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EXACT FORMULAS FOR THE MOMENTS OF THE FIRST PASSAGE TIME OF REWARD PROCESSES

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Abstract:

- Let $\{\mathcal{Z}_\rho(t),\ t \geq 0\}$ be a reward process based on a semi-Markov process $\{\mathcal{J}(t),\ t \geq 0\}$ and a reward function ρ. Let T_z be the first passage time of $\{\mathcal{Z}_\rho(t),\ t \geq 0\}$ from $\mathcal{Z}_\rho(0) = 0$ to a prespecified level z. In this article we provide the Laplace transform of the $E[T_z^k]$ and obtain the exact formulas for ET_z, ET_z^2 and $\mathrm{var}(T_z)$. Formulas for certain type I counter models are given.

Key-Words:

- *Semi-Markov process; reward process; Laplace transform; first passage time.*

AMS Subject Classification:

- 49A05, 78B26.

1. INTRODUCTION

Let $\{\mathcal{J}(t),\ t \geq 0\}$ be a semi-Markov process with a Markov renewal process $\{(\mathcal{J}_n, \mathcal{T}_n),\ n = 0, 1, 2, ...\}$. The state space of $\{\mathcal{J}_n\}$ is assumed to be $\mathcal{N} = \{0, 1, 2, ..., N\}$. A reward process is a certain functional that is defined on a semi-Markov process (Markov renewal process) by

$$\mathcal{Z}_\rho(t) \;=\; \sum_{n:\ \mathcal{T}_{n+1}<t} \rho(\mathcal{J}_n,\, \mathcal{T}_{n+1} - \mathcal{T}_n) \;+\; \rho(\mathcal{J}(t), X(t))\ , \tag{1}$$

where $X(t)$ is the age process. The function ρ in (1) is a real function of two variables; $\rho : \mathcal{N} \times R \to R$, and $\rho(i, \tau)$ measures the excess reward when time τ is spent in the state i. The process $\mathcal{Z}_\rho(t)$ given by (1) provides the cumulative reward at time t, under the given reward function ρ. This process was introduced and studied in [4], for general ρ. For $\rho(i, \tau) = i\tau$, the reward process $\mathcal{Z}_\rho(t)$ has been treated by different authors, see [1] [2] [5]. Let T_z be the first passage time of $\mathcal{Z}_\rho(t)$ from $\mathcal{Z}_\rho(0) = 0$ to a prespecified level z. Asymptotic behaviors of ET_z, ET_z^2 as $z \to \infty$, were obtained in [5] for $\rho(i, x) = ix$, and in [3] for general ρ. In this article we provide exact formulas for ET_z, ET_z^2 and $\mathrm{var}(T_z)$, under general ρ. We apply our formulas to certain type I counter models and provide precise results. The main results are Theorems 2.1, 3.1, Corollary 3.1, Remark 3.1, and formulas (23), (24).

2. NOTATION AND PRELIMINARIES

Let $\{\mathcal{J}(t),\ t \geq 0\}$ be a semi-Markov process and $\{(\mathcal{J}_n, \mathcal{T}_n),\ n = 0, 1, 2, ...\}$ be a Markov renewal process, where $\mathcal{J}_n$ is a Markov chain in discrete time on state space $\mathcal{N} = \{0, 1, 2, ..., N\}$, and $\mathcal{T}_n$ is the n-th transition epoch with $\mathcal{T}_0 = 0$. The behavior of the Markov renewal process is governed by a semi-Markov matrix $A(x) = [A_{ij}(x)]$, where

$$A_{ij}(x) \;=\; P\Big\{\, \mathcal{J}_{n+1} = j,\ \mathcal{T}_{n+1} - \mathcal{T}_n \;\leq\; x \;\mid\; \mathcal{J}_n = i \Big\}\ . \tag{2}$$

We assume that the stochastic matrix $P = [P_{ij}] = A(\infty)$ governing the embedded Markov chain $\{\mathcal{J}_n : n = 0, 1, 2, ...\}$ is aperiodic and irreducible. For convenience let,

$$\begin{aligned} A_{k:ij} \;&=\; \int_0^\infty x^k A_{ij}(dx)\ , \\ A_{k:i} \;&=\; \int_0^\infty x^k A_i(dx)\ , \qquad k = 0, 1, 2, \ldots\ , \end{aligned} \tag{3}$$

if they exist, where

$$A_i(x) = \sum_{j \in \mathcal{N}} A_{ij}\ , \qquad \overline{A}_i(x) = 1 - A_i(x)\ .$$

We note that $A_i(x) = P\{\mathcal{T}_{n+1} - \mathcal{T}_n \le x \mid \mathcal{J}_n = i\}$ is the cumulative distribution function of the dwell time of the semi-Markov process at state i, and $\overline{A}_i(x)$ is the corresponding survival function. Let $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$. We define,

$$A_D(x) = [\delta_{ij} A_j(x)] \ , \qquad \overline{A}_D(x) = [\delta_{ij} \overline{A}_j(x)] \ ,$$
$$A_k = [A_{k:ij}], \quad A_{D:k} = [\delta_{ij} A_{k:i}] \ , \qquad k = 0, 1, 2, \dots \ . \tag{4}$$

Note that $A_{D:0} = I$. The Laplace–Stieltjes transform of $A(x)$ is denoted by

$$\alpha(s) = [\alpha_{ij}(s)] \ , \qquad \alpha_{ij}(s) = \int_0^\infty e^{-sx} A_{ij}(dx) \ , \tag{5}$$

Laplace–Stieltjes transforms $\alpha_i(s)$, $\alpha_D(s)$, etc. are defined similarly. We define n-fold convolution $A(x)$ by

$$A^{(n)}(x) \ = \ \int_0^x A(dx')\, A^{(n-1)}(x - x') \ ,$$

$$A^{(0)}_{jk}(t) \ = \ \begin{cases} 0 & \text{if } t < 0 \\ \delta_{jk} & \text{if } t \ge 0 \end{cases}$$

and

$$A^{(n)}_{jk}(t) \ = \ \begin{cases} 0 & \text{if } t < 0 \\ \sum_\nu \int_0^t A_{j\nu}(dy)\, A^{(n-1)}_{\nu k}(t - y) & \text{if } t \ge 0 \end{cases}$$

if M is a matrix of measures and N is a matrix of measurable functions, the convolution of M and N (written $M * N$) is defined by $M * N(t) = [(M * N)_{ik}(t)]$, where

$$M * N_{jk}(t) \ = \ \sum_\nu \int_0^t M_{j\nu}(dy)\, N_{\nu k}(t - y) \ .$$

Let $A(x)$ be a semi-Markov matrix. Then

$$\mathcal{A}(x) \ = \ \sum_{n=0}^{\infty} A^{(n)}(x)$$

is called the Markov renewal matrix corresponding to $A(x)$. Also denote the Laplace transform of the Markov renewal matrix by

$$\mathcal{L}_s[\mathcal{A}] = \frac{1}{s}\, [I - \alpha(s)]^{-1} \ .$$

The transition probability matrix of $J(t)$ is denoted by $P(t)$, i.e.,

$$P(t) = [P_{ij}(t)] \ , \qquad P_{ij}(t) = P\Big\{ J(t) = j \ \mid \ J(0) = i \Big\} \ . \tag{6}$$

The state probability vector at time t, $p'(t) = (\ p_0(t),\ p_1(t), ..., p_N(t)\ )$, is given by $p'(t) = p'(0)P(t)$, where $p'(0)$ is the initial probability vector. In this article $\underline{e}$ is the unit vector, i.e., $\underline{e} = (1, ..., 1)'$.

Let $X(t)$ be the age process, i.e., the time elapsed at time t since the last transition of $J(t)$, $X(t) = t - \mathcal{T}_n$, where $n = \sup\{m : \mathcal{T}_m \le t\}$. The joint distributions corresponding to the bivariate process $\{(\mathcal{J}(t), X(t)),\ t \ge 0\}$ and the trivariate process $\{(\mathcal{J}(t), X(t), \mathcal{Z}_\rho(t)),\ t \ge 0\}$, respectively, are given by

$$
\begin{aligned}
G_{ij}(x,t) &= P\Big\{\mathcal{J}(t) = j,\ X(t) \le x \mid \mathcal{J}(0) = i\Big\}\ , \\
F_{ij}(x,z,t) &= P\Big\{\mathcal{J}(t) = j,\ X(t) \le x,\ \mathcal{Z}_\rho(t) \le z \mid \mathcal{J}(0) = i\Big\}\ .
\end{aligned} \tag{7}
$$

The Laplace transform of $F_{ij}(x,z,t)$ is denoted by

$$
\phi_{ij}(v,\omega,s) = \int_0^\infty \int_{-\infty}^\infty \int_0^\infty e^{-vx-\omega z-st}\, F_{ij}(dx,dz,t)\, dt\ , \tag{8}
$$

in the matrix form $\phi(v,\omega,s) = [\phi_{ij}(v,\omega,s)]$. It is demonstrated in [4] that the following informative transform formula plays a crucial role in studying the statistical properties of the reward process (1), see also [5],

$$
\phi(v,\omega,s) = [I - C(\omega,s)]^{-1}\, E_D(\omega, v+s)\ , \tag{9}
$$

where

$$
\begin{aligned}
C(w,s) &= [C_{kj}(\omega,s)]\ , \qquad C_{kj}(w,s) = \int_0^\infty e^{-\omega\rho(k,x)-sx} A_{kj}(dx)\ , \\
E_D(\omega,s) &= [\delta_{kj} E_j(\omega,s)]\ , \qquad E_j(\omega,s) = \int_0^\infty e^{-\omega\rho(j,x)-sx}\, \overline{A}_j(x)\, dx\ .
\end{aligned} \tag{10}
$$

Let z be a given level, then the first passage time of the level z for $\mathcal{Z}_\rho(t)$, given $\mathcal{Z}_\rho(0) = 0$, is defined by

$$
T_z = \inf\Big\{t > 0 : \mathcal{Z}_\rho(t) = z \mid \mathcal{Z}_\rho(0) = 0\Big\}\ .
$$

Clearly

$$
P\{T_z > t\} = P\{\mathcal{Z}_\rho(t) < z\}\ . \tag{11}
$$

Let $H(z,t)$ be the distribution of T_z, and denote the Laplace transform of $E[e^{-sT_z}]$ by

$$
\psi(\omega,s) = \int_0^\infty e^{-\omega z}\, E[e^{-sT_z}]\, dz\ . \tag{12}
$$

Similarly we denote the Laplace transform of the survival function $\overline{H}(z,t) = P\{T_z > t\}$ by $\overline{\psi}(\omega,s)$. We recall from [5] that,

$$
\psi(\omega,s) = 1 - s\,\overline{\psi}(\omega,s)\ , \tag{13}
$$

where $s \in D_0 = \{u\colon \mathrm{Re}(u) > 0\}$ and $\omega \in \mathrm{Im} = \{u\colon u = it,\ t \in R\}$. The following theorem was provided in [2] for $\rho(i,x) = ix$, and in [3] for general ρ.

Theorem 2.1.

$$\overline{\psi}(\omega,s) \;=\; \frac{1}{\omega}\,p'(0)\,[I-C(\omega,s)]^{-1}\,E_D(\omega,s)\,\underline{e}\;, \qquad \omega,s\in D_0\;. \tag{14}$$

For deriving the moments of T_z, we first note that

$$\left(\frac{\partial}{\partial s}\right)^k \overline{\psi}(\omega,s) \;=\; (-1)^k \int_0^\infty e^{-\omega z}\left(\int_0^\infty t^k\, e^{-st} P\{T_z>t\}\,dt\right)dz\;,$$

$$(-1)^k\,(k+1)\left(\frac{\partial}{\partial s}\right)^k \overline{\psi}(\omega,s)\big|_{s=0} \;=\; \int_0^\infty e^{-\omega z} E[T_z^{k+1}]\,dz\;,$$

$0\le k\le K$, where $E[T_z^{K+1}]<\infty$ is assumed. Hence from the formula given above and Theorem 2.1,

$$\int_0^\infty e^{-\omega z}\,E[T_z^{k+1}]\,dz \;=\; (k+1)\,(-1)^k\,\frac{1}{\omega}\,p'(0)\left\{\left(\frac{\partial}{\partial s}\right)^k \phi(0,\omega,s)\big|_{s=0}\right\}\underline{e}\;. \tag{15}$$

In next section we use (15) to derive exact formulas for $E[T_z^k]$.

3. EXACT FORMULAS

In this section we apply (15) in order to derive formulas for ET_z, ET_z^2, and $\mathrm{var}(T_z)$. Throughout this section we assume that ρ satisfies the following condition.

(**A**) For each k, $\rho(k,x)\colon[0,\infty)\to[0,\infty)$ is one to one, admits a continuously differential inverse, and $\rho(k,0)=0$.

We also introduce the following matrices:

$$F(t) \;=\; \Big[\delta_{kj}(\rho^{-1}(k,t))'\,\overline{A}_k(\rho^{-1}(k,t))\Big]\;,$$

$$\mathcal{B}(t) \;=\; \sum_{n=0}^{\infty} B^{(n)}(t)\;,$$

$$K(t) \;=\; \Big[\delta_{kj}\,\rho^{-1}(k,t)\,(\rho^{-1}(k,t))'\,\overline{A}_k(\rho^{-1}(k,t))\Big]\;,$$

$$D(t) \;=\; \left[\int_0^t \rho^{-1}(k,x)\,dB_{kj}(x)\right],$$

where B is the matrix with entries $B_{kj}(z)=A_{kj}(\rho^{-1}(k,z))$ and $B^{(n)}$ is n-fold convolution of B.

Theorem 3.1. *Let T_z be the first passage time of the reward process $\mathcal{Z}_\rho(t)$, $t \geq 0$, given by (1) with a reward function $\rho(k,x)$, $k \in \mathcal{N}$, $x \geq 0$, that satisfying condition (A). If $\mathcal{B}(t)$ exist then*

$$\textbf{(a)} \qquad ET_z \;=\; p'(0)\left\{\int_0^z \mathcal{B} * F(x)\,dx\right\}\underline{e}\;,$$

$$\textbf{(b)} \qquad ET_z^2 \;=\; 2\,p'(0)\left\{\int_0^z \mathcal{B}{*}K(x)\,dx\right\}\underline{e} \,+\, 2\,p'(0)\left\{\int_0^z \mathcal{B} * D * \mathcal{B} * F(x)\,dx\right\}\underline{e}\;,$$

$$\textbf{(c)} \quad \operatorname{var}(T_z) \;=\; 2\,p'(0)\left\{\int_0^z \mathcal{B} * K(x)\,dx\right\}\underline{e} \,+\, 2\,p'(0)\left\{\int_0^z \mathcal{B} * D * \mathcal{B} * F(x)\,dx\right\}\underline{e} \\ -\left\{p'(0)\left\{\int_0^z \mathcal{B} * F(x)\,dx\right\}\underline{e}\right\}^2 .$$

Proof: (**a**): By using (15) and (9) we obtain that

$$\mathcal{L}_\omega(ET_z) \;=\; \frac{1}{\omega}\,p'(0)\,[I - C(\omega,0)]^{-1}\,E_D(\omega,0)\,\underline{e}\;, \tag{16}$$

where

$$C(\omega,0) = [C_{kj}(\omega,0)]\;,$$

with

$$C_{kj}(\omega,0) \;=\; \int_0^\infty e^{-\omega\rho(k,x)}\,dA_{kj}(x)\;.$$

Now for each k, j, let $B_{kj}(\Delta) = A_{kj}\{x \in [0,\infty)\colon\, \rho(k,x) \in \Delta\}$, $\Delta \subset [0,\infty)$, then $B_{kj}(.)$ is a probability distribution on $[0,\infty)$ and it follows by change of variable that,

$$C_{kj}(\omega,0) \;=\; \int_0^\infty e^{-\omega t}\,dB_{kj}(t) \\ =\; \beta_{kj}(\omega)\;.$$

Therefore $C_{kj}(\omega,0)$ is the Laplace transform of the distribution B_{kj}, and in matrix form

$$[I - C(\omega,0)]^{-1} \;=\; [I - \beta(\omega)]^{-1}\;. \tag{17}$$

Also note that

$$E_D(\omega,0) = [\delta_{ij}E_j(\omega,0)]\;,$$

where

$$E_j(\omega,0) \;=\; \int_0^\infty e^{-\omega\rho(j,x)}\,\overline{A}_j(x)\,dx\;,$$

and it follows by change of variable that

$$E_j(\omega,0) \;=\; \int_0^\infty e^{-\omega t}(\rho^{-1}(j,t))'\,\overline{A}_j(\rho^{-1}(j,t))\,dt \\ =\; \int_0^\infty e^{-\omega t}F(j,t)\,dt\;.$$

Therefore in matrix form we have

$$E_D(\omega,0) \;=\; \int_0^\infty e^{-\omega t} F(t)\,dt \;. \tag{18}$$

If we replace (17) and (18) in (16) we obtain

$$\begin{aligned}\mathcal{L}_\omega(ET_z) \;&=\; p'(0)\,\frac{1}{\omega}\,[I-\beta(\omega)]^{-1}\,\mathcal{L}_\omega(F(t))\,\underline{e}\\ &=\; p'(0)\,\frac{1}{\omega}\,\mathcal{L}_\omega(\mathcal{B}(t))\,\mathcal{L}_\omega(F(t))\,\underline{e}\;,\end{aligned}$$

or equivalently

$$ET_z \;=\; p'(0)\left\{\int_0^z \mathcal{B} * F(t)\,dt\right\}\underline{e}\;,$$

giving (a).

(**b**): It follows from (15) that

$$\mathcal{L}_\omega E[T_z^2] \;=\; -2\,\frac{1}{\omega}\;p'(0)\left\{\frac{\partial}{\partial s}\phi(0,\omega,s)\Big|_{s=0}\right\}\underline{e}\;. \tag{19}$$

But from (9),

$$\begin{aligned}\frac{\partial\phi(0,\omega,s)}{\partial s} \;&=\; [I-C(\omega,s)]^{-1}\,\frac{\partial C(\omega,s)}{\partial s}\;[I-C(\omega,s)]^{-1}\,E_D(\omega,s)\\ &\quad+\;[I-C(\omega,s)]^{-1}\,\frac{\partial E_D(\omega,s)}{\partial s}\;,\end{aligned} \tag{20}$$

where

$$\begin{aligned}C_{kj}(\omega,s) \;&=\; \int_0^\infty e^{-\omega\rho(k,x)-sx}\,dA_{kj}(x)\;,\\ \frac{\partial C_{kj}(\omega,s)}{\partial s}\Big|_{s=0} \;&=\; -\int_0^\infty x\,e^{-\omega\rho(k,x)}\,dA_{kj}(x)\;.\end{aligned}$$

Again it follows by change of variable that

$$\frac{\partial C_{kj}(\omega,s)}{\partial s}\Big|_{s=0} \;=\; -\int_0^\infty e^{-\omega t}\rho^{-1}(k,t)\,dB_{kj}(t)\;.$$

Therefore in matrix form

$$\begin{aligned}\frac{\partial C(\omega,s)}{\partial s}\Big|_{s=0} \;&=\; -\int_0^\infty e^{-\omega t}\rho_D^{-1}(t)\,dB(t)\\ &=\; -\mathcal{L}_\omega(D)\;,\end{aligned} \tag{21}$$

where

$$\begin{aligned}D(\Delta) \;&=\; \int_\Delta \rho_D^{-1}(t)\,dB(t)\;,\\ \rho_D^{-1}(t) \;&=\; \left[\delta_{kj}\,\rho^{-1}(k,t)\right]\;.\end{aligned}$$

On the other hand

$$\frac{\partial E_k(\omega,s)}{\partial s}\Big|_{s=0} = -\int_0^\infty x\, e^{-\omega\rho(k,x)}\, \overline{A}_k(x)\, dx \ ,$$

and using change of variable

$$\begin{aligned}\frac{\partial E_k(\omega,s)}{\partial s}\Big|_{s=0} &= -\int_0^\infty e^{-\omega t}\rho^{-1}(k,t)\,(\rho^{-1}(k,t))'\ \overline{A}_k(\rho^{-1}(k,t))\ dt \\ &= -\int_0^\infty e^{-\omega t} K(k,t)\, dt\ .\end{aligned}$$

Therefore in matrix form

$$\begin{aligned}\frac{\partial E_D(\omega,s)}{\partial s}\Big|_{s=0} &= -\int_0^\infty e^{-\omega t} K(t)\, dt \\ &= -\mathcal{L}_\omega(K)\ .\end{aligned} \tag{22}$$

By replacing (17), (18), (21) and (22) in (20), we obtain from (19) that

$$\begin{aligned}\mathcal{L}_\omega(ET_z^2) &= 2\,p'(0)\,\frac{1}{\omega}\,[I-\beta(\omega)]^{-1}\,\mathcal{L}_\omega(D(t))\,[I-\beta(\omega)]^{-1}\mathcal{L}_\omega(F(t))\,\underline{e} \\ &\quad + 2\,p'(0)\,\frac{1}{\omega}\,[I-\beta(\omega)]^{-1}\mathcal{L}_\omega(K(t))\,\underline{e}\ ,\end{aligned}$$

or

$$ET_z^2 = 2\,p'(0)\left\{\int_0^z \mathcal{B} * K(x)\,dx\right\}\underline{e} + 2\,p'(0)\left\{\int_0^z \mathcal{B} * D * \mathcal{B} * F(x)\,dx\right\}\underline{e}\ .$$

Part (**c**) Follows from (a) and (b). □

Corollary 3.1. *Let $\rho(k,x) = g_n(k)x^n,\ k \in \mathcal{N},\ x \in [0,\infty)$ and $g_n(k) > 0$. If $\mathcal{B}(t)$ exists, then the formulas (a), (b) and (c) of Theorem 3.1 are satisfied. Moreover*

$$\begin{aligned}F(t) &= \left[\delta_{ij}\,\frac{1}{n\,\sqrt[n]{\rho_j\, t^{n-1}}}\ \overline{A}_j\left(\sqrt[n]{\frac{t}{\rho_j}}\right)\right]\ , \\ B(t) &= [B_{ij}]\ , \qquad B_{ij}(t) = A_{ij}\left(\sqrt[n]{\frac{t}{\rho_j}}\right)\ , \\ K(t) &= \left[\delta_{ij}\,\frac{1}{n\,\sqrt[n]{\rho_j^2\, t^{n-2}}}\ \left(1 - A_j\left(\sqrt[n]{\frac{t}{\rho_j}}\right)\right)\right]\ , \\ D(t) &= \left[\int_0^t \frac{1}{n\,\sqrt[n]{\rho_j^2\, x^{n-2}}}\, dA_{ij}\left(\sqrt[n]{\frac{x}{\rho_j}}\right)\right]\ .\end{aligned}$$

Proof: The reward function satisfies condition (A), therefore Theorem 3.1 can be applied. □

Remark 3.1. Let $n = 1$ in Corollary 3.1, i.e., the reward function is linear. Then Corollary 3.1 holds with $n = 1$.

4. APPLICATIONS TO CERTAIN TYPE I COUNTERS MODELS

Arrivals at a counter form a Poisson process with rate q. An arriving particle that finds the counter free gets registered and locks it for a random duration with distribution function $F(t)$. Arrivals during a locked periods have no effect whatsover. Suppose a registration occurs at $T_0 = 0$, and write $T_0, T_1, T_2, ...$ for the successive epochs of changes in the state of the counter. Write $X_n = 1$ or 0 according as the n-th change locks or frees the counter. Clearly $X_0 = 1$, $X_1 = 0$, $X_2 = 1$, $X_3 = 0$, ... and (X_n, T_n) is a Markov renewal process. Its semi-Markov matrix is

$$A(x) = \begin{bmatrix} 0 & 1-e^{-qx} \\ F(x) & 0 \end{bmatrix}.$$

Let $F(x) = 1 - e^{-2qx}$ and $\mathcal{Z}_\rho(t)$ be the reward process that is defined by (1) with reward function $\rho(k, x) = \rho_k\, x$, $\rho_0 = 1$, $\rho_1 = 2$. Let T_z be the first passage time reward process $\mathcal{Z}_\rho(t)$ from $\mathcal{Z}_\rho(0) = 0$ to a prespecified level z. We apply the formulas of the previous section to give explicit expressions for ET_z and ET_z^2. Note that for each k, j

$$\begin{aligned} B_{kj}(t) &= A_{kj}\left(\frac{t}{\rho_k}\right), \\ B(t) &= \begin{bmatrix} 0 & 1-e^{-qt} \\ 1-e^{-qt} & 0 \end{bmatrix}, \\ B^{(0)}(t) &= I\ . \end{aligned}$$

By induction it follows that

$$B^{(2n+1)}(t) = \begin{bmatrix} 0 & B_{01}^{(2n+1)} \\ B_{10}^{(2n+1)} & 0 \end{bmatrix},$$

where

$$B_{01}^{(2n+1)} = B_{10}^{(2n+1)} = 1 - e^{-qt} - qte^{-qt} - \frac{q^2\, t^2}{2!}e^{-qt} - \cdots - \frac{q^{2n}\, t^{2n}}{2\, n!}e^{-qt}\ ,$$

and

$$B^{(2n)}(t) = \begin{bmatrix} B_{00}^{(2n)} & 0 \\ 0 & B_{11}^{(2n)} \end{bmatrix},$$

$$B_{00}^{(2n)} = B_{11}^{(2n)} = 1 - e^{-qt} - qte^{-qt} - \frac{q^2\,t^2}{2}e^{-qt} - \frac{q^3\,t^3}{3!}e^{-qt} - \cdots - \frac{q^{2n-1}\,t^{2n-1}}{(2n-1)!}e^{-qt} .$$

Therefore

$$\begin{aligned}
\mathcal{B}_{00}(t) &= \sum_{n=0}^{\infty} B_{00}^{(n)}(t) = 1 + \sum_{n=1}^{\infty}\left[1 - e^{-qt}\sum_{k=0}^{2n-1}\frac{(qt)^k}{k!}\right], \\
\mathcal{B}_{00}(t) &= \mathcal{B}_{11}(t) , \\
\mathcal{B}_{01}(t) &= \sum_{n=0}^{\infty} B_{01}^{(n)}(t) = \sum_{n=0}^{\infty}\left[1 - e^{-qt}\sum_{k=0}^{2n}\frac{(qt)^k}{k!}\right], \\
\mathcal{B}_{01}(t) &= \mathcal{B}_{10}(t) , \\
\mathcal{B}_{00}(t) &= 1 + \sum_{n=0}^{\infty}\Big[1 - P(Y \le 2n+1)\Big] \\
&= 1 + \sum_{n=0}^{\infty} P(Y > 2n+1) , \\
\mathcal{B}_{01}(t) &= \sum_{n=0}^{\infty}\Big[1 - P(Y \le 2n)\Big] \\
&= \sum_{n=0}^{\infty} P(Y > 2n) ,
\end{aligned}$$

where Y is a Poisson random variable with $\lambda = qt$. Therefore

$$\mathcal{B}(t) = \begin{bmatrix} 1 + \sum_{n=0}^{\infty} P(Y > 2n+1) & \sum_{n=0}^{\infty} P(Y > 2n) \\ \sum_{n=0}^{\infty} P(Y > 2n) & 1 + \sum_{n=0}^{\infty} P(Y > 2n+1) \end{bmatrix}.$$

The derivation of $\mathcal{B}(t)$ can be simplified by noting that if

$$p_k = \frac{\lambda^k\,e^{-\lambda}}{k!}$$

where $\lambda = qt$, then

$$P_E \equiv P\{Y \text{even}\} = \sum_{k\in\{0,2,4,\ldots\}} p_k$$

and

$$P_O \equiv P\{Y \text{odd}\} = \sum_{k\in\{1,3,5,\ldots\}} p_k$$

implying that (after simplication)

$$\begin{aligned}\sum_{n=0}^{\infty} P(Y > 2n+1) &= (p_2 + 2p_4 + 3p_6 + ...) + (p_3 + 2p_5 + 3p_7 + ...) \\ &= \frac{\lambda}{2} P_O + \left\{ \frac{\lambda}{2}(P_E - e^{-\lambda}) - \frac{1}{2}(P_O - \lambda e^{-\lambda}) \right\} \\ &= \frac{\lambda}{2} - \frac{P_O}{2} ,\end{aligned}$$

similarly

$$\begin{aligned}\sum_{n=0}^{\infty} P(Y > 2n) &= (p_1 + 2p_3 + 3p_5 + ...) + (p_2 + 2p_4 + 3p_6 + ...) \\ &= \frac{1}{2}\left\{ \lambda P_E + P_O \right\} + \frac{\lambda}{2} P_O \\ &= \frac{\lambda}{2} + \frac{P_O}{2} .\end{aligned}$$

Now if $P(s) = \sum_{k=0}^{\infty} p_k \, s^k = e^{-\lambda + \lambda s}$, then

$$\begin{aligned}P(1) &= p_0 + p_1 + p_2 + p_3 + ... = 1 = P_O + P_E , \\ P(-1) &= p_0 - p_1 + p_2 - p_3 + ... = e^{-2\lambda} = P_E - P_O ,\end{aligned}$$

implying $P_E = \frac{1}{2}(1 + e^{-2\lambda})$ and $P_O = \frac{1}{2}(1 - e^{-2\lambda})$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty} P(Y > 2n+1) &= \frac{\lambda}{2} - \frac{1}{4} + \frac{e^{-2\lambda}}{4} = \frac{qt}{2} - \frac{1}{4} + \frac{e^{-2qt}}{4} , \\ \sum_{n=0}^{\infty} P(Y > 2n) &= \frac{\lambda}{2} + \frac{1}{4} - \frac{e^{-2\lambda}}{4} = \frac{qt}{2} + \frac{1}{4} - \frac{e^{-2qt}}{4} ,\end{aligned}$$

and

$$\mathcal{B}(t) = \begin{bmatrix} \frac{qt}{2} + \frac{3}{4} + \frac{e^{-2qt}}{4} & \frac{qt}{2} + \frac{1}{4} - \frac{e^{-2qt}}{4} \\ \frac{qt}{2} + \frac{1}{4} - \frac{e^{-2qt}}{4} & \frac{qt}{2} + \frac{3}{4} + \frac{e^{-2qt}}{4} \end{bmatrix} ,$$

$$F(t) = \begin{bmatrix} e^{-qt} & 0 \\ 0 & \frac{1}{2} e^{-qt} \end{bmatrix} , \qquad K(t) = \begin{bmatrix} te^{-qt} & 0 \\ 0 & \frac{t}{4} e^{-qt} \end{bmatrix} ,$$

$$dD(t) = \begin{bmatrix} 0 & qte^{-qt} \\ \frac{qt}{2} e^{-qt} & 0 \end{bmatrix} ,$$

$$\mathcal{B} * F(t) = \int_0^t d\mathcal{B}(x) \, F(t-x) ,$$

hence

$$\mathcal{B} * F(t) = \begin{bmatrix} \frac{1}{2}\{1 - 2e^{-qt} + e^{-2qt}\} & \frac{1}{4}\{1 - e^{-2qt}\} \\ \frac{1}{2}\{1 - e^{-2qt}\} & \frac{1}{4}\{1 - 2e^{-qt} + e^{-2qt}\} \end{bmatrix},$$

and

$$\int_0^z \mathcal{B} * F(x)\,dx = \begin{bmatrix} \frac{z}{2} - \frac{3}{4q} + \frac{1}{q}e^{-qz} - \frac{1}{4q}e^{-2qz} & \frac{z}{4} - \frac{1}{8q} + \frac{1}{8q}e^{-2qz} \\ \frac{z}{2} - \frac{1}{4q} + \frac{1}{4q}e^{-2qz} & \frac{z}{4} - \frac{3}{8q} + \frac{1}{2q}e^{-qz} - \frac{1}{8q}e^{-2qz} \end{bmatrix}.$$

In the example $X_0 = 1$, the initial probability vector is clearly $p'(0) = (1,0)$, then

$$ET_z = \frac{3}{4}z - \frac{7}{8q} + \frac{1}{q}\,e^{-qz} - \frac{1}{8q}\,e^{-2qz}\,. \tag{23}$$

$$\mathcal{B} * D * \mathcal{B} * F(x) = \begin{bmatrix} \mathcal{B} * D * \mathcal{B} * F_{00}(x) & \mathcal{B} * D * \mathcal{B} * F_{01}(x) \\ \mathcal{B} * D * \mathcal{B} * F_{10}(x) & \mathcal{B} * D * \mathcal{B} * F_{11}(x) \end{bmatrix},$$

where

$$\mathcal{B} * D * \mathcal{B} * F_{00}(x) = \frac{1}{8}\left\{3x - \frac{9}{q} + \frac{9}{q}e^{-2qx} + 12xe^{-qx} + 3xe^{-2qx}\right\},$$

$$\mathcal{B} * D * \mathcal{B} * F_{01}(x) = \frac{1}{16}\left\{3x - \frac{10}{q} + \frac{12}{q}e^{-qx} - \frac{10}{q}e^{-2qx} - 3xe^{-2qx} + 4qx^2e^{-qx}\right\},$$

$$\mathcal{B} * D * \mathcal{B} * F_{10}(x) = \frac{1}{8}\left\{3x - \frac{8}{q} + \frac{16}{q}e^{-qx} - \frac{8}{q}e^{-2qx} - 3xe^{-2qx} + 2qx^2e^{-qx}\right\},$$

$$\mathcal{B} * D * \mathcal{B} * F_{11}(x) = \frac{1}{16}\left\{3x - \frac{9}{q} + \frac{9}{q}e^{-2qx} + 12xe^{-qx} + 3xe^{-2qx}\right\}.$$

Also

$$\mathcal{B} * K(x) = \begin{bmatrix} \frac{1}{2}\left\{\frac{1}{q} - \frac{1}{q}e^{-2qx} - 2xe^{-qx}\right\} & \frac{1}{8}\left\{\frac{1}{q} - \frac{2}{q}e^{-qx} + \frac{1}{q}e^{-2qx}\right\} \\ \frac{1}{2}\left\{\frac{1}{q} + \frac{1}{q}e^{-2qx} - \frac{2}{q}e^{-qx}\right\} & \frac{1}{8}\left\{\frac{1}{q} - 2xe^{-qx} - \frac{1}{q}e^{-2qx}\right\} \end{bmatrix}.$$

If we replace $\mathcal{B} * D * \mathcal{B} * F(x)$ and $\mathcal{B} * K(x)$ in formula (b) of Corollary 3.1, we get

(24)

$$ET_z^2 = \frac{1}{16}\left\{9z^2 - \frac{36}{q}z + \frac{103}{2q^2} - \frac{48}{q^2}e^{-qz} - \frac{7}{2q^2}e^{-2qz} - \frac{32}{q}ze^{-qz} + \frac{3}{q}ze^{-2qz} + 8z^2e^{-qz}\right\}.$$

Remark 4.1. The asymptotic behaviors of ET_z, ET_z^2 were derived in [5] for $\rho(k,x) = \rho_k x$, and in [3] for general ρ. For the case considered in the Example given above,

$$ET_z = \frac{m_1}{m_1^{**}}z + p'(0)\left\{H_0^{**}A_{D:1} - \frac{1}{2}H_1^{**}\rho_{D:1}A_{D:2}\right\}\underline{e} + o(1)\,,$$

$$ET_z^2 = \left\{\frac{m_1}{m_1^{**}}\right\}^2 z^2 - p'(0)\left\{2\,V_1^{**}A_{D:1} - [V_2^{**} + H_1^{**}A_{D:2}]\right\}\underline{e}\,z + o(z)\,,$$

as $z \to \infty$, where $m_1 = \pi' A_1 \underline{e}$, $\rho_{D:1}$ = diagonal matrix of ρ_i,

$$B_k = \rho_{D:k} A_k \ , \qquad m_1^{**} = \pi' B_1 \underline{e} \ , \qquad H_1^{**} = \frac{1}{m_1^{**}} \underline{e} \pi' \ , \qquad Z_0 = [I - P + e\pi']^{-1} \ ,$$

$$H_0^{**} = \frac{1}{m_1^{**}} \underline{e} \, \pi' \left\{ -B_1 + \frac{1}{2m_1^{**}} B_2 \, \underline{e} \, \pi' \right\} + \left\{ Z_0 - \frac{1}{m_1^{**}} \underline{e} \, \pi' B_1 Z_0 \right\} \left\{ P - \frac{1}{m_1^{**}} B_1 \, \underline{e} \, \pi' \right\},$$

$$V_1^{**} = (H_1^{**} \rho_{D:1} A_2 - H_0^{**} A_1) H_1^{**} - H_1^{**} A_1 H_0^{**} \ ,$$

$$V_2^{**} = -H_1^{**} A_1 H_1^{**} \rho_{D:1} A_{D:2} \ .$$

For the semi-Markov $A(x)$ defined above

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad A_1 = \begin{bmatrix} 0 & \frac{1}{q} \\ \frac{1}{2q} & 0 \end{bmatrix} , \qquad A_{D:1} = \begin{bmatrix} \frac{1}{q} & 0 \\ 0 & \frac{1}{2q} \end{bmatrix} .$$

$$A_2 = \begin{bmatrix} 0 & \frac{2}{q^2} \\ \frac{1}{2q^2} & 0 \end{bmatrix} , \qquad A_{D:2} = \begin{bmatrix} \frac{2}{q^2} & 0 \\ 0 & \frac{1}{2q^2} \end{bmatrix} ,$$

$$\pi' P = \pi' \implies \pi' = (0.5, \, 0.5) \ ,$$

$$m_1 = \pi' A_1 \, \underline{e} = \frac{3}{4q} \ ,$$

$$\rho_{D:1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} ,$$

$$B_1 = \rho_{D:1} \, A_1 = \begin{bmatrix} 0 & \frac{1}{q} \\ \frac{1}{q} & 0 \end{bmatrix} ,$$

$$m_1^{**} = \pi' B_1 \, \underline{e} = \frac{1}{q} \ ,$$

$$B_2 = \rho_{D:2} \, A_2 = \begin{bmatrix} 0 & \frac{2}{q^2} \\ \frac{2}{q^2} & 0 \end{bmatrix} ,$$

$$Z_0 = [I - P + e \, \pi']^{-1} \ ,$$

therefore

$$Z_0 = \frac{1}{2} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} ,$$

$$H_1^{**} = \frac{1}{m_1^{**}} \underline{e} \, \pi' \ ,$$

therefore

$$H_1^{**} = q \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} ,$$

$$H_0^{**} = \frac{1}{m_1^{**}} \, \underline{e} \, \pi' \left\{ -B_1 + \frac{1}{2m_1^{**}} \, B_2 \, \underline{e} \, \pi' \right\} + \left\{ Z_0 - \frac{1}{m_1^{**}} \, \underline{e} \, \pi' \, B_1 Z_0 \right\} \left\{ P - \frac{1}{m_1^{**}} \, B_1 \, \underline{e} \, \pi' \right\},$$

hence

$$H_0^{**} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix},$$

$$V_1^{**} = (H_1^{**} \rho_{D:1} A_2 - H_0^{**} A_1) \, H_1^{**} - H_1^{**} A_1 H_0^{**} ,$$

therefore

$$V_1^{**} = \begin{bmatrix} \frac{12}{16} & \frac{14}{16} \\ \frac{10}{16} & \frac{12}{16} \end{bmatrix},$$

$$V_2^{**} = -H_1^{**} A_1 H_1^{**} \rho_{D:1} A_{D:2} ,$$

$$V_2^{**} = -\begin{bmatrix} \frac{3}{4q} & \frac{3}{8q} \\ \frac{3}{4q} & \frac{3}{8q} \end{bmatrix}.$$

In the example, $X_0 = 1$, so that the initial probability vector is clearly $p'(0) = (1, 0)$. Then by replacing values in ET_z, ET_z^2, we have

$$\begin{aligned} ET_z &= \frac{3}{4} z - \frac{7}{8q} + o(1) , \\ ET_z^2 &= \frac{9}{16} z^2 - \frac{18}{8q} z + o(z) , \end{aligned}$$

as $z \to \infty$, which also can be observed from the formulas (23), (24), as $z \to \infty$.

Remark 4.2. If one wishes to compare ET_z with the asymptotic behaviour it is sensible to allow for a general initial probability vector say $p'(0) = (p_0(0), \; p_1(0))$. In this case

$$\begin{aligned} ET_z &= \frac{3}{4} z - \frac{7p_0(0) + 5p_1(0)}{8q} + \frac{2p_0(0) + p_1(0)}{2q} \, e^{-qz} - \frac{p_1(0) + p_0(0)}{8q} \, e^{-2qz} \\ &= \frac{3}{4} z - \frac{7p_0(0) + 5p_1(0)}{8q} + o(1) . \end{aligned}$$

This last result is also obtained for the asymptotic expression for ET_z with a general initial probability vector.

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