.0824	1.0995	1.1577

Example 2

The considered data, taken from (Lawless, 2002), is analyzed by (Gupta and Kundu, 2001) with Gamma, Weibull and EE distributions. The data arose from results of tests on the endurance of deep groove ball bearings. We fit the Weibull distribution to this data set and observe that the Weibull distribution can be a plausible model.

The descriptive statistics of the resulting residuals from LS1, LS2, GLS1 and GLS2 are given in Table 6. As can be seen from Table 6, while autocorrelation among the residuals of LS1 and LS2 may be present according to the DW test, with p values less than 0.05, the p-values of 0.6004 and 0.6457 for the DW test suggests that the GLS1 and GLS2 residuals are not autocorrelated.

Table 6: Descriptive statistics and normality test results of regression residuals for the LS1, LS2 and GLS1 for the Weibull distribution.

Methods	Descriptive statistics				JB Test		DW Test	
	min	max	skew.	kurt.	p-val.	Test val.	p-val.	Test val.
LS1	−1.9634	1.8236	−0.0884	2.6113	0.5000	0.1747	0.0003	0.8454
LS2	−1.7711	2.2547	0.1630	2.8357	0.5000	0.1277	0.0004	0.8856
GLS1	−1.2475	2.1430	0.6942	2.4689	0.1289	2.1178	0.6004	2.3730
GLS2	−1.2786	2.1049	0.6090	2.3733	0.1708	1.7982	0.6457	2.3439

The estimates of the scale and shape parameters of the Weibull distribution, using estimation methods mentioned in this study, are provided in Table 7.

Table 7: Parameter estimates for the Weibull distribution.

Param.	ML	LS1	LS2	WLS (Z&S)	WLS (Hung)	WLS (Lu)	WLS	GLS1	GLS2
Scale	81.8958	82.2138	81.6037	81.1957	84.1501	81.0844	81.6037	82.8795	82.9189
Shape	2.1030	2.0430	2.1037	1.8748	1.8743	1.8863	2.1037	1.8756	2.0167

7. CONCLUSIONS

In this article, we consider generalized least squares (GLS1 and GLS2) and weighted least squares (WLS) estimation methods, based on an easily-calculated approximation of the covariance matrix, for estimating the parameters of a distribution that can be converted to a linear regression model. The considered GLS1, GLS2 and WLS methods, which are computationally easy and provide explicit estimators of the parameters, are then applied to the estimation of the parameters of different distributions, such as the Weibull, Pareto and log-logistic. The simulation results show that the considered GLS1, GLS2 and WLS estimation methods, for the shape parameters of Pareto and log-logistic distributions, show better performance than ML, LS and certain alternative estimation methods in terms of RMSE for most of the considered sample sizes and shape cases. In addition, GLS1 and GLS2 apparently provide an improvement over ML, LS and certain alternative WLS for the shape parameter of the Weibull distribution in terms of RMSE and bias. In conclusion, the results of the simulations and real life examples demonstrate that the considered GLS1 and GLS2 for the shape parameters of log-logistic, Pareto and Weibull distributions can be considered as good alternative estimators.

Moreover, it is also emphasized that the considered estimation methods can be applied to Gumbel, Burr XII, Fréchet and other distributions, which have explicit cumulative distribution functions, after calculation of the covariance matrix concerning them.

In a future study, we plan to investigate the performance of the GLS estimation method in the case of right censored data and contaminated data.

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REFERENCES

[1] Abernethy, R.B. (1996). *The New Weibull Handbook*, R.B. Abernethy North Palm Beach.

[2] Ahmad, M.I.; Sinclair, C.D. and Werrity, A. (1988). Log-logistic flood frequency analysis, *J. Hydrol.*, **98**, 205–224.

[3] Ashkar, F. and Mahdi, S. (2006). Fitting the log-logistic distribution by generalized moments, *Journal of Hydrology*, **3-4**, 328, 694–703.

[4] Baxter, M.A. (1980). Minimum variance unbiased estimation of the Pareto distribution, *Metrika*, **27**, 133–138.

[5] Bergman, B. (1986). Estimation of Weibull parameters using a weight function, *Journal of Materials Science Letters*, **5**, 611–614.

[6] Blom, G. (1962). *Nearly best linear estimators of scale and location parameters.* Contributions to Order Statistics. (A.E. Sarhan and B.G. Greenberg, Eds.), New York: John Wiley, Ch. 4B.

[7] Boudt, K.; Caliskan, D. and Croux, C. (2011). Robust explicit estimators of Weibull parameters, *Metrika*, **73**, 187–209.

[8] Chen, Z. (2006). Estimating the shape parameter of the log-logistic distribution, *Int. J. Rel., Qual. and Safety Eng.*, **13**, 257–266.

[9] Clark, D.R. (2013). A Note on the Upper-Truncated Pareto Distribution, *Actuarial Society E-Forum*.

[10] Engeman, R.M. and Keefe, T.J. (1982). On generalized least squares estimation of the Weibull distribution, *Communications in Statistics – Theory and Methods*, **19**, 11, 2181–2193.

[11] Gebizlioglu, O.L.; Senoglu, B. and Kantar, Y.M. (2011). Comparison of certain value-at-risk estimation methods for the two-parameter Weibull loss distribution, *Journal of Computational and Applied Mathematics*, **235**, 3304–3314.

[12] Genschel, U. and Meeker, W.Q. (2010). A Comparison of Maximum Likelihood and Median-Rank Regression for Weibull Estimation, *Quality engineering*, **4**, 22, 236–255.

[13] Gupta, R.D. and Kundu, D. (2001). Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions, *Biometrical Journal*, **43**, 117–130.

[14] Hassanein, K.M. (1971). Percentile estimators for the parameters of the Weibull distribution, *Biometrika*, **3**, 58, 673–676.

[15] Hossain, A. and Zimmer, W.J. (2000). Comparisons of methods of estimation for Pareto of the first kind, *Communications in Statistics – Theory and Methods*, **4**, 29, 859–878.

[16] Hossain, A. and Zimmer, W.J. (2003). Comparison of estimation methods for Weibull parameters: complete and censored samples, *Journal of Statistical Computation and Simulation*, **2**, 73, 145–153.

[17] Hossain, A. and Howlader, H.A. (1996). Unweighted least squares estimation of Weibull parameters, *Journal of Statistical Computation and Simulation*, **54**, 265–271.

[18] Hung, W.L. (2001). Weighted least squares estimation of the shape parameter of the Weibull distribution, *Qual. Reliab. Eng. Int.*, **6**, 17, 467–469.

[19] Kantar, Y.M. and Senoglu, B. (2008). A Comparative study for the location and scale parameters of the Weibull distribution with a given shape parameter, *Computers & Geosciences*, **12**, 34, 1900–1909.

[20] Kantar, Y.M. and Usta, I. (2008). Analysis of wind speed distributions: wind distribution function derived from minimum cross entropy principles as better alternative to Weibull function, *Energy Convers. Manage.*, **49**, 962–973.

[21] Kantar, Y.M. and Arik, I. (2014). M-estimation of log-logistic distribution parameters with outliers, *International Journal of Agricultural and Statistical Sciences*, Accepted manuscript.

[22] Kantar, Y.M. and Yildirim, V. (2015). Robust Estimation for Parameters of the Extended Burr Type III Distribution, *Communications in Statistics – Theory and Methods, Communications in Statistics – Simulation and Computation*, **44**, 7, 1901–1930.

[23] Koenker, R. and Bassett, G. (1978). Regression quantiles, *Econometrica*, **46**, 33–50.

[24] Lawless, J.F. (2002). *Statistical Models and Methods for Lifetime Data*, John Wiley and Sons, New York.

[25] Likes, J. (1969). Minimum variance unbiased estimates of the Pareto distribution, *Metrika*, **15**, 126–132.

[26] Lu, H.L.; Chen, C.H. and Wu, J.W. (2004). A Note on weighted least-squares estimation of the shape parameter of the Weibull distribution, *Qual. Reliab. Eng. Int.*, **6**, 20, 579–586.

[27] Lu, H.L. and Tao, S.H. (2007). The Estimation of Pareto Distribution by a Weighted Least Square Method, *Quality & Quantity*, **41**, 913–926.

[28] Malik, H. (1970). Estimation of the parameters of the Pareto distribution, *Metrika*, **1**, 15, 126–132.

[29] Marks, N.B. (2005). Estimation of Weibull parameters from common percentiles, *J. Appl. Stat.*, **1**, 32, 17–24.

[30] Nelson, W. (2004). *Applied Life Data Analysis*, Wiley, USA.

[31] Quandt, R.E. (1966). Old and New methods of estimation and the Pareto distribution, *Metrika*, **1**, 10, 55–82.

[32] Prakash, G. and Singh, D.C. (2009). A Bayesian shrinkage approach in Weibull Type-II censored data using prior point information, *Revstat – Statistical Journal*, **2**, 7, 171–187.

[33] Saksena, S.K. and Johnson, A.M. (1984). Best unbiased estimators for the parameters of a two-parameter Pareto distribution, *Metrika*, **1**, 31, 77–83.

[34] Teimouri, M. and Nadarajah, S. (2012). A simple estimator for the Weibull shape parameter, *Int. J. Str. Stab. Dyn.*, **395**, DOI: 10.1142S0219455412500095.

[35] Tiryakioglu, M. and Hudak, D. (2007). On estimating Weibull modulus by the linear regression method, *Mater. Sci.*, **24**, 42, 10173–10179.

[36] Usta, I. (2013). Different estimation methods for the parameters of the extended Burr XII distribution, *J. App. Stat.*, **2**, 40, 397–414.

[37] Wang, F.K. and Cheng, Y.F. (2010). Robust regression for estimating the Burr XII parameters with outliers, *J. App. Stat.*, **5**, 37, 807–819.

[38] White, J.S. (1969). The Moments of Log-Weibull Order Statistics, *Technometrics*, **2**, 11, 373–386.

[39] Zhang, L.F.; Xie, M. and Tang, L.C. (2008). On weighted least squares estimation for the parameters of Weibull distribution, *Recent Advances in Reliability and Quality in Design Springer Series in Reliability Engineering*, **2**, 11, Chapter 57–84.

[40] Zhang, L.F.; Xie, M. and Tang, L.C. (2007). A study of two estimation approaches for parameters of Weibull distribution based on WPP, *Reliab. Eng. Syst. Safe.*, **92**, 360–368.

[41] Zyl, J.M.V. (2012). A Median Regression Model to Estimate the Parameters of the Three-Parameter Generalized Pareto Distribution, *Commun. Stat.-Simul. C.*, **41**, 544–553.

[42] Zyl, J.M.V. and Schall, R. (2012). Parameter Estimation through Weighted Least-Squares Rank Regression with Specific Reference to the Weibull and Gumbel Distributions, *Commun. Stat.-Simul. C.*, **41**, 1654–1666.