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PORT-ESTIMATION OF A SHAPE SECOND-ORDER PARAMETER

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Abstract:

- In this paper we study, under a semi-parametric framework and for heavy right tails, a class of location invariant estimators of a shape second-order parameter, ruling the rate of convergence of the normalised sequence of maximum values to a non-degenerate limit. This class is based on the PORT methodology, with PORT standing for peaks over random thresholds. Asymptotic normality of such estimators is achieved under a third-order condition on the right-tail of the underlying model F and for suitable large intermediate ranks. An illustration of the finite sample behaviour of the estimators is provided through a small-scale Monte-Carlo simulation study.

Key-Words:

- *asymptotic properties; location/scale invariant estimation; Monte-Carlo simulation;* PORT *methodology; sample of excesses; semi-parametric estimation; shape second-order parameters; statistics of extremes; third-order framework.*

AMS Subject Classification:

- 62G32, 62E20; 65C05.

1. INTRODUCTION AND MOTIVATION

Let $\underline{X}_n = (X_1, ..., X_n)$ denote a random sample of n independent, identically distributed (i.i.d.) random variables (r.v.'s) with distribution function (d.f.) F. We are interested in heavy-tailed models, i.e. in d.f.'s with a regularly varying right-tail. This means that, for $\xi > 0$, the right tail-function

$$\overline{F} := 1 - F$$

is such that

$$\lim_{t\to\infty} \overline{F}(tx)/\overline{F}(t) \;=\; x^{-1/\xi}, \quad \text{for all } x > 0. \tag{1.1}$$

We then say that $\overline{F}$ is of regular variation at infinity with an index equal to $-1/\xi$, and define

$$G_\xi(x) := \begin{cases} \exp\left(-(1+\xi\ x)^{-1/\xi}\right), & 1+\xi\ x > 0, \text{ if } \xi \neq 0 \\ \exp(-\exp(-x)), & x \in \mathbb{R}, \quad\quad\quad \text{if } \xi = 0, \end{cases} \tag{1.2}$$

the general extreme-value (EV) distribution function. If (1.1) holds, we are in the domain of attraction for maxima of G_ξ, with $\xi > 0$, and we write $F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0})$, meaning that it is possible to find sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum $X_{n:n} := \max(X_1, ..., X_n)$, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate r.v. with d.f. $G_\xi(x)$, in (1.2), with $\xi > 0$. This type of heavy-tailed models arises often in practice, in fields like telecommunication traffic, finance, insurance, economics, ecology and biometry, among others. The parameter ξ, in (1.2), is the extreme-value index (EVI), one of the primary parameters of extreme events.

Let $F^{\leftarrow}$ denote the generalised inverse function of F, defined by

$$F^{\leftarrow}(t) := \inf\{x : F(x) \geq t\}, \tag{1.3}$$

and let U be the associated (reciprocal) quantile function, defined as

$$U(t) := F^{\leftarrow}(1 - 1/t), \qquad t \geq 1. \tag{1.4}$$

1.1. First and second-order conditions for heavy-tailed models

In a heavy-tailed framework, i.e. if (1.1) holds, with the usual notation RV_a for the class of regularly varying functions at infinity with an index $a \in \mathbb{R}$, and on the basis of the results in Gnedenko (1943), for the right-tail function $\overline{F} = 1 - F$, and in de Haan (1984), for U in (1.4), the following first-order conditions are equivalent,

$$F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0}) \iff \overline{F} \in RV_{-1/\xi} \iff U \in RV_{\xi}. \tag{1.5}$$

Now we need to say something about the rate of convergence in (1.5), and assume that the following limiting relation holds for every $x > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if } \rho < 0 \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{1.6}$$

where $|A|$ must then be in RV_ρ (Geluk and de Haan, 1987). The second-order parameter $\rho \le 0$ rules the rate of convergence provided by (1.6), which increases with $|\rho|$. Note further that in the scope of applications, the most common models depend on a location or shift parameter $s \in \mathbb{R}$, not necessarily null, i.e. $F(x) \equiv F_s(x) = F_0(x - s)$. Then, $U(t) \equiv U_s(t) = U_0(t) + s$ and also both A and ρ depend obviously on s, i.e. $A = A_s$ and $\rho = \rho_s$, with

$$\rho_s := \begin{cases} -\xi, & \text{if } \xi + \rho_0 < 0 \ \wedge s \neq 0 \\ \rho_0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Among the literature specifically devoted to the estimation of the second-order parameter ρ, in (1.6), we mention Gomes *et al.* (2002), Fraga Alves *et al.* (2003a), and the more recent articles by Goegebeur *et al.* (2008; 2010), Ciuperca and Mercadier (2010) and Caeiro and Gomes (2012a,b). Indeed, most of the research devised to improve the classical EVI-estimators tries to reduce the dominant component of their asymptotic bias, deals with second-order reduced-bias (SORB) EVI-estimators, and an adequate estimation of ρ is needed, for an adequate reduction of the bias. Some of the pioneering papers in the area of SORB-estimation are the ones by Beirlant *et al.* (1999), Feuerverger and Hall (1999), Gomes *et al.* (2000) and Gomes and Martins (2001; 2002). More recently, the minimum-variance reduced-bias (MVRB) EVI-estimators, studied in Caeiro *et al.* (2005), Gomes *et al.* (2007) and Gomes *et al.* (2008c), among others, also call for an adequate estimation of ρ. An overview of the subject can be found in Chapter 6 of the book by Reiss and Thomas (2007). See also Gomes *et al.* (2008a) and Beirlant *et al.* (2012) in this respect. However, despite of scale-invariant, all these MVRB EVI-estimators are not location-invariant.

1.2. The PORT methodology

Let $X_{i:n}$, $1 \le i \le n$, be the o.s.'s associated with the random sample $\underline{X}_n = (X_1, ..., X_n)$ with common d.f. F_0. The class of estimators suggested here is a function of the sample of excesses over a random threshold $X_{n_q:n}$, with $n_q = \lfloor nq \rfloor + 1$, where $\lfloor x \rfloor$ stands for the integer part of x. Such a sample is denoted by

$$\underline{X}_n^{(q)} := \left(X_{n:n} - X_{n_q:n}, X_{n-1:n} - X_{n_q:n}, ..., X_{n_q+1:n} - X_{n_q:n}\right), \tag{1.8}$$

where, we can have

- $0 < q < 1$, for any $F_0 \in D_{\mathcal{M}}(G_{\xi>0})$ (the random threshold, $X_{n_q:n}$, is an empirical quantile);
- $q = 0$, for d.f.'s with a finite left endpoint $x_F := \inf\{x : F_0(x) > 0\}$, (the random threshold is the minimum, $X_{1:n}$).

Any statistical inference methodology based on the sample of excesses $\underline{X}_n^{(q)}$, in (1.8), will be called a PORT-methodology, with PORT standing for peaks over random thresholds, a term coined by Araújo Santos *et al.* (2006). This methodology enabled the introduction and study of classical location/scale invariant EVI-estimators, like the PORT-Hill and the PORT-Moment estimators, studied for finite-samples in Gomes *et al.* (2008b). This methodology was also applied in the estimation of high quantiles in Henriques-Rodrigues and Gomes (2009).

Such a methodology leads to location-invariant estimation, where the unshifted model F_0 thus plays a central role. In what follows, we use the notation χ_q for the q-quantile of the d.f. F_0, i.e. the value

$$\chi_q := F_0^{\leftarrow}(q) \tag{1.9}$$

(by convention $\chi_0 := x_F$, the left endpoint of F_0), with $F^{\leftarrow}(\cdot)$ defined in (1.3). Since $n_q/n \to q$, as $n \to \infty$, we then know that the o.s. $X_{n_q:n}$, associated with a sample from F_0, is a consistent estimator for $F_0^{\leftarrow}(q)$ (Mosteller, 1946, under stronger assumptions on F; van der Vaart, 1998, p.308), i.e. we have the following convergence in probability:

$$X_{n_q:n} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \chi_q = F_0^{\leftarrow}(q), \quad \text{for} \quad 0 \le q < 1 \quad \left(\chi_0 = x_F\right). \tag{1.10}$$

1.3. Scope of the paper

We shall make use of the above-mentioned PORT methodology for heavy tails. Henceforth $\xi > 0$ denotes the first-order parameter of the model underlying the available data, $\rho_0 \le 0$ is the second-order parameter of the associated unshifted model, and χ_q has been provided in the limit of (1.10), in order to introduce a class of location-invariant semi-parametric estimators of the so-called PORT-ρ second-order parameter,

$$\rho_q := \begin{cases} -\xi, \text{ if } \xi + \rho_0 < 0 \ \wedge \ \chi_q \neq 0 \\ \rho_0, \quad \text{otherwise.} \end{cases} \tag{1.11}$$

Note that when applying the PORT-methodology, we are working with the sample of excesses in (1.8), and we can assume that we are dealing with an unshifted d.f. F_0 underlying the r.v. X_0, to which we are inducing a random shift, strictly related to χ_q, in (1.9). More precisely, we have a shift $s = -\chi_q$, i.e. we are working with $X_q := X_0 - \chi_q$, and use the simpler notation ρ_q for $\rho_{-\chi_q}$, with ρ_s defined in (1.7). Hence $\rho_q = -\xi \neq \rho_0$ if and only if $\chi_q \neq 0$ and the underlying model is such that $\xi + \rho_0 < 0$, just as written in (1.11), i.e. $\rho_q \neq \rho_0$ if and only if $s = 0$, $\chi_q \neq 0$ and $\xi + \rho_0 < 0$.

The main motivation for a class of estimators of the shape second-order parameter ρ_q, in (1.11), is related to its possible use, concomitantly with a class of PORT estimators of the functional A, in (1.6), or at least of an adequate location-invariant estimator of the scale parameter of such a A-function, in the building of second-order PORT-MVRB EVI-estimators, invariant for changes in location. The study of the asymptotic behaviour of such EVI-estimators is a challenging theoretical open subject, out of the scope of this paper, but already dealt with by Monte-Carlo simulation, in Gomes *et al.* (2011, 2013).

The building block of our estimators of the shape second-order parameter ρ_q, defined in (1.11) are of the same kind as the statistics used in Dekkers *et al.* (1989), Gomes *et al.* (2002), Fraga Alves *et al.* (2003a) and Caeiro and Gomes (2006), among others, i.e. for $\alpha > 0$ we consider the moment statistics

$$(1.12)\qquad M_{n,k}^{(\alpha)} \equiv M_{n,k}^{(\alpha)}(\underline{X}_n) := \frac{1}{k}\sum_{i=1}^{k}\left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^{\alpha},$$

but now applied to the sample of excesses $\underline{X}_n^{(q)}$, $0 \le q < 1$, in (1.8). For an intermediate k-sequence, i.e. a sequence $k = k_n$ of positive integers such that

$$(1.13)\qquad k = k_n \to \infty \quad \text{and} \quad k = o(n) \quad \text{as} \quad n \to \infty,$$

we shall thus consider the location and scale-invariant statistics,

$$(1.14)\qquad M_{n,k}^{(\alpha,q)} \equiv M_{n,k}^{(\alpha)}(\underline{X}_n^{(q)}) := \frac{1}{k}\sum_{i=1}^{k}\left(\ln \frac{X_{n-i+1:n}-X_{n_q:n}}{X_{n-k:n}-X_{n_q:n}}\right)^{\alpha},$$

defined for $k < n - n_q$, with $M_{n,k}^{(\alpha)}(\underline{X}_n)$ given in (1.12), $\alpha > 0$.

Regarding the *tuning* parameters $\tau_q \in \mathbb{R}$, $\alpha, \theta_1, \theta_2 \in \mathbb{R}^+$, $\theta_1, \theta_2 \neq 1$ and $\theta_1 < \theta_2$, we shall consider the PORT-versions of the statistics used in Fraga Alves *et al.* (2003a) for the estimation of ρ, in (1.6), i.e.

$$(1.15)\qquad T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} := \frac{\left(\frac{M_{n,k}^{(\alpha,q)}}{\Gamma(\alpha+1)}\right)^{\tau_q} - \left(\frac{M_{n,k}^{(\alpha\theta_1,q)}}{\Gamma(\alpha\theta_1+1)}\right)^{\tau_q/\theta_1}}{\left(\frac{M_{n,k}^{(\alpha\theta_1,q)}}{\Gamma(\alpha\theta_1+1)}\right)^{\tau_q/\theta_1} - \left(\frac{M_{n,k}^{(\alpha\theta_2,q)}}{\Gamma(\alpha\theta_2+1)}\right)^{\tau_q/\theta_2}} =: \frac{D_{n,k}^{(\alpha,1,\theta_1,\tau_q,q)}(\xi)}{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)},$$

with $\Gamma(t)$ denoting the complete Gamma function. As detailed in Section 3.1, under adequate conditions upon the growth of $k = k_n$, $T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ converges in probability to

$$t_{\alpha,\theta_1,\theta_2}(\rho_q) := \theta_2 \frac{(\theta_1-1)(1-\rho_q)^{\alpha\theta_2}-\theta_1(1-\rho_q)^{\alpha(\theta_2-1)}+(1-\rho_q)^{\alpha(\theta_2-\theta_1)}}{(\theta_2-\theta_1)(1-\rho_q)^{\alpha\theta_2}-\theta_2(1-\rho_q)^{\alpha(\theta_2-\theta_1)}+\theta_1}. \tag{1.16}$$

Remark 1.1. Note that the function $t_{\alpha,\theta_1,\theta_2}(\rho_q)$, defined for $\rho_q \le 0$, $\alpha > 0$, $\theta_1, \theta_2 \in \mathbb{R}^+ \setminus \{1\}$, $\theta_1 < \theta_2$, is a decreasing function of ρ_q if $\theta_1, \theta_2 > 1$ or $\theta_1, \theta_2 < 1$ and increasing otherwise. Since $t_{\alpha,\theta_1,\theta_2}(\rho_q)$ is always monotone continuous then it is invertible. Moreover,

$$\lim_{\rho_q \to -\infty} t_{\alpha,\theta_1,\theta_2}(\rho_q) = \frac{\theta_2(\theta_1-1)}{\theta_2-\theta_1} \quad \text{and} \quad \lim_{\rho_q \to 0} t_{\alpha,\theta_1,\theta_2}(\rho_q) = \frac{\theta_1-1}{\theta_2-\theta_1}.$$

The general class of consistent ρ_q-estimators, invariant for changes in location, already introduced and validated under a second-order framework in Henriques-Rodrigues and Gomes (2012), and named PORT-ρ class of estimators, it is now written as

$$\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} := -\left| t^{\leftarrow}_{\alpha,\theta_1,\theta_2}\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}\right)\right|. \tag{1.17}$$

with $T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ given in (1.15).

The simplest choice of tuning control parameters suggested in Fraga Alves *et al.* (2003a) for the classical ρ-estimators, $(\alpha,\theta_1,\theta_2) = (1,2,3)$, gives rise to an explicit ρ-estimator, denoted $\widehat{\rho}_k^{(\tau)}$ in the aforementioned paper, and leads us to a simpler class of PORT-ρ estimators of the shape second-order parameter ρ_q, because it only depends on the tuning parameter τ_q. With ρ_q defined in (1.11), we have that

$$t(\rho_q) = t_{1,2,3}(\rho_q) = \frac{3(1-\rho_q)}{3-\rho_q} = \begin{cases} \frac{3(1+\xi)}{3+\xi}, & \text{if } \xi+\rho_0<0 \ \wedge \chi_q \ne 0, \\ \frac{3(1-\rho_0)}{3-\rho_0}, & \text{otherwise.} \end{cases}$$

Thus the PORT-ρ estimator associated with $(\alpha,\theta_1,\theta_2) = (1,2,3)$ is explicitly given by

$$\widehat{\rho}_k^{(\tau_q,q)} \equiv \widehat{\rho}_{n,k|T}^{(1,2,3,\tau_q,q)} := -\left| \frac{3\left(T_{n,k}^{(1,2,3,\tau_q,q)}-1\right)}{T_{n,k}^{(1,2,3,\tau_q,q)}-3}\right|, \tag{1.18}$$

where

$$T_{n,k}^{(1,2,3,\tau_q,q)} = \frac{\left(M_{n,k}^{(1,q)}\right)^{\tau_q} - \left(M_{n,k}^{(2,q)}/2\right)^{\tau_q/2}}{\left(M_{n,k}^{(2,q)}/2\right)^{\tau_q/2} - \left(M_{n,k}^{(3,q)}/6\right)^{\tau_q/3}},$$

for any $\tau_q \in \mathbb{R}$, with $M_{n,k}^{(\alpha,q)}$ given in (1.14). The notation $a^{b\tau_q} = b\ln a$ is used for $\tau_q = 0$.

In Section 2 of this paper we present preliminary asymptotic results related to the PORT-methodology. In Section 3 we justify the class of PORT-ρ estimators of the shape second-order parameter ρ_q, in (1.11), addressing the possibility of shifted heavy-tailed models, and refer the conditions required for their consistency and asymptotic normality. In Section 4, we illustrate the finite sample behaviour of the new estimators through a small-scale Monte-Carlo simulation study. Finally, in Section 5, we present the proofs of the results in Section 3.

2. TECHNICAL RESULTS RELATED TO THE PORT-METHODOLOGY

2.1. The second-order PORT-framework for heavy-tailed models

Under the aforementioned set-up in Section 1.3, the transformed r.v., $X_q = X_0 - \chi_q$, has an associated quantile function given by $U_q(t) = U_0(t) - \chi_q$. The second-order condition in (1.6) translates as

$$\lim_{t\to\infty} \frac{\ln U_q(tx) - \ln U_q(t) - \xi \ln x}{A_q(t)} = \begin{cases} \frac{x^{\rho_q}-1}{\rho_q}, & \text{if } \rho_q < 0 \\ \ln x, & \text{if } \rho_q = 0, \end{cases} \tag{2.1}$$

for all $x > 0$. Moreover, $|A_q| \in RV_{\rho_q}$, $\rho_q \le 0$, and A_q relates to A_0 according to the following lemma.

Lemma 2.1. *Assume $U_0 \in RV_\xi$ satisfies the second order condition in* (1.6) *with $\rho = \rho_0$ and $A = A_0$. Then $U_q(t) = U_0(t) - \chi_q$, with χ_q defined in* (1.9), *is such that $U_q \in RV_\xi$ and* (2.1) *holds with ρ_q given in* (1.11) *and*

$$A_q(t) := \begin{cases} \xi\chi_q/U_0(t), & \text{if } \xi + \rho_0 < 0 \ \wedge\ \chi_q \ne 0 \\ A_0(t), & \text{if } \xi + \rho_0 > 0 \ \vee\ \chi_q = 0 \\ A_0(t) + \xi\chi_q/U_0(t), & \text{if } \xi + \rho_0 = 0 \ \wedge\ \chi_q \ne 0. \end{cases} \tag{2.2}$$

2.2. Third-order framework and asymptotic behaviour of auxiliary statistics

Next, we present the asymptotic behaviour of the statistics $M_{n,k}^{(\alpha,q)}$ defined in (1.14), based on the sample of excesses $\underline{X}_n^{(q)}$, $0 \le q < 1$, defined in (1.8). This requires a third-order framework because we further need to know the rate of convergence in (1.6). It is common to assume a third-order condition that rules such a rate of convergence through the shape third-order parameter $\rho' \le 0$, assuming that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx)-\ln U(t)-\xi\ln x}{A(t)} - \frac{x^\rho-1}{\rho}}{B(t)} = \frac{x^{\rho+\rho'}-1}{\rho+\rho'}, \tag{2.3}$$

with $|A| \in RV_\rho$ and $|B| \in RV_{\rho'}$. For technical simplicity, we shall assume that $\rho,\ \rho' < 0$, i.e. we assume to be working in a class $\mathcal{H}$ of heavy-tailed models, such that, as $t \to \infty$,

$$U(t) = Ct^\xi \left\{1 + D_1 t^\rho + D_2 t^{\rho+\rho'} + o(t^{\rho+\rho'})\right\}, \tag{2.4}$$

where $C > 0$. Details on the third-order condition in (2.3) can be found in Fraga Alves *et al.* (2003b, 2006) and more generally in Wang and Cheng (2005).

Note that the statistics $M_{n,k}^{(\alpha,q)}$, in (1.14), depend on q through χ_q, in (1.9) (see also (1.10)), but are obviously independent on any shift s imposed to the data. We can thus assume throughout that $s = 0$.

Let $\mathbb{E}$ and $\mathbb{V}ar$ denote the mean value and variance operators, respectively, and let E denote a unit exponential random variable. For any real $\alpha > 0$, with $\xi > 0$ and $\rho < 0$, let us define

$$\mu_\alpha^{(1)}(\xi) := \mathbb{E}\Big(E^\alpha e^{-\xi E}\Big) = \frac{\Gamma(\alpha+1)}{(1+\xi)^{\alpha+1}}, \qquad \mu_\alpha^{(1)} := \mu_\alpha^{(1)}(0) = \Gamma(\alpha+1), \tag{2.5}$$

$$\sigma_\alpha^{(1)} := \sqrt{\mathbb{V}\mathrm{ar}(E^\alpha)} = \sqrt{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}, \tag{2.6}$$

$$\mu_\alpha^{(2)}(\xi,\rho) := \mathbb{E}\big(E^{\alpha-1}\, e^{-\xi E}\, (e^{\rho E}-1)/\rho\big) = \frac{\Gamma(\alpha)}{\rho}\left(\frac{(1+\xi)^\alpha - (1+\xi-\rho)^\alpha}{(1+\xi-\rho)^\alpha(1+\xi)^\alpha}\right),$$

$$\mu_\alpha^{(2)}(\rho) := \mu_\alpha^{(2)}(0,\rho) = \frac{\Gamma(\alpha)}{\rho}\left(\frac{1-(1-\rho)^\alpha}{(1-\rho)^\alpha}\right),$$

$$\sigma_\alpha^{(2)}(\rho) := \sqrt{\mathbb{V}\mathrm{ar}\big(E^{\alpha-1}(e^{\rho E}-1)/\rho\big)} = \sqrt{\mu_{2\alpha}^{(3)}(\rho) - \big(\mu_\alpha^{(2)}(\rho)\big)^2},$$

$$\eta_\alpha^{(3)}(\xi,\rho) := \mathbb{E}\Big(E^{\alpha-2}\ \big((e^{-\xi E}-1)/(-\xi)\big)\ \big((e^{\rho E}-1)/\rho\big)\Big)$$
$$= \begin{cases} -\frac{1}{\xi\rho}\ln\frac{(1+\xi)(1-\rho)}{1+\xi-\rho}, & \text{if } \alpha = 1 \\ -\frac{\Gamma(\alpha)}{\xi\rho(\alpha-1)}\left\{\frac{1}{(1+\xi-\rho)^{\alpha-1}} - \frac{1}{(1+\xi)^{\alpha-1}} - \frac{1}{(1-\rho)^{\alpha-1}} + 1\right\}, & \text{if } \alpha \ne 1, \end{cases}$$

and

$$\mu_\alpha^{(3)}(\rho) := \mathbb{E}\Big(E^{\alpha-2}\ \big((e^{\rho E}-1)/\rho\big)^2\Big)$$
$$= \begin{cases} \frac{1}{\rho^2}\ln\frac{(1-\rho)^2}{1-2\rho}, & \text{if } \alpha = 1 \\ \frac{\Gamma(\alpha)}{\rho^2(\alpha-1)}\left\{\frac{1}{(1-2\rho)^{\alpha-1}} - \frac{2}{(1-\rho)^{\alpha-1}} + 1\right\}, & \text{if } \alpha \ne 1. \end{cases}$$

Let us further introduce the notations:

$$\overline{\mu}_\alpha^{(j)}(\rho) := \frac{\mu_\alpha^{(j)}(\rho)}{\mu_\alpha^{(1)}},\ j=2,3, \quad \overline{\mu}_\alpha^{(2)}(\xi,\rho) := \frac{\mu_\alpha^{(2)}(\xi,\rho)}{\mu_\alpha^{(1)}}, \tag{2.7}$$

$$\overline{\eta}_\alpha^{(3)}(\xi,\rho) := \frac{\eta_\alpha^{(3)}(\xi,\rho)}{\mu_\alpha^{(1)}}, \tag{2.8}$$

$$\overline{\sigma}_\alpha^{(1)} := \frac{\sigma_\alpha^{(1)}}{\mu_\alpha^{(1)}}, \qquad \overline{\sigma}_\alpha^{(2)}(\rho) := \frac{\sigma_\alpha^{(2)}(\rho)}{\mu_\alpha^{(1)}}, \tag{2.9}$$

and for any $\theta_1,\ \theta_2 > 0$, define

$$d_{\alpha,\theta_1,\theta_2}(\rho) := \overline{\mu}_{\alpha\theta_1}^{(2)}(\rho) - \overline{\mu}_{\alpha\theta_2}^{(2)}(\rho). \tag{2.10}$$

Recall that $E_i,\ i \geq 1$, are i.i.d. unit exponential r.v.'s, and, with $\sigma_\alpha^{(1)}$ given in (2.6), define the asymptotically standard normal r.v.'s

$$Z_k^{(\alpha)} := \sqrt{k}\Big(\tfrac{1}{k}\sum_{i=1}^{k} E_i^\alpha - \Gamma(\alpha+1)\Big)/\sigma_\alpha^{(1)}. \tag{2.11}$$

Now, together with (2.9), we can combine these as follows:

$$W_k^{(\alpha,\theta_1,\theta_2)} := \overline{\sigma}_{\alpha\theta_1}^{(1)} Z_k^{(\alpha\theta_1)}/\theta_1 - \overline{\sigma}_{\alpha\theta_2}^{(1)} Z_k^{(\alpha\theta_2)}/\theta_2. \tag{2.12}$$

Finally, for $\tau \in \mathbb{R}$, $\alpha, \theta > 0$, and with $\big(\overline{\mu}_\alpha^{(2)}(\rho), \overline{\mu}_\alpha^{(2)}(\xi,\rho)\big)$ and $\overline{\eta}_\alpha^{(3)}(\xi,\rho)$ defined in (2.7) and (2.8), respectively, we define

$$c_{\alpha,\theta,\tau}(\rho) := (\alpha\theta-1)\overline{\mu}_{\alpha\theta}^{(3)}(\rho) + \alpha(\tau-\theta)\big(\overline{\mu}_{\alpha\theta}^{(2)}(\rho)\big)^2, \tag{2.13}$$

$$g_{\alpha,\theta,\tau}(\xi,\rho) := \overline{\mu}_{\alpha\theta}^{(2)}(\xi,\rho) + (\alpha\theta-1)\overline{\eta}_{\alpha\theta}^{(3)}(\xi,\rho) + \alpha(\tau-\theta)\overline{\mu}_{\alpha\theta}^{(2)}(\rho)\overline{\mu}_{\alpha\theta}^{(2)}(-\xi), \tag{2.14}$$

$$h_{\alpha,\theta,\tau}(\xi) := 2\overline{\mu}_{\alpha\theta}^{(2)}(-2\xi) + (\alpha\theta-1)\overline{\mu}_{\alpha\theta}^{(3)}(-\xi) + \alpha(\tau-\theta)\left(\overline{\mu}_{\alpha\theta}^{(2)}(-\xi)\right)^2. \tag{2.15}$$

We first state Proposition 2.1, related to the behaviour of $M_{n,k}^{(\alpha)}$, in (1.12), now needed only for $s=0$ $(\rho=\rho_0)$, proved in Gomes *et al.* (2002), also under a third-order framework.

Proposition 2.1 (Gomes *et al.*, 2002)**.** *Under the third-order condition* (2.3)*, with $\rho_0,\ \rho_0' < 0$, for intermediate sequences $k=k_n$, i.e. sequences of positive integers such that* (1.13) *holds, and with $M_{n,k}^{(\alpha)}$, $\mu_\alpha^{(1)}$, $\overline{\mu}_\alpha^{(j)}(\rho)$, $j=2,3$, $\overline{\sigma}_\alpha^{(1)}$ and $Z_k^{(\alpha)}$ defined in* (1.12), (2.5), (2.7), (2.9) *and* (2.11)*, respectively,*

$$\begin{aligned} M_{n,k}^{(\alpha)} &\overset{d}{=} \xi^\alpha \mu_\alpha^{(1)}\Big\{1 + \overline{\sigma}_\alpha^{(1)}\ \frac{Z_k^{(\alpha)}}{\sqrt{k}} + \tfrac{\alpha}{\xi}\ \overline{\mu}_\alpha^{(2)}(\rho_0) A_0(n/k) \\ &+ \Big(\tfrac{\alpha(\alpha-1)}{2\xi^2}\ \overline{\mu}_\alpha^{(3)}(\rho_0) A_0^2(n/k) + \tfrac{\alpha}{\xi}\ \overline{\mu}_\alpha^{(2)}(\rho_0+\rho_0') A_0(n/k) B_0(n/k)\Big)(1+o_p(1))\Big\}. \end{aligned}$$

We next provide, under the third-order framework in (2.3), the behaviour of $M_{n,k}^{(\alpha,q)}$, in (1.14).

Proposition 2.2. *Let us assume that* (1.13) *holds, as well as the third-order condition in* (2.3), *with* $\rho_0, \rho_0' < 0$. *We then get for* $M_{n,k}^{(\alpha,q)}$, *in* (1.14), $\alpha > 0$, $k < n - n_q$, *with* χ_q *and* $M_{n,k}^{(\alpha)}$ (*for* $s = 0$), *given in* (1.10) *and* (1.12), *respectively,* $\mu_\alpha^{(1)}$ *and* $\big(\overline{\mu}_\alpha^{(2)}(\rho), \overline{\mu}_\alpha^{(2)}(\xi,\rho), \overline{\mu}_\alpha^{(3)}(\rho)\big)$ *and* $\overline{\eta}_\alpha^{(3)}(\xi,\rho)$ *respectively given in* (2.5), (2.7) *and* (2.8), *the distributional representation,*

$$
\begin{aligned}
(2.16) \quad M_{n,k}^{(\alpha,q)} \stackrel{d}{=} {} & M_{n,k}^{(\alpha)} + \frac{\alpha\xi^\alpha \mu_\alpha^{(1)}\chi_q}{U_0(n/k)} \Big\{ \overline{\mu}_\alpha^{(2)}(-\xi) \\
& + \frac{\overline{\mu}_\alpha^{(2)}(\xi,\rho_0) + (\alpha-1)\, \overline{\eta}_\alpha^{(3)}(\xi,\rho_0)}{\xi}\, A_0(n/k)(1+o_p(1)) \\
& + \frac{\chi_q}{U_0(n/k)} \Big(\overline{\mu}_\alpha^{(2)}(-2\xi) + \frac{(\alpha-1)}{2} \overline{\mu}_\alpha^{(3)}(-\xi) \Big)(1+o_p(1)) \Big\}.
\end{aligned}
$$

3. ASYMPTOTIC BEHAVIOUR OF THE PORT-ρ ESTIMATORS

3.1. Consistency of the PORT-ρ estimators

For $\alpha > 0$, let us consider the statistics $M_{n,k}^{(\alpha,q)} = M_{n,k}^{(\alpha)}\big(\underline{X}_n^{(q)}\big)$, in (1.14), defined for $k < n - n_q$, with $\underline{X}_n^{(q)}$ the sample of excesses in (1.8). Under the third-order framework in (2.3), if (1.13) holds, on the basis of the results in Propositions 2.1 and 2.2, similarly to the developments in Fraga Alves *et al.* (2003a), and for real tuning parameters $\tau_q \in \mathbb{R}$ and $\theta \neq 0$,

$$
\begin{aligned}
(3.1) \quad & \left(\frac{M_{n,k}^{(\alpha\theta,q)}}{\mu_{\alpha\theta}^{(1)}}\right)^{\tau_q/\theta} \stackrel{d}{=} \xi^{\alpha\tau_q} \Bigg(1 + \frac{\tau_q}{\theta} \frac{\overline{\sigma}_{\alpha\theta}^{(1)}}{\sqrt{k}} Z_k^{(\alpha\theta)} + \frac{\alpha\tau_q\, \overline{\mu}_{\alpha\theta}^{(2)}(\rho_0) A_0(n/k)}{\xi} + \frac{\alpha\tau_q\, \chi_q\, \overline{\mu}_{\alpha\theta}^{(2)}(-\xi)}{U_0(n/k)} \\
& + \left\{ \frac{\alpha\tau_q\, c_{\alpha,\theta,\tau_q}(\rho_0)}{2\xi^2} A_0^2(n/k) + \frac{\alpha\tau_q\, \overline{\mu}_{\alpha\theta}^{(2)}(\rho_0+\rho_0')}{\xi} A_0(n/k) B_0(n/k) \right\} (1+o_p(1)) \\
& + \left\{ \frac{\alpha\tau_q\chi_q}{\xi} g_{\alpha,\theta,\tau_q}(\xi,\rho_0) \frac{A_0(n/k)}{U_0(n/k)} + \frac{\alpha\tau_q\chi_q^2}{2}\, h_{\alpha,\theta,\tau_q}(\xi) \frac{1}{U_0^2(n/k)} \right\} (1+o_p(1)) \Bigg).
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\left(\frac{M_{n,k}^{(\alpha\theta,q)}}{\mu_{\alpha\theta}^{(1)}}\right)^{\tau_q/\theta} \stackrel{d}{=} {} & \left(\frac{M_{n,k}^{(\alpha\theta)}}{\mu_{\alpha\theta}^{(1)}}\right)^{\tau_q/\theta} + \frac{\alpha\tau_q\xi^{\alpha\tau_q}\chi_q}{U_0(n/k)} \Big\{ \overline{\mu}_{\alpha\theta}^{(2)}(-\xi) \\
& + \frac{g_{\alpha,\theta,\tau_q}(\xi,\rho_0)}{\xi} A_0(n/k)(1+o_p(1)) + \frac{\chi_q\; h_{\alpha,\theta,\tau_q}(\xi)}{2} \frac{1}{U_0(n/k)} (1+o_p(1)) \Big\},
\end{aligned}
$$

with $M_{n,k}^{(\alpha,q)}$, $\mu_\alpha^{(1)}$, $\overline{\mu}_\alpha^{(j)}(\rho)$, $j = 2,3$, $\overline{\sigma}_\alpha^{(1)}$, $Z_k^{(\alpha)}$, $c_{\alpha,\theta,\tau}(\rho)$, $g_{\alpha,\theta,\tau}(\xi,\rho)$ and $h_{\alpha,\theta,\tau}(\xi)$ given in (1.14), (2.5), (2.7), (2.9), (2.11), (2.13), (2.14) and (2.15), respectively.

Let us next introduce the notations,

$$u_{\alpha,\theta_1,\theta_2,\tau}(\rho) := \big\{c_{\alpha,\theta_1,\tau}(\rho) - c_{\alpha,\theta_2,\tau}(\rho)\big\}/(2\xi), \tag{3.2}$$

$$v_{\alpha,\theta_1,\theta_2}(\rho,\rho') := \overline{\mu}^{(2)}_{\alpha\theta_1}(\rho+\rho') - \overline{\mu}^{(2)}_{\alpha\theta_2}(\rho+\rho') \;\equiv\; d_{\alpha,\theta_1,\theta_2}(\rho+\rho'), \tag{3.3}$$

$$w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho) := \{g_{\alpha,\theta_1,\tau}(\xi,\rho) - g_{\alpha,\theta_2,\tau}(\xi,\rho)\}/\xi, \tag{3.4}$$

$$y_{\alpha,\theta_1,\theta_2,\tau}(\xi) := \{h_{\alpha,\theta_1,\tau}(\xi) - h_{\alpha,\theta_2,\tau}(\xi)\}/2, \tag{3.5}$$

with $d_{\alpha,\theta_1,\theta_2}(\rho)$, $c_{\alpha,\theta,\tau}(\rho)$, $g_{\alpha,\theta,\tau}(\xi,\rho)$ and $h_{\alpha,\theta,\tau}(\xi)$ defined in (2.10), (2.13), (2.14) and (2.15), respectively. On the basis of (3.1), using the notation $W_k^{(\alpha,\theta_1,\theta_2)}$ in (2.12), and with $D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)$ defined in (1.15), we can write

$$\begin{aligned} D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi) \stackrel{d}{=} \xi^{\alpha\tau_q}\Bigg(\frac{\tau_q}{\sqrt{k}}\, W_k^{(\alpha,\theta_1,\theta_2)} + \frac{\alpha\tau_q\; A_0(n/k)}{\xi}\Big\{d_{\alpha,\theta_1,\theta_2}(\rho_0) \\ + u_{\alpha,\theta_1,\theta_2,\tau}(\rho_0)A_0(n/k)(1+o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1))\Big\} \\ + \frac{\alpha\tau_q\chi_q}{U_0(n/k)}\Big\{d_{\alpha,\theta_1,\theta_2}(-\xi) + w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho_0)A_0(n/k)(1+o_p(1)) \\ + \frac{\chi_q\; y_{\alpha,\theta_1,\theta_2,\tau}(\xi)}{U_0(n/k)}(1+o_p(1))\Big\}\Bigg), \end{aligned} \tag{3.6}$$

i.e.

$$\begin{aligned} D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi) \;\stackrel{d}{=}\; D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi) + \frac{\alpha\tau_q\chi_q\xi^{\alpha\tau_q}}{U_0(n/k)}\Big\{d_{\alpha,\theta_1,\theta_2}(-\xi) \\ + w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho_0)A_0(n/k)(1+o_p(1)) + \frac{\chi_q\; y_{\alpha,\theta_1,\theta_2,\tau}(\xi)}{U_0(n/k)}(1+o_p(1))\Big\}. \end{aligned}$$

The dominant component of the right hand-side of (3.6) depends on the relative behaviour of the functions $A_0(t)$ and $1/U_0(t)$. We shall thus consider three different regions related to χ_q, in (1.9), and the vector (ξ,ρ_0) of the unshifted model F_0 associated with the available data:

- $\mathcal{R}_1 := \{F_0 : \xi+\rho_0 < 0 \wedge \chi_q \neq 0\}$,
- $\mathcal{R}_2 := \{F_0 : \xi+\rho_0 > 0 \vee \chi_q = 0\}$,
- $\mathcal{R}_3 := \{F_0 : \xi+\rho_0 = 0 \wedge \chi_q \neq 0\}$.

We now state the following:

Theorem 3.1 (Henriques-Rodrigues and Gomes, 2013, Theorem 1)**.** *Under the validity of the second-order condition in* (1.6)*, with* $\rho=\rho_0<0$*,* ρ_q *defined in* (1.11)*,* $\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ *defined in* (1.17)*, and with an explicit expression given in* (1.18) *for the particular case* $(\alpha,\theta_1,\theta_2)=(1,2,3)$*, is consistent for the estimation of* ρ_q*, i.e.*

$$\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \;\underset{n\to\infty}{\overset{p}{\longrightarrow}}\; \rho_q,$$

for any real $\alpha > 0$, $\tau_q \in \mathbb{R}$, $\theta_1, \theta_2 \in \mathbb{R}^+ \backslash \{1\}$, $\theta_1 < \theta_2$ *and* $0 < q < 1$ *or* $q = 0$ *if* $\chi_0 = x_{_F}$, *the left endpoint of the underlying parent, is finite, provided that* k *is an intermediate sequence, and moreover, with* A_q *defined in* (2.2),

$$\sqrt{k} A_q(n/k) \to \infty, \text{ as } n \to \infty. \tag{3.7}$$

Remark 3.1. Note that when we consider models $F_0 \in \mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$ and with $A_q(t) = \xi \chi_q / U_0(t)$, by (2.2), condition (3.7) corresponds to $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$. For models $F_0 \in \mathcal{R}_2$, $1/U_0(t) = o(A_0(t))$ and since $A_q(t) = A_0(t)$, condition (3.7) is equivalent to $\sqrt{k} A_0(n/k) \to \infty$, as $n \to \infty$. Finally, for models $F_0 \in \mathcal{R}_3$, $1/U_0(t) = O(A_0(t))$ and since $A_q(t) = A_0(t) + \xi \chi_q / U_0(t)$, condition (3.7) is equivalent to $\sqrt{k} A_0(n/k) \to \infty$ or $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$.

3.2. Non-degenerate asymptotic behaviour of the PORT-ρ estimators

In this section, and under a third-order framework, we derive the non-degenerate asymptotic properties of the PORT-ρ classes of estimators introduced with all the generality in (1.17), and particularised in (1.18). We first state the following result:

Proposition 3.1 (Fraga Alves *et al*., 2003)**.** *Under the validity of the second-order condition in* (1.6), *with* $\rho < 0$, *if* (1.13) *holds and* $\sqrt{k} A(n/k) \to \infty$, *as* $n \to \infty$, *the asymptotic variance of* $W_k^{(\alpha,\theta_1,\theta_2)}$, *in* (2.12), *is*

$$\sigma^2_{W|\alpha,\theta_1,\theta_2} = \frac{2}{\alpha}\left(\frac{\Gamma(2\alpha\theta_1)}{\theta_1^3 \Gamma^2(\alpha\theta_1)} + \frac{\Gamma(2\alpha\theta_2)}{\theta_2^3 \Gamma^2(\alpha\theta_2)} - \frac{(\theta_1+\theta_2)\Gamma(\alpha(\theta_1+\theta_2))}{\theta_1^2\theta_2^2\Gamma(\alpha\theta_1)\Gamma(\alpha\theta_2)}\right) - \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)^2, \tag{3.8}$$

and the asymptotic covariance of $(W_k^{(\alpha,1,\theta_1)}, W_k^{(\alpha,\theta_1,\theta_2)})$ *is given by*

$$\begin{aligned} \sigma_{W|\alpha,1,\theta_1,\theta_2} = \frac{1}{\alpha}\Bigg(&\frac{(\theta_1+1)\Gamma(\alpha(\theta_1+1))}{\theta_1^2\Gamma(\alpha)\Gamma(\alpha\theta_1)} - \frac{(\theta_2+1)\Gamma(\alpha(\theta_2+1))}{\theta_2^2\Gamma(\alpha)\Gamma(\alpha\theta_2)} - \frac{2\Gamma(2\alpha\theta_1)}{\theta_1^3\Gamma^2(\alpha\theta_1)} \\ &+ \frac{(\theta_1+\theta_2)\Gamma(\alpha(\theta_1+\theta_2))}{\theta_1^2\theta_2^2\Gamma(\alpha\theta_1)\Gamma(\alpha\theta_2)}\Bigg) - \left(1 - \frac{1}{\theta_1}\right)\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right). \end{aligned} \tag{3.9}$$

Note that $t'_{\alpha,\theta_1,\theta_2}(\rho) := dt_{\alpha,\theta_1,\theta_2}(\rho)/d\rho$, with $t_{\alpha,\theta_1,\theta_2}(\rho_q)$ defined in (1.16), is given by

$$\begin{aligned} t'_{\alpha,\theta_1,\theta_2}(\rho)(1-\rho)&\left((\theta_2-\theta_1)(1-\rho)^{\alpha\theta_2} - \theta_2(1-\rho)^{\alpha(\theta_2-\theta_1)} + \theta_1\right)^2 \\ = \alpha\theta_1\theta_2 &\Big\{\theta_1(\theta_2-1)(1-\rho)^{\alpha(\theta_2-1)}\left(1+(1-\rho)^{\alpha(\theta_2-\theta_1+1)}\right) \\ &- (\theta_2-\theta_1)(1-\rho)^{\alpha(\theta_2-\theta_1)}\left(1+(1-\rho)^{\alpha(\theta_2-\theta_1-1)}\right) \\ &- \theta_2(\theta_1-1)(1-\rho)^{\alpha\theta_2}\left(1+(1-\rho)^{\alpha(\theta_2-\theta_1-1)}\right)\Big\}. \end{aligned} \tag{3.10}$$

Let us further use the notations,

$$(3.11)\quad y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{y_{\alpha,1,\theta_1,\tau}(\xi)-t_{\alpha,\theta_1,\theta_2}(\rho)y_{\alpha,\theta_1,\theta_2,\tau}(\xi)}{d_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$y_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.12)\quad z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{d_{\alpha,1,\theta_1}(\rho)-t_{\alpha,\theta_1,\theta_2}(-\xi)d_{\alpha,\theta_1,\theta_2}(\rho)}{\xi d_{\alpha,\theta_1,\theta_2}(-\xi)},$$

$$z_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(-\xi)},$$

$$(3.13)\quad u_T^{(\alpha,\theta_1,\theta_2,\tau)}(\rho) := \frac{u_{\alpha,1,\theta_1,\tau}(\rho)-t_{\alpha,\theta_1,\theta_2}(\rho)u_{\alpha,\theta_1,\theta_2,\tau}(\rho)}{d_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$u_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\rho) := \frac{u_T^{(\alpha,\theta_1,\theta_2,\tau)}(\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.14)\quad v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho') := \frac{v_{\alpha,1,\theta_1}(\rho,\rho')-t_{\alpha,\theta_1,\theta_2}(\rho)v_{\alpha,\theta_1,\theta_2}(\rho,\rho')}{d_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$v_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\rho,\rho') := \frac{v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.15)\quad f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{\xi\left\{d_{\alpha,1,\theta_1}(-\xi)-t_{\alpha,\theta_1,\theta_2}(\rho)d_{\alpha,\theta_1,\theta_2}(-\xi)\right\}}{d_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$f_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.16)\quad g_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{w_{\alpha,1,\theta_1,\tau}(\xi,\rho)-t_{\alpha,\theta_1,\theta_2}(\rho)w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho)}{d_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$g_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{g_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

with $t_{\alpha,\theta_1,\theta_2}(\rho)$, $d_{\alpha,\theta_1,\theta_2}(\rho)$, $u_{\alpha,\theta_1,\theta_2,\tau}(\rho)$, $v_{\alpha,\theta_1,\theta_2,\tau}(\rho,\rho')$, $w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho)$, $y_{\alpha,\theta_1,\theta_2,\tau}(\xi)$ and $t'_{\alpha,\theta_1,\theta_2}(\rho)$ given in (1.16), (2.10), (3.2), (3.3), (3.4), (3.5) and (3.10), respectively.

We can finally derive the non-degenerate asymptotic behaviour of the class of PORT-ρ estimators, in (1.17).

Theorem 3.2. *Let us assume that the third-order condition in* (2.3) *holds, with* ρ_0, $\rho'_0 < 0$ *and consider the* PORT-ρ *class of estimators,* $\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$, *defined in* (1.17), *with* ρ_q *given in* (1.11). *Then, with* $\theta_1 < \theta_2$, *real numbers different from* 1, $\alpha > 0$, $\tau_q \in \mathbb{R}$ *and* $0 < q < 1$ *or* $q = 0$ *provided that* $\chi_0 = x_F$ *is finite, and intermediate sequences of positive integers* $k = k_n$, *as in* (1.13), *such that* (3.7) *holds, we have:*

i) *In $\mathcal{R}_1$, let us consider the regions $\mathcal{R}_{11} := \{\rho_0 < -2\xi \wedge \chi_q \neq 0\}$, $\mathcal{R}_{12} := \{\rho_0 = -2\xi \wedge \chi_q \neq 0\}$ and $\mathcal{R}_{13} := \{-2\xi < \rho_0 < -\xi \wedge \chi_q \neq 0\}$. If we further assume that $\lim_{n\to\infty} \sqrt{k} A_0(n/k) = \lambda$ and $\lim_{n\to\infty} \sqrt{k}/U_0^2(n/k) = \lambda_U$, we get*

$$\frac{\sqrt{k}}{U_0(n/k)} \left(\widehat{\rho}_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - \rho_q\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\overset{\bullet}{\mu}{}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}, \overset{\bullet}{\sigma}{}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q}\right),$$

with

$$\overset{\bullet}{\mu}{}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = \begin{cases} \chi_q\, \lambda_U\, y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi), & \text{in } \mathcal{R}_{11} \\ \frac{\lambda\, z_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0) + \chi_q^2\, \lambda_U\, y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)}{\chi_q}, & \text{in } \mathcal{R}_{12} \\ \frac{\lambda\, z_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{\chi_q}, & \text{in } \mathcal{R}_{13}, \end{cases}$$

$y_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ and $z_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ defined in (3.11) and (3.12), respectively. Moreover,

$$\overset{\bullet}{\sigma}{}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q} \equiv \overset{\bullet}{\sigma}{}^2_{\rho_0|T,\alpha,\theta_1,\theta_2} = \left\{\overset{\bullet}{\sigma}_{T|\alpha,\theta_1,\theta_2} / t'_{\alpha,\theta_1,\theta_2}(-\xi)\right\}^2,$$

where

$$\begin{aligned} \overset{\bullet}{\sigma}{}^2_{T|\alpha,\theta_1,\theta_2} &= \left(\frac{1}{\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}\right)^2 \mathbb{V}\mathrm{ar}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi) W_k^{(\alpha,\theta_1,\theta_2)}\right) \\ &= \frac{\sigma^2_{W|\alpha,1,\theta_1} + t^2_{\alpha,\theta_1,\theta_2}(-\xi)\sigma^2_{W|\alpha,\theta_1,\theta_2} - 2t_{\alpha,\theta_1,\theta_2}(-\xi)\sigma_{W|\alpha,1,\theta_1,\theta_2}}{\left(\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\right)^2}, \end{aligned}$$

with $\sigma^2_{W|\alpha,\theta_1,\theta_2}$, $\sigma_{W|\alpha,1,\theta_1,\theta_2}$ and $t'_{\alpha,\theta_1,\theta_2}(\rho)$ given in (3.8), (3.9) and (3.10), respectively.

ii) *In $\mathcal{R}_2$, let us consider the regions $\mathcal{R}_{21} := \{-\xi < \rho_0 < -\frac{\xi}{2} \wedge \chi_q \neq 0\}$, $\mathcal{R}_{22} := \{\rho_0 = -\frac{\xi}{2} \wedge \chi_q \neq 0\}$ and $\mathcal{R}_{23} := \{\frac{\xi}{2} < \rho_0 < 0 \vee (\xi > -\rho_0 \wedge \chi_q = 0)\}$. If we further assume that $\lim_{n\to\infty} \sqrt{k} A_0^2(n/k) = \lambda_A$, $\lim_{n\to\infty} \sqrt{k} A_0(n/k) B_0(n/k) = \lambda_B$ and $\lim_{n\to\infty} \sqrt{k}/U_0(n/k) = \lambda'$, we get*

$$\sqrt{k} A_0(n/k) \left(\widehat{\rho}_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - \rho_q\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}, \sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q}\right),$$

where with $\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)} := u_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B$, and $u_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\rho)$, $v_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ and $f_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ given in (3.13), (3.14) and (3.15), respectively,

$$\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = \begin{cases} \chi_q \lambda' f_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0), & \text{in } \mathcal{R}_{21} \\ \mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)} + \chi_q \lambda' f_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0), & \text{in } \mathcal{R}_{22} \\ \mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}, & \text{in } \mathcal{R}_{23}. \end{cases}$$

Additionally,

$$\sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q} = \sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2} = \left\{\sigma_{T|\alpha,\theta_1,\theta_2}/t'_{\alpha,\theta_1,\theta_2}(\rho_0)\right\}^2,$$

with $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ *given by*

$$\begin{aligned}\sigma^2_{T|\alpha,\theta_1,\theta_2} &= \left(\tfrac{\xi}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)}\right)^2 \mathbb{V}\mathrm{ar}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)\\ &= \frac{\xi^2\left(\sigma^2_{W|\alpha,1,\theta_1} + t^2_{\alpha,\theta_1,\theta_2}(\rho_0)\sigma^2_{W|\alpha,\theta_1,\theta_2} - 2t_{\alpha,\theta_1,\theta_2}(\rho_0)\sigma_{W|\alpha,1,\theta_1,\theta_2}\right)}{\left(\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\right)^2}, \qquad (3.17)\end{aligned}$$

$\sigma^2_{W|\alpha,\theta_1,\theta_2}$ *and* $\sigma_{W|\alpha,1,\theta_1,\theta_2}$ *defined in* (3.8) *and* (3.9), *respectively.*

iii) *In* $\mathcal{R}_3$, *if we further assume that* $\lim_{n\to\infty}\sqrt{k}A_0^2(n/k) = \lambda_A$, $\lim_{n\to\infty}\sqrt{k}A_0(n/k)B_0(n/k) = \lambda_B$ *and* $\lim_{n\to\infty}\sqrt{k}A_0(n/k)/U_0(n/k) = \lambda_{AU}$, *we get*

$$\sqrt{k}A_0(n/k)\left(\widehat{\rho}_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - \rho_q\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(\widetilde{\mu}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}, \widetilde{\sigma}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q}\right),$$

where, with $\widetilde{\lambda} = \lim_{n\to\infty} 1/\bigl(A_0(n/k)U_0(n/k)\bigr) \neq 0$, $w_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)} := g_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) + \chi_q\widetilde{\lambda}\; y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $y_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $g_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ *and* $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ *defined in* (3.11), (3.16) *and* (3.17), *respectively,*

$$\widetilde{\mu}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = \frac{u_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B + \xi\chi_q w_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}\lambda_{AU}}{1+\xi\widetilde{\lambda}\chi_q},$$

$$\widetilde{\sigma}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q} = \widetilde{\sigma}^2_{\rho_0|T,\alpha,\theta_1,\theta_2} = \frac{\sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2}}{(1+\xi\widetilde{\lambda}\chi_q)^2} = \left\{\frac{\sigma_{T|\alpha,\theta_1,\theta_2}}{(1+\xi\widetilde{\lambda}\chi_q)\; t'_{\alpha,\theta_1,\theta_2}(\rho_0)}\right\}^2.$$

We finally present the non-degenerate behaviour of the PORT-ρ estimators, in (1.18).

Corollary 3.1. *Under the validity of the third-order condition in* (2.3), *with* $\rho = \rho_0$, $\rho' = \rho_0' < 0$, *and for the particular case* $(\alpha,\theta_1,\theta_2) = (1,2,3)$, *we have the validity of the following asymptotic distributional representation for the* PORT-ρ *estimator,* $\widehat{\rho}_k^{(\tau_q,q)}$, *in* (1.18).

i) *In* $\mathcal{R}_1$, *and with the same notation as before for* $\mathcal{R}_{11}$, $\mathcal{R}_{12}$ *and* $\mathcal{R}_{13}$,

$$\begin{aligned}\widehat{\rho}_k^{(\tau_q,q)} &\overset{d}{=} \rho_q + \frac{\overset{\bullet}{\sigma}_{\rho_0,q}}{\sqrt{k}/U_0(n/k)} W_k^{R_1}\\ &+ \begin{cases} \frac{\chi_q\; y_{\rho_0|T}(\xi)}{U_0(n/k)}(1+o_p(1)), & \text{in } \mathcal{R}_{11}\\ \left(\frac{z_{\rho_0|T}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q} + \frac{\chi_q\; y_{\rho_0|T}(\xi)}{U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{12}\\ \frac{z_{\rho_0|T}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q}(1+o_p(1)), & \text{in } \mathcal{R}_{13},\end{cases}\end{aligned}$$

where $W_k^{R_1}$ *is asymptotically standard normal,*

$$\begin{aligned} y_{\rho_0|T}(\xi) &= \tfrac{6\xi\left(-4+\xi\left(-13+2\xi\left(-3+2\xi(2+\xi)^2\right)\right)\right)-\xi(3+\xi)(1+2\xi)^3(3+2\xi)\tau}{12(1+\xi)^2(1+2\xi)^3}, \\ z_{\rho_0|T}(\xi,\rho_0) &= -\tfrac{(1+\xi)^3\rho_0(\xi+\rho_0)}{\xi^2(1-\rho_0)^3} \end{aligned}$$

and $\overset{\bullet}{\sigma}{}^2_{\rho_0,q} = (1+\xi)^6\left(2\,\xi^2+2\,\xi+1\right)/(\xi\chi_q)^2$.

ii) *In* $\mathcal{R}_2$, *and again with the same notation as before for* $\mathcal{R}_{21}$, $\mathcal{R}_{22}$ *and* $\mathcal{R}_{23}$,

$$\begin{aligned} \widehat{\rho}_k^{(\tau_q,q)} &\overset{d}{=} \rho_q + \tfrac{\sigma_{\rho_0,q}}{\sqrt{k}A_0(n/k)}W_k^{R_2} \\ &+ \begin{cases} \left(\tfrac{\chi_q f_{\rho_0|T}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{21} \\ \left(m_{\rho_0,\rho_0'|T} + \tfrac{\chi_q f_{\rho_0|T}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{22} \\ m_{\rho_0,\rho_0'|T}(1+o_p(1)), & \text{in } \mathcal{R}_{23}, \end{cases} \end{aligned}$$

where $m_{\rho,\rho'|T} = u_{\rho|T}(\rho)A_0(n/k) + v_{\rho|T}(\rho,\rho')B_0(n/k)$, *with* $u_{\rho|T}(\rho) \equiv u_\rho(\tau=\tau_q)$ *and* $v_{\rho|T}(\rho,\rho') \equiv v_{\rho,\rho'}$, *given by*

(3.18)
$$u_\rho \equiv u_\rho(\tau) = \tfrac{\rho\left(\rho\,(42-45\,\tau)+\rho^3\,(96-44\,\tau)+8\,\rho^4\,(\tau-3)+9\,\tau+2\,\rho^2\,(37\,\tau-60)\right)}{12\,\xi\,(1-3\,\rho+2\,\rho^2)^2}$$

and

(3.19) $$v_{\rho,\rho'} = (1-\rho)^3\,\rho'\,\left(\rho+\rho'\right)/\left\{\rho\,\left(1-\rho-\rho'\right)^3\right\},$$

respectively. Moreover, $W_k^{R_2}$ *is asymptotically standard normal,*

$$\begin{aligned} \sigma^2_{\rho_0,q} &\equiv \sigma^2_{\rho_0} = \xi^2(1-\rho_0)^6\left(2\rho_0^2-2\rho_0+1\right)/{\rho_0}^2, \\ f_{\rho_0|T}(\xi,\rho_0) &= \tfrac{\xi^2(1-\rho_0)^3(\xi+\rho_0)}{(1+\xi)^3\rho_0}. \end{aligned}$$

iii) *In* $\mathcal{R}_3$, *and with* $\widetilde{\lambda} = \lim_{n\to\infty} 1/\big(A_0(n/k)U_0(n/k)\big) = (\xi\beta_0 C)^{-1} \neq 0$, *with* C *given in* (2.4),

$$\begin{aligned} \widehat{\rho}_k^{(\tau_q,q)} &\overset{d}{=} \rho_q + \tfrac{\widetilde{\sigma}_{\rho_0,q}}{\sqrt{k}A_0(n/k)}W_k^{R_3} \\ &+\left(\widetilde{u}_{\rho_0|T}A_0(n/k) + \widetilde{v}_{\rho_0,\rho_0'|T}B_0(n/k) + \xi\chi_q\tfrac{\widetilde{g}_{\xi,\rho_0|T}+\chi_q\widetilde{\lambda}\;\widetilde{y}_{\xi,\rho_0|T}}{U_0(n/k)}\right)(1+o_p(1)), \end{aligned}$$

where $W_k^{R_3}$ *is an asymptotically standard normal r.v.,* $u_{\rho|T} \equiv u_\rho(\tau=\tau_q)$ *and* $v_{\rho,\rho'|T} \equiv v_{\rho,\rho'}$, *defined in* (3.18) *and* (3.19), *respectively,* $\widetilde{u}_{\rho|T} = u_{\rho|T}/(1+\xi\widetilde{\lambda}\chi_q)$, $\widetilde{v}_{\rho,\rho'|T} = v_{\rho,\rho'|T}/(1+\xi\widetilde{\lambda}\chi_q)$, *and* $\widetilde{\bullet}_{\xi,\rho|T} = \bullet_{\xi,\rho|T}/(1+\xi\widetilde{\lambda}\chi_q)$, *with* $\bullet = g, y$, *with*

$$\begin{aligned} g_{\xi,\rho_0|T} = g_{-\rho_0,\rho_0|T} &\equiv g_{\rho_0|T} \\ &= -\tfrac{6\left(4+\rho_0\left(-13+2\rho_0\left(3+2\rho_0(2-\rho_0)^2\right)\right)\right)+(3-\rho_0)(3-2\rho_0)(1-2\rho_0)^3\tau}{6(1-\rho_0)^2(1-2\rho_0)^3}, \end{aligned}$$

$$y_{\xi,\rho_0|T} = y_{-\rho_0,\rho_0|T} \;\equiv\; y_{\rho_0|T} = \tfrac{(3-\rho_0)(1-\rho_0)^3}{2\rho_0} b(\rho_0,\tau),$$

$$b(\rho,\tau) = -\tfrac{(\rho-2)^2(\tau-2)}{4(1-\rho)^4} + \tfrac{\tau-1}{(1-\rho)^2} - \tfrac{2(1-\rho)}{(1-2\rho)^2} + \tfrac{2}{1-2\rho} - \tfrac{1}{1-\rho(3-2\rho)}$$
$$+ \tfrac{(1-\rho)\rho\left\{-(\rho+3)(5\rho(\rho+3)+12)(2\rho+1)^3\tau-6(6+\rho(3+2\rho)(4\rho^5+24\rho^4+42\rho^3+31\rho^2+14\rho+9))\right\}}{12(3-\rho)(1+\rho)^6(1+2\rho)^3}$$

$$\text{and } \widetilde{\sigma}^2_{\rho_0,q} = (1-\rho_0)^6\left(2\,{\rho_0}^2 - 2\,\rho_0 + 1\right)/\left(1-\widetilde{\lambda}\chi_q\,\rho_0\right)^2.$$

3.3. A few comments and conclusions

- We consider that the class of PORT-ρ estimators introduced and studied in this article is, from a theoretical point of view, a nice alternative to the classical ρ-estimators whenever, in a real data analysis, we are led to a bad performance of the classical estimators. Such a bad performance is usually due to the existence of a location $s \neq 0$ in the available data, associated with unshifted models with $\xi + \rho_0 < 0$, a quite common situation in practical applications.
- Concomitantly, the development and the theoretical study of a new class of PORT-estimators of the functional A, in (1.6), can lead us to SORB EVI-estimators, invariant for changes in location and MVRB for an adequate choice of q, i.e. EVI-estimators of the type of the ones in Caeiro *et al.* (2005), Gomes *et al.* (2007) and Gomes *et al.* (2008c), but invariant for changes in location, the so-called PORT-MVRB EVI-estimators. Note that these PORT-MVRB EVI-estimators have already been studied for finite samples in Gomes *et al.* (2011, 2012), and exhibit a quite interesting performance.

4. A SMALL-SCALE MONTE-CARLO SIMULATION

We next present in Figures 1 and 2, respectively the mean values (E) and the root mean squared errors (RMSE), of the classical estimator $\widehat{\rho}_k^{(0)}$ and the PORT-ρ estimators $\left\{\widehat{\rho}_k^{(0,q)}\right\}_{q=0,0.1,0.25}$, as defined in Eq. (1.18), as a function of the sample fraction k/n, for sample sizes $n = 5000$ and $n = 10000$. The results are associated with the output of a small-scale simulation, of size 5000, related to underlying Fréchet parents $F_0(x) = \exp(-x^{-1/\xi}), x > 0$, with $\xi = 0.25$, and the shifted model $F_s(x) = \exp\left(-(x-s)^{-1/\xi}\right),\ x > s$, with $s = 1$.

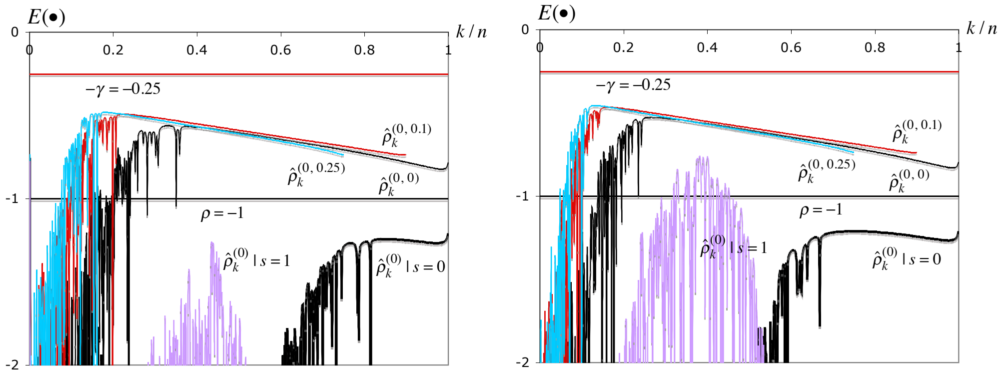

Figure 1: Mean values of the estimators under consideration for Fréchet unshifted ($s = 0$) and shifted ($s = 1$) parents, with $\xi = 0.25$, and sample size $n = 5000$ (left) and $n = 10000$ (right).

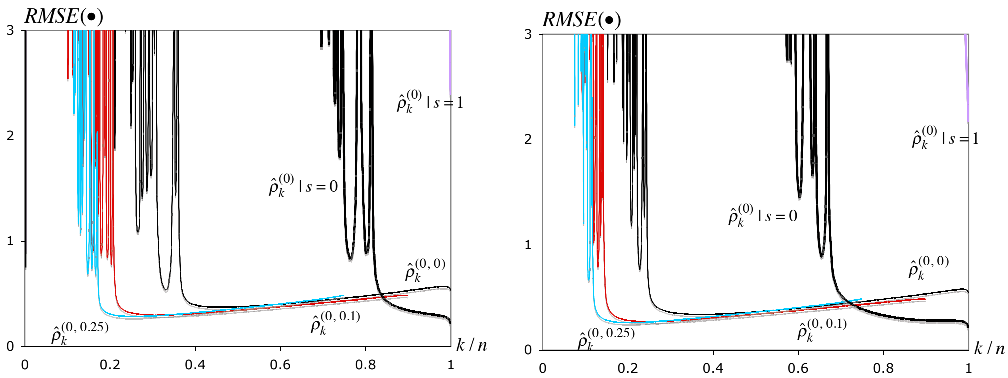

Figure 2: RMSEs of the estimators under consideration for Fréchet unshifted ($s = 0$) and shifted ($s = 1$) parents, with $\xi = 0.25$, and sample size $n = 5000$ (left) and $n = 10000$ (right).

There is indeed only a light improvement in all estimators as the sample size increases, and a high volatility of the classical ρ-estimators for shifted models, as can be seen, in either Figure 1 or in Figure 2, where the RMSE of such estimator is above 2, even for $n = 10000$. For smaller values of n, the sample paths of all estimators are even more volatile, particularly for small sample fractions k/n. But if we consider a much larger sample size, $n = 100000$, there is a clear improvement only in the classical ρ-estimators for shifted models, as can be seen, in Figure 3.

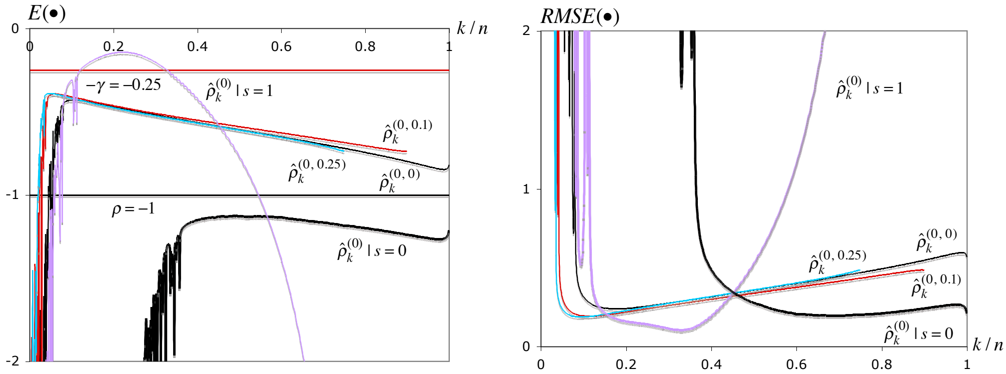

Figure 3: Mean values (left) and RMSEs (right) of the estimators under consideration for Fréchet unshifted ($s = 0$) and shifted ($s = 1$) parents, with $\xi = 0.25$, and sample size $n = 100000$.

We now would like to emphasise the following points:

- The stability of the classical ρ-estimators around the 'target' for large k can be fictitious or even non-existent, unless the model is an unshifted model. As can be seen in Figures 1 and 3, left, the classical ρ-estimator associated with the unshifted model, $\widehat{\rho}_k^{(0)}|s = 0$ is close to -1 for large values of k, as expected, but the ρ-estimator associated with the shifted model, $\widehat{\rho}_k^{(0)}|s = 1$, that should converge to -0.25, exhibits no stability in the sample paths.

- We are in the region $\xi + \rho_0 < 0$ ($\xi = 0.25,\ \rho_0 = -1$). Consequently, the PORT-ρ estimator should converge to $-\xi = -0.25$ for $\chi_q \neq 0$ and to $\rho_0 = -1$ for $\chi_q = 0$. Unfortunately, the pattern of the PORT-ρ estimators does not depend strongly on χ_q. If we decide for a large value of k, we obtain a value close to -1 if $\chi_q = 0$, but a value not a long way from -1 when $\chi_q \neq 0$. But if we look at the region of k/n close to 0.2, the PORT-ρ estimators associated with $\chi_q \neq 0$ are reasonably close to $-\xi = -0.25$, with a not too large RMSE. We shall thus be again confronted with an adequate choice of the threshold k.

- This means that for shifted models or PORT-ρ estimators associated with $\chi_q \neq 0$, the optimal level is clearly attained for not very large k, as can be seen in Figures 2 and 3, right, when we look at the minimal RMSE.

- For $\chi_q = 0$, the PORT-ρ estimator is able to beat the classical one regarding minimum RMSE, even for very large sample sizes.

- Similar comments apply to other simulated underlying models.

- The choice of the tuning parameters τ and τ_q is also crucial. We have here used $\tau_q = \tau = 0$. The choice $\tau = 0$ has been heuristically suggested and used before for the ρ-estimation and the region $|\rho| \le 1$, but it is possibly not the most adequate choice for the PORT-ρ estimation. This is another interesting topic out of the scope of this paper.

5. PROOFS

Proof: [**Lemma 2.1**]. We begin by writing

$$\begin{aligned}\ln U_q(tx) - \ln U_q(t) &= \ln \tfrac{U_0(tx)-\chi_q}{U_0(t)-\chi_q} = \ln\left(\tfrac{U_0(tx)}{U_0(t)} \, \tfrac{1-\frac{\chi_q}{U_0(tx)}}{1-\frac{\chi_q}{U_0(t)}} \right) \\ &= \xi \ln x + \ln\left(x^{-\xi} \tfrac{U_0(tx)}{U_0(t)}\right) + \ln\left(1 - \tfrac{\chi_q}{U_0(tx)}\right) - \ln\left(1 - \tfrac{\chi_q}{U_0(t)}\right).\end{aligned}$$

Using Taylor's expansion of $\ln(1+x)$, as $x \to 0$, we obtain

$$\begin{aligned}\ln U_q(tx) - \ln U_q(t) &= \xi \ln x + \ln\left(x^{-\xi} \tfrac{U_0(tx)}{U_0(t)}\right) - \tfrac{\chi_q}{U_0(tx)} + \tfrac{\chi_q}{U_0(t)} + o\left(\tfrac{1}{U_0(t)}\right), \\ &= \xi \ln x + \ln\left(x^{-\xi} \tfrac{U_0(tx)}{U_0(t)}\right) + \tfrac{\chi_q}{U_0(t)}\left(1 - \tfrac{U_0(t)}{U_0(tx)}\right) + o\left(\tfrac{1}{U_0(t)}\right),\end{aligned}$$

as $t \to \infty$. Since $U_0(tx) \sim x^\xi U_0(t)$, $t \to \infty$, we thus have that

$$\begin{aligned}&\ln U_q(tx) - \ln U_q(t) - \xi \ln x \\ &= \ln\left(x^{-\xi} \tfrac{U_0(tx)}{U_0(t)}\right) + \tfrac{\chi_q}{U_0(t)}(1 - x^{-\xi}) - \tfrac{\chi_q}{U_0(t)}\left(\tfrac{U_0(t)}{U_0(tx)} - x^{-\xi}\right) + o\left(\tfrac{1}{U_0(t)}\right).\end{aligned}$$

Now, condition (1.6) with U, A and ρ replaced with U_0, A_0 and ρ_0, respectively, ascertains

$$\begin{aligned}\ln U_q(tx) - \ln U_q(t) - \xi \ln x = A_0(t)\, \tfrac{x^{\rho_0}-1}{\rho_0} &+ \tfrac{\chi_q}{U_0(t)}(1 - x^{-\xi}) \\ &- \tfrac{\chi_q}{U_0(t)}\left(\tfrac{U_0(t)}{U_0(tx)} - x^{-\xi}\right) + o\left(\tfrac{1}{U_0(t)}\right) + o\big(A_0(t)\big).\end{aligned}$$

The precise result thus follows by noting that $1/U_0 \in RV_{-\xi}$ (hence χ_q/U_0 is also in $RV_{-\xi}$) and that $x^\xi U_0(t)/U_0(tx) - 1$ divided by $A_0(t)$ has the same limit as in (1.6), with the same second order parameter ρ_0 (cf. Proposition 6 and Corollary 7 of Neves, 2009). This result confirms a similar one for the rate of convergence of $U_q(tx)/U_q(t)$ to x^ξ as obtained in Araújo Santos *et al.* (2006, Lemma 2.1). □

Proof: [**Proposition 2.2**]. Using the same arguments as in Fraga Alves *et al.* (2009), bearing in mind the unshifted model ($s = 0$), we can write the PORT log-excesses of the observations over the random quantile $X_{n_q:n}$, i.e. $X_{n-i+1:n} - X_{n_q:n}$, for $i = 1, ..., k$, in terms of the POT log-excesses, $X_{n-i+1:n} - \chi_q$, over $\chi_q := F_0^{\leftarrow}(q) = U_0(1/(1-q))$, as follows:

$$\ln\left(X_{n-i+1:n} - X_{n_q:n}\right) = \ln\left(X_{n-i+1:n} - \chi_q\right) + \ln\left(1 - \tfrac{X_{n_q:n}-\chi_q}{X_{n-i+1:n}-\chi_q}\right).$$

Now for the second term holds the inequality

$$\ln\left(1-\frac{X_{n_q:n}-\chi_q}{X_{n-i+1:n}-\chi_q}\right) \le \ln\left(1-\frac{X_{n_q:n}-\chi_q}{X_{n:n}-\chi_q}\right).$$

Since we are assuming $\xi>0$ we have that $X_{n:n}-\chi_q \xrightarrow[n\to\infty]{p} \infty$, which in conjunction with the asymptotical normality of the empirical quantile $\sqrt{n}\left(X_{n_q:n}-\chi_q\right) = O_p(1)$ ascertains

$$\begin{aligned}\sqrt{k}\ln\left(1-\frac{X_{n_q:n}-\chi_q}{X_{n:n}-\chi_q}\right) &= \sqrt{k}\,\frac{X_{n_q:n}-\chi_q}{X_{n:n}-\chi_q}(1+o_p(1)) = \sqrt{k/n}\,o_p\left(\sqrt{n}(X_{n_q:n}-\chi_q)\right)\\ &= o_p\left(\sqrt{k/n}\right) \xrightarrow[n\to\infty]{p} 0\,.\end{aligned}$$

Then it is easily seen that, for any $\alpha>0$, the PORT-moment statistics $M_{n,k}^{(\alpha,q)}$ provided in (1.14) are asymptotically identically distributed to their POT-moment counterparts

$$\widetilde{M}_{n,k}^{(\alpha,q)} = \frac{1}{k}\sum_{i=1}^{k}\left(\ln\frac{X_{n-i+1:n}-\chi_q}{X_{n-k:n}-\chi_q}\right)^{\alpha}.$$

In fact, $\widetilde{M}_{n,k}^{(\alpha,q)}$ differs from $M_{n,k}^{(\alpha)} = \frac{1}{k}\sum_{i=1}^{k}\left(\ln\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{\alpha}$ by a deterministic shift $-\chi_q = -U_0(1/(1-q))$ in the observations X_i, $1\le i\le n$. Then the asymptotic results for $\widetilde{M}_{n,k}^{(\alpha,q)} \equiv \frac{1}{k}\sum_{i=1}^{k}\left(\ln\frac{\widetilde{X}_{n-i+1:n}}{\widetilde{X}_{n-k:n}}\right)^{\alpha}$ can be obtained in view of the shifted observations from $\widetilde{X} := X_q = X_0-\chi_q$, with associated $U_q(t)=U_0(t)-\chi_q$.

Let us begin with the first moment of the log-excesses. With $\{Y_i\}_{i=1,\dots,n}$ i.i.d. unit Pareto r.v.'s, we have the equality in distribution

$$\{\widetilde{X}_{n-i+1:n}\}_{i=1}^{n} := \{X_{n-i+1:n}-\chi_q\}_{i=1}^{n} \stackrel{d}{=} \{U_q(Y_{n-i+1:n})\}_{i=1}^{n},$$

and we can write,

$$\begin{aligned}\text{(5.1)}\quad \widetilde{M}_{n,k}^{(1,q)} &= \frac{1}{k}\sum_{i=1}^{k}\ln\widetilde{X}_{n-i+1:n}-\ln\widetilde{X}_{n-k:n}\\ &\stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k}\ln U_q\left(Y_{n-i+1:n}\right)-\ln U_q(Y_{n-k:n}).\end{aligned}$$

We note that

$$\begin{aligned}&\ln U_q(tx)-\ln U_q(t)-\big(\ln U_0(tx)-\ln U_0(t)\big)\\ &=\ln\frac{\frac{U_0(tx)}{U_0(t)}-\frac{\chi_q}{U_0(t)}}{1-\frac{\chi_q}{U_0(t)}}-(\ln U_0(tx)-\ln U_0(t))\\ &=\ln\Big(\big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1\big)-x^{-\xi}\tfrac{\chi_q}{U_0(t)}+1\Big)-\ln\Big(\big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1\big)+1\Big)-\ln\big(1-\tfrac{\chi_q}{U_0(t)}\big).\end{aligned}$$

Next, we deal with the first two terms in the above. Towards this end, we define for each $x>0$,

$$\begin{aligned}y_1(t) &:= \big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1\big)-x^{-\xi}\tfrac{\chi_q}{U_0(t)},\\ y_2(t) &:= x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1,\end{aligned}$$

with $y_1(t)$ and $y_2(t)$ converging to zero as $t \to \infty$ (see text in the end of the proof of lemma 2.1). MacLaurin's expansion of the logarithm, i.e. $\ln(1+y) = y - y^2/2 + o(y^2)$, applied to both $y_1(t)$ and $y_2(t)$ now yields

$$\begin{aligned}&\ln U_q(tx) - \ln U_q(t) - \big(\ln U_0(tx) - \ln U_0(t)\big)\\ &= -x^{-\xi}\tfrac{\chi_q}{U_0(t)} - \frac{1}{2}\Big(x^{-\xi}\tfrac{\chi_q}{U_0(t)}\Big)^2\big(1+o(1)\big) + \big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)} - 1\big)x^{-\xi}\tfrac{\chi_q}{U_0(t)}\big(1+o(1)\big)\\ &\quad - \ln\big(1 - \tfrac{\chi_q}{U_0(t)}\big).\end{aligned}$$

In order to have a grasp at the remainder $o(1)$-terms, we require the following uniform bounds, which arise in connection with the third-order framework in (2.3) and Remark B.3.12 of de Haan and Ferreira (2006): for any ε, $\delta > 0$, there exists a $t_0 = t_0(\varepsilon, \delta)$ such that for $t \geq t_0$, $x \geq 1$,

$$\left|\frac{\frac{x^{-\xi}\frac{U_0(tx)}{U_0(t)} - 1}{A_0(t)} - \frac{x^{\rho_0}-1}{\rho_0}}{B_0(t)} - \frac{x^{\rho_0+\rho_0'}-1}{\rho_0+\rho_0'}\right| \leq \varepsilon x^{\rho_0+\rho_0'+\delta}.$$

Furthermore, since $0 < -\ln(1-v) - v - v^2/2 < v^3/\big(3(1-v)\big)$, $v \in (0,1)$, we can set $v = \chi_q/U_0$ in order to establish the upper bound

$$\begin{aligned}&\ln U_q(tx) - \ln U_q(t) - \big(\ln U_0(tx) - \ln U_0(t)\big)\\ &\qquad - \xi\big(\tfrac{x^{-\xi}-1}{-\xi}\big)\tfrac{\chi_q}{U_0(t)} - \xi\big(\tfrac{x^{-2\xi}-1}{-2\xi}\big)\big(\tfrac{\chi_q}{U_0(t)}\big)^2 - x^{-\xi}\big(\tfrac{x^{\rho_0}-1}{\rho_0}\big)\tfrac{\chi_q A_0(t)}{U_0(t)}\\ &\leq \tfrac{\chi_q^3}{3}\Big(U_0^3(t)\big(1-\tfrac{\chi_q}{U_0(t)}\big)\Big)^{-1} + x^{-\xi}\tfrac{x^{\rho_0+\rho_0'}-1}{\rho_0+\rho_0'}\chi_q\tfrac{A_0(t)}{U_0(t)}B_0(t) + \varepsilon\big|\tfrac{A_0(t)}{U_0(t)}B_0(t)\big|x^{-\xi+\rho_0+\rho_0'+\delta}.\end{aligned}$$

We can also establish a similar lower bound. In this development, and with respect to the right hand-side of (5.1), assuming $k = k_n$ an intermediate sequence of positive integers, i.e. such that (1.13) holds, then taking average across $i = 1, 2, ..., k$, for arbitrary ε, $\delta > 0$, the weak law of large numbers ensures that

$$M_{n,k}^{(1,q)} - M_{n,k}^{(1)} = \tfrac{\chi_q}{U_0(n/k)}\Big(\tfrac{\xi}{1+\xi} + \tfrac{\xi}{1+2\xi}\tfrac{\chi_q}{U_0(n/k)}(1+o_p(1)) + \tfrac{A_0(n/k)}{(1+\xi)(1+\xi-\rho_0)}(1+o_p(1))\Big).$$

We are then led to (2.16) for $\alpha = 1$ where

$$\tfrac{\xi}{1+\xi} = \xi\overline{\mu}_1^{(2)}(-\xi), \quad \tfrac{1}{(1+\xi)(1+\xi-\rho_0)} = \overline{\mu}_1^{(2)}(\xi,\rho_0) \quad \text{and} \quad \tfrac{\xi}{1+2\xi} = \xi\overline{\mu}_1^{(2)}(-2\xi).$$

Let us next consider a general α. Similarly as before, we can write

$$\begin{aligned}&\Big(\ln U_q(tx) - \ln U_q(t)\Big)^\alpha - \Big(\ln U_0(tx) - \ln U_0(t)\Big)^\alpha = \tfrac{\alpha(\xi\ln x)^\alpha\chi_q}{U_0(t)}\Bigg(\tfrac{1}{\ln x}\ \Big(\tfrac{x^{-\xi}-1}{-\xi}\Big)\\ &\qquad + \tfrac{1}{\xi}\Big(\tfrac{x^{-\xi}}{\ln x}\Big(\tfrac{x^{\rho_0}-1}{\rho_0}\Big) + \tfrac{(\alpha-1)}{(\ln x)^2}\Big(\tfrac{x^{\rho_0}-1}{\rho_0}\Big)\Big(\tfrac{x^{-\xi}-1}{-\xi}\Big)\Big)A_0(t)\\ &\qquad\qquad + \tfrac{1}{\ln x}\ \tfrac{\chi_q}{U_0(t)}\Bigg(\Big(\tfrac{x^{-2\xi}-1}{-2\xi}\Big) + \tfrac{\alpha-1}{2\ln x}\Big(\tfrac{x^{-\xi}-1}{-\xi}\Big)^2\Bigg)\Bigg) + o(1/U_0^2(t)).\end{aligned}$$

Considering again $k = k_n$ as an intermediate sequence of integers, i.e. (1.13) holds, the same type of arguments of the previous case ($\alpha = 1$), and the weak law of large numbers enable us to write (2.16) for any $\alpha > 0$. □

Proof: [**Theorem 3.2**]. (**i**) In the region $\mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$, as $t \to \infty$, the third and last term of the right-hand side of (3.6) is the dominant one, and the r.v.'s $D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)/(1/U_0(n/k))$ converge in probability to $\alpha\ \tau_q \xi^{\alpha\tau_q} \chi_q\ d_{\alpha,\theta_1,\theta_2}(-\xi)$ provided that (3.7) holds, i.e. if $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$ (see Remark 3.1). Moreover, we get

$$\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)}{1/U_0(n/k)} \stackrel{d}{=} \xi^{\alpha\tau_q}\Bigg(\alpha\ \tau_q\chi_q\ d_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{\tau_q W_k^{(\alpha,\theta_1,\theta_2)} U_0(n/k)}{\sqrt{k}} + \alpha\tau_q\left\{\frac{d_{\alpha,\theta_1,\theta_2}(\rho_0) A_0(n/k) U_0(n/k)(1+o_p(1))}{\xi} + \frac{\chi_q^2 y_{\alpha,\theta_1,\theta_2,\tau_q}(\xi)(1+o_p(1))}{U_0(n/k)}\right\}\Bigg).$$

For levels k such that (1.13) holds, with $W_k^{(\alpha,\theta_1,\theta_2)}$ given in (2.12), and with $T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ defined in (1.15), we can say that if (3.7) holds,

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\Big(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi) W_k^{(\alpha,\theta_1,\theta_2)}\Big)}{\alpha\chi_q\sqrt{k}/U_0(n/k)} + \frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0) A_0(n/k) U_0(n/k)(1+o_p(1))}{\chi_q} + \frac{\chi_q y_T^{(\alpha,\theta_1,\theta_2\tau_q)}(\xi,-\xi)(1+o_p(1))}{U_0(n/k)}.$$

For sequences of positive intermediate integers $k = k_n$ such that $k_n = o(n)$, $\sqrt{k}/U_0(n/k) \to \infty$, $\sqrt{k}A_0(n/k) \to \lambda$ and $\sqrt{k}/U_0^2(n/k) \to \lambda_U$, as $n \to \infty$, let us consider the following cases:

- if $\xi + \rho_0 < -\xi$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\Big(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi) W_k^{(\alpha,\theta_1,\theta_2)}\Big)}{\alpha\chi_q\sqrt{k}/U_0(n/k)} + \frac{\chi_q y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)(1+o_p(1))}{U_0(n/k)},$$

  and

$$\frac{\sqrt{k}}{U_0(n/k)}\Big(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(-\xi)\Big) \xrightarrow[n\to\infty]{d} \mathcal{N}(\dot{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \dot{\sigma}^2_{T|\alpha,\theta_1,\theta_2}),$$

  where $\dot{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \lambda_U \chi_q y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)$, with $y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ defined in (3.11).

- if $\xi + \rho_0 = -\xi$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\Big(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi) W_k^{(\alpha,\theta_1,\theta_2)}\Big)}{\alpha\chi_q\sqrt{k}/U_0(n/k)} + \frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\ A_0(n/k) U_0(n/k)(1+o_p(1))}{\chi_q} + \frac{\chi_q\ y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)(1+o_p(1))}{U_0(n/k)},$$

and

$$\frac{\sqrt{k}}{U_0(n/k)}\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(-\xi)\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \overset{\bullet}{\sigma}{}^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \frac{\lambda z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{\chi_q} + \lambda_U \chi_q y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)$, with $y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ and $z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ defined in (3.11) and (3.12), respectively.

- if $\xi+\rho_0 > -\xi$ and $\chi_q \neq 0$, then

$$\begin{aligned} T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \overset{d}{=}\ & t_{\alpha,\theta_1,\theta_2}(-\xi) \\ &+ \frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi)W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\chi_q\sqrt{k}/U_0(n/k)} \\ &+ \frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)A_0(n/k)U_0(n/k)(1+o_p(1))}{\chi_q}, \end{aligned}$$

and

$$\frac{\sqrt{k}}{U_0(n/k)}\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(-\xi)\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \overset{\bullet}{\sigma}{}^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \frac{\lambda z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{\chi_q}$, with $z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ defined in (3.12).

(ii) In the region $\xi+\rho_0>0$, where $1/U_0(t) = o(A_0(t))$, as $t\to\infty$, or more generally in the region $\mathcal{R}_2$, the second term of the right-hand side of (3.6) is the dominant one. In $\mathcal{R}_2$, $A_q(t)=A_0(t)$, so condition (3.7) can be rewritten as $\sqrt{k}A_0(n/k)\to\infty$, as $n\to\infty$ and if we assume that this condition holds,

$$\begin{aligned} \frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)}{A_0(n/k)} \overset{d}{=} \xi^{\alpha\tau_q}\Bigg(&\frac{\alpha\tau_q\ d_{\alpha,\theta_1,\theta_2}(\rho_0)}{\xi} + \frac{\tau_q W_k^{(\alpha,\theta_1,\theta_2)}}{\sqrt{k}A(n/k)} \\ &+ u_{\alpha,\theta_1,\theta_2,\tau_q}(\rho_0)A_0(n/k)(1+o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1)) \\ &+ \frac{\alpha\tau_q\chi_q}{A_0(n/k)U_0(n/k)}d_{\alpha,\theta_1,\theta_2}(-\xi)\Bigg). \end{aligned}$$

If $\xi > -\rho_0$ or ($\xi\le-\rho_0$, $\chi_q=0$), and (3.7) holds,

$$\begin{aligned} T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \overset{d}{=}\ & t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\sqrt{k}A_0(n/k)} \\ &+ \left(u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k) + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)\right)(1+o_p(1)) \\ &+ \frac{\chi_q\ f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1)). \end{aligned}$$

For sequences of positive intermediate integers $k=k_n$ such that $k_n=o(n)$, $\sqrt{k}A_0(n/k)\to\infty$, $\sqrt{k}A_0^2(n/k)\to\lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k)\to\lambda_B$ and $\sqrt{k}/U_0(n/k)$

$\to \lambda'$, as $n \to \infty$, let us consider the following cases:

- if $0 < \xi + \rho_0 < -\rho_0$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\sqrt{k}A_0(n/k)} + \frac{\chi_q\ f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1)),$$

and

$$\sqrt{k}A_0(n/k)\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \sigma^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \chi_q\ f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda'$, with $f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ and $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ defined in (3.15) and (3.17), respectively.

- if $\xi + \rho_0 = -\rho_0$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\sqrt{k}A_0(n/k)} + \left(u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k) + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)\right)(1+o_p(1)) + \frac{\chi_q\ f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1)),$$

and

$$\sqrt{k}A_0(n/k)\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \sigma^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B + \chi_q\ f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda'$, $u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho)$, $v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$, $f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ and $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ defined in (3.13), (3.14), (3.15) and (3.17), respectively.

- if $\xi + \rho_0 > -\rho_0$ or $(\xi + \rho_0 > 0 \wedge \chi_q = 0)$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\sqrt{k}A_0(n/k)} + \left(u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k) + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)\right)(1+o_p(1)),$$

and

$$\sqrt{k}A_0(n/k)\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \sigma^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \mu_{T|\alpha,\theta_1,\theta_2,\tau_q} = u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B$, with $u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho)$ and $v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ defined in (3.13) and (3.14), respectively, and $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ is defined in (3.17).

(iii) In the region $\mathcal{R}_3$, $A_0(t)$ and $1/U_0(t)$ are of the same order, i.e. the dominant terms of the right-hand side of (3.6) are the second and the third. In $\mathcal{R}_3$, $A_q(t) = A_0(t) + \xi\chi_q/U_0(t)$, so condition (3.7) can be rewritten as $\sqrt{k}A_0(n/k) \to \infty$, as $n \to \infty$. If we assume that this condition holds with $\widetilde{\lambda} = \lim_{n\to\infty} 1/(A_0(n/k)U_0(n/k)) \neq 0$, then

$$\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)}{A_0(n/k)} \stackrel{d}{=} \xi^{\alpha\tau_q}\Bigg(\frac{\alpha\tau_q}{\xi}\Big\{ d_{\alpha,\theta_1,\theta_2}(\rho_0) + \xi\,\widetilde{\lambda}\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\Big\} + \frac{\tau_q W_k^{(\alpha,\theta_1,\theta_2)}}{\sqrt{k}A_0(n/k)}$$
$$+ \frac{\alpha\tau_q}{\xi}\Big\{u_{\alpha,\theta_1,\theta_2,\tau_q}(\rho_0)A_0(n/k)(1+o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1))\Big\}$$
$$+ \frac{\alpha\tau_q\chi_q}{U_0(n/k)}\Big\{w_{\alpha,\theta_1,\theta_2,\tau_q}(\xi,\rho_0) + y_{\alpha,\theta_1,\theta_2,\tau_q}(\xi)\widetilde{\lambda}\chi_q(1+o_p(1))\Big\}\Bigg),$$

If $\xi + \rho_0 = 0$ and $\chi_q \neq 0$, if we consider levels k such that (1.13) and (3.7) hold,

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\Big(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)W_k^{(\alpha,\theta_1,\theta_2)}\Big)}{\alpha(1+\xi\widetilde{\lambda}\chi_q)\sqrt{k}A_0(n/k)}$$
$$+ \frac{u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k) + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)}{1+\xi\widetilde{\lambda}\chi_q}(1+o_p(1))$$
$$+ \left\{\frac{\xi\chi_q\; g_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,\rho_0)}{(1+\xi\widetilde{\lambda}\chi_q)U_0(n/k)} + \frac{\xi\chi_q^2\;\widetilde{\lambda}\; y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,\rho_0)}{(1+\xi\widetilde{\lambda}\chi_q)U_0(n/k)}\right\}(1+o_p(1)),$$

with $y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $u_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $v_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ and $g_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ defined in (3.11), (3.13), (3.14) and (3.16), respectively. The proof of the theorem follows for sequences of positive intermediate integers $k = k_n$ such that $k_n = o(n)$, $\sqrt{k}A_0(n/k) \to \infty$, $\sqrt{k}A_0^2(n/k) \to \lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$ and $\sqrt{k}A_0(n/k)/U_0(n/k) \to \lambda_{AU}$, as $n \to \infty$. □

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REFERENCES

[1] Araújo Santos, P.; Fraga Alves, M.I. and Gomes, M.I. (2006). Peaks over random threshold methodology for tail index and quantile estimation, *Revstat*, **4**(3), 227–247.

[2] Beirlant, J.; Dierckx, G.; Goegebeur, Y. and Matthys, G. (1999). Tail index estimation and an exponential regression model, *Extremes*, **2**, 177–200.

[3] Beirlant, J.; Caeiro, F. and Gomes, M.I. (2012). An overview and open research topics in the field of statistics of univariate extremes, *Revstat*, **10**(1), 1–31.

[4] Caeiro, C. and Gomes, M.I. (2006). A new class of estimators of a scale second order parameter, *Extremes*, **9**, 193–211.

[5] Caeiro, F. and Gomes, M.I. (2012a). *A Semi-Parametric Estimator of a Shape Second Order Parameter*. In "New Advances in Statistical Modeling and Application. Studies in Theoretical and Applied Statistics" (A. Pacheco, M.R. Oliveira, R. Santos and C.D. Paulino, Eds.), Selected Papers of the Statistical Societies, Springer-Verlag, Berlin and Heidelberg, 137–144.

[6] Caeiro, F. and Gomes, M.I. (2012b). Bias reduction in the estimation of a shape second-order parameter of a heavy right tail model, *J. Statist. Comput. Simul.*, `http://dx.doi.org/10.1080/00949655.2014.975707`

[7] Caeiro, C.; Gomes, M.I. and Pestana, D. (2005). Direct reduction of bias of the classical Hill estimator, *Revstat*, **3**(2), 113–136.

[8] Ciuperca, G. and Mercadier, C. (2010). Semi-parametric estimation for heavy tailed distributions, *Extremes*, **13**(1), 55–87.

[9] Dekkers, A.; Einmahl, J. and de Haan, L. (1989). A moment estimator for the index of an extreme-value distribution, *Annals of Statistics*, **17**, 1833–1855.

[10] Feuerverger, A. and Hall, P. (1999). Estimating a tail exponent by modelling departure from a Pareto distribution, *Annals of Statistics*, **27**, 760–781.

[11] Fraga Alves, M.I.; Gomes, M.I. and Haan, L. de (2003a). A new class of semi-parametric estimators of the second order parameter, *Portugaliae Mathematica*, **60**(2), 193–213.

[12] Fraga Alves, M.I.; de Haan, L. and Lin, T. (2003b). Estimation of the parameter controlling the speed of convergence in extreme value theory, *Mathematical Methods of Statistics*, **12**(2), 155–176.

[13] Fraga Alves, M.I.; Haan, L. de and Lin, T. (2006). Third order extended regular variation, *Publications de l'Institut Mathématique*, **80**(94), 109–120.

[14] Fraga Alves, M.I.; Gomes, M.I.; de Haan, L. and Neves, C. (2009). The mixed moment estimator and location invariant alternatives, *Extremes*, **12**, 149–185.

[15] Geluk, J. and Haan, L. de (1987). *Regular Variation, Extensions and Tauberian Theorems*, CWI Tract 40, Centre for Mathematics and Computer Science, Amsterdam, The Netherlands.

[16] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire, *Annals of Mathematics*, **44**(6), 423–453.

[17] Goegebeur, Y; Beirlant, J. and de Wet, T. (2008). Linking Pareto-tail kernel goodness-of-fit statistics with tail index at optimal threshold and second order estimation, *Revstat*, **6**(1), 51–69.

[18] Goegebeur, Y.; Beirlant, J. and de Wet, T. (2010). Kernel estimators for the second order parameter in extreme value statistics, *J. Statist. Planning and Inference*, **140**, 2632–2652.

[19] Gomes, M.I. and Martins, M.J. (2001). Generalizations of the Hill estimator: asymptotic versus finite sample behaviour, *J. Statist. Planning and Inference*, **93**, 161–180.

[20] Gomes, M.I. and Martins, M.J. (2002). "Asymptotically unbiased estimators of the tail index based on external estimation of the second order parameter, *Extremes*, **5**(1), 5–31.

[21] Gomes, M.I.; Martins, M.J. and Neves, M. (2000). Alternatives to a semi-parametric estimator of parameters of rare events — the Jackknife methodology, *Extremes*, **3**(3), 207–229.

[22] Gomes, M.I.; Haan, L. de and Peng, L. (2002). Semi-parametric estimation of the second order parameter — asymptotic and finite sample behaviour, *Extremes*, **5**(4), 387–414.

[23] Gomes, M.I.; Martins, M.J. and Neves, M. (2007). Improving second order reduced bias extreme value index estimation, *Revstat*, **5**(2), 177–207.

[24] Gomes, M.I.; Canto e Castro, L.; Fraga Alves, M.I. and Pestana, D. (2008a). Statistics of extremes for iid data and breakthroughs in the estimation of the extreme value index: Laurens de Haan leading contributions, *Extremes*, **11**(1), 3–34.

[25] Gomes, M.I.; Fraga Alves, M.I. and Araújo Santos, P. (2008b). PORT Hill and Moment Estimators for Heavy-Tailed Models, *Commun. Statist. — Simul. and Comput.*, **37**, 1281–1306.

[26] Gomes, M.I.; de Haan, L. and Henriques-Rodrigues, L. (2008c). Tail Index estimation for heavy-tailed models: accommodation of bias in weighted log-excesses, *J. Royal Statistical Society*, **B70**(1), 31–52.

[27] Gomes, M.I.; Henriques-Rodrigues, L. and Miranda, C. (2011). Reduced-bias location-invariant extreme value index estimation: a simulation study, *Commun. Statist. — Simul. and Comput.*, **40**(3), 424–447.

[28] Gomes, M.I.; Henriques-Rodrigues, L.; Fraga Alves, M.I. and Manjunath, B.G. (2013). Adaptive PORT-MVRB estimation: an empirical comparison of two heuristic algorithms, *J. Statist. Comput. Simul.*, **83**(6), 1129–1144.

[29] Haan, L. de (1984). *Slow variation and characterization of domains of attraction.* In "Statistical Extremes and Applications" (Tiago de Oliveira, Ed.), 31–48, D. Reidel, Dordrecht, Holland.

[30] Haan, L. de and Ferreira, A. (2006). *Extreme Value Theory: An Introduction*, Springer, New York.

[31] Henriques-Rodrigues, L. and Gomes, M.I. (2009). High quantile estimation and the PORT methodology, *Revstat*, **7**(3), 245–264.

[32] Henriques-Rodrigues, L. and Gomes, M.I. (2013). *A note on the PORT methodology in the estimation of a shape second-order parameter.* In "Studies in Theoretical and Applied Statistics: Subseries B: Recent Developments in Modeling and Applications in Statistics" (P. Oliveira, M.G. Temido, C. Henriques and M. Vichi, Eds.), 127–137, Springer.

[33] Mosteller, F. (1946). On some useful "inefficient" statistics, *Ann. Math. Statist.*, **17**, 377–408.

[34] Neves, C. (2009). From extended regular variation to regular variation with application in extreme value statistics, *J. Mathematical Analysis and Applications*, **355**, 216–230.

[35] Reiss, R.-D. and Thomas, M. (2001; 2007). *Statistical Analysis of Extreme Values, with Application to Insurance, Finance, Hydrology and Other Fields*, 2nd edition; 3rd edition, Birkhäuser Verlag.

[36] van der Vaart, A.W. (1998). *Asymptotic Statistics*, Cambridge University Press.

[37] Wang, X. and Cheng, S. (2005). General regular variation of n-th order and the 2nd order edgeworth expansion of the extreme value distribution, *Acta Mathematica Sinica*, **21**(5), 1121–1130.