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THE TAYLOR PROPERTY IN BILINEAR MODELS

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Abstract:

- The aim of this paper is to discuss the presence of the Taylor property in the class of simple bilinear models. Considering strictly and weakly stationary models, we deduce autocorrelations of the process and of its square and analyze the presence of the Taylor property in non-negative bilinear models considering several error process distributions, which are chosen according to the kurtosis value. For each one of these error process distributions, the class of parameterizations for the corresponding bilinear model satisfying Taylor property is obtained. The analysis of the relationship between the Taylor property and leptokurtosis in these bilinear processes allows to conclude that this property is a consequence of heavy tailed model distributions.

  With the goal of extending this research to real valued bilinear models, a simulation study is developed in a class of such models with symmetrical innovations.

Key-Words:

- *bilinear models; nonlinear time series; stationarity; Taylor property.*

AMS Subject Classification:

- 62M10.

1. INTRODUCTION

The search for non-trivial empirical regularities in time series, usually called stylized facts, has been the subject of several studies in order to identify classes of time series models that conveniently capture such empirical properties. A stylized fact detected by Taylor ([9]) when he analyzed 40 returns series is known as the Taylor effect. He observed that, for most of the returns series, denoted by X_t for instant t, the sample autocorrelations of the absolute returns, $\hat{\rho}_{|X|}(n) = \widehat{\mathrm{corr}}(|X_t|, |X_{t-n}|)$, were larger than those of the squared returns, $\hat{\rho}_{X^2}(n) = \widehat{\mathrm{corr}}(X_t^2, X_{t-n}^2)$, for $n \in \{1, ..., 30\}$. More recently, Gonçalves *et al.* ([2]) also recorded Taylor effect in the physical time series of plage region areas describing solar activity.

We point out that there is still little research on the theoretical counterpart on this empirical property due to the difficulty of handling the true autocorrelations of time series models. For example, this theoretical counterpart was studied by He and Teräsvirta ([5]) on conditionally Gaussian absolute value generalized ARCH (AVGARCH) models, assuring its presence for some of these models. More precisely, they called the theoretical relation $\rho_{|X|}(n) > \rho_{X^2}(n)$, $n \geq 1$, the Taylor property and concentrated their study on the autocorrelation of lag 1. Analogously, Gonçalves, Leite and Mendes-Lopes ([1]) studied the presence of the Taylor property in TARCH models, concluding that this property is satisfied when $n = 1$, for some first-order models. Generalizing these papers, Haas ([4]) proposed a methodology for identifying the Taylor property in AVGARCH$(1, 1)$ models at all lags.

The research of this property within heteroskedastic models is mainly related to the empirical facts observed and the good fit of those models to financial time series. The established results have shown a strong connection between the Taylor property and the kurtosis of the process; in fact, its presence seems to be more related to the leptokurtic character of those models than to its conditional heteroskedascity. This interpretation is consistent with the leptokurtic nature of the real series presenting such stylized fact. Thus, we believe that it is important to assess the presence of the Taylor property in other classes of processes with relevance in time series analysis as it is the case of bilinear ones, which have also been proven to be suitable in financial and physical time series modeling ([3], p. 181).

In this paper we consider the simple bilinear diagonal model

$$X_t = \beta X_{t-k} \varepsilon_{t-k} + \varepsilon_t \;, \qquad k > 0 \;, \tag{1.1}$$

where β is a real parameter and $(\varepsilon_t, t \in \mathbb{Z})$ a sequence of i.i.d. random variables, designated here by error process.

We state sufficient conditions for the strict and weak stationarity of the processes $X = (X_t, t \in \mathbb{Z})$ and $X^2 = (X_t^2, t \in \mathbb{Z})$, and we derive expressions for the moments of X up to the 4th order. We also consider the study of the Taylor property assuming that $\beta > 0$ and that the error process is non-negative. In fact, there has been considerable interest in non-negative bilinear models. For instance, Pereira and Scotto ([7]) studied some properties of the simple first-order bilinear diagonal model ($k = 1$) driven by exponentially distributed innovations. Also Zhang and Tong ([10]) have examined some distributional properties of a simple first-order non-negative bilinear model considering for the error process the uniform distribution in $(0, 1)$.

The remainder of the paper is organized as follows. In Section 2 we establish sufficient conditions under which X and X^2 are strictly and weakly stationary. Moreover, the moments of X up to 4th order are evaluated and a working example on this matter is presented in appendix. In Section 3, the Taylor property in first-order bilinear diagonal models with non-negative error process is analyzed. This study is developed considering several distributions for the error process with significantly different kurtosis values. A simulation study evaluating the Taylor property in other real-valued simple bilinear models is presented in Section 4. Some concluding remarks and future developments are given in Section 5.

2. STATIONARITY AND MOMENTS OF X AND X^2

In this section we consider the simple bilinear model defined by (1.1) and we denote $\mu_i = E(\varepsilon_t^i)$, $i \in \mathbb{N}$.

Proposition 2.1. *Suppose that μ_4 and $E(\ln|\varepsilon_t|)$ exist. If $\beta^2\,\mu_2 < 1$ then the process X is strictly and weakly stationary.*

Proof: The strict stationarity of the process X is achieved by proving that $X_t = Y_t$, a.s., with

$$Y_t = \varepsilon_t + \sum_{n=1}^{+\infty} T_n\ ,$$

where, for each $n \in \mathbb{N}$, $T_n = T_n(t)$ is given by

$$T_n = \beta^n \varepsilon_{t-nk} \prod_{j=1}^{n} \varepsilon_{t-jk}\ .$$

The proof of this result is similar to that of Theorem 1 in Quinn ([8]), as the condition $\beta^2\mu_2 < 1$ implies Quinn's condition $\ln|\beta| + E(\ln|\varepsilon_t|) < 0$ by applying Jensen's inequality to the random variable ε_t^2.

To prove the weak stationarity, we now verify that $E(Y_t^2) < +\infty$. We have

$$\begin{aligned} E(Y_t^2) &= E\left[\left(\varepsilon_t + \sum_{i=1}^{+\infty} T_i\right)^2\right] \\ &\le E(\varepsilon_t^2) + 2\sum_{i=1}^{\infty} E(|\varepsilon_t|\,|T_i|) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E(|T_i T_j|) \ . \end{aligned} \tag{2.1}$$

Under the given conditions, each series in (2.1) is convergent. In fact, let us consider, for example, the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E(|T_i T_j|)$.

For each $i, j \in \mathbb{N}$, we have

$$\begin{aligned} E(|T_i T_j|) &\le |\beta|^{i+j}\left[E\Big(\varepsilon_{t-ik}^4\,\varepsilon_{t-k}^2\,\varepsilon_{t-2k}^2 \cdots \varepsilon_{t-(i-1)k}^2\Big)\right]^{1/2} \\ &\qquad \left[E\Big(\varepsilon_{t-jk}^4\,\varepsilon_{t-k}^2\,\varepsilon_{t-2k}^2 \cdots \varepsilon_{t-(j-1)k}^2\Big)\right]^{1/2} \\ &= \mu_4\,\mu_2^{-1}\left[\left(\beta^2\mu_2\right)^{1/2}\right]^{i+j} , \end{aligned}$$

by Schwarz's inequality and the independence of the r.v.'s ε_t, $t\in\mathbb{Z}$. As $\left(\beta^2\mu_2\right)^{1/2} < 1$, the series is convergent.

Taking into account the equality $X_t = Y_t$, a.s., and the strict stationarity of the process X, we conclude that $E(X_t^2)$ exists and that X is weakly stationary. □

Proposition 2.2. *Suppose that $E(\ln|\varepsilon_t|)$ and μ_8 exist. If $\beta^4\mu_4 < 1$ then the process X^2 is strictly and weakly stationary.*

Proof: The condition $\beta^4\mu_4 < 1$ implies $\beta^2\mu_2 < 1$, by Schwarz's inequality, which implies in turn the strict stationarity of X and, consequently, of X^2. The proof of the weak stationarity of X^2 is analogous to the previous one. We have

$$\begin{aligned} E(Y_t^4) \le\ & E(\varepsilon_t^4) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} E(|T_i T_j T_p T_q|) + 4\sum_{i=1}^{\infty} E(|\varepsilon_t^3|\,|T_i|) \\ &+ 4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{p=1}^{\infty} E(|\varepsilon_t|\,|T_i T_j T_p|) + 6\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E(\varepsilon_t^2\,|T_i T_j|) \ . \end{aligned}$$

Let us consider, for example, the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} E(|T_i T_j T_p T_q|)$, which is a sum of series of the types

(i) $\displaystyle\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=p+1}^{\infty} E(|T_iT_jT_pT_q|)\,,$

(ii) $\displaystyle\sum_{i=1}^{\infty}\sum_{p=1}^{\infty} E(T_i^2T_p^2)\,,$

(iii) $\displaystyle\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{p=1}^{\infty} E(|T_iT_j|\,T_p^2)\,.$

Concerning (i), as $j>i$ and $q>p$, we have

$$\begin{aligned} E(|T_iT_jT_pT_q|) \;&=\; E\Big[(|T_iT_j|)\,(|T_pT_q|)\Big]\\ &\le\; \Big[E\Big(\varepsilon^4_{t-ik}\varepsilon^4_{t-l}\varepsilon^4_{t-l-k}\cdots\varepsilon^4_{t-l-(i-1)k}\varepsilon^2_{t-jk}\varepsilon^2_{t-l-ik}\cdots\varepsilon^2_{t-l-(j-1)k}\Big)\Big]^{1/2}\\ &\quad\;\Big[E\Big(\varepsilon^4_{t-pk}\varepsilon^4_{t-l}\varepsilon^4_{t-l-k}\cdots\varepsilon^4_{t-l-(p-1)k}\varepsilon^2_{t-qk}\varepsilon^2_{t-l-pk}\cdots\varepsilon^2_{t-l-(q-1)k}\Big)\Big]^{1/2}, \end{aligned}$$

by Schwarz's inequality.

Taking into account the independence of the random variables ε_t, we have, for $i,j\in\mathbb{N},\ j>i$,

$$E\Big(\varepsilon^4_{t-ik}\,\varepsilon^4_{t-l}\,\varepsilon^4_{t-l-k}\cdots\varepsilon^4_{t-l-(i-1)k}\,\varepsilon^2_{t-jk}\,\varepsilon^2_{t-l-ik}\cdots\varepsilon^2_{t-l-(j-1)k}\Big) \;=\; \mu_4^{i+1}\mu_2^{j-i+1}\,.$$

Then

$$\begin{aligned} &\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=p+1}^{\infty} E(|T_iT_jT_pT_q|)\\ &\qquad\le \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=p+1}^{\infty} |\beta|^{i+j+p+q}\left(\mu_4^{i+p+2}\mu_2^{j-i+q-p+2}\right)^{1/2}\\ &\qquad= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=p+1}^{\infty} \mu_2\,\mu_4\Big[\big(\beta^4\mu_4\big)^{1/2}\Big]^{i+p}\Big[\big(\beta^2\mu_2\big)^{1/2}\Big]^{[(j+q)-(i+p)]}. \end{aligned}$$

As $(\beta^4\mu_4)^{1/2}<1$ and $(\beta^2\mu_2)^{1/2}<1$, the series in (i) is convergent. The convergence of the series (ii) and (iii) is proved in a similar way. Then we conclude that $E(X_t^4)<+\infty,\ t\in\mathbb{Z}$. As the process X^2 is strictly stationary and $E(X_t^4)$ exists, then it is weakly stationary. $\square$

Let us now evaluate the moments up to the 4th order of the process X given by (1.1).

Proposition 2.3. *If $\beta^4\mu_4 < 1$ and μ_8 exists then the nth moment of X_t, $n \le 4$, can be expressed as*

$$E(X_t^n) = \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i}\, \mu_i\, E(X_t^{n-i}\varepsilon_t^{n-i}) \;, \tag{2.2}$$

where

$$E(X_t^n\varepsilon_t^n) = \frac{1}{1-\beta^n\mu_n} \sum_{i=1}^{n} \binom{n}{i} \beta^{n-i}\, \mu_{n+i}\, E(X_t^{n-i}\varepsilon_t^{n-i}) \;, \qquad n \le 4 \;. \tag{2.3}$$

Proof: For $n \le 4$, we have

$$\begin{aligned} E(X_t^n) &= \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i}\, E\Big[\varepsilon_t^i\, (X_{t-k}\varepsilon_{t-k})^{n-i}\Big] \\ &= \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i}\, \mu_i\, E(X_t^{n-i}\varepsilon_t^{n-i}) \;, \end{aligned}$$

since the process $(X_t\varepsilon_t,\, t \in \mathbb{Z})$ is strictly stationary due to the fact that $X_t\varepsilon_t$ is a measurable function of $\varepsilon_t, \varepsilon_{t-1}, \ldots$. Now we need to evaluate $E(X_t^n\varepsilon_t^n)$, $n \le 4$.

$$\begin{aligned} E(X_t^n\varepsilon_t^n) &= \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i}\, E\Big[\varepsilon_t^i\, (X_{t-k}\varepsilon_{t-k})^{n-i}\, \varepsilon_t^n\Big] \\ &= \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i}\, E(\varepsilon_t^{n+i})\, E(X_t^{n-i}\varepsilon_t^{n-i}) \\ &= \beta^n\, \mu_n\, E(X_t^n\varepsilon_t^n) + \sum_{i=1}^{n} \binom{n}{i} \beta^{n-i}\, \mu_{n+i}\, E(X_t^{n-i}\varepsilon_t^{n-i}) \end{aligned}$$

and the result follows. □

It is easy to verify that $E(X_t\varepsilon_t) = \mu_2/(1-\beta\mu_1)$. The values $E(X_t^n\varepsilon_t^n)$, $n = 1, 2, 3$, are obtained recursively by using the previous equation; and finally, we achieve $E(X_t^n)$, $n \le 4$. A working example to illustrate these evaluations is developed in appendix for a first order bilinear model with exponentially-distributed error process.

We note that $\beta^4\mu_4 < 1$ implies $|\beta^n\mu_n| < 1$, $n = 1, 2, 3$, by Schwarz's inequality.

3. THE TAYLOR PROPERTY IN FIRST-ORDER NON-NEGATIVE BILINEAR MODELS

3.1. Preliminary results

In this section we consider the first-order non-negative bilinear model

$$X_t = \beta X_{t-1}\varepsilon_{t-1} + \varepsilon_t\ , \qquad t\in\mathbb{Z}\ , \tag{3.1}$$

where $\beta>0$ and $(\varepsilon_t, t\in\mathbb{Z})$ is a sequence of non-negative i.i.d. random variables.

We assume that $E(\ln\varepsilon_t)$ and μ_8 exist and that $\beta^4\mu_4<1$ in order to guarantee that both processes, X and X^2, are strictly and weakly stationary.

In this context, the Taylor property for $n=1$ establishes that $\rho_X(1)>\rho_{X^2}(1)$, where $\rho_X(1)$ and $\rho_{X^2}(1)$ denote, respectively, the autocorrelations of lag 1 of the processes X and X^2. It is enough to evaluate $E(X_tX_{t-1})$ and $E(X_t^2X_{t-1}^2)$ in order to obtain these autocorrelations since we derived $E(X_t^i)$, $i=1,2,3,4$, in the previous section. Using (3.1) and the stationarity of the involved processes, we have

$$\begin{aligned} E(X_tX_{t-1}) &= \beta E(X_t^2\varepsilon_t) + E(X_{t-1}\varepsilon_t)\\ &= \beta E\big(\beta^2X_{t-1}^2\varepsilon_{t-1}^2\varepsilon_t + 2\beta X_{t-1}\varepsilon_{t-1}\varepsilon_t^2 + \varepsilon_t^3\big) + E(X_{t-1}\varepsilon_t)\ . \end{aligned}$$

Taking into account the independence of the random variables ε_t, $t\in\mathbb{Z}$, and the strict stationarity of the related processes, we have $E(X_{t-1}^2\varepsilon_{t-1}^2\varepsilon_t)=\mu_1E(X_t^2\varepsilon_t^2)$ and $E(X_{t-1}\varepsilon_{t-1}\varepsilon_t^2)=\mu_2E(X_t\varepsilon_t)$. Then

$$E(X_tX_{t-1}) = \beta^3\mu_1E(X_t^2\varepsilon_t^2) + 2\beta^2\mu_2E(X_t\varepsilon_t) + \mu_1E(X_t) + \beta\mu_3\ .$$

Using an analogous procedure, we obtain

$$\begin{aligned} E(X_t^2X_{t-1}^2) &= \beta^4E_1 + 2\beta^3E_2 + 2\beta^3\mu_1E_3 + 4\beta^2\mu_1E_4 + \beta^2E_5 + 2\beta\mu_1E_6\\ &\quad + \beta^2\mu_2E(X_t^2\varepsilon_t^2) + 2\beta\mu_1\mu_2E(X_t\varepsilon_t) + \mu_2^2\ , \end{aligned}$$

where

$$\begin{aligned} E_1 &= E(X_t^2X_{t-1}^2\varepsilon_t^2\varepsilon_{t-1}^2) = \beta^2\mu_2E(X_t^4\varepsilon_t^4) + 2\beta\mu_3E(X_t^3\varepsilon_t^3) + \mu_4E(X_t^2\varepsilon_t^2)\ ,\\ E_2 &= E(X_t^2X_{t-1}\varepsilon_t^3\varepsilon_{t-1}) = \beta^2\mu_3E(X_t^3\varepsilon_t^3) + 2\beta\mu_4E(X_t^2\varepsilon_t^2) + \mu_5E(X_t\varepsilon_t)\ ,\\ E_3 &= E(X_tX_{t-1}^2\varepsilon_t\varepsilon_{t-1}^2) = \beta\mu_1E(X_t^3\varepsilon_t^3) + \mu_2E(X_t^2\varepsilon_t^2)\ ,\\ E_4 &= E(X_tX_{t-1}\varepsilon_t^2\varepsilon_{t-1}) = \beta\mu_2E(X_t^2\varepsilon_t^2) + \mu_3E(X_t\varepsilon_t)\ ,\\ E_5 &= E(X_t^2\varepsilon_t^4) = \beta^2\mu_4E(X_t^2\varepsilon_t^2) + 2\beta\mu_5E(X_t\varepsilon_t) + \mu_6\ ,\\ E_6 &= E(X_t\varepsilon_t^3) = \beta\mu_3E(X_t\varepsilon_t) + \mu_4\ . \end{aligned}$$

Finally, the results of the previous section allow us to obtain the values of $E(X_tX_{t-1})$ and $E(X_t^2X_{t-1}^2)$ in terms of the moments of ε_t.

3.2. The Taylor property and the error process

In the following, we investigate the presence of the Taylor property in Model (3.1), considering some non-negative distributions for the error process, namely, the uniform distribution in $]0,\alpha[$, the exponential distribution in $]0,+\infty[$ with mean α, and the Pareto distribution with density $f(x)=\dfrac{\nu\,\alpha^{\nu}}{x^{\nu+1}}\,\mathbb{I}_{]\alpha,+\infty[}(x)$, for $\nu=12$ and $\nu=9$. In all cases, α is a non-negative parameter and the condition $E(|\ln\varepsilon_t|)<+\infty$ is satisfied.

The choice of these distributions takes into account the fact that the Taylor property seems to be related with the kurtosis value of the process. In this paper, we consider that the kurtosis of a random variable Z is given by $K_Z = M_4/M_2^2-3$, where M_n is the n^{th} central moment of Z, $n=2,4$, providing that M_4 exists (K_Z is also called "excess kurtosis"). The uniform distribution is platykurtic with a constant kurtosis value equal to -1.2, while the exponential distribution is leptokurtic with a constant kurtosis value equal to 6. On the other hand, the kurtosis of the Pareto distribution depends on the parameter ν and it is given by $\frac{6(\nu^3+\nu^2-6\nu-2)}{\nu(\nu-3)(\nu-4)}$, $\nu>4$. This is a decreasing function of ν that goes to 6 when ν tends to infinity, and to infinity when ν tends to 4. So, the Pareto distribution is leptokurtic, no matter what is the value of ν.

We also point out that, in all cases, the condition $\beta^4\mu_4<1$ and the values of $\rho_X(1)$ and $\rho_{X^2}(1)$ can be written in terms of $r=\alpha\beta$.

In each case, we also present the value of the kurtosis of the process X given by (3.1), which also depends on $r=\alpha\beta$, as well as the corresponding graphic representation as a function of r. We point out that, in all these models, the leptokurtosis of the error process implies the same property for the process X. In what concerns the Taylor property and kurtosis of X, comparisons are made separately between the first two distributions, uniform and exponential, and also between the two referred Pareto distributions.

3.2.1. Error process with uniform distribution in $]0,\alpha[$

In this case, the condition $\beta^4\mu_4<1$ is equivalent to $0<r<\sqrt[4]{5}\simeq 1.495$ and we obtain

$$\rho_X(1) = \frac{r\big(-180+120r-51r^2-4r^3+r^4\big)}{-180+180r-177r^2+12r^3+7r^4}\ ,$$

$$\rho_{X^2}(1) = -\frac{r}{12}\,\frac{N_U(r)}{D_U(r)}\ ,$$

with

$$\begin{aligned} N_U(r) &= -604\,800 - 480\,600\,r - 155\,700\,r^2 - 257\,400\,r^3 - 2\,490\,r^4 + 48\,525\,r^5 \\ &\quad - 6\,270\,r^6 + 6\,810\,r^7 + 10\,620\,r^8 + 11\,384\,r^9 + 4\,012\,r^{10} - 586\,r^{11} \\ &\quad + 94\,r^{12} - 53\,r^{13} + 6\,r^{14} \;, \\ D_U(r) &= 50\,400 + 12\,600\,r + 35\,700\,r^2 + 40\,200\,r^3 + 13\,490\,r^4 + 14\,015\,r^5 + 8\,360\,r^6 \\ &\quad - 5\,210\,r^7 - 5\,999\,r^8 - 2\,407\,r^9 - 720\,r^{10} + 114\,r^{11} + 177\,r^{12} - 8\,r^{13} \;. \end{aligned}$$

From Figure 1(a), we can see that the Taylor property is present for values of r in the interval[1] $\left]1.186\,8987, \sqrt[4]{5}\right[$. So, for a fixed α, the Taylor property is achieved for parameterizations of Model (3.1) such that

$$\beta \in \left] \frac{1.186\,8987}{\alpha}, \frac{\sqrt[4]{5}}{\alpha} \right[\;.$$

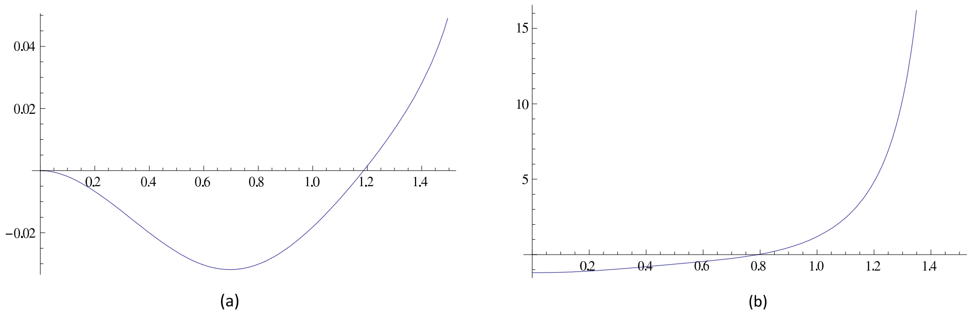


Figure 1: Graphs from $\rho_X(1) - \rho_{X^2}(1)$ (a) and $K_U(r)$ (b), $0 < r < \sqrt[4]{5}$.

For Model (3.1) with such an error process, the kurtosis is given by

$$K_U(r) \;=\; \frac{-3\,(-3+r^2)}{7\,(-4+r^3)\,(-5+r^4)}\,\frac{N_U^*(r)}{D_U^*(r)} - 3 \;,$$

where

$$\begin{aligned} N_U^*(r) &= 907\,200 - 1\,814\,400\,r + 4\,284\,000\,r^2 - 4\,510\,800\,r^3 + 3\,254\,460\,r^4 \\ &\quad - 2\,030\,520\,r^5 + 1\,973\,540\,r^6 - 617\,175\,r^7 - 185\,700\,r^8 + 371\,005\,r^9 \\ &\quad - 236\,308\,r^{10} + 78\,747\,r^{11} - 11\,496\,r^{12} + 511\,r^{13} \;, \\ D_U^*(r) &= \left(-180 + 180\,r - 177\,r^2 + 12\,r^3 + 7\,r^4\right)^2 . \end{aligned}$$

From Figure 1(b), we observe that the kurtosis of this model is an increasing function of r and that the model is leptokurtic for $r > 0.8$ (approx.). We also observe that the Taylor property occurs for large values of the kurtosis, namely for $K_U(r) > 4.403$ ($\simeq K_U(1.186\,8987)$).

[1]The value 1.186 8987 was obtained with an approximation error inferior to 5×10^{-9}.

3.2.2. Error process with exponential distribution with mean α (in $]0,+\infty[$)

The condition $\beta^4\mu_4 < 1$ is now equivalent to $0 < r < \frac{1}{\sqrt[4]{24}} \simeq 0.4518$. In this case,

$$\rho_X(1) = \frac{2r\,(2 - 3r + 7r^2 - 6r^3 + 2r^4)}{1 - 2r + 19r^2 - 20r^3 + 6r^4} ,$$

$$\rho_{X^2}(1) = 2r\,\frac{N_E(r)}{D_E(r)} .$$

with

$$\begin{aligned} N_E(r) = {} & -5 - 80\,r + 65\,r^2 - 112\,r^3 - 1184\,r^4 - 5774\,r^5 + 10848\,r^6 + 12720\,r^7 \\ & - 9408\,r^8 - 17880\,r^9 - 16272\,r^{10} + 52992\,r^{11} + 9216\,r^{12} \\ & - 46656\,r^{13} + 17280\,r^{14} , \end{aligned}$$

$$\begin{aligned} D_E(r) = {} & -5 + 2\,r - 21\,r^2 - 602\,r^3 - 9060\,r^4 + 11126\,r^5 + 13252\,r^6 - 26448\,r^7 \\ & + 16368\,r^8 + 13896\,r^9 - 12192\,r^{10} + 13824\,r^{11} - 12672\,r^{12} + 4032\,r^{13} . \end{aligned}$$

So, when the errors are exponentially distributed with mean α, Model (3.1) presents the Taylor property for parameterizations such that[2]

$$\beta \in \left]0, \frac{0.0695566}{\alpha}\right[\cup \left]\frac{0.1437879}{\alpha}, \frac{1}{\sqrt[4]{24}\,\alpha}\right[.$$

This conclusion is illustrated in Figure 2(a). In Figure 2(b), we have the graphic representation of the kurtosis of Model (3.1) with exponential errors, which is given by

$$K_E(r) = \frac{-3\,(-1 + 2r^2)}{(-1 + 6r^3)\,(-1 + 24r^4)}\,\frac{N_E^*(r)}{D_E^*(r)} - 3 ,$$

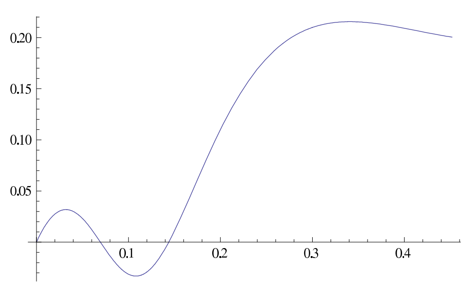
where

$$\begin{aligned} N_E^*(r) = {} & 3 - 12\,r + 52\,r^2 - 134\,r^3 + 11815\,r^4 - 36752\,r^5 + 44802\,r^6 + 1062\,r^7 \\ & - 42648\,r^8 + 17028\,r^9 + 12240\,r^{10} + 5616\,r^{11} - 17280\,r^{12} + 6048\,r^{13} , \end{aligned}$$

$$D_E^*(r) = \left(1 - 2\,r + 19\,r^2 - 20\,r^3 + 6\,r^4\right)^2 .$$

As in the previous case, the kurtosis of Model (3.1) is an increasing function of r but the process X is always leptokurtic in this case. Again, we observe that large kurtosis values correspond to large values of the difference $\rho_X(1) - \rho_{X^2}(1)$. In fact, the Taylor property is clearly present in this model for kurtosis values greater than 13 ($\simeq K_E(0.16)$).

[2]The values 0.0695566 and 0.1437879 were obtained with an approximation error inferior to 5×10^{-8}.



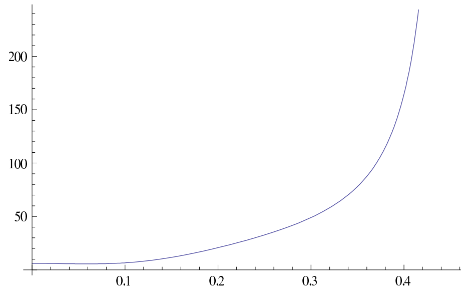


Figure 2: Graphs from $\rho_X(1) - \rho_{X^2}(1)$ (a) and $K_E(r)$ (b), $0 < r < \frac{1}{\sqrt[4]{24}}$.

We also observe that the kurtosis of the process X is larger when the errors are exponentially distributed than when they are uniformly distributed, corresponding to an analogous relation between the kurtosis of those error processes. The Taylor property seems to emerge in a relatively stronger way when the kurtosis of X increases.

3.2.3. Error process with Pareto density $f(x) = \dfrac{12\,\alpha^{12}}{x^{13}}\,\mathbb{I}_{]\alpha,+\infty[}(x)$

The region of existence of the autocorrelations in terms of $r = \alpha\beta$ is now defined by $0 < r < \sqrt[4]{\frac{2}{3}} \simeq 0.9036$. We have

$$\rho_X(1) = \frac{44\,r\,(6050 - 10230\,r + 13035\,r^2 - 7524\,r^3 + 1296\,r^4)}{3\,(36300 - 79200\,r + 219255\,r^2 - 171160\,r^3 + 29472\,r^4)}\,,$$

$$\rho_{X^2}(1) = \frac{r}{55}\,\frac{N_{P12}(r)}{D_{P12}(r)}\,,$$

with

$$\begin{aligned} N_{P12}(r) = {} & -7043652000 - 5638479000\,r - 1900483200\,r^2 - 6228372150\,r^3 \\ & - 3064649280\,r^4 + 2622844140\,r^5 + 24533447400\,r^6 \\ & + 19854650865\,r^7 + 11360213480\,r^8 - 16340416020\,r^9 \\ & - 30235824828\,r^{10} + 23037530976\,r^{11} + 7650162960\,r^{12} \\ & - 11215587456\,r^{13} + 2802615552\,r^{14}\,, \end{aligned}$$

$$\begin{aligned} D_{P12}(r) = {} & -58697100 + 14229600\,r - 142425360\,r^2 - 468153840\,r^3 \\ & - 218936564\,r^4 + 536116224\,r^5 + 616017864\,r^6 \\ & + 374454192\,r^7 + 130906149\,r^8 - 805701976\,r^9 \\ & - 15605040\,r^{10} + 401099652\,r^{11} \\ & - 245871648\,r^{12} + 48736320\,r^{13}\,. \end{aligned}$$

Concerning the kurtosis of this model, it is given by

$$K_{P12}(r) \;=\; \frac{-2\,(-5+6r^2)}{49\,(-3+4r^3)\,(-2+3r^4)}\,\frac{N^*_{P12}(r)}{D^*_{P12}(r)} - 3\ ,$$

where

$$\begin{aligned} N^*_{P12}(r) \;=\;& 599\,933\,276\,250 - 2\,617\,890\,660\,000\,r + 4\,970\,166\,270\,300\,r^2 \\ & - 5\,546\,727\,078\,200\,r^3 + 59\,041\,720\,498\,845\,r^4 - 161\,234\,870\,633\,760\,r^5 \\ & + 126\,074\,334\,149\,694\,r^6 + 2\,238\,307\,939\,140\,r^7 + 25\,296\,348\,317\,400\,r^8 \\ & - 57\,875\,913\,071\,352\,r^9 - 89\,078\,826\,937\,116\,r^{10} + 180\,941\,306\,693\,040\,r^{11} \\ & - 102\,607\,682\,886\,720\,r^{12} + 19\,713\,391\,884\,288\,r^{13} \\ D^*_{P12}(r) \;=\;& \left(36\,300 - 79\,200\,r + 219\,255\,r^2 - 171\,160\,r^3 + 29\,472\,r^4\right)^2\ . \end{aligned}$$

As can be seen in Figure 3(a), the Taylor property is now achieved for all considered parameterizations of Model (3.1). From Figure 3(b), we conclude that the process X is always leptokurtic.

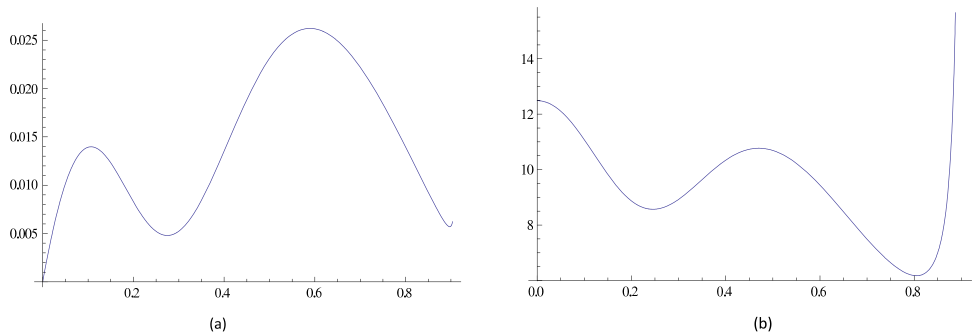


Figure 3: Graphs from $\rho_X(1) - \rho_{X^2}(1)$ (a) and $K_{P12}(r)$ (b), $0 < r < \sqrt[4]{\frac{2}{3}}$.

3.2.4. Error process with Pareto density $f(x) = \dfrac{9\alpha^9}{x^{10}}\,\mathbb{I}_{]\alpha,+\infty[}(x)$

We have

$$\beta^4\mu_4 < 1 \quad\Longleftrightarrow\quad 0 < r < \sqrt[4]{\tfrac{5}{9}} \simeq 0.863 \quad \text{and}$$

$$\rho_X(1) \;=\; \frac{8\,r\,(15\,680 - 27\,720\,r + 39\,564\,r^2 - 27\,864\,r^3 + 6\,561\,r^4)}{47\,040 - 105\,840\,r + 343\,119\,r^2 - 315\,504\,r^3 + 73\,791\,r^4}$$

$$\rho_{X^2}(1) \;=\; \frac{r}{48}\,\frac{N_{P9}(r)}{D_{P9}(r)}\ ,$$

with

$$
\begin{aligned}
N_{P9}(r) \;=\; & -67\,737\,600 - 83\,339\,200\,r + 19\,038\,600\,r^2 - 88\,401\,600\,r^3 \\
& - 148\,138\,920\,r^4 - 511\,287\,075\,r^5 + 1\,466\,330\,040\,r^6 + 1\,499\,354\,145\,r^7 \\
& - 1\,537\,629\,480\,r^8 - 1\,966\,005\,837\,r^9 - 602\,608\,896\,r^{10} \\
& + 3\,869\,347\,563\,r^{11} - 61\,620\,912\,r^{1}2 - 2\,818\,841\,796\,r^{1}3 + 1\,179\,090\,432\,r^{14}
\end{aligned}
$$

$$
\begin{aligned}
D_{P9}(r) \;=\; & -627\,200 + 235\,200\,r - 1\,650\,600\,r^2 - 8\,601\,600\,r^3 - 13\,809\,280\,r^4 \\
& + 31\,729\,095\,r^5 + 27\,010\,080\,r^6 - 23\,002\,305\,r^7 - 21\,773\,448\,r^8 \\
& - 24\,182\,469\,r^9 + 58\,517\,640\,r^{10} + 9\,248\,823\,r^{11} \\
& - 50\,143\,536\,r^{12} + 19\,665\,504\,r^{13}\;.
\end{aligned}
$$

The Taylor property is also present for all considered parameterizations of Model (3.1), as it is illustrated in Figure 4(a), and we point out that the magnitude of the difference $\rho_X(1) - \rho_{X^2}(1)$ is greater in this case than in the case $\nu = 12$.

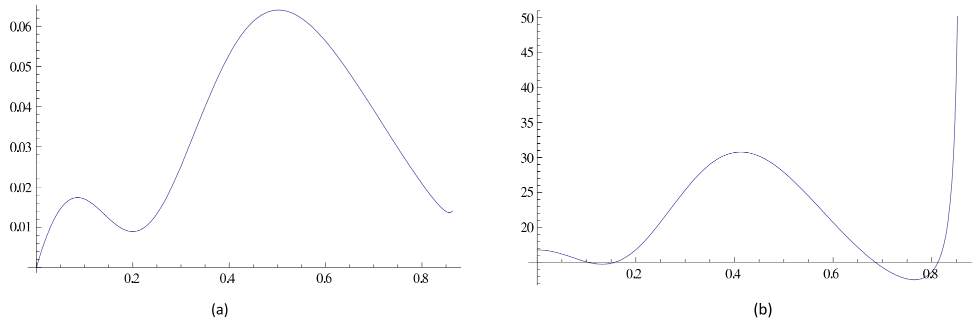


Figure 4: Graphs from $\rho_X(1) - \rho_{X^2}(1)$ (a) and $K_{P9}(r)$ (b), $0 < r < \sqrt[4]{\frac{5}{9}}$.

The kurtosis of Model (3.1) is now given by

$$
K_{P9}(r) \;=\; \frac{7 - 9r^2}{9(-2+3r^3)\,(-5+9r^4)}\,\frac{N^*_{P9}(r)}{D^*_{P9}(r)} - 3\;,
$$

where

$$
\begin{aligned}
N^*_{P9}(r) \;=\; & 62\,449\,049\,600 - 281\,020\,723\,200\,r + 532\,657\,440\,000\,r^2 - 582\,241\,598\,400\,r^3 \\
& + 25\,718\,506\,014\,670\,r^4 - 92\,872\,063\,045\,440\,r^5 + 100\,396\,353\,649\,230\,r^6 \\
& - 6\,337\,711\,636\,725\,r^7 - 8\,536\,591\,340\,550\,r^8 - 41\,782\,534\,519\,365\,r^9 \\
& - 62\,336\,742\,758\,694\,r^{10} + 195\,729\,014\,255\,481\,r^{11} \\
& - 145\,385\,404\,543\,008\,r^{12} + 35\,664\,808\,109\,193\,r^{13}
\end{aligned}
$$

$$
D^*_{P9}(r) \;=\; \left(15\,680 - 35\,280\,r + 114\,373\,r^2 - 105\,168\,r^3 + 24\,597\,r^4\right)^2\;.
$$

The process X is also leptokurtic for all considered values of r. We observe that the kurtosis of the process X is greater when $\nu = 9$ than when $\nu = 12$, corresponding to an analogous relation between the kurtosis of the respective error processes. In these two examples, it is seen again how the Taylor property emerges when the process X is leptokurtic.

As regards the Pareto distribution, graphic representations for several values of ν also suggest that the presence of the Taylor property is stronger for higher values of the kurtosis of the process X. In fact, as functions of ν, the difference $\rho_X(1) - \rho_{X^2}(1)$ seems to increase when $K_{P\nu}(r)$ increases, for all values of r that satisfy the condition $\beta^4\mu_4 < 1$. This situation is illustrated in Figure 5 and strongly contributes to conjecture that the Taylor property and leptokurtosis are highly related in time series.

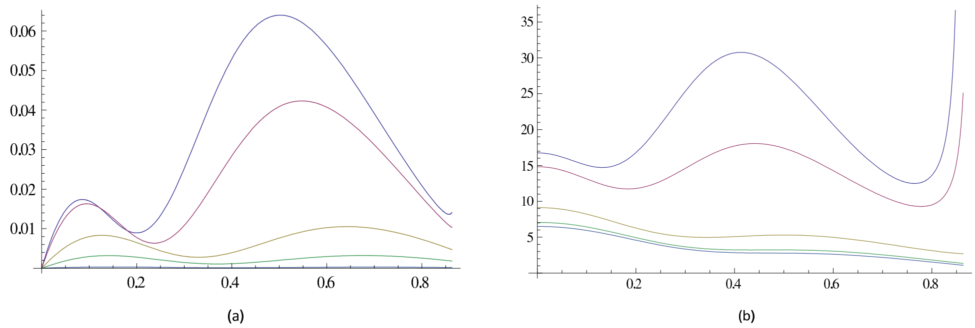


Figure 5: Graphs from $\rho_X(1) - \rho_{X^2}(1)$ (a) and $K_{P\nu}(r)$ (b), $\nu = 100, 50, 20, 10, 9$ (from bottom to top), $0 < r < \sqrt[4]{\frac{5}{9}}$.

4. THE TAYLOR PROPERTY IN THE CASE OF SYMMETRICALLY DISTRIBUTED ERRORS: SIMULATION STUDY

When the errors are symmetrically distributed, the autocorrelation function of X^2 for Model (1.1) verifies $\rho_{X^2}(1) = 0$, if $k > 1$ (Martins, [6]). So, in this case, the property $\rho_{|X|}(1) > \rho_{X^2}(1)$ is equivalent to $\rho_{|X|}(1) > 0$. However, the autocorrelation function of the process $(|X_t|,\, t \in \mathbb{Z})$ is not available when the error process is allowed to assume negative values. To investigate the presence of the Taylor property in Model (3.1) with symmetrically distributed errors, we perform a simulation study considering the simple first-order bilinear diagonal model with an i.i.d. error process $(\varepsilon_t,\, t \in \mathbb{Z})$ with four symmetrical distributions with unit variance, namely, the uniform distribution in $]-\sqrt{3}, \sqrt{3}[$, the standard normal distribution, and the distribution of a variable $\varepsilon = \sqrt{\frac{\nu-2}{\nu}}\, Y$, where Y has a Student distribution with ν degrees of freedom ($\nu = 30$ and $\nu = 9$). In each case, the condition $E(|\ln|\varepsilon_t||) < +\infty$ is satisfied and parameterizations that satisfy $\beta^4\mu_4 < 1$ are considered in the simulations. For each value of the parameter β and each one of the considered distributions, we generate 500 observations according to the corresponding model and obtain the 95% confidence intervals for the probability that such a model satisfies the Taylor property. The results appear in Table 1 (where NA means "Not Applicable", due to the fact that the corresponding value of β does not satisfy the condition $\beta^4\mu_4 < 1$). The special values 0.69, 0.74, 0.75 and 0.863 are the greatest values of β such that $\beta^4\mu_4 < 1$ for each one of the considered distributions.

Table 1: 95% confidence intervals for the probability that the model with symmetrical innovations presents the Taylor property.

β	$U(]-\sqrt{3},\sqrt{3}[)$	$N(0,1)$	$\sqrt{\frac{14}{15}}\,Y,\ Y\sim T(30)$	$\sqrt{\frac{7}{9}}\,Y,\ Y\sim T(9)$
0.01	$[0.373, 0.627]$	$[0.459, 0.708]$	$[0.459, 0.708]$	$[0.476, 0.724]$
0.05	$[0.357, 0.610]$	$[0.373, 0.627]$	$[0.373, 0.627]$	$[0.407, 0.660]$
0.1	$[0.140, 0.360]$	$[0.292, 0.541]$	$[0.214, 0.453]$	$[0.260, 0.506]$
0.2	$[0, 0]$	$[0, 0.105]$	$[0, 0.049]$	$[0, 0.049]$
0.3	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0.079]$
0.4	$[0, 0]$	$[0, 0]$	$[0, 0.079]$	$[0.260, 0.506]$
0.5	$[0, 0]$	$[0.155, 0.379]$	$[0.292, 0.541]$	$[0.699, 0.901]$
0.6	$[0, 0]$	$[0.566, 0.801]$	$[0.603, 0.831]$	$[0.781, 0.953]$
0.69	$[0, 0]$	$[0.802, 0.965]$	$[0.802, 0.965]$	$[0.951, 1]$
0.74	$[0, 0.079]$	$[0.847, 0.987]$	$[0.870, 0.996]$	NA
0.75	$[0.004, 0.130]$	$[0.847, 0.987]$	NA	NA
0.863	$[0.566, 0.801]$	NA	NA	NA

We can observe that the Taylor property seems to be present for high values of β and that this presence increases with the kurtosis of the error process, as we have established and observed in non-negative bilinear models.

The confidence intervals corresponding to small values of β do not allow us to infer about the presence of the Taylor property, as they certainly correspond to values of β for which the difference $\rho_X(1) - \rho_{X^2}(1)$ is close to zero.

5. CONCLUSIONS

In this paper, we analyze the presence of the Taylor property in first-order bilinear time series models. For this analysis we evaluate the autocorrelations of the process X and of X^2. Considering X non-negative, we discuss the presence of the Taylor property taking several distributions for the error process, chosen according to the kurtosis value as this property is strongly related with the value of this parameter. More precisely, the Taylor property seems to emerge when the process X is leptokurtic.

Based on a simulation study, we also analyze the presence of the Taylor property in the class of real valued first-order bilinear diagonal models with symmetrical innovations.

The studies presented here show that bilinear models are able to reproduce the Taylor effect. They also reinforce the connection of the Taylor property to leptokurtic models which has been observed in the few theoretical studies developed until now. In fact, He and Teräsvirta ([5]), Gonçalves, Leite and Mendes-Lopes ([1]) and Haas ([4]) show the presence of this property in some conditional heteroskedastic models, which are leptokurtic processes. Moreover, all the cases considered in this paper also show that, when the Taylor property occurs, the model is leptokurtic.

We still observe that leptokurtosis is not enough to induce the Taylor property. Examples of bilinear models that are leptokurtic but do not have the Taylor property are $X_t = X_{t-1}\varepsilon_{t-1} + \varepsilon_t$, where ε_t is uniformly distributed in $[0, 1]$, and $X_t = 0.5\,X_{t-1}\varepsilon_{t-1} + \varepsilon_t$, where ε_t is exponentially distributed with mean 0.2. This is in line with the simulation results of He and Teräsvirta ([5]) suggesting that the Taylor property is not present for the standard GARCH$(1, 1)$ process with normal errors.

In conclusion, our study allows to conjecture that a general assessment of the Taylor property in the bilinear process is strongly dependent on the magnitude of its tails weight.

6. APPENDIX

A working example to illustrate the results of Section 2, namely evaluation of $E(X_t^n \varepsilon_t^n)$ and $E(X_t^n)$, $n \le 4$, for a first-order bilinear process is now presented.

Let us suppose that ε_t, $t \in \mathbb{Z}$, is exponentially distributed with density $f(x) = \frac{1}{\alpha}\, e^{-x/\alpha}\, \mathbb{I}_{]0,+\infty[}(x)$. Then $\mu_n = n!\,\alpha^n$, $n \in \mathbb{N}$. In this case, the condition $\beta^4 \mu_4 < 1$ is equivalent to $0 < r < \frac{1}{\sqrt[4]{24}}$, where $r = \alpha\beta$. Under this hypothesis, and taking into account that ε_t is independent of $X_{t-1}^n \varepsilon_{t-1}^n$, $t \in \mathbb{Z}$, and that the process $(X_t \varepsilon_t,\, t \in \mathbb{Z})$ is strictly stationary, we have

$$E(X_t \varepsilon_t) \;=\; E(\beta X_{t-1}\varepsilon_{t-1}\varepsilon_t) + E(\varepsilon_t^2) \;=\; \beta E(X_t \varepsilon_t)\, \mu_1 + \mu_2$$

which is equivalent to

$$E(X_t \varepsilon_t) \;=\; \frac{2\alpha^2}{1-r}\;. \tag{6.1}$$

Then, by (2.3), we have

$$
\begin{aligned}
E(X_t^2\varepsilon_t^2) &= \frac{1}{1-\beta^2\mu_2}\big(2\beta\mu_3\,E(X_t\varepsilon_t)+\mu_4\big)\\
&= 24\,\alpha^4\,\frac{1}{(1-r)\,(1-2r^2)}\ .
\end{aligned}
\tag{6.2}
$$

Taking into account (2.3), (6.1) and (6.2), we now obtain

$$
\begin{aligned}
E(X_t^3\varepsilon_t^3) &= \frac{1}{1-\beta^3\mu_3}\big(3\beta^2\mu_4\,E(X_t^2\varepsilon_t^2)+3\beta\mu_5\,E(X_t\varepsilon_t)+\mu_6\big)\\
&= \frac{1}{1-6r^3}\left(\frac{1728\,\alpha^6r^2}{(1-r)\,(1-r^2)}+\frac{720\,\alpha^6r}{1-r}+720\,\alpha^6\right)\\
&= 144\,\alpha^6\,\frac{2r^2+5}{(1-r)\,(1-2r^2)\,(1-6r^3)}\ .
\end{aligned}
\tag{6.3}
$$

Finally, we evaluate $E(X_t^4\varepsilon_t^4)$ using (2.3), (6.1), (6.2) and (6.3).

$$
\begin{aligned}
E(X_t^4\varepsilon_t^4) &= \frac{1}{1-\beta^4\mu_4}\big(4\beta^3\mu_5\,E(X_t^3\varepsilon_t^3)+6\beta^2\mu_6\,E(X_t^2\varepsilon_t^2)+4\beta\mu_7\,E(X_t\varepsilon_t)+\mu_8\big)\\
&= \frac{1}{1-24\,r^4}\left(\frac{69120\,\alpha^8\,r^3\,(2r^2+5)}{(1-r)\,(1-r^2)\,(1-r^3)}\right.\\
&\quad\left.+\ \frac{103680\,\alpha^8r^2}{(1-r)\,(1-r^2)}+\frac{40320\,\alpha^8r}{1-r}+40320\,\alpha^8\right)\\
&= 5760\,\alpha^8\,\frac{18r^3+4r^2+7}{(1-r)\,(1-2r^2)\,(1-6r^3)\,(1-24r^4)}\ .
\end{aligned}
\tag{6.4}
$$

The values of $E(X_t^n)$, $n\le 4$, are then given by (2.2). More precisely,

$$
E(X_t) \;=\; \beta E(X_t\varepsilon_t)+\mu_1 \;=\; \alpha\,\frac{1+r}{1-r}\ ,
\tag{6.5}
$$

$$
\begin{aligned}
E(X_t^2) &= \beta^2E(X_t^2\varepsilon_t^2)+2\,\mu_1E(X_t\varepsilon_t)+\mu_2\\
&= \frac{24\,\alpha^4\beta^2}{(1-r)\,(1-2r^2)}+\frac{4\alpha^3\beta}{1-r}+2\,\alpha^2\\
&= 2\alpha^2\,\frac{1+r+10r^2-2r^3}{(1-r)\,(1-2r^2)}\ ,
\end{aligned}
\tag{6.6}
$$

$$
\begin{aligned}
E(X_t^3) &= \beta^3E(X_t^3\varepsilon_t^3)+3\,\beta^2\mu_1E(X_t^2\varepsilon_t^2)+3\,\beta\mu_2\,E(X_t\varepsilon_t)+\mu_3\\
&= \frac{144\,\alpha^6\beta^3(2r^2+5)}{(1-r)\,(1-2r^2)\,(1-6r^3)}+\frac{72\,\alpha^5\beta^2}{(1-r)\,(1-2r^2)}+\frac{12\,\alpha^4\beta}{1-r}+6\,\alpha^3\\
&= 6\alpha^3\,\frac{1+r+10r^2+112r^3-6r^4-12r^5+12r^6}{(1-r)\,(1-2r^2)\,(1-6r^3)}
\end{aligned}
\tag{6.7}
$$

and

$$
\begin{aligned}
E(X_t^4) &= \beta^4 E(X_t^4\varepsilon_t^4) + 4\beta^3\mu_1 E(X_t^3\varepsilon_t^3) + 6\beta^2\mu_2 E(X_t^2\varepsilon_t^2) + 4\beta\mu_3 E(X_t\varepsilon_t) + \mu_4 \\
&= \frac{15\,760\,\alpha^8\beta^4(8r^3+4r^2+7)}{(1-r)\,(1-2r^2)\,(1-6r^3)\,(1-24r^4)} + \frac{576\,\alpha^7\beta^3(2r^2+5)}{(1-r)\,(1-2r^2)\,(1-6r^3)} \\
&\quad + \frac{288\,\alpha^6\beta^2}{(1-r)\,(1-2r^2)} + \frac{48\,\alpha^5\beta}{1-r} + 24\,\alpha^4 \\
&= \frac{24\,\alpha^4\,D(r)}{(1-r)\,(1-2r^2)\,(1-6r^3)\,(1-24r^4)}\,,
\end{aligned}
\tag{6.8}
$$

with

$$
\begin{aligned}
D(r) \;=\; & 1 + r + 10\,r^2 + 112\,r^3 + 1650\,r^4 - 36\,r^5 + 732\,r^6 + 1632\,r^7 \\
& + 144\,r^8 + 288\,r^9 - 288\,r^{10}\;.
\end{aligned}
$$

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