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CONTROL CHARTS FOR MULTIVARIATE NONLINEAR TIME SERIES

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Abstract:

- In this paper control charts for the simultaneous monitoring of the means and the variances of multivariate nonlinear time series are introduced. The underlying target process is assumed to be a constant conditional correlation process (cf. [3]). The new schemes make use of local measures of the means and the variances based on current observations, conditional moments, or residuals. Exponential smoothing and cumulative sums are applied to these characteristic quantities. Distances between these quantities and target values are measured by the Mahalanobis distance. The introduced schemes are compared via a simulation study. As a measure of performance the average run length is used.

Key-Words:

- *statistical process control; multivariate CUSUM charts; multivariate EWMA charts; conditional correlation model.*

AMS Subject Classification:

- 62L10, 62M10, 62P20, 91B84.

1. INTRODUCTION

Nowadays, the methods of statistical process control (SPC) are frequently used in order to detect changes in model parameters of multivariate time series. Many applications can be found in various scientific fields, e.g. engineering, economics, medicine, and environmental sciences. The main idea of SPC is to detect deviations of an observed process from a predefined target process as soon as possible after their occurrence. The most important tools of SPC are control charts. A control chart consists of the control statistic and the control limits. The data are sequentially examined. If at a certain point of time the control statistic lies within the control limits, we conclude the process is still in control and the procedure continues at the next point of time. If the control statistic exceeds the control limit, the algorithm stops and the process is considered to be out of control.

In current literature on SPC the underlying multivariate process is assumed to consist of independent random vectors and the parameter of interest is chosen to be the mean vector of the process (cf. [4, 14, 12, 13, 8]). Mean charts for multivariate time series are considered in [17, 11, 1]. In [10] multivariate control charts for nonlinear autocorrelated processes are introduced using the support vector regression approach.

The monitoring of the covariance matrix of multivariate time series is discussed only in a few papers. In [15, 16] several types of exponentially weighted moving average (EWMA) charts are proposed. The underlying process is assumed to be either a multivariate Gaussian process or a multivariate GARCH process in the sense of [5].

However, in the present paper the aim is to jointly monitor the means and the variances of multivariate time series. The target process is assumed to be a constant conditional correlation (CCC) model (cf. [3]). A CCC process is a multivariate nonlinear time series turning out to be quite attractive for practical purposes. Although the amount of parameters is not too high, the model is still sufficiently flexible.

Subsequently, we introduce several new control charts. They are based on combining local measures for the means and the variances of the target process or the residual process, e.g. current observations and conditional variances with an EWMA recursion or a cumulative sum (CUSUM). In order to avoid the curse of dimensionality variances are monitored using the squared Euclidean distance of present observations and residuals from their in-control mean as well as the trace of the conditional covariance matrix. These schemes seem to be very useful even for higher dimensions. Via an extensive Monte Carlo study the charts are compared with schemes proposed in [6]. In order to assess the performance of the schemes the average run length (ARL) is applied.

The paper is structured as follows. In Sect. 2 we describe the CCC model and the modeling of the out-of-control process. In Sect. 3 characteristic quantities are presented. Their in-control means as well as their in-control covariances are derived. In Sect. 4 multivariate EWMA and CUSUM type schemes based on the former characteristic quantities are described. In Sect. 5 the results of the simulation study are presented. Finally, we draw a conclusion in Sect. 6.

2. THE MODEL

Below, the target process is denoted by $\{Y_t\}$ and the observed process by $\{X_t\}$. Next, we describe the modeling of both processes.

2.1. Modeling dynamics of the conditional covariance matrix

The p-dimensional target process $\{Y_t\}$ assumed to be a nonlinear CCC model is given by

$$Y_t \;=\; \mu + \Sigma_t^{1/2} \varepsilon_t \;. \tag{2.1}$$

Hence, ε_t is assumed to be an independent and normally distributed random sequence with zero mean and a covariance matrix equal to the identity matrix. Further, μ denotes the overall mean vector which is independent of time. The matrix $\Sigma_t = \mathrm{Cov}(Y_t \,|\, \mathcal{I}_{t-1}) = E_{t-1}\big[(Y_t-\mu)\,(Y_t-\mu)'\big]$ denotes the conditional covariance matrix of $\{Y_t\}$ conditioned on the information set $\mathcal{I}_{t-1}$. $\mathcal{I}_{t-1}$ is equal to the smallest σ-algebra generated by $Y_{t-1}, Y_{t-2}, \ldots$. As a consequence,

$$Y_t \,|\, \mathcal{I}_{t-1} \;\sim\; \mathcal{N}_p\left(\mu, \Sigma_t\right) \;, \tag{2.2}$$

i.e. the conditional distribution of $Y_t = (Y_{1t}, \ldots, Y_{pt})'$ is a normal distribution.

The CCC model is introduced in [3] where the conditional correlation matrix is assumed to be time invariant. The conditional covariance matrix of Y_t is given by

$$\Sigma_t \;=\; \mathrm{D}_t\, \mathrm{R}\, \mathrm{D}_t \;= \big(\sigma_{it}\, \varrho_{ij}\, \sigma_{jt}\big)_{i,j=1,\ldots,p} \tag{2.3}$$

with $\sigma_{it}^2 = \mathrm{Var}(Y_{it} \,|\, \mathcal{I}_{t-1})$ for $i = 1, \ldots, p$ and $\mathrm{D}_t = \mathrm{diag}(\sigma_{1t}, \ldots, \sigma_{pt})$. The conditional variances of Y_{it} usually follow a GARCH model (cf. [2])

$$\sigma_{it}^2 \;=\; \omega_i + \sum_{m=1}^{M_i} \alpha_{im}\, (Y_{i,t-m} - \mu_i)^2 + \sum_{n=1}^{N_i} \beta_{in}\, \sigma_{i,t-n}^2 \qquad \forall\, i \in \{1, \ldots, p\} \;. \tag{2.4}$$

The conditional correlation matrix $\mathrm{R} = (\varrho_{ij})_{i,j=1,\ldots,p}$ of Y_t is assumed to be time invariant and positive definite.

A unique weakly stationary solution of (2.1), strictly stationary and ergodic as well, exists if the polynomials fulfill the condition

$$1 - \sum_{m=1}^{M_i} \alpha_{im}\, z^m - \sum_{n=1}^{N_i} \beta_{in}\, z^n \neq 0 \qquad \text{for } \; |z| \leq 1 \tag{2.5}$$

with $i = 1, ..., p$ (cf. [9]). Moreover, $E(Y_t) = \mu$ and

$$\mathrm{Var}(Y_{it}) \;=\; \frac{\omega_i}{1 - \sum\limits_{m=1}^{M_i} \alpha_{im} - \sum\limits_{n=1}^{N_i} \beta_{in}} \qquad \forall\, i \in \{1, ..., p\}\ . \tag{2.6}$$

2.2. Modeling the out-of-control behavior

We are faced with a sequential problem where we check at each point of time t whether a shift in mean or variances has occurred or not. The respective decision problem is therefore

$$\begin{array}{ccc} H_{0,t}:\ E(X_t) = \mu & \wedge & \mathrm{Cov}(X_t) = \Sigma \\ & \text{against} & \\ H_{1,t}:\ E(X_t) \neq \mu & \vee & \mathrm{Cov}(X_t) \neq \Sigma \end{array} \tag{2.7}$$

where $\Sigma = \mathrm{Cov}(Y_t)$. The relationship between the target process $\{Y_t\}$ and the observed process $\{X_t\}$ is given by

$$X_t = \begin{cases} Y_t & \text{for } \; t < \tau \\ \mu + a + \Delta(Y_t - \mu) & \text{for } \; t \geq \tau \end{cases} . \tag{2.8}$$

The parameters $a \in \mathbb{R}^p \backslash \{0\}$ and $\Delta = \mathrm{diag}(d_1, d_2, ..., d_p) \neq \mathrm{I}_p$, where I_p denotes the identity matrix of dimension $p \times p$, are unknown. Here we focus on the detection of increases in variances. For that reason we assume that $d_i \geq 1$ for $i = 1, ..., p$. The position of the change point is denoted by $\tau \in \mathbb{N} \cup \{\infty\}$. If a change is present, i.e. $\tau < \infty$, the process is said to be out of control. Hence, changes in mean or in variances can be observed. On the contrary, $\tau = \infty$ means that the change point never occurs and therefore the process is in control.

3. CHARACTERISTIC QUANTITIES

In order to monitor the means and the variances of a multivariate process we need several local measures for these characteristics. We reduce the number of characteristic quantities because we monitor the sum of variances. Since we are exclusively interested in detecting increases of variances, the following procedures can be applied. Below, the characteristic quantities are denoted by T_t. We derive several properties of these quantities in this section (see Propositions 3.1 to 3.3). In order to shorten the paper we do not present the proofs here. They can be found in [7]. Further, the notation $\mu = (\mu_i)_{i=1,\dots,p}$, $\Sigma_t = (\sigma_{ij,t})$, $\Sigma = (\sigma_{ij})$, and $\sigma_i^2 = \sigma_{ii} = \mathrm{Var}(Y_{it})$, $i = 1, \dots, p$, is used.

3.1. Characteristics based on current observations and residuals

As already mentioned, current observations are local measures for the mean vector. To monitor the covariance matrix we can use the squared Euclidean distance between X_t and μ. This leads to

$$T_t^{(1)} = \begin{pmatrix} X_t - \mu \\ (X_t - \mu)' (X_t - \mu) \end{pmatrix} . \tag{3.1}$$

Further, the in-control mean vector and the in-control covariance matrix have to be derived. If $\{Y_t\}$ is a weakly stationary process with the mean μ and the covariance matrix $\mathrm{Var}(Y_t) = \Sigma = (\sigma_{ij})$,

$$E\big(T_t{}^{(1)}\big) = \begin{pmatrix} 0 \\ \sum\limits_{i=1}^{p} \sigma_i^2 \end{pmatrix} \quad \text{for} \ \ t < \tau \qquad \text{and} \qquad E\big(T_t{}^{(1)}\big) = \begin{pmatrix} a \\ \sum\limits_{i=1}^{p} d_i^2 \sigma_i^2 \end{pmatrix} \quad \text{for} \ \ t \geq \tau \ .$$

Apparently, the quantity $T_t^{(1)}$ reflects changes in the mean and the variances of $\{Y_t\}$ but no changes in the covariances of $\{Y_t\}$. If values smaller than 1 are permitted for d_i, values larger and smaller than 1 might overlap. As a consequence, a change in $\sum_{i=1}^{p} d_i^2 \sigma_i^2$ is not observed. Since we only consider increases in variances, we avoid this problem. Next, the underlying target process is a CCC process.

Proposition 3.1. *Assume that (2.1) and (2.2) hold and that $E(Y_{it}^4) < \infty$ for all i and t. Then*

$$\mathrm{Cov}_{\tau=\infty}\big(T_t^{(1)}\big) = \begin{pmatrix} \Sigma & 0 \\ 0' & a_{1t} \end{pmatrix} \quad \text{and} \quad \mathrm{Cov}_{\tau=\infty}\big(T_s^{(1)}, T_t^{(1)}\big) = \begin{pmatrix} \mathrm{O}_{p\times p} & 0 \\ c_{1,st} & b_{1,st} \end{pmatrix}$$

for $s < t$ where

$$a_{1t} = \sum_{i=1}^{p} \sum_{j=1}^{p} \Big[E_{\tau=\infty}\big(\sigma_{it}^2 \sigma_{jt}^2 + 2\, \sigma_{ij,t}^2\big) - \sigma_i^2 \sigma_j^2 \Big] ,$$

$$b_{1,st} = \sum_{i=1}^{p} \sum_{j=1}^{p} \Big[E_{\tau=\infty}\big(X_{is} - \mu_i\big)^2 \sigma_{jt}^2 - \sigma_i^2 \sigma_j^2 \Big] \quad \text{and}$$

$$c_{1,st} = E_{\tau=\infty}\left[(X_s - \mu) \sum_{i=1}^{p} \sigma_{it}^2 \right] .$$

These quantities can be explicitly calculated only for less complex processes, otherwise they have to be estimated via a simulation study. However, in order to apply these results the underlying process has to be strictly stationary. Then these quantities do not depend on t. Note that $\mathrm{Cov}_{\tau=\infty}\big(T_s^{(1)}, T_t^{(1)}\big) = \Big[\mathrm{Cov}_{\tau=\infty}\big(T_t^{(1)}, T_s^{(1)}\big)\Big]'$ so that the covariances can be computed for $s > t$ as well.

Another simple characteristic quantity is based on the transformed observed process $\eta_t = (\eta_{it})_{i=1,\ldots,p}$. In this case the respective mean and the covariance matrix of the residual vector are monitored. The residual vector is given by

$$\eta_t \;=\; \Sigma_t^{-1/2}(X_t - \mu) \;=\; \begin{cases} \varepsilon_t\,, & t < \tau\,, \\ \Sigma_t^{-1/2} a + \Sigma_t^{-1/2} \Delta \Sigma_t^{1/2}\, \varepsilon_t\,, & t \geq \tau\,. \end{cases} \tag{3.2}$$

Note that $\eta_t \,|\, \Sigma_t \sim \mathcal{N}_p\left(\Sigma_t^{-1/2} a\,,\, \Sigma_t^{-1/2} \Delta\Sigma_t\, \Delta\Sigma_t^{-1/2}\right)$. Thus, $E\big(\eta_t\big) = E\big(\Sigma_t^{-1/2}\big)\, a$.

The characteristic quantity based on residuals is given by

$$T_t^{(2)} = \begin{pmatrix} \eta_t \\ \eta_t' \eta_t \end{pmatrix} . \tag{3.3}$$

In the following proposition we compute the in-control mean and the in-control covariance matrix.

Proposition 3.2. *Assume that (2.1) and (2.2) hold and that* R *is positive definite. Then*

$$E_{\tau=\infty}\big({T_t}^{(2)}\big) \;=\; \begin{pmatrix} 0 \\ p \end{pmatrix}$$

as well as

$$\mathrm{Cov}_{\tau=\infty}\big(T_t^{(2)}\big) \;=\; \begin{pmatrix} \mathrm{I} & 0 \\ 0' & 2p \end{pmatrix} \quad \textit{and} \quad \mathrm{Cov}_{\tau=\infty}\big(T_s^{(2)}, T_t^{(2)}\big) \;=\; \mathrm{O}_{(p+1)\times(p+1)}$$

for $s \neq t$.

The application of control charts to residuals is much easier than the application to the original process, because η_t is independent and normally distributed with zero mean and a covariance matrix equal to the identity matrix as long as the process is in control. In the out-of-control state the process η_t is neither independent nor identically distributed. In Proposition 3.2 only the existence of the first two moments of the target process is required while in Proposition 3.1 the first four moments are needed.

3.2. Characteristics based on the conditional variances

Regarding characteristics based on conditional variances we compute the trace of Σ_t at each point of time t. The characteristic quantity referring to the trace of the conditional covariance matrix is given by

$$T_t^{(3)} = \begin{pmatrix} X_t - \mu \\ \operatorname{tr}(\Sigma_t) \end{pmatrix} = \begin{pmatrix} X_t - \mu \\ \sum\limits_{i=1}^{p} \sigma_{it}^2 \end{pmatrix} . \tag{3.4}$$

Note that $E_{\tau=\infty}\left(\sum_{i=1}^{p}\sigma_{it}^2\right) = \sum_{i=1}^{p} E\left[\operatorname{Var}\left(Y_{it} \,|\, \mathcal{I}_{t-1}\right)\right] = \sum_{i=1}^{p}\sigma_i^2$. Thus, the quantity is able to detect changes in variances of the target process. The derivation of the in-control mean vector and the in-control covariance matrix of $T_t^{(3)}$ is straight forward.

Proposition 3.3. *Assume that (2.1) and (2.2) hold and that* R *is positive definite then*

$$E_{\tau=\infty}\left(T_t{}^{(3)}\right) = \begin{pmatrix} 0 \\ \sum\limits_{i=1}^{p} \sigma_i^2 \end{pmatrix} .$$

If additionally $E(Y_{it}^4) < \infty$ *for all* i *and* t,

$$\operatorname{Cov}_{\tau=\infty}\left(T_t^{(3)}\right) = \begin{pmatrix} \Sigma & 0 \\ 0' & a_{3t} \end{pmatrix} \quad \text{and} \quad \operatorname{Cov}_{\tau=\infty}\left(T_s^{(3)}, T_t^{(3)}\right) = \begin{pmatrix} \mathrm{O}_{p\times p} & 0 \\ c_{3,st} & b_{3,st} \end{pmatrix}$$

for $s < t$, *where*

$$a_{3t} = \sum_{i=1}^{p}\sum_{j=1}^{p}\left[E_{\tau=\infty}\left(\sigma_{it}^2\sigma_{jt}^2\right) - \sigma_i^2\sigma_j^2\right] ,$$

$$b_{3,st} = \sum_{i=1}^{p}\sum_{j=1}^{p}\left[E_{\tau=\infty}\left(\sigma_{is}^2\sigma_{jt}^2\right) - \sigma_i^2\sigma_j^2\right]$$

and $c_{3,st} = c_{1,st}$.

As in the case of present observations these quantities can be explicitly determined only for special cases. For more general processes they can be estimated using a simulation study if the underlying target process is strictly stationary.

4. CONTROL SCHEMES FOR MULTIVARIATE TIME SERIES

In this section we propose several new control charts. They are obtained applying univariate or multivariate EWMA recursions and cumulative sums to the characteristic quantities considered in Sect. 3. The control statistics are based on the Mahalanobis distance between the weighted characteristics and the corresponding in-control means.

Since these control statistics are distance measures, the charts are one-sided. Therefore, the charts give a signal if the control statistic exceeds the control limit. The first signal occurs at a certain point of time which is said to be the run length. The control limit is usually chosen such that the in-control expectation of the run length is equal to a pre-specified value $\mathcal{A}$. In practice $\mathcal{A}$ is frequently chosen to be equal to 500. Below, the quantity T_t stands for one of the characteristic quantities $T_t^{(i)}$ for $i \in \{1, 2, 3\}$.

4.1. Multivariate EWMA charts

Here we follow the procedure in [15, 16]. We apply a multivariate EWMA (MuE) recursion to T_t. This leads to

$$Z_t \;=\; (\mathrm{I} - \Lambda)\, Z_{t-1} + \Lambda\, T_t \qquad \text{for} \quad t \geq 1\,. \tag{4.1}$$

Hence, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_p, \lambda_{p+1})$ is a diagonal matrix of smoothing parameters of dimension $(p+1) \times (p+1)$. It is assumed that $0 < \lambda_i \leq 1$ for $i \in \{1, ..., p, p+1\}$. We presume that the starting value is equal to the target value such that $Z_0 = E_{\tau=\infty}(T_t)$. The mean vector of the considered quantity Z_t is then

$$E_{\tau=\infty}(Z_t) \;=\; E_{\tau=\infty}(T_t)\;. \tag{4.2}$$

Introducing $\operatorname{Cov}_{\tau=\infty}(T_{t-i}, T_{t-j}) = \Gamma(j - i)$ and assuming $\Lambda = \lambda\, \mathrm{I}_{p+1}$ the respective limit of the covariance matrix for $t \to \infty$ simplifies to

$$\lim_{t\to\infty} \operatorname{Cov}_{\tau=\infty}(Z_t) \;=\; \frac{\lambda}{2-\lambda} \left[\Gamma(0) + \sum_{v=1}^{\infty} (1-\lambda)^v \left[\Gamma(v) + \Gamma(v)'\right]\right]. \tag{4.3}$$

Since $\{Y_t\}$ is assumed to be a strictly stationary CCC model with existing fourth moments, the quantities $T_t^{(1)}$ and $T_t^{(3)}$ are weakly stationary. Since $\operatorname{Cov}_{\tau=\infty}(\eta_s, \eta_t) = \mathrm{O}_{(p+1)\times(p+1)}$ for $s \neq t$ in the case $\Lambda = \lambda\, \mathrm{I}_{p+1}$, the respective limit equals

$$\lim_{t\to\infty} \operatorname{Cov}_{\tau=\infty}\big(Z_t^{(2)}\big) \;=\; \frac{\lambda}{2-\lambda} \begin{pmatrix} \mathrm{I}_p & 0 \\ 0' & 2p \end{pmatrix}. \tag{4.4}$$

Eventually, the control statistic equals the Mahalanobis distance between Z_t and its in-control mean. This leads to

$$T_t \;=\; \big[Z_t - E_{\tau=\infty}(Z_t)\big]' \left[\operatorname{Cov}_{\tau=\infty}(Z_t)\right]^{-1} \big[Z_t - E_{\tau=\infty}(Z_t)\big]\,. \tag{4.5}$$

In order to implement a less time-consuming procedure one may use the asymptotic covariance matrix instead of the exact one

$$T_t \;=\; \big[Z_t - E_{\tau=\infty}(Z_t)\big]' \left\{\lim_{t\to\infty} \left[\operatorname{Cov}_{\tau=\infty}(Z_t)\right]\right\}^{-1} \big[Z_t - E_{\tau=\infty}(Z_t)\big]\,. \tag{4.6}$$

On the contrary, the Mahalanobis EWMA (MaE) chart scheme is based on the Mahalanobis distance of the vector T_t from its in-control mean $E_{\tau=\infty}(T_t)$. The control statistic is specified as

$$Z_t \;=\; (1-\lambda)\, Z_{t-1} + \lambda T_t \qquad \text{for} \quad t \geq 1 \tag{4.7}$$

where

$$T_t = \left[T_t - E_{\tau=\infty}(T_t)\right]' \left[\mathrm{Cov}_{\tau=\infty}(T_t)\right]^{-1} \left[T_t - E_{\tau=\infty}(T_t)\right] . \tag{4.8}$$

The quantity $\lambda \in (0,1]$ is said to be the memory parameter. The starting value Z_0 is chosen to be equal to the in-control expectation of T_t, i.e. $Z_0 = E_{\tau=\infty}(T_t) = p+1$.

4.2. Multivariate CUSUM control schemes

In [14] two multivariate control schemes based on the cumulative sum of $\{X_t\}$ are introduced. We extend this approach to the CCC model. Regarding the first multivariate CUSUM (MC1) chart the cumulative sum is determined before computing the respective standardized distance which represents a quadratic form. The cumulative sum is specified as

$$S_{t-n_t,t} = \sum_{i=t-n_t+1}^{t} \left[T_i - E_{\tau=\infty}(T_i)\right] , \qquad t \geq 1 . \tag{4.9}$$

Accordingly, the relevant control statistic is based on a suitable norm of the cumulative sum given by

$$\|S_{t-n_t,t}\|_{\Gamma(0)} = \sqrt{S'_{t-n_t,t}\, \Gamma(0)^{-1}\, S_{t-n_t,t}} . \tag{4.10}$$

The control statistic is equal to the norm of the cumulative sum subtracted by a reference value

$$MC1_t = \max\left\{0,\, \|S_{t-n_t,t}\|_{\Gamma(0)} - k\, n_t\right\} , \qquad t \geq 1 . \tag{4.11}$$

Thus, $k \geq 0$ is said to be the reference parameter. Further, the quantity n_t denotes the number of observations since the last restart given by

$$n_t = \begin{cases} n_{t-1}+1\,, & MC1_{t-1} > 0\,, \\ 1\,, & MC1_{t-1} = 0\,, \end{cases} \tag{4.12}$$

where $t \geq 1$ with $MC1_0 = 0$.

Regarding the second multivariate CUSUM (MC2) control chart we have to determine the cumulative sum after computing the standardized distance, the Mahalanobis distance of the quantity T_t from its in-control mean

$$D_t^2 = \left[T_t - E_{\tau=\infty}(T_t)\right]' \Gamma(0)^{-1} \left[T_t - E_{\tau=\infty}(T_t)\right] . \tag{4.13}$$

Eventually, the control statistic of the second multivariate CUSUM scheme equals

$$MC2_t = \max\left\{0,\, MC2_{t-1} + D_t^2 - (p+1) - 2\,k^2\right\} \tag{4.14}$$

where $k \geq 0$ and $MC2_0 = 0$.

5. COMPARISON STUDY

We intend to jointly monitor the mean vector and the variances of a bivariate nonlinear process. Initially, a CCC model must be chosen for a simulation study. Via a Monte Carlo simulation explicitly dominated control schemes should be identified. Eventually, the detection speed of the control schemes presented in the previous section should be evaluated. As a performance measure for control charts the ARL is used. In order to compute the ARLs we implement a program written in C++. The solutions are obtained using the bisection algorithm where 10^6 Monte Carlo replications are submitted for each algorithm iteration. This algorithm is interrupted when the numerical error of the ARL is less than $\pm 10^{-6}$ or the change in the control limits does not exceed $\pm 10^{-6}$.

5.1. Configuration of the Monte Carlo study

First of all, we need to calibrate the considered control charts such that the ARL in the in-control state is equal to a pre-specified value $\mathcal{A}$. Here we choose $\mathcal{A} = 120$, i.e. we consider approximately a half a year on the capital market. In the following section we want to focus on charts based on residuals. The control limits do not depend on the parameters of the underlying target process but only on the smoothing parameter for EWMA type charts and the reference value for CUSUM type schemes. Therefore, the calculation of the control design appears to be much easier for these schemes. In this comparison study we choose the smoothing parameter $\lambda \in \{0.1, 0.2, ..., 1.0\}$ and the reference value $k \in \{0.0, 0.1, ..., 1.0\}$. Here we present our results for the target process with

$$\sigma_{1t}^2 = 0.2 + 0.2\, Y_{1,t-1}^2 + 0.1\, \sigma_{1,t-1}^2 \tag{5.1}$$

$$\sigma_{2t}^2 = 0.1 + 0.1\, Y_{2,t-1}^2 + 0.2\, \sigma_{2,t-1}^2\ . \tag{5.2}$$

The constant conditional correlation ϱ equals 0.5. We take into account shifts in the mean vector, shifts in the covariance matrix, as well as simultaneous shifts. Although we study all three cases only one table referring to the joint monitoring of means and variances is presented. Moreover, we choose $a_1 = a_2$ and $d_1 = d_2$. The elements of the vector a take the values $a_1 \in \{0.0, 0.25, ..., 2.0\}$. Further, the scale transformation Δ is specified by the parameter $d_1 \in \{1.0, 1.1, ..., 1.4\}$.

5.2. Detection of changes in means and variances

The out-of-control ARLs for the considered process are given in Table 1. The smallest out-of-control ARLs are printed in bold. The parameter values for each scheme and each type of change leading to the smallest out-of-control ARLs are presented in parentheses.

Table 1: Out-of-control ARL of the CCC model based on an in-control ARL equal to $\mathcal{A} = 120$; ARLs refer to different shifts in means and variances ($m = 10^6$, $p = 2$).

a_1/d_1	1.0		1.1		1.2		1.3		1.4	
	MuE		**36.00**	(0.3)	**18.98**	(0.3)	**11.90**	(0.3)	**8.40**	(0.3)
0.0	MaE		40.70	(0.1)	20.03	(0.1)	12.29	(0.1)	8.57	(0.1)
	MC1		44.15	(0.4)	21.85	(0.4)	13.29	(0.5)	9.20	(0.5)
	MC2		40.99	(0.2)	21.19	(0.4)	13.32	(0.6)	9.36	(0.7)
	16.73	(0.1)	13.50	(0.2)	**10.16**	(0.2)	**7.94**	(0.2)	**6.37**	(0.3)
0.25	39.81	(0.1)	20.45	(0.1)	12.80	(0.1)	9.01	(0.1)	6.84	(0.1)
	14.80	(0.3)	**12.68**	(0.3)	10.21	(0.4)	8.19	(0.5)	6.62	(0.5)
	39.55	(0.2)	21.58	(0.4)	13.90	(0.5)	9.88	(0.6)	7.49	(0.7)
	5.94	(0.2)	5.36	(0.2)	4.84	(0.2)	4.36	(0.3)	**3.91**	(0.3)
0.5	10.14	(0.1)	7.66	(0.1)	6.12	(0.1)	5.10	(0.1)	4.37	(0.1)
	5.56	(0.5)	**5.15**	(0.5)	**4.74**	(0.5)	**4.34**	(0.6)	3.95	(0.6)
	11.19	(0.5)	8.50	(0.6)	6.76	(0.7)	5.59	(0.8)	4.75	(0.8)
	3.19	(0.3)	3.01	(0.3)	2.85	(0.3)	2.71	(0.3)	**2.58**	(0.3)
0.75	4.21	(0.1)	3.70	(0.1)	3.34	(0.1)	3.05	(0.1)	2.82	(0.1)
	3.06	(0.7)	**2.94**	(0.7)	**2.81**	(0.7)	**2.70**	(0.7)	2.59	(0.7)
	4.72	(0.8)	4.11	(0.8)	3.64	(0.9)	3.29	(1.0)	3.01	(1.0)
	2.01	(0.3)	1.97	(0.4)	1.93	(0.4)	1.89	(0.4)	1.86	(0.4)
1.0	2.28	(0.1)	2.18	(0.1)	2.10	(0.1)	2.02	(0.1)	1.96	(0.1)
	1.96	(0.9)	**1.93**	(0.9)	**1.91**	(0.8)	**1.88**	(0.9)	**1.85**	(0.9)
	2.49	(1.0)	2.35	(1.0)	2.23	(1.0)	2.14	(1.0)	2.06	(1.0)
	1.42	(0.4)	1.43	(0.4)	1.44	(0.4)	**1.44**	(0.4)	1.45	(0.4)
1.25	1.49	(0.1)	1.49	(0.1)	1.49	(0.1)	1.49	(0.1)	1.48	(0.1)
	1.41	(1.0)	**1.42**	(1.0)	**1.43**	(1.0)	**1.44**	(1.0)	**1.44**	(1.0)
	1.56	(1.0)	1.56	(1.0)	1.55	(1.0)	1.54	(1.0)	1.54	(1.0)
	1.14	(0.5)	**1.16**	(0.5)	**1.18**	(0.5)	**1.20**	(0.5)	**1.21**	(0.5)
1.5	1.16	(0.1)	1.18	(0.1)	1.20	(0.1)	1.21	(0.1)	1.23	(0.1)
	1.15	(1.0)	1.17	(1.0)	**1.18**	(1.0)	**1.20**	(1.0)	1.22	(1.0)
	1.18	(1.0)	1.21	(1.0)	1.22	(1.0)	1.24	(1.0)	1.25	(1.0)
	1.04	(0.6)	**1.05**	(0.6)	**1.06**	(0.5)	**1.08**	(0.5)	**1.09**	(0.5)
1.75	**1.04**	(0.2)	**1.05**	(0.3)	1.07	(0.2)	**1.08**	(0.1)	**1.09**	(0.2)
	1.04	(1.0)	**1.05**	(1.0)	1.07	(1.0)	**1.08**	(1.0)	**1.09**	(1.0)
	1.05	(1.0)	1.06	(1.0)	1.08	(1.0)	1.09	(1.0)	1.11	(1.0)
	1.01	(0.6)	**1.01**	(0.6)	**1.02**	(0.5)	**1.03**	(0.5)	**1.03**	(0.5)
2.0	**1.01**	(0.4)	**1.01**	(0.3)	**1.02**	(0.3)	**1.03**	(0.4)	**1.03**	(0.4)
	1.01	(1.0)	**1.01**	(1.0)	**1.02**	(1.0)	**1.03**	(1.0)	1.04	(1.0)
	1.01	(1.0)	1.02	(1.0)	**1.02**	(1.0)	**1.03**	(1.0)	1.04	(1.0)

Regarding Table 1 none of the introduced charts exclusively dominates all the other schemes. For small changes in means and variances the MuE chart provides the smallest out-of-control ARLs. For medium-sized changes, the MC1 chart turns out to be the best scheme. Finally, for larger changes again the MuE scheme appears to be the best control chart. Since the deviations between MC1 and MuE are small for larger changes, the MC1 control chart seems to be the most suitable scheme.

Accordingly, the optimal values of the parameters λ and k increase with increasing mean and variance changes. If we concentrate on the MC1 and the MC2 scheme, in many situations the optimal k is on the boundary $k = 1.0$. Consequently, their performance could be improved choosing a higher upper bound for k.

Additionally, we compare these findings with the results in [6]. In that paper we propose control charts where changes in means, variances, and covariances are taken into account. The characteristic quantities of these schemes are of dimension $p(p+3)/2$. Neglecting covariances in the present approach we reduce the dimension to $p+1$. Nevertheless, the residual charts monitoring means and variances provide smaller out-of-control ARLs. Consequently, the reduction of the dimension does not lead to a loss of efficiency. In our study we analyze processes up to dimension 5. The charts seem to be useful for higher dimensions as well. However, further research is necessary in order to assess their behavior for high-dimensional processes.

6. CONCLUSION

Multivariate nonlinear time series are very attractive for practical applications in finance because of their time dependent conditional covariance matrix. In this paper we propose new control charts for the joint surveillance of the means and the variances of such processes. Therefore, characteristic quantities based on current observations and residuals as well as characteristics based on conditional variances are introduced. Several multivariate EWMA and CUSUM schemes in connection with these characteristic quantities are proposed. The dimension of the characteristics is reduced compared to the schemes proposed in [6]. As a consequence, these control charts can be applied to nonlinear processes with an explicitly higher dimension. Since the schemes based on residuals are much easier to handle, we recommend the application of residual charts. First the control design does not depend on the parameters of the target process. Therefore, the control limits can be easily determined. Second the residual schemes can be applied assuming weaker conditions on the underlying process. Eventually, financial processes usually do not fulfill conditions on higher moments.

Via a Monte Carlo simulation study we compare the detection speed of each control scheme using the out-of-control ARL as a reliable indicator. In many cases we find that the MC1 control chart appears to be the best chart for joint shifts in means and variances.

REFERENCES

[1] Bodnar, O. and Schmid, W. (2007). Surveillance of the mean behavior of multivariate time series, *Stat. Neerl.*, **61**, 1–24.

[2] Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity, *J. Econometrics*, **31**(3), 307–327.

[3] Bollerslev, T. (1990). Modeling the coherence in short-run nominal exchange rates: a multivariate generalized ARCH model, *Rev. Econ. Stat.*, **72**, 498–505.

[4] Crosier, R.B. (1988). Multivariate generalizations of cumulative sum quality control schemes, *Technometrics*, **30**, 291–303.

[5] Engle, R.F. and Kroner, K.F. (1995). Multivariate simultaneous generalized ARCH, *Economet. Theor.*, **11**, 122–150.

[6] Garthoff, R.; Okhrin I. and Schmid W. (2014). Statistical surveillance of the mean vector and the covariance matrix of nonlinear time series, *AStA – Advances in Statistical Analysis*, **98**(3), 225–255.

[7] Garthoff, R.; Okhrin, I. and Schmid, W. (2012). Control charts for multivariate nonlinear time series. European University Viadrina Frankfurt (Oder), Department of Business Administration and Economics, Discussion Paper.

[8] Golosnoy, V.; Ragulin S. and Schmid W. (2009). Multivariate CUSUM chart: properties and enhancements, *Advances in Statistical Analysis*, **93**, 263–279.

[9] Jeantheau, T. (1998). Strong consistency of estimators for multivariate ARCH models, *Economet. Theor.*, **14**, 70–86.

[10] Khediri, I.B.; Weihs, C. and Limam, C. (2010). Support vector regression control charts for multivariate nonlinear autocorrelated processes, *Chemometr. Intell. Lab.*, **103**, 76–81.

[11] Kramer, H. and Schmid, W. (1997). EWMA charts for multivariate time series, *Sequential Analysis*, **16**, 131–154.

[12] Lowry, C.A.; Woodall, W.H.; Champ, C.W. and Rigdon, S.E. (1992). A multivariate exponentially weighted moving average control chart, *Technometrics*, **34**, 46–53.

[13] Ngai, H.-M. and Zhang, J. (2001). Multivariate cumulative sum control charts based on projection pursuit, *Stat. Sinica*, **11**, 747–766.

[14] Pignatiello, J.J. and Runger, G.C. (1990). Comparison of multivariate CUSUM charts, *J. Qual. Technol.*, **22**, 173–186.

[15] Śliwa, P. and Schmid, W. (2005). Monitoring the cross-covariances of a multivariate time series, *Metrika*, **61**, 89–115.

[16] Śliwa, P. and Schmid, W. (2005). Surveillance of the covariance matrix of multivariate nonlinear time series, *Statistics*, **39**, 221–246.

[17] Theodossiou, P.T. (1993). Predicting shifts in the mean of a multivariate time series process: an application in predicting business failures, *J. Am. Stat. Assoc.*, **88**, 441–449.