REVSTAT – Statistical Journal
Volume 12, Number 3, November 2014, 199–220

LIMIT THEORY FOR JOINT GENERALIZED ORDER STATISTICS

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Received: October 2012 Revised: April 2013 Accepted: September 2013

Abstract:

- In Kamps [7] generalized order statistics (gos) have been introduced as a unifying theme for several models of ascendingly ordered random variables (rv's). The main aim of this paper is to study the limit joint distribution function (df) of any two statistics in a wide subclass of the gos model known as m-gos. This subclass contains many important practical models of gos such as ordinary order statistics (oos), order statistics with non-integer sample size, and sequential order statistics (sos). The limit df's of lower-lower extreme, upper-upper extreme, lower-upper extreme, central-central and lower-lower intermediate m-gos are obtained. It is revealed that the convergence of the marginals m-gos implies the convergence of the joint df. Moreover, the conditions, under which the asymptotic independence between the two marginals occurs, are derived.

Key-Words:

- *generalized order statistics; generalized extreme order statistics; generalized central order statistics; generalized intermediate order statistics.*

AMS Subject Classification:

- 60F05, 62E20, 62E15, 62G30.

1. INTRODUCTION

Generalized order statistics have been introduced as a unified distribution theoretical set-up which contains a variety of models of ordered rv's. Since Kamps [7] had introduced the concept of gos as a unification of several models of ascendingly ordered rv's, the use of such concept has been steadily growing along the years. This is due to the fact that such concept includes important well-known concepts that have been separately treated in statistical literature. Theoretically, many of the models of ordered rv's contained in the gos model, such as oos, order statistics with non-integral sample size, sos, record values, Pfeifer's record model and progressive type II censored order statistics (pos). These models can be applied in reliability theory. For instance, the sos model is an extension of the oos model and serves as a model describing certain dependencies or interactions among the system components caused by failures of components and the pos model is an important method of obtaining data in lifetime tests. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter. The concept of gos enables a common approach to structural similarities and analogies. Known results in submodels can be subsumed, generalized, and integrated within a general framework. Kamps [7] defined gos by first defining what he called uniform gos and then using the quantile transformation to obtain the general gos $X(r,n,\tilde{m},k)$, $r=1,2,...,n$, based on a df F, which are defined by their probability density function (pdf)

$$f^{(\tilde{m},k)}_{1,2,...,n:n}(x_1,x_2,...,x_n) = \\ = \left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}\big(1-F(x_j)\big)^{\gamma_j-\gamma_{j+1}-1}f(x_j)\right)\big(1-F(x_n)\big)^{\gamma_n-1}f(x_n)\;,$$

where $F^{-1}(0)\le x_1\le ...\le x_n\le F^{-1}(1)$, $\gamma_n=k>0$, $\gamma_r=k+n-r+\sum_{j=r}^{n-1}m_j$, $r=1,2,...,n-1$, and $\tilde{m}=(m_1,\,m_2,...,m_{n-1})\in\mathbb{R}^{n-1}$. Particular choices of the parameters $\gamma_1,\gamma_2,...,\gamma_n$ lead to different models, e.g., m-gos ($\gamma_r=k+(n-r)(m+1)$, $r=1,2,...,n-1$), oos ($k=1,\gamma_r=n-r+1, r=1,2,...,n-1$) and sos ($k=\alpha_n$, $\gamma_r=(n-r+1)\,\alpha_r$, $r=1,2,...,n-1$)[1].

Nasri-Roudsari [10] (see also Barakat [2]) has derived the marginal df of the rth m-gos, $m\ne-1$, in the form $\Phi^{(m,k)}_{r:n}(x)=I_{G_m(x)}(r,N-r+1)$, where $G_m(x)=1-(1-F(x))^{m+1}=1-\bar{F}^{m+1}(x)$, $I_x(a,b)=\frac{1}{\beta(a,b)}\int_o^x t^{a-1}(1-t)^{b-1}dt$ denotes the incomplete beta ratio function and $N=\frac{k}{m+1}+n-1$. By using the well-known relation $I_x(a,b)=1-I_{\bar{x}x}(b,a)$, where $\bar{x}=1-x$, the marginal df of the $(n-r+1)$th m-gos, $m\ne-1$, is given by $\Phi^{(m,k)}_{n-r+1:n}(x)=I_{G_m(x)}(N-R_r+1,R_r)$, where $R_r=\frac{k}{m+1}+r-1$. Moreover, by using the results of Kamps [7], we can write explicitly the joint pdf of the rth and sth m-gos $m\ne-1$, $1\le r<s\le n$,

[1] See, for instance, Kamps ([7]).

as:

$$f_{r,s:n}^{(m,k)}(x,y) = \frac{C_{s-1,n}}{\Gamma(r)\,\Gamma(s-r)}\,\bar{F}^m(x)\,g_m^{r-1}(F(x))$$

$$\times\,\big(g_m(F(y)) - g_m(F(x))\big)^{s-r-1}\,\bar{F}^{\gamma_s-1}(y)\,f(x)\,f(y)\;, \qquad (1.1)$$

$$-\infty < x < y < \infty\;,$$

where $C_{s-1,n} = \prod_{j=1}^{s}\gamma_j$. In the present paper we develop the limit theory for gos, by revealing the asymptotic dependence structural between the members of gos, with fixed and variable ranks. Namely, the limit joint df of the m-gos $X(r,n,m,k)$ and $X(s,n,m,k)$, when $m \neq -1$, is derived in the following three cases:

(1) Lower extremes, where r,s are fixed w.r.t. n and upper extremes, where $\grave{r} = n-r+1$, $\grave{s} = n-s+1$, where r,s are fixed w.r.t. n.

(2) Central case, where $r,s \to \infty$ and $\frac{r}{N} \to \lambda_1$, $\frac{s}{N} \to \lambda_2$, where $0 < \lambda_1 < \lambda_2 < 1$, as $N \to \infty$ (or equivalently, as $n \to \infty$). A remarkable example of the central oos the pth sample quantile, where $r_n = [np]$, $0 < p < 1$, and $[x]$ denotes the largest integer not exceeding x.

(3) Intermediate case, where $r,s \to \infty$ and $\frac{r}{N}, \frac{s}{N} \to 0$, as $N \to \infty$ (or equivalently, as $n \to \infty$). The intermediate oos have many applications, e.g., in the theory of statistics, they can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extreme relative to the available sample size, see Pickands [12]. Many authors, e.g., Teugels [14] and Mason [9] have also found estimates that are based, in part, on intermediate order statistics.

Everywhere in what follows the symbols ($\xrightarrow[n]{}$) and ($\xrightarrow[n]{w}$) stand for convergence, as $n \to \infty$ and the weak convergence, as $n \to \infty$.

2. THE JOINT df OF EXTREME m-gos

The following two lemmas, which are originally derived by Nasri-Roudsari [10] and Nasri-Roudsari and Cramer [11] (see also Barakat [2]), extend the well-known results concerning the asymptotic theory of extreme oos to the extreme m-gos. These lemmas can be easily proved by applying the following asymptotic relations, due to Smirnov [13]:

$$\Gamma_r(nA_n) - \delta_{1n} \;\leq\; I_{A_n}(r, n-r+1) \;\leq\; \Gamma_r(nA_n) - \delta_{2n}\;,$$

if $nA_n \sim A < \infty$, as $n \to \infty$, and

$$1 - \Gamma_r(n\bar{A}_n) - \delta_{2n} \;\leq\; I_{A_n}(n-r+1, r) \;\leq\; 1 - \Gamma_r(n\bar{A}_n) - \delta_{1n}\;,$$

if $n\bar{A}_n \sim \bar{A} < \infty$, as $n \to \infty$, where $\Gamma_r(x) = \frac{1}{\Gamma(r)} \int_0^x t^{r-1} e^{-t} dt$ is the incomplete gamma function (Gamma df with parameter r), $\delta_{in} > 0$, $\delta_{in} \xrightarrow[n]{} 0$, $i = 1, 2$, and $0 < A_n < 1$.

Lemma 2.1. *Let $m > -1$ and $r \in \{1, 2, ..., n\}$. Then, there exist normalizing constants $c_n > 0$ and d_n, for which*

$$\Phi_{r:n}^{(m,k)}(c_n x + d_n) = I_{G_m(c_n x + d_n)}(r, N - r + 1) \xrightarrow[n]{w} \Phi_r^{(m,k)}(x) , \tag{2.1}$$

where $\Phi_r^{(m,k)}(x)$ is nondegenerate df if, and only if, there exist normalizing constants $\alpha_n > 0$ and β_n, for which $\Phi_{r:n}^{(0,1)}(\alpha_n x + \beta_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $\beta > 0$. In this case $\Phi_r^{(m,k)}(x) = \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $j \in \{1, 2, 3\}$, where $\mathcal{V}_1(x) = \mathcal{V}_{1;\beta}(x) = e^x$, $\forall\ x$;

$$\mathcal{V}_{2;\beta}(x) = \begin{cases} (-x)^{-\beta}, & x \le 0\,, \\ \infty, & x > 0\,; \end{cases} \qquad \mathcal{V}_{3;\beta}(x) = \begin{cases} 0, & x \le 0\,, \\ x^{\beta}, & x > 0\,. \end{cases}$$

Moreover, c_n and d_n may be chosen such that $c_n = \alpha_{\psi(n)}$ and $d_n = \beta_{\psi(n)}$, where $\psi(n) = n(m+1)$. Finally, (2.1) holds if, and only if, $NG_m(c_n x + d_n) \xrightarrow[n]{} \mathcal{V}_{j,\beta}(x)$ (note that $N \sim n$, as $n \to \infty$).

Lemma 2.2. *Let $m > -1$ and $r \in \{1, 2, ..., n\}$. Then, there exist normalizing constants $a_n > 0$ and b_n, for which*

$$\Phi_{n-r+1:n}^{(m,k)}(a_n x + b_n) = I_{G_m(a_n x + b_n)}(N - R_r + 1, R_r) \xrightarrow[n]{w} \hat{\Phi}_r^{(m,k)}(x) , \tag{2.2}$$

where $\hat{\Phi}_r^{(m,k)}(x)$ is nondegenerate df if, and only if, there exist normalizing constants $\hat{\alpha}_n > 0$ and $\hat{\beta}_n$, for which $\Phi_{n-r+1:n}^{(0,1)}(\hat{\alpha}_n x + \hat{\beta}_n) \xrightarrow[n]{w} 1 - \Gamma_r(\mathcal{U}_{i,\alpha}(x))$, $\alpha > 0$. In this case $\hat{\Phi}_r^{(m,k)}(x) = 1 - \Gamma_{R_r}(\mathcal{U}_{i,\alpha}^{m+1}(x))$, $i \in \{1, 2, 3\}$, where $\mathcal{U}_1(x) = \mathcal{U}_{1;\alpha}(x) = e^{-x}$, $\forall\ x$;

$$\mathcal{U}_{2;\alpha}(x) = \begin{cases} \infty, & x \le 0\,, \\ x^{-\alpha}, & x > 0\,; \end{cases} \qquad \mathcal{U}_{3;\alpha}(x) = \begin{cases} (-x)^{\alpha}, & x \le 0\,, \\ 0, & x > 0\,. \end{cases}$$

Moreover, a_n and b_n may be chosen such that $a_n = \hat{\alpha}_{\phi(n)}$ and $b_n = \hat{\beta}_{\phi(n)}$, where $\phi(n) = n^{\frac{1}{m+1}}$. Finally, (2.2) holds if, and only if, $N\bar{G}_m(a_n x + b_n) \xrightarrow[n]{} \mathcal{U}_{i,\alpha}^{m+1}(x)$.

We need the following three lemmas proved in the Appendix and individually express interesting and practically useful facts. These lemmas provide us with the asymptotic lower and upper bounds for the joint df's of extreme gos. Therefore, they can be applied to estimate the error committed by the replacement of the exact joint df's of extreme gos by their limiting (see Remark 2.1). Throughout Lemma 2.3, we assume that $1 \le r < s \le n$, while we assume $1 \le s < r \le n$ and $1 \le r, s \le n$, $\grave{s} = n - s + 1$ in Lemma 2.4 and Lemma 2.5, respectively.

Lemma 2.3. *Let $c_n > 0$ and d_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{r:n}(x_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x))$ and $\Phi^{(m,k)}_{s:n}(y_n) \xrightarrow[n]{w} \Gamma_s(\mathcal{V}_{j,\beta}(y))$, $j \in \{1,2,3\}$, hold, where $x_n = c_n x + d_n$ and $y_n = c_n y + d_n$. Then the normalized joint df $\Phi^{(m,k)}_{r,s:n}(x_n, y_n)$ of the rth and sth m-gos, $m \neq -1$, satisfies the relations*

$$\begin{aligned} &\frac{(1-\sigma_N)}{(r-1)!} \int_0^{NG_m(x_n)} \Gamma_{s-r}\big(NG_m(y_n) - u\big)\, u^{r-1} e^{-u}\, du \;\leq \\ &\qquad \leq\; \Phi^{(m,k)}_{r,s:n}(x_n, y_n) \\ &\qquad \leq\; \frac{(1+\rho_N)}{(r-1)!} \int_0^{NG_m(x_n)} \Gamma_{s-r}(NG_m(y_n) - u) u^{r-1} e^{-u} du\,, \qquad \forall x \leq y\,, \end{aligned} \tag{2.3}$$

where $\rho_N, \sigma_N \xrightarrow[n]{} 0$.

Lemma 2.4. *Let $a_n > 0$ and b_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{\grave{r}:n}(x_n) \xrightarrow[n]{w} 1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))$ and $\Phi^{(m,k)}_{\grave{s}:n}(y_n) \xrightarrow[n]{w} 1 - \Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y))$, $i \in \{1,2,3\}$, hold, where $x_n = a_n x + b_n$, $y_n = a_n y + b_n$ and $\grave{r} = n - r + 1 < n - s + 1 = \grave{s}$. Then the joint df of the $\grave{r}$th and $\grave{s}$th m-gos, $m \neq -1$, satisfies the relation*

$$\begin{aligned} &\frac{\grave{C}_n}{(N+R_s)^{R_r}} \int_{(N+R_s)\frac{\bar{G}_m(x_n)}{G_m(x_n)}}^{(N+R_s)} \int_{(N+R_s)\frac{\bar{G}_m(y_n)}{G_m(y_n)}}^{\phi} e^{-\phi} \theta^{R_s-1} \;\times \\ &\qquad\qquad \times \left(1 + \frac{\theta}{N+R_s}\right)^{-R_r} (\phi-\theta)^{R_r-R_s-1}\, d\theta\, d\phi \;\leq \\ &\leq\; \Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n, y_n) \\ &\leq\; 1 - \Gamma_{R_r}\big(N\bar{G}_m(x_n)\big) - \frac{1}{\Gamma(R_r)} \int_{N\bar{G}_m(x_n)}^{N} I_{\frac{N\bar{G}_m(y_n)}{t}}(R_s, R_r - R_s)\, t^{R_r-1} e^{-t}\, dt\,, \end{aligned} \tag{2.4}$$

where $\grave{C}_n = \dfrac{\Gamma(N+1)}{\Gamma(N - R_r + 1)\, \Gamma(R_r - R_s)\, \Gamma(R_s)}$.

Lemma 2.5. *Let $a_n, c_n > 0$ and b_n, d_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{r:n}(x_n) \xrightarrow[n]{w} \Phi^{(m,k)}_{r}(x) = \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $j \in \{1,2,3\}$, and $\Phi^{(m,k)}_{\grave{s}:n}(y_n) \xrightarrow[n]{w} \hat{\Phi}^{(m,k)}_{s}(y) = 1 - \Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y))$, $i \in \{1,2,3\}$, hold, where $x_n = c_n x + d_n$ and $y_n = a_n y + b_n$. Then, for all large n and for all x and y, for which $\mathcal{V}_{j,\beta}(x) < \infty$, i.e., $\Phi^{(m,k)}_{r}(x) < 1$ and $\mathcal{U}_{i,\alpha}(y) < \infty$, i.e., $\hat{\Phi}^{(m,k)}_{s}(y) > 0$, respectively, the joint df of the rth and $\grave{s}$th m-gos, $m \neq -1$, satisfies the relation*

$$\begin{aligned} \Phi^{(m,k)}_{r:n}(x_n)\, \Phi^{(m,k)}_{\grave{s}:n}(y_n) \;&\leq\; \Phi^{(m,k)}_{r,\grave{s}:n}(x_n, y_n) \\ &\leq\; \Gamma_r\big(NG_m(x_n)\big)\Big(\Gamma_{R_s}(N) - \Gamma_{R_s}\big(N\bar{G}_m(y_n)\big)\Big)\,. \end{aligned} \tag{2.5}$$

The first inequality of (2.5) holds for all x, y.

Theorem 2.1. *Under the conditions of Lemma 2.3, 2.4 and 2.5, we get respectively*

$$(2.6)\quad \Phi^{(m,k)}_{r,s:n}(x_n,y_n) \xrightarrow[n]{w} \begin{cases} \Gamma_s\big(\mathcal{V}_{j,\beta}(y)\big)\,, & x\ge y\,,\\ \dfrac{1}{(r-1)!}\displaystyle\int_0^{\mathcal{V}_{j,\beta}(x)} \Gamma_{s-r}\big(\mathcal{V}_{j,\beta}(y)-u\big)u^{r-1}e^{-u}\,du\,, & x\le y\,,\end{cases}$$

(2.7)

$$\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n,y_n) \xrightarrow[n]{w} \begin{cases} 1-\Gamma_{R_s}\big(\mathcal{U}^{m+1}_{i,\alpha}(y)\big)\,, & x\ge y\,,\\ 1-\Gamma_{R_r}\big(\mathcal{U}^{m+1}_{i,\alpha}(x)\big)-\dfrac{1}{\Gamma(R_r)}\times & \\ \quad\times\int_{\mathcal{U}^{m+1}_{i,\alpha}(x)}^{\infty} I_{\frac{\mathcal{U}^{m+1}_{i,\alpha}(y)}{t}}(R_s,R_r-R_s)\,t^{R_r-1}e^{-t}\,dt\,, & x\le y\,,\end{cases}$$

and

$$(2.8)\quad \Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) \xrightarrow[n]{w} \Phi^{(m,k)}_{r}(x)\,\hat\Phi^{(m,k)}_{s}(y) = \Gamma_r\big(\mathcal{V}_{j,\beta}(x)\big)\Big[1-\Gamma_{R_s}\big(\mathcal{U}^{m+1}_{i,\alpha}(y)\big)\Big].$$

Proof: By noting that $\Phi^{(m,k)}_{r,s:n}(x_n,y_n)=\Phi^{(m,k)}_{s:n}(y_n)$, if $y\le x$, the relation (2.6) follows by applying Lemmas 2.1 and 2.3. In view of (2.2), (1.1) and the condition of Lemma 2.4, the relation (2.7) follows in the case of $y\le x$. On the other hand, since both of the lower and upper bounds of (2.4) are equivalent to (as $n\to\infty$) $1-\Gamma_{R_r}(N\bar G_m(x_n))-\frac{1}{\Gamma(R_r)}\int_{N\bar G_m(x_n)}^{N} I_{\frac{N\bar G_m(y_n)}{t}}(R_s,R_r-R_s)\,t^{R_r-1}e^{-t}dt$, then the relation (2.7) in the case $x\le y$, follows by applying Lemmas 2.2 and 2.4. Finally, by combining Lemmas 2.1, 2.2 and 2.5, the relation (2.8) follows immediately. □

Remark 2.1. One of the referees of the paper suggests a dexterous short proof of Theorem 2.1 based on the result of Cramer [5]. Namely, we get with the notations of Cramer [5] for two lower gos $X_t=X(t,n,m,k)$, $t=r,s$, $r<s$ (Z_j are iid standard exponential rv's; $u(x)=-\log(1-F(x))$; and $\gamma_{j,n}=k+(m+1)(n-j)$)

$$\begin{aligned} P\big(X_r\le x_n, X_s\le y_n\big) &= P\left(\sum_{j=1}^{r}\frac{Z_j}{\gamma_{j,n}}\le u(x_n)\,,\ \sum_{j=1}^{s}\frac{Z_j}{\gamma_{j,n}}\le u(y_n)\right)\\ &= P\Big(\Lambda_{r,n}\le n(m+1)\,u(x_n)\,,\ \Lambda_{r,n}+\Delta_{r,s,n}\le n(m+1)\,u(y_n)\Big)\,,\end{aligned}$$

where $\Lambda_{r,n}=n(m+1)\sum\limits_{j=1}^{r}\frac{Z_j}{\gamma_{j,n}}$ converges to a Gamma distribution with parameter r and $\Delta_{r,s,n}$ converges to a Gamma distribution with parameter $s-r+1$. Moreover, $\Lambda_{r,n}$ and $\Delta_{r,s,n}$ are independent for any n. Provided that F is in the domain of attraction of a minimum-stable distribution we get that $n(m+1)\,u(x_n)$ ($n(m+1)\,u(y_n)$) converges appropriately to some function $\mathcal{V}(x)$ ($\mathcal{V}(y)$). Hence, the limit df is of the type $P(\Lambda_r\le\mathcal{V}(x),\ \Lambda_r+\Delta_{s-r+1}\le\mathcal{V}(y))$, where Λ_r and Δ_{s-r+1}

are independent gamma distributed rv's with parameters given above, respectively. This proves the result in (2.6). Similar arguments can be used in proving (2.7) and (2.8). Although, this short method directly results the limit joint df's, but our lengthy method provides more informative results (Lemmas 2.3–2.5), which enable us to estimate the error committed by the replacement of the exact joint df's of extreme gos by their limiting. Actually, in view of the slow rate of convergence of oos (and consequently the gos) (cf. Arnold *et al.* [1], Page 216), Lemmas 2.3–2.5 are of a remarkable practically importance.

Example 2.1 (The limit df's of the generalized range and midrange)**.** Under the conditions of Lemma 2.5 the left and the right extreme m-gos, is asymptotically independent. Therefore, if there exist normalizing constants $a_n, c_n > 0$ and b_n, d_n, for which $a_n/c_n \xrightarrow[n]{} c > 0$ and the limit relations $\Phi^{(m,k)}_{\grave{r}:n}(a_n x + b_n) \xrightarrow[n]{w} 1-\Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))$, $i\in\{1,2,3\}$, and $\Phi^{(m,k)}_{r:n}(c_n x + d_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $j\in\{1,2,3\}$, hold, then in view of Lemma 2.9.1 in Galambos [6], the generalized quasi-ranges $R(r,n,m,k) = X(\grave{r},n,m,k) - X(r,n,m,k)$ and the generalized quasi-midranges $M(r,n,m,k) = \frac{1}{2}\big(X(\grave{r},n,m,k) + X(r,n,m,k)\big)$, $r=1,2,...$, satisfy the relations

$$P\big(R(r,n,m,k) \le a_n x + b_n - d_n\big) \xrightarrow[n]{w} \big[1-\Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))\big] \star \big[1-\Gamma_r(\mathcal{V}_{j,\beta}(-cx))\big]$$

and

$$P\big(2M(r,n,m,k) \le a_n x + b_n + d_n\big) \xrightarrow[n]{w} \big[1-\Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))\big] \star \big[\Gamma_r(\mathcal{V}_{j,\beta}(cx))\big]\,,$$

respectively, where the symbol $\star$ denotes the convolution operation.

3. LIMIT df's OF THE JOINT CENTRAL m-gos

Consider a variable rank sequence $r = r_n \xrightarrow[n]{} \infty$ and $\sqrt{n}\left(\frac{r}{n} - \lambda\right) \xrightarrow[n]{} 0$, where $0 < \lambda < 1$. Smirnov [13] showed that if there exist normalizing constants $\alpha_n > 0$ and β_n such that

$$\Phi^{(0,1)}_{r:n}(\alpha_n x + \beta_n) = I_{F(\alpha_n x + \beta_n)}(r, n-r+1) \xrightarrow[n]{w} \Phi^{(0,1)}(x;\lambda)\,, \tag{3.1}$$

where $\Phi^{(0,1)}(x;\lambda)$ is some nondegenerate df, then $\Phi^{(0,1)}(x;\lambda)$ must have one and only one of the types $\mathcal{N}(W_{i;\beta}(x))$, $i = 1,2,3,4$, where $\mathcal{N}(\cdot)$ denotes the standard normal df,

$$W_{1;\beta}(x) = \begin{cases} -\infty\,, & x \le 0\,,\\ c x^{\beta}\,, & x > 0\,, \end{cases} \qquad W_{2;\beta}(x) = \begin{cases} -c|x|^{\beta}\,, & x \le 0\,,\\ \infty\,, & x > 0\,, \end{cases}$$

$$W_{3;\beta}(x) = \begin{cases} -c_1|x|^{\beta}\,, & x \le 0\,,\\ c_2 x^{\beta}\,, & x > 0\,, \end{cases} \qquad W_{4;\beta}(x) = W_4(x) = \begin{cases} -\infty\,, & x \le -1\,,\\ 0\,, & -1 < x \le 1\,,\\ \infty\,, & x > 1\,, \end{cases}$$

and $\beta, c, c_1, c_2 > 0$. In this case we say that F belongs to the λ-normal domain of attraction of the limit df $\Phi^{(0,1)}(x;\lambda)$, written $F \in \mathcal{D}_\lambda(\Phi^{(0,1)}(x;\lambda))$. Moreover, (3.1) is satisfied with $\Phi^{(0,1)}(x;\lambda) = \mathcal{N}(W_{i;\beta}(x))$, for some $i \in \{1,2,3,4\}$ if, and only if,

$$\sqrt{n}\,\frac{F(\alpha_n x + \beta_n) - \lambda}{C_\lambda} \underset{n}{\longrightarrow} W_{i,\beta}(x)\ ,$$

where $C_\lambda = \sqrt{\lambda(1-\lambda)}$. It is worth to mention that the condition $\sqrt{n}\left(\frac{r}{n} - \lambda\right) \underset{n}{\longrightarrow} 0$ is necessary to have a unique limit law for any two ranks r, r', for which $\lim\limits_{n\to\infty} \frac{r}{n} = \lim\limits_{n\to\infty} \frac{r'}{n}$ (see Smirnov [13]).

Barakat [2], in Theorem 2.2, characterized the possible limit laws of the df $\Phi^{(m,k)}_{n-r+1:n}(x)$. The following corresponding lemma characterizes the possible limit laws of the df $\Phi^{(m,k)}_{r:n}(x)$. The proof of this lemma follows by using the same argument which is applied in the proof of Theorem 2.2 of Barakat [2].

Lemma 3.1. *Let $r = r_n$ be such that $\sqrt{n}\left(\frac{r}{n} - \lambda\right) \underset{n}{\longrightarrow} 0$, where $0 < \lambda < 1$. Furthermore, let $m_1 = m_2 = ... = m_{n-1} = m > -1$. Then, there exist normalizing constants $a_n > 0$ and b_n for which*

$$\Phi^{(m,k)}_{r:n}(a_n x + b_n) \xrightarrow[n]{w} \Phi^{(m,k)}(x;\lambda)\ , \tag{3.2}$$

where $\Phi^{(m,k)}(x;\lambda)$ is a nondegenerate df if, and only if,

$$\sqrt{n}\,\frac{G_m(a_n x + b_n) - \lambda}{C_\lambda} \underset{n}{\longrightarrow} W(x)\ ,$$

where $\Phi^{(m,k)}(x;\lambda) = \mathcal{N}(W(x))$. Moreover, (3.2) is satisfied for some nondegenerate df $\Phi^{(m,k)}(x;\lambda)$ if, and only if, $F \in \mathcal{D}_{\lambda(m)}(\mathcal{N}(W_{i;\beta}(x)))$, for some $i \in \{1,2,3,4\}$, where $\lambda(m) = 1 - \bar{\lambda}^{\frac{1}{m+1}}$ and $\bar{\lambda} = 1 - \lambda$. In this case we have $W(x) = \frac{C^\star_{\lambda(m)}}{C^\star_\lambda}(m+1)\cdot$ $\cdot W_{i;\beta}(x)$, where $C^\star_\lambda = \frac{C_\lambda}{\lambda}$ (note that, when $m = 0$, we get $W(x) = W_{i;\beta}(x)$).

We assume that in this section in all time that $r = r_n$, $s = s_n \underset{n}{\longrightarrow} \infty$ and $\sqrt{n}(\frac{r}{n} - \lambda_1), \sqrt{n}(\frac{s}{n} - \lambda_2) \underset{n}{\longrightarrow} 0$, where $0 < \lambda_1 < \lambda_2 < 1$. Moreover, we assume that there are suitable normalizing constants $a_n, c_n > 0$ and b_n, d_n, for which $\Phi^{(m,k)}_{r:n}(a_n x + b_n) \xrightarrow[n]{w} \Phi^{(m,k)}(x;\lambda_1)$ and $\Phi^{(m,k)}_{s:n}(c_n y + d_n) \xrightarrow[n]{w} \Phi^{(m,k)}(y;\lambda_2)$, where $\Phi^{(m,k)}(x;\lambda_1)$ and $\Phi^{(m,k)}(y;\lambda_2)$ are nondegenerate df's. Let $\Phi^{(m,k)}_{r,s:n}(x,y)$ be the joint df's of rth and sth m-gos, $m \neq -1$, in view of (1.1) we get $\Phi^{(m,k)}_{r,s:n}(x,y) = \Phi^{(m,k)}_{s:n}(y)$, $y \le x$, and

$$\begin{aligned}\Phi^{(m,k)}_{r,s:n}(x,y) \;=\; & C^\star_n \int_0^{F(x)}\!\!\int_\xi^{F(y)} \bar{\xi}^m\, \bar{\eta}^{\gamma_s - 1}\big(1 - \bar{\xi}^{m+1}\big)^{r-1} \\ & \times \big(\bar{\xi}^{m+1} - \bar{\eta}^{m+1}\big)^{s-r-1} d\eta\, d\xi\ , \qquad x \le y\ ,\end{aligned} \tag{3.3}$$

where $C^\star_n = \frac{(m+1)^2\,\Gamma(N+1)}{\Gamma(N-s+1)\,(r-1)!\,(s-r-1)!}$. The following lemma proved in the Appendix is an essential tool in studying the limit df of the joint central m-gos.

Lemma 3.2. *Let* $\lambda_i = \frac{i}{N+1}$, $\nu_i = 1-\lambda_i$, $\tau_i = \sqrt{\frac{\lambda_i \nu_i}{N+1}}$, $i=r,s$, $0 < R_{rs} = \sqrt{\frac{\lambda_r(1-\lambda_s)}{\lambda_s(1-\lambda_r)}} < 1$, $U_n^{(1)}(x) = \frac{G_m(x_n)-\lambda_r}{\tau_r}$, $U_n^{(2)}(y) = \frac{G_m(y_n)-\lambda_s}{\tau_s}$, $x_n = a_n x + b_n$ *and* $y_n = c_n y + d_n$. *Then*

$$\left| \Phi_{r,s:n}^{(m,k)}(x_n, y_n) - \int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta \right| \xrightarrow[n]{} 0$$

uniformly with respect to x *and* y, *where* $W_{r,s}(\xi,\eta) = \frac{1}{2\pi\sqrt{1-R_{rs}^2}}\, e^{-\frac{(\xi^2+\eta^2-2\xi\eta R_{rs})}{2(1-R_{rs}^2)}}$.

Lemma 3.2 directly yields the following interesting theorem.

Theorem 3.1. *The convergence of the two marginals* $\Phi_{r:n}^{(m,k)}(x_n)$ *and* $\Phi_{s:n}^{(m,k)}(y_n)$ *to nondegenerate df's* $\Phi^{(m,k)}(x;\lambda_1) = \mathcal{N}(W(x))$ *and* $\Phi^{(m,k)}(y;\lambda_2) = \mathcal{N}(\tilde{W}(y))$, *respectively, are necessary and sufficient condition for the convergence of the joint df* $\Phi_{r,s:n}^{(m,k)}(x_n,y_n)$ *to the nondegenerate limit*

$$\Phi^{(m,k)}(x,y;\lambda_1,\lambda_2) \;=\; \frac{1}{2\pi\sqrt{1-R^2}} \int_{-\infty}^{W(x)} \int_{-\infty}^{\tilde{W}(y)} e^{-\frac{(\xi^2+\eta^2-2\xi\eta R)}{2(1-R^2)}}\, d\xi\, d\eta \;,$$

where $R = \sqrt{\frac{\lambda_1(1-\lambda_2)}{\lambda_2(1-\lambda_1)}}$. *Moreover, in view of Lemma 3.1, we deduce that the convergence of the joint df* $\Phi_{r,s:n}^{(m,k)}(x_n,y_n)$, *as well as the convergence of the two marginals* $\Phi_{r:n}^{(m,k)}(x_n)$ *and* $\Phi_{s:n}^{(m,k)}(y_n)$, *occurs if, and only if, with the same normalizing constants, we have* $F \in \mathcal{D}_{\lambda_1(m)}(\mathcal{N}(W_{i;\beta}))$ *and* $F \in \mathcal{D}_{\lambda_2(m)}(\mathcal{N}(W_{j;\beta'}))$, *for some* $i,j \in \{1,2,3,4\}$, *where* $\lambda_t(m) = 1 - \bar{\lambda}_t^{\frac{1}{m+1}}$ *and* $\bar{\lambda}_t = 1-\lambda_t$, $t=1,2$. *In this case we have* $W(x) = \frac{C^\star_{\lambda_1(m)}}{C^\star_{\lambda_1}}(m+1)\,W_{i;\beta}(x)$ *and* $\tilde{W}(y) = \frac{C^\star_{\lambda_2(m)}}{C^\star_{\lambda_2}}(m+1)\,W_{j;\beta'}(y)$, *where* $C^\star_{\lambda_t} = \frac{C_{\lambda_t}}{\lambda}$, $t=1,2$.

4. LIMIT df's OF THE JOINT INTERMEDIATE m-gos

Chibisov [4] studied a wide class of intermediate oos, where $r = r_n = \ell^2 n^\alpha(1+\circ(1))$, $0<\alpha<1$, and he showed that if there are normalizing constants $\alpha_n > 0$ and β_n such that

$$\Phi_{r:n}^{(0,1)}(\alpha_n x + \beta_n) \;=\; \mathrm{I}_{F(\alpha_n x+\beta_n)}(r, n-r+1) \xrightarrow[n]{w} \Phi^{(0,1)}(x) \;, \tag{4.1}$$

where $\Phi^{(0,1)}(x)$ is a nondegenerate df, then $\Phi^{(0,1)}(x)$ must have one and only one of the types $\mathcal{N}(V_i(x))$, $i=1,2,3$, where $V_1(x) = x$, $\forall x$, and

$$V_2(x) = \begin{cases} -\beta \ln|x|\,, & x \le 0\,, \\ \infty\,, & x > 0\,, \end{cases} \qquad V_3(x) = \begin{cases} -\infty\,, & x \le 0\,, \\ \beta \ln|x|\,, & x > 0\,, \end{cases} \tag{4.2}$$

where β is some positive constant. In this case F belongs to the domain of attraction of the df $\Phi^{(0,1)}(x)$, written $F \in \mathcal{D}(\Phi^{(0,1)}(x))$. Moreover, (4.1) is satisfied with $\Phi^{(0,1)}(x) = \mathcal{N}(V_i(x))$, for some $i \in \{1,2,3\}$ if, and only if,

$$\frac{nF(\alpha_n x + \beta_n) - r_n}{\sqrt{r_n}} \underset{n}{\longrightarrow} V_i(x) \ . \tag{4.3}$$

Wu [15] generalized the Chibisov result for any nondecreasing intermediate rank sequence and proved that the only possible types for the limit df of the intermediate oos are those defined in (4.2).

Barakat [2], in Lemma 2.2 and Theorem 2.3, characterized the possible limit laws of the df of the upper intermediate m-gos. The following corresponding lemma characterizes the possible limit laws of the df of the lower intermediate m-gos. The proof of this lemma follows by using the same argument which is applied in the proof of Lemma 2.2 and Theorem 2.3 of Barakat [2].

Lemma 4.1. *Let* $m_1 = m_2 = ... = m_{n-1} = m > -1$, *and let* r_n *be a nondecreasing intermediate rank sequence. Then, there exist normalizing constants* $a_n > 0$ *and* b_n *such that*

$$\Phi^{(m,k)}_{r_n:n}(a_n x + b_n) \xrightarrow[n]{w} \Phi^{(m,k)}(x) \ , \tag{4.4}$$

where $\Phi^{(m,k)}(x)$ *is a nondegenerate df if, and only if,* $\frac{NG_m(a_n x+b_n)-r_N}{\sqrt{r_N}} \underset{n}{\longrightarrow} V(x)$, *where* $\Phi^{(m,k)}(x) = \mathcal{N}(V(x))$. *Furthermore, let* $r^\star_n$ *be a variable rank sequence defined by* $r^\star_n = r_{\theta^{-1}(N)}$, *where* $\theta(n) = (m+1)N$ *(remember that* $N = \frac{k}{m+1} + n - 1$, *then* $\theta(n) = n$, *if* $m = 0$, $k = 1$, *i.e., in the case of oos). Then, there exist normalizing constants* $a_n > 0$ *and* b_n *for which (4.4) is satisfied for some nondegenerate df* $\Phi^{(m,k)}(x)$ *if, and only if, there are normalizing constants* $\alpha_n > 0$ *and* β_n *for which* $\Phi^{(0,1)}_{r^\star_n:n}(\alpha_n x + \beta_n) \xrightarrow[n]{w} \Phi^{(0,1)}(x)$, *where* $\Phi^{(0,1)}(x)$ *is some nondegenerate df, or equivalently* $\frac{nF(\alpha_n x+\beta_n)-r^\star_n}{\sqrt{r^\star_n}} \underset{n}{\longrightarrow} V_i(x)$, $i \in \{1,2,3\}$, *and* $\Phi^{(0,1)}(x) = \mathcal{N}(V_i(x))$. *In this case, we can take* $a_n = \alpha_{\theta(n)}$ *and* $b_n = \beta_{\theta(n)}$. *Moreover,* $\Phi^{(m,k)}(x)$ *must has the form* $\mathcal{N}(V_i(x))$, *i.e.,* $V(x) = V_i(x)$.

In this section we consider the limit df of the two intermediate m-gos $\eta_r = \frac{X(r,n,m,k)-b_n}{a_n}$ and $\zeta_s = \frac{X(s,n,m,k)-d_n}{c_n}$, where $\frac{r}{n^{\alpha_1}} \underset{n}{\longrightarrow} l_1^2$, $\frac{s}{n^{\alpha_2}} \underset{n}{\longrightarrow} l_2^2$, $0<\alpha_1, \alpha_2<1$, $l_1, l_2 > 0$, and $a_n, c_n > 0$, b_n, d_n are suitable normalizing constants. The main aim of this section is to:

1 – Prove that the weak convergence of the df's of η_r and ζ_s implies the convergence of the joint df of η_r and ζ_s;

2 – Obtain the limit joint df of η_r and ζ_s and derive the condition under which the two statistics η_r and ζ_s are asymptotically independent.

We can distinguish the following distinct and exhausted two cases:

$$A)\ s-r \underset{n}{\longrightarrow} c,\ \ 0\le c<\infty\,, \qquad \text{and} \qquad B)\ s-r \underset{n}{\longrightarrow} \infty\ .$$

Remark 4.1. Under the condition $A)$, we clearly have $l_1{=}l_2$, $\alpha_1{=}\alpha_2=\alpha$. Moreover $\frac{r}{s}\underset{n}{\longrightarrow}1$. Finally, under the condition $B)$ we have the following three distinct and exhausted cases:

$B_1)$ $\alpha_2>\alpha_1$, which implies $\frac{r}{s}\underset{n}{\longrightarrow}0$.

$B_2)$ $\alpha_2=\alpha_1=\alpha$, $l_2>l_1$, which implies $\frac{r}{s}\underset{n}{\longrightarrow}\frac{l_1^2}{l_2^2}$.

$B_3)$ $\alpha_2=\alpha_1=\alpha$, $l_2=l_1$, which implies $\frac{r}{s}\underset{n}{\longrightarrow}1$.

The following, corresponding lemma (proved in the Appendix) to Lemma 3.2, characterizes the possible limit laws of the joint intermediate m-gos.

Lemma 4.2. *Let* $\Phi^{(m,k)}_{r,s:n}(x_n,y_n){=}P(\eta_r{<}x,\,\zeta_s{<}y)$, $0{<}R_{rs}{=}\sqrt{\frac{\lambda_r(1-\lambda_s)}{\lambda_s(1-\lambda_r)}}{<}1$, $\frac{r}{s}\underset{n}{\longrightarrow}R$, $R_{rs}\underset{n}{\longrightarrow}\sqrt{R}$, $0{\le}R{<}1$, $x_n{=}\,a_nx+b_n$, $y_n{=}\,c_ny+d_n$, $U^{(1)}_n(x)=\frac{G_m(x_n)-\lambda_r}{\tau_r}$, $U^{(2)}_n(y)=\frac{G_m(y_n)-\lambda_s}{\tau_s}$, $\lambda_i=\frac{i}{N+1}$, $\tau_i=\sqrt{\frac{\lambda_i\nu_i}{N+1}}$ *and* $\nu_i=1-\lambda_i$, $i=r,s$. *Then*

$$\left|\Phi^{(m,k)}_{r,s:n}(x_n,y_n)-\frac{1}{2\pi\sqrt{1-R^2_{rs}}}\int_{-\infty}^{U^{(1)}_n(x)}\int_{-\infty}^{U^{(2)}_n(y)} e^{-\frac{(\xi^2+\eta^2-2\,\xi\eta R_{rs})}{2(1-R^2_{rs})}}\,d\xi\,d\eta\right|$$

converges to zero uniformly with respect to x and y.

Lemma 4.2 leads to the following theorem.

Theorem 4.1. *Let* $x_n{=}\,a_nx+b_n$, $y_n{=}\,c_ny+d_n$, $\frac{r}{n},\frac{s}{n}\underset{n}{\longrightarrow}0$, $\frac{r}{s}\underset{n}{\longrightarrow}R$, *and* $R_{rs}\underset{n}{\longrightarrow}\sqrt{R}$, $0\le R<1$. *Then the convergence of the two marginals* $\Phi^{(m,k)}_{r:n}(x_n)$ *and* $\Phi^{(m,k)}_{s:n}(y_n)$ *to nondegenerate limit df's* $\Phi^{(m,k)}(x)=\mathcal{N}(V(x))$ *and* $\Phi^{(m,k)}(y)=\mathcal{N}(\tilde{V}(y))$, *respectively, are necessary and sufficient condition for the convergence of the joint df* $\Phi^{(m,k)}_{r,s:n}(x_n,y_n)$ *to the nondegenerate limit*

$$\Phi^{(m,k)}_{r,s:n}(x_n,y_n)\ \underset{n}{\overset{w}{\longrightarrow}}\ \frac{1}{2\pi\sqrt{1-R}}\int_{-\infty}^{V(x)}\int_{-\infty}^{\tilde{V}(y)} e^{-\frac{(\xi^2+\eta^2-2\xi\eta\sqrt{R})}{2(1-R)}}\,d\xi\,d\eta\ .$$

Moreover, in view of Lemma 4.1, we deduce that the convergence of the joint df $\Phi^{(m,k)}_{r,s:n}(x_n,y_n)$, *as well as the convergence of the two marginals* $\Phi^{(m,k)}_{r:n}(x_n)$ *and* $\Phi^{(m,k)}_{s:n}(y_n)$, *occurs if, and only if, there are normalizing constants* $\alpha_n,\gamma_n>0$ *and* β_n,δ_n *for which* $\Phi^{(0,1)}_{r^\star_n:n}(\alpha_nx+\beta_n)\underset{n}{\overset{w}{\longrightarrow}}\Phi^{(0,1)}(x)=\mathcal{N}(V_i(x))$ *and* $\Phi^{(0,1)}_{s^\star_n:n}(\gamma_ny+\delta_n)\underset{n}{\overset{w}{\longrightarrow}}\Phi^{(0,1)}(y)=\mathcal{N}(V_j(y))$, *for some* $i,j\in\{1,2,3\}$, *where* $r^\star_n=r_{\theta^{-1}(N)}$, $s^\star_n=s_{\theta^{-1}(N)}$ *and* $\theta(n)=(m{+}1)N$. *In this case, we can take* $a_n{=}\,\alpha_{\theta(n)}$, $c_n{=}\,\gamma_{\theta(n)}$, $b_n{=}\,\beta_{\theta(n)}$ *and* $d_n=\delta_{\theta(n)}$. *Moreover,* $V(x)=V_i(x)$ *and* $\tilde{V}(y)=V_j(y)$. *Finally, the two marginals are asymptotically independent if, and only if,* $\frac{r}{s}\underset{n}{\longrightarrow}0$, *i.e.,* $R=0$.

APPENDIX

Proof of Lemma 2.3: In (1.1), consider the transformation $\xi = F(u)$, $\eta = F(v)$, we get

$$\Phi_{r,s:n}^{(m,k)}(x_n, y_n) = \tag{A.1}$$
$$= C_n^\star \int_0^{F(x_n)} \int_\xi^{F(y_n)} \bar\xi^m \, \bar\eta^{\gamma_s - 1} \, (1 - \bar\xi^{m+1})^{r-1} \, (\bar\xi^{m+1} - \bar\eta^{m+1})^{s-r-1} \, d\eta \, d\xi \ ,$$

where $\bar\eta = 1-\eta$, $\bar\xi = 1-\xi$ and $C_n^\star = \frac{C_{s-1,n}}{(m+1)^{s-2}\,(r-1)!\,(s-r-1)!}$. Again, by using the transformation $1-\bar\xi^{m+1} = z$, $1-\bar\eta^{m+1} = w$, we get

$$\Phi_{r,s:n}^{(m,k)}(x_n, y_n) = C_n^{\star\star} \int_0^{G_m(x_n)} \int_z^{G_m(y_n)} (1-w)^{\frac{\gamma_s - m - 1}{m+1}} \, z^{r-1} (w-z)^{s-r-1} \, dw \, dz \, , \tag{A.2}$$

where $C_n^{\star\star} = \frac{C_n^\star}{(m+1)^2}$. On the other hand, we have $\frac{\gamma_s - m - 1}{m+1} = N - s$ and

$$\frac{(r-1)!\,(s-r-1)!\,C_n^{\star\star}}{(N-s)^s} = \frac{\prod_{j=1}^s \gamma_j}{(N-s)^s\,(m+1)^s} = \frac{\prod_{j=1}^s (N-j+1)}{(N-s)^s} =$$
$$= \frac{\prod_{j=1}^s (1-\frac{j-1}{N})}{(1-\frac{s}{N})^s} = \left(1 + \frac{s^2}{N}\big(1+o(1)\big)\right)\left(1 - \sum_{j=2}^s \frac{j-1}{N}\big(1+o(1)\big)\right) =$$
$$= 1 + \frac{s^2}{N} - \frac{1}{N}\left(\frac{s^2-s}{2}\right)\big(1+o(1)\big) = 1 + \rho_N \ ,$$

where $0 < \rho_N = \frac{1}{2N}\,(s^2+s)\,(1+o(1)) \underset{N}{\longrightarrow} 0$. Therefore, by using the transformation $w = \frac{\theta}{N-s}$, $z = \frac{\phi}{N-s}$ and the inequality $(1-z)^n \le e^{-nz}$, $\forall\, 0 \le z \le 1$ (cf. Lemma 1.3.1 in Galambos [6]), we get

$$\Phi_{r,s:n}^{(m,k)}(x_n, y_n) =$$
$$= \frac{C_n^{\star\star}}{(N-s)^s} \int_0^{(N-s)G_m(x_n)} \int_\phi^{(N-s)G_m(y_n)} \left(1 - \frac{\theta}{N-s}\right)^{N-s} \phi^{r-1} (\theta - \phi)^{s-r-1} \, d\theta \, d\phi$$
$$\le \frac{(1+\rho_N)}{(r-1)!\,(s-r-1)!} \int_0^{NG_m(x_n)} \int_\phi^{NG_m(y_n)} e^{-\theta} \phi^{r-1} (\theta-\phi)^{s-r-1} \, d\theta \, d\phi$$
$$= \frac{(1+\rho_N)}{(r-1)!} \int_0^{NG_m(x_n)} \Gamma_{s-r}\big(NG_m(y_n) - u\big) \, u^{r-1} \, e^{-u} \, du \ .$$

On the other hand, by using the transformation $\frac{w}{1-w} = \frac{\theta}{N+r}$, $\frac{z}{1-z} = \frac{\phi}{N+r}$ in (A.2), and noting that $\frac{(r-1)!\,(s-r-1)!\,C_n^{\star\star}}{(N+r)^s} = \frac{\prod_{j=1}^s (1-\frac{j-1}{N})}{(1+\frac{r}{N})^s} = \big(1 - \frac{rs}{N}(1+o(1))\big)\big(1 - \sum_{j=2}^s \frac{j-1}{N}(1+o(1))\big) = 1 - \big(\frac{rs}{N} + \sum_{j=2}^s \frac{j-1}{N}\big)\big(1+o(1)\big) = 1 - \frac{1}{N}\big(rs + \frac{s^2-s}{2}\big)\big(1+o(1)\big) =$

$1-\sigma_N^\star$, we get, by using the inequality $e^{-nz} \le (1+z)^{-n}$, $\forall\, 0 \le z \le 1$,

$$\begin{aligned}
&\Phi_{r,s:n}^{(m,k)}(x_n,y_n) = \\
&= \frac{C_n^{\star\star}}{(N+r)^s} \int_0^{(N+r)G_m(x_n)/\bar G_m(x_n)} \int_\phi^{(N+r)G_m(y_n)/\bar G_m(y_n)} (\theta-\phi)^{s-r-1} \\
&\quad \times \phi^{r-1} \left(1+\frac{\theta}{N+r}\right)^{-(N+r)} \left(1+\frac{\theta}{N+r}\right)^{2r-1} \left(1+\frac{\phi}{N+r}\right)^{-s} d\theta\, d\phi \\
&\ge \frac{(1-\sigma_N^\star)\,\bar F^{(m+1)s}(x_n)}{(r-1)!\,(s-r-1)!} \int_0^{NG_m(x_n)} \int_\phi^{NG_m(y_n)} (\theta-\phi)^{s-r-1} \phi^{r-1} \left(1+\frac{\theta}{N+r}\right)^{-(N+r)} d\theta\, d\phi \\
&\ge \frac{(1-\sigma_N)}{(r-1)!\,(s-r-1)!} \int_0^{NG_m(x_n)} \int_\phi^{NG_m(y_n)} (\theta-\phi)^{s-r-1} \phi^{r-1}\, e^{-\theta}\, d\theta\, d\phi \\
&= \frac{(1-\sigma_N)}{(r-1)!} \int_0^{NG_m(x_n)} \Gamma_{s-r}\big(NG_m(y_n)-u\big)\, u^{r-1} e^{-u}\, du\ ,
\end{aligned}$$

where $\sigma_N = 1-(1-\sigma_N^\star)\,\bar F^{(m+1)s}(x_n) \xrightarrow[N]{} 0$ (note that $\bar F^{(m+1)s}(x_n) \sim 1$). The lemma is proved. □

Proof of Lemma 2.4: We begin with the relation (A.1), after replacing r and s by $\grave r$ and $\grave s$, respectively. By using the transformation $\bar\xi^{m+1} = z$, $\bar\eta^{m+1} = w$ and noting that $n-r = N-R_r$, $n-s = N-R_s$, $\gamma_{n-s+1} = (m+1)R_s$ and $C_{\grave s-1,n} = C_{N-R_s,n} = (m+1)^{N-R_s+1} \prod_{j=1}^{N-R_s+1}(N-j+1) = (m+1)^{N-R_s+1} \frac{\Gamma(N+1)}{\Gamma(R_s)}$, we get

$$\text{(A.3)} \quad \Phi_{\grave r,\grave s:n}^{(m,k)}(x_n,y_n) = \grave C_n \int_{\bar G_m(x_n)}^1 \int_{\bar G_m(y_n)}^z w^{R_s-1} (1-z)^{N-R_r} (z-w)^{R_r-R_s-1}\, dw\, dz\,,$$

where $\grave C_n = \frac{\Gamma(N+1)}{\Gamma(N-R_r+1)\,\Gamma(R_r-R_s)\,\Gamma(R_s)}$. Again by using the transformation $w = \frac{\theta}{N-R_r}$, $z = \frac{\phi}{N-R_r}$ and the inequality $(1-z)^n \le e^{-nz}$, $\forall\, 0 \le z \le 1$, we get

$$\begin{aligned}
&\Phi_{\grave r,\grave s:n}^{(m,k)}(x_n,y_n) \le \\
&\qquad \le \frac{\grave C_n}{(N-R_r)^{R_r}} \int_{(N-R_r)\bar G_m(x_n)}^{(N-R_r)} \int_{(N-R_r)\bar G_m(y_n)}^{\phi} e^{-\phi}\, \theta^{R_s-1} (\phi-\theta)^{R_r-R_s-1}\, d\theta\, d\phi\,.
\end{aligned}$$

Now, by using Stirling's formula (cf. Lebedev [8]), we have $\frac{\Gamma(R_r-R_s)\,\Gamma(R_s)\,\grave C_n}{(N-R_r)^{R_r}} \sim e^{-R_r}(1-\frac{R_r}{N})^{-(N+\frac12)} \sim 1$, as $N\to\infty$ (i.e., as $n\to\infty$), and noting that $(N-R_r)\cdot \cdot \bar G_m(x_n) \sim N\bar G_m(x_n)$, $(N-R_r)\bar G_m(y_n) \sim N\bar G_m(y_n)$, as $N\to\infty$, we get

$$\begin{aligned}
\Phi_{\grave r,\grave s:n}^{(m,k)}(x_n,y_n) &\le \frac{1}{\Gamma(R_r-R_s)\,\Gamma(R_s)} \int_{N\bar G_m(x_n)}^N \int_{N\bar G_m(y_n)}^\phi e^{-\phi}\, \theta^{R_s-1} (\phi-\theta)^{R_r-R_s-1} d\theta\, d\phi \\
&= \frac{1}{\Gamma(R_r)} \int_{N\bar G_m(x_n)}^N \phi^{R_r-1} e^{-\phi} \Big(1 - I_{\frac{N\bar G_m(y_n)}{\phi}}(R_s, R_r-R_s)\Big)\, d\phi \\
&= 1 - \Gamma_{R_r}\big(N\bar G_m(x_n)\big) - \frac{1}{\Gamma(R_r)} \int_{N\bar G_m(x_n)}^N I_{\frac{N\bar G_m(y_n)}{t}}(R_s, R_r-R_s)\, t^{R_r-1} e^{-t}\, dt\ .
\end{aligned}$$

On the other hand, by using the transformation $\frac{w}{1-w} = \frac{\theta}{N+R_s}, \frac{z}{1-z} = \frac{\phi}{N+R_s}$ in (A.3) and the inequality $e^{-nz} \le (1+z)^{-n}$, $\forall\, 0 \le z \le 1$, we get

$$\begin{aligned}
\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n,y_n) \;&=\; \frac{\grave{C}_n}{(N+R_s)^{R_r}} \int_{(N+R_s)\frac{\bar{G}_m(x_n)}{G_m(x_n)}}^{\infty} \int_{(N+R_s)\frac{\bar{G}_m(y_n)}{G_m(y_n)}}^{\phi} \theta^{R_s-1} \\
&\quad \times \left(1+\frac{\phi}{N+R_s}\right)^{-(N+R_s)+2R_s-1} \left(1+\frac{\theta}{N+R_s}\right)^{-R_r} (\phi-\theta)^{R_r-R_s-1}\, d\theta\, d\phi \\
&\ge\; \frac{\grave{C}_n}{(N+R_s)^{R_r}} \int_{(N+R_s)\frac{\bar{G}_m(x_n)}{G_m(x_n)}}^{(N+R_s)} \int_{(N+R_s)\frac{\bar{G}_m(y_n)}{G_m(y_n)}}^{\phi} e^{-\phi}\, \theta^{R_s-1} \\
&\quad \times \left(1+\frac{\theta}{N+R_s}\right)^{-R_r} (\phi-\theta)^{R_r-R_s-1}\, d\theta\, d\phi\;.
\end{aligned}$$

The lemma is proved. □

Proof of Lemma 2.5: The proof of the lower bound follows from the fact that the gos are positively quadrant dependent (see Barakat [3]). To prove the upper bound, in view of (1.1), we have

$$\begin{aligned}
&\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) \;=\; \\
\text{(A.4)}\qquad &\;=\; D_n \int_0^{F(x_n)}\!\!\int_{\xi}^{F(y_n)} \bar{\xi}^m\, \bar{\eta}^{\gamma_{n-s+1}-1}\, (1-\bar{\xi}^{m+1})^{r-1} (\bar{\xi}^{m+1}-\bar{\eta}^{m+1})^{n-s-r} d\eta\, d\xi\,,
\end{aligned}$$

$\forall\, x_n \le y_n$, where $D_n = \frac{C_{n-s,n}}{(m+1)^{n-s-1}\,(r-1)!\,(n-s-r)!}$. Now, in view of the conditions of the lemma, it is easy to show that $\forall (x,y)$, for which $\mathcal{V}_{j,\beta}(x), \mathcal{U}_{i,\alpha}(y) < \infty$, we have $y_n \xrightarrow[n]{} \omega(F) = \sup\{x : F(x) < 1\} > \inf\{x : F(x) > 0\} = \alpha(F) \xleftarrow[n]{} x_n$. Therefore, for all large n, the relation (A.4) holds, $\forall x, y$, for which $\mathcal{V}_{j,\beta}(x), \mathcal{U}_{i,\alpha}(y) < \infty$. Now, by using the transformation $1-\bar{\xi}^{m+1} = v$, $\bar{\eta}^{m+1} = u$ and noting that $\frac{\gamma_{n-s+1}-m-1}{m+1} = R_s - 1$, we get

$$\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) \;=\; \frac{D_n}{(m+1)^2} \int_0^{G_m(x_n)}\!\!\int_{\bar{G}_m(y_n)}^{1-v} u^{R_s-1}\, v^{r-1}\, (1-u-v)^{n-s-r}\, du\, dv\;.$$

Therefore, by using the transformation $u = \frac{w}{N-R_s-r}$, $v = \frac{z}{N-R_s-r}$ and the inequality $(1-z)^n \le e^{-nz}$, $\forall\, 0 \le z \le 1$, we get

$$\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) \;\le\; \tilde{C}_n \int_0^{NG_m(x_n)}\!\!\int_{(N-R_s-r)\bar{G}_m(y_n)}^{N} w^{R_s-1}\, z^{r-1}\, e^{-(w+z)}\, dw\, dz\;,$$

where $\tilde{C}_n = \frac{D_n}{(m+1)^2\,(N-R_s-r)^{R_s+r}}$. On the other hand, by using Stirling's formula, we get

$$\begin{aligned}
\Gamma(r)\,\tilde{C}_n \;&=\; \frac{C_{N-R_s,n}}{(m+1)^{N-R_s+1}\,(N-R_s-r)^{R_s+r}\,\Gamma(N-R_s-r+1)} \\
&=\; \frac{\Gamma(N+1)}{\Gamma(N-R_s-r+1)\,(N-R_s-r)^{R_s+r}\,\Gamma(R_s)} \;\sim\; \frac{1}{\Gamma(R_s)}\;.
\end{aligned}$$

Therefore, since $(N-R_s-r)\,\bar{G}_m(y_n) \sim N\bar{G}_m(y_n)$, we get the upper bound of (2.5). The lemma is proved. □

Proof of Lemma 3.2: For given $\epsilon > 0$, choose T large enough to satisfy the inequalities $\frac{1}{T^2} < \epsilon$ and $\mathcal{N}(-T) < \epsilon$. If $U_n^{(1)}(x) \le -T$. Thus, for sufficiently large n, we get $1 - \bar{F}^{m+1}(x_n) \le \lambda_r - \tau_r T < 1$. Therefore, after routine calculations, we can show that

$$\begin{aligned}
\Phi_{r:n}^{(m,k)}(x_n) &= \frac{1}{\beta(r, N-r+1)} \int_0^{1-\overline{F}^{m+1}(x_n)} \xi^{r-1}(1-\xi)^{N-r}\, d\xi \\
&\le \frac{1}{\beta(r, N-r+1)} \int_0^{\lambda_r - \tau_r T} \xi^{r-1}(1-\xi)^{N-r}\, d\xi \\
&\le \frac{1}{\beta(r, N-r+1)} \int_0^{1} \frac{(\xi-\lambda_r)^2}{\tau_r^2\, T^2}\, \xi^{r-1}(1-\xi)^{N-r} d\xi \\
&= \frac{N+1}{(N+2)\, T^2} < \frac{1}{T^2} < \epsilon\, .
\end{aligned}$$

Since $\Phi_{r,s:n}^{(m,k)}(x_n, y_n) \le \Phi_{r:n}^{(m,k)}(x_n)$, then $\Phi_{r,s:n}^{(m,k)}(x_n, y_n) < \epsilon$. Similarly, if $U_n^{(2)}(y) \le -T$, we can prove that $\Phi_{r,s:n}^{(m,k)}(x_n, y_n) \le \Phi_{s:n}^{(m,k)}(y_n) < \epsilon$. On the other hand, we have

$$\int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta \le \min\Big(\mathcal{N}\big(U_n^{(1)}(x)\big), \mathcal{N}\big(U_n^{(2)}(y)\big)\Big) < \epsilon\, .$$

Therefore, if $U_n^{(1)}(x) \le -T$ or $U_n^{(2)}(y) \le -T$, we get

$$\left| \Phi_{r,s:n}^{(m,k)}(x_n, y_n) - \int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta \right| \le 2\,\epsilon\, .$$

Now, if $U_n^{(1)}(x) \ge T$, then $1 - \bar{F}^{m+1}(x_n) \ge \lambda_r + \tau_r T$. Therefore, after routine calculations, we get

$$\begin{aligned}
1 - \Phi_{r:n}^{(m,k)}(x_n) &\le \frac{1}{\beta(r, N-r+1)} \int_{\lambda_r+\tau_r T}^{1} \xi^{r-1}(1-\xi)^{N-r}\, d\xi \\
&\le \frac{1}{\beta(r, N-r+1)} \int_0^{1} \frac{(\xi-\lambda_r)^2}{\tau_r^2\, T^2}\, \xi^{r-1}(1-\xi)^{N-r}\, d\xi \\
&= \frac{N+1}{(N+2)\, T^2} < \frac{1}{T^2} < \epsilon\, .
\end{aligned}$$

Thus, we also get

$$\Phi_{s:n}^{(m,k)}(y_n) - \Phi_{r,s:n}^{(m,k)}(x_n, y_n) \le 1 - \Phi_{r:n}^{(m,k)}(x_n) < \epsilon\, . \tag{A.5}$$

On the other hand, in view of our assumptions and Lemma 3.1, we get

$$\begin{aligned}
\mathcal{N}\big(U_n^{(2)}(y)\big) - \int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta &= \\
&= \int_{U_n^{(1)}(x)}^{\infty} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta \qquad \text{(A.6)} \\
&\le \frac{1}{\sqrt{2\pi}} \int_{U_n^{(1)}(x)}^{\infty} e^{\frac{-\xi^2}{2}}\, d\xi \le \frac{1}{\sqrt{2\pi}} \int_T^{\infty} e^{\frac{-\xi^2}{2}}\, d\xi < \epsilon\, ,
\end{aligned}$$

for sufficiently large n, and

$$\text{(A.7)} \qquad \left|\Phi^{(m,k)}_{s:n}(y_n) - \mathcal{N}\big(U_n^{(2)}(y)\big)\right| < \epsilon \ ,$$

for sufficiently large n. The relations (A.5), (A.6) and (A.7) show that when $U_n^{(1)}(x) \geq T$, we have $|\Phi^{(m,k)}_{r,s:n}(x_n, y_n) - \int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta| < 3\,\epsilon$. Similarly, we can show that the last inequality holds for sufficiently large n, if $U_n^{(2)}(y) \geq T$. In order to complete the proof of the lemma, we have to consider the case $|U_n^{(1)}(x)|, |U_n^{(2)}(y)| < T$. First, we note that, since $G_m(x_n) \xrightarrow[n]{} \lambda_1 < \lambda_2 \xleftarrow[n]{} G_m(y_n)$, we have $x_n \leq y_n$, for sufficiently large n. Therefore, for sufficiently large n, $\Phi^{(m,k)}_{r,s:n}(x_n, y_n)$ is given by (3.3). Moreover, in this case we have $1 - \bar{F}^{m+1}(x_n) > \lambda_r - \tau_r T \geq 0$ and $1 - \bar{F}^{m+1}(y_n) > \lambda_s - \tau_s T \geq 0$. Thus,

$$\text{(A.8)} \qquad \begin{aligned} \Phi^{(m,k)}_{r,s:n}(x_n, y_n) &= \int_0^{1-\bar{F}^{m+1}(x_n)} \int_z^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \\ &= \int_0^{\lambda_r - \tau_r T} \int_z^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \\ &\quad + \int_{\lambda_r - \tau_r T}^{1-\bar{F}^{m+1}(x_n)} \int_z^{\lambda_s - \tau_s T} \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \\ &\quad + \int_{\lambda_r - \tau_r T}^{1-\bar{F}^{m+1}(x_n)} \int_{\lambda_s - \tau_s T}^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \ , \end{aligned}$$

where $\varphi^{(m,k)}_{r,s:n}(w,z) = \frac{C_n^\star}{(m+1)^2}\ z^{r-1}(1-w)^{N-s}\,(w-z)^{s-r-1}$. We shall separately consider, each of the integrals in the summation (A.8):

$$\text{(A.9)} \qquad \begin{aligned} &\int_0^{\lambda_r - \tau_r T} \int_z^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \ \leq \\ &\quad \leq \int_0^{\lambda_r - \tau_r T} \int_z^1 \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \\ &\quad = \frac{C_n^\star}{(m+1)^2} \int_0^{\lambda_r - \tau_r T} \int_z^1 z^{r-1}(1-w)^{N-s}\,(w-z)^{s-r-1}\, dw\, dz \\ &\quad = \frac{\Gamma(N+1)}{\Gamma(N-r+1)\,\Gamma(r)} \int_0^{\lambda_r - \tau_r T} z^{r-1}(1-z)^{N-r}\, dz \ < \ \frac{1}{T^2} \ < \ \epsilon \ , \end{aligned}$$

$$\text{(A.10)} \qquad \begin{aligned} &\int_{\lambda_r - \tau_r T}^{1-\bar{F}^{m+1}(x_n)} \int_z^{\lambda_s - \tau_s T} \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \ \leq \\ &\quad \leq \int_0^{\lambda_s - \tau_s T} \int_z^{\lambda_s - \tau_s T} \varphi^{(m,k)}_{r,s:n}(w,z)\ dw\, dz \\ &\quad = \frac{\Gamma(N+1)}{\Gamma(N-s+1)\,(s-1)!} \int_0^{\lambda_s - \tau_s T} w^{s-1}(1-w)^{N-s}\, dw \ < \ \frac{1}{T^2} \ < \ \epsilon \ , \end{aligned}$$

and by using the transformation $z = \lambda_r + \xi\,\tau_r$, $w = \lambda_s + \eta\,\tau_s$, the third integral takes the form

$$\int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_{\lambda_s-\tau_s T}^{1-\bar{F}^{m+1}(y_n)} \varphi_{r,s:n}^{(m,k)}(w,z)\,dw\,dz = \\ = A_{r,s:n}\int_{-T}^{U_n^{(1)}(x)}\int_{-T}^{U_n^{(2)}(y)} g_{r,s:n}(\xi,\eta)\,d\eta\,d\xi\ ,$$

where

$$A_{r,s:n} = \frac{\Gamma(N+1)\,\tau_r\,\tau_s\,\lambda_r^{r-1}\,\nu_s^{N-s}(\lambda_s-\lambda_r)^{s-r-1}}{\Gamma(N-s+1)\,(r-1)!\,(s-r-1)!}$$

and

$$g_{r,s:n}(\xi,\eta) = \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r-1}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r-1}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s}.$$

On the other hand, by using Stirling's formula $\Gamma(M+1)=e^{-M}\sqrt{2\pi M}\,\cdot$ $\cdot\,M^M(1+\circ(1))$, as $M\to\infty$, we get

$$\begin{aligned} A_{r,s:n} &= \frac{(N+1)^2\,\Gamma(N+1)\,\tau_r\,\tau_s\,\lambda_r^{r}\,\nu_s^{N-s}(\lambda_s-\lambda_r)^{s-r}}{\Gamma(N-s+1)\,r!\,(s-r)!} \\ &= \frac{1+\circ(1)}{2\pi\sqrt{\frac{(N+1)\,(s-r)}{s(N-r+1)}}} = \frac{1+\circ(1)}{2\pi\sqrt{1-R_{rs}^2}}\ . \end{aligned}$$

Also, it is easy to show that

$$\begin{aligned} g_{r,s:n}(\xi,\eta) &= \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s} \\ &\quad\times\left[\left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{-1}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{-1}\right] \\ &= \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s} \\ &\quad\times\left[\left(1-\frac{\xi\tau_r}{\lambda_r}\,(1+o(1))\right)\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\,(1+\circ(1))\right)\right] \\ &= \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s}\big(1+\rho_n(\xi,\eta)\big)\,, \end{aligned} \tag{A.11}$$

where $\rho_n(\xi,\eta)\xrightarrow[n]{}0$, uniformly in any finite interval $(-T,T)$ of the value ξ and η. Furthermore, we have

$$\begin{aligned} r\ln\left(1+\frac{\xi\tau_r}{\lambda_r}\right) &= r\left(\frac{\xi\tau_r}{\lambda_r}-\frac{\xi^2\tau_r^2}{2\lambda_r^2}+\frac{\xi^3\tau_r^3}{3\lambda_r^3}+\cdots\right) \\ &= \xi\,\tau_r(N+1)-\frac{\xi^2\nu_r}{2}+\circ\left(\frac{T^3}{\sqrt{r}}\right), \end{aligned} \tag{A.12}$$

$$\begin{aligned}(s-r)\,\ln\!\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right) &= \\ &= (\eta\tau_s-\xi\tau_r)\,(N{+}1)-\frac{1}{2}\,\frac{(\eta\tau_s-\xi\tau_r)^2}{\lambda_s-\lambda_r}\,(N{+}1)+\circ\!\left(\frac{T^3}{\sqrt{s}}\right)\end{aligned} \tag{A.13}$$

and

$$(N-s)\,\ln\!\left(1-\frac{\eta\tau_s}{\nu_s}\right) = -\eta\tau_s(N{+}1)-\frac{1}{2}\,\eta^2\lambda_s+\circ\!\left(\frac{\lambda_s^{\frac{3}{2}}\,T^3}{\sqrt{N}}\right). \tag{A.14}$$

Therefore, by combining (A.11)–(A.14), as $n\to\infty$ (or equivalently as $N\to\infty$), we get

$$\begin{aligned}\ln g_{r,s:n}(\xi,\eta) &= r\,\ln\!\left(1+\frac{\xi\tau_r}{\lambda_r}\right)+(s-r)\,\ln\!\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)+(N-s)\,\ln\!\left(1-\frac{\eta\tau_s}{\nu_s}\right)\\ &\sim -\frac{\xi^2\nu_r}{2}-\frac{\eta^2\tau_s^2-2\,\xi\,\eta\,\tau_r\tau_s+\xi^2\tau_r^2}{2(\lambda_s-\lambda_r)}\,(N{+}1)-\frac{1}{2}\,\eta^2\lambda_s\\ &= -\frac{\xi^2\nu_r}{2}\left(1+\frac{\lambda_r}{\lambda_s-\lambda_r}\right)-\frac{1}{2}\,\eta^2\lambda_s\left(1+\frac{\nu_s}{\lambda_s-\lambda_r}\right)-\frac{1}{2}\left(-2\,\xi\,\eta\,\frac{\tau_r\,\tau_s}{\lambda_s-\lambda_r}\right)\\ &= -\frac{1}{2}\,\frac{\lambda_s(1-\lambda_r)}{\lambda_s-\lambda_r}\left(\xi^2+\eta^2-2\,\xi\,\eta\sqrt{\frac{\lambda_r(1-\lambda_s)}{\lambda_s(1-\lambda_r)}}\right),\end{aligned}$$

which implies $g_{r,s:n}(\xi,\eta)=e^{-\frac{\left(\xi^2+\eta^2-2\,\xi\eta R_{rs}\right)}{2\left(1-R_{rs}^2\right)}}\big(1+\circ(1)\big)$. Therefore, for sufficiently large n (or equivalently for large N), we get

$$\left|\int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_{\lambda_s-\tau_s T}^{1-\bar{F}^{m+1}(y_n)}\varphi_{r,s:n}^{(m,k)}(w,z)\;dw\,dz-\int_{-T}^{U_n^{(1)}(x)}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\right|<\epsilon\,.$$

Since,

$$\int_{-\infty}^{-T}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta+\int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{-T}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\;<\;2\mathcal{N}(-T)\;<\;2\,\epsilon$$

and

$$\begin{aligned}\int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta &= \\ &= \int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{-T}W_{r,s}(\xi,\eta)\;d\xi\,d\eta+\int_{-\infty}^{-T}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\\ &\quad+\int_{-T}^{U_n^{(1)}(x)}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\,,\end{aligned}$$

then

$$\left|\int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_{\lambda_s-\tau_s T}^{1-\bar{F}^{m+1}(y_n)}\varphi_{r,s:n}^{(m,k)}(w,z)\;dw\,dz-\int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\right|<3\,\epsilon\,.$$

By combining the last inequality with (A.9) and (A.10) we get, for sufficient large n, the inequality

$$\left|\Phi^{(m,k)}_{r,s:n}(x_n,y_n) - \int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta\right| < 5\,\epsilon\ ,$$

which proves the lemma in the case $\left|U_n^{(1)}(x)\right|, \left|U_n^{(2)}(y)\right| < T$. This completes the proof. □

Proof of Lemma 4.2: Under the condition of the lemma $(0 \le R < 1)$, we consider only the cases B_1) and B_2). On the other hand, the proof is very close to the proof of Lemma 3.2. Therefore, we only show the necessary changes in the proof of Lemma 3.2. For given $\epsilon > 0$, we choose T, large enough to satisfy both of the inequalities $\frac{1}{T^2} < \epsilon$, and $\mathcal{N}(-T) < \epsilon$. In this case it is easy to see that the proof of the two lemmas coincides in the cases $U_n^{(t)}(\cdot) \le -T$ and $U_n^{(t)}(\cdot) \ge T$, $t = 1,2$. Therefore, we only prove the lemma under the case $|U_n^{(1)}(x)| < T$ and $|U_n^{(2)}(y)| < T$. In this case we have $1 - \bar{F}^{m+1}(x_n) > \lambda_r - \tau_r T \ge 0$ and $1 - \bar{F}^{m+1}(y_n) > \lambda_s - \tau_s T \ge 0$. Thus, we get

$$\begin{aligned}\Phi^{(m,k)}_{r,s:n}(x_n,y_n) \;=\; & \int_0^{\lambda_r-\tau_r T}\int_z^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz \\ & + \int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_z^{\lambda_s-\tau_s T} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz \\ & + \int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_{\lambda_s-\tau_s T}^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz\ ,\end{aligned} \tag{A.15}$$

where $\varphi^{(m,k)}_{r,s:n}(w,z) = \frac{C_n^\star}{(m+1)^2}\, z^{r-1}(1-w)^{N-s}(w-z)^{s-r-1}$. We shall separately consider, each of the integrals in the summation (A.15).

$$\begin{aligned}\int_0^{\lambda_r-\tau_r T}\int_z^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz \;\le\; \int_0^{\lambda_r-\tau_r T}\int_z^{1} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz \;= \\ = \frac{\Gamma(N+1)}{\Gamma(N-r+1)\,(r-1)!}\int_0^{\lambda_r-\tau_r T} z^{r-1}(1-z)^{N-r}\, dz \;<\; \frac{1}{T^2} \;<\; \epsilon\ .\end{aligned}$$

Since $|U_n^{(1)}(x)| < T$, for large N, we get

$$1 - \bar{F}^{m+1}(x_n) < \lambda_r + \tau_r T\ . \tag{A.16}$$

On the other hand, we have

$$\frac{\lambda_r + \tau_r T}{\lambda_s - \tau_s T} \;\xrightarrow[n]{}\; \begin{cases} 0\,, & \text{in the case } B_1)\,, \\ \dfrac{l_l^2}{l_2^2}\,, & \text{in the case } B_2)\,. \end{cases} \tag{A.17}$$

Therefore, for large N, the relations (A.16) and (A.17) imply the inequality $1 - \bar{F}^{m+1}(x_n) < \lambda_s - \tau_s T$, which in turn leads to the following estimate for the 2nd integral in (A.15):

$$\begin{aligned}
\int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x)} \int_{z}^{\lambda_s-\tau_s T} \varphi_{r,s:n}^{(m,k)}(w,z)\, dw\, dz \;\le& \\
\le\; & \int_{0}^{\lambda_s-\tau_s T} \int_{z}^{\lambda_s-\tau_s T} \varphi_{r,s:n}^{(m,k)}(w,z)\, dw\, dz \\
=\; & \int_{0}^{\lambda_s-\tau_s T} \int_{0}^{w} \varphi_{r,s:n}^{(m,k)}(w,z)\, dz\, dw \\
=\; & \frac{\Gamma(N+1)}{\Gamma(N-s+1)\,(s-1)!} \int_{0}^{\lambda_s-\tau_s T} w^{s-1}(1-w)^{N-s}\, dw \;<\; \frac{1}{T^2} \;<\; \epsilon\;.
\end{aligned}$$

It is easy to show that, under the cases B_1) and B_2), the mathematical treatments of the third integral of the summation, as well as the remaining part of the proof, is exactly the same as in the proof of Lemma 3.2. This completes the proof. □

ACKNOWLEDGMENTS

The authors would like to thank the Associate Editor as well as the anonymous referees for constructive suggestions leading to improvement of the representation of the paper.

REFERENCES

[1] Arnold, B.C.; Balakrishnan, N. and Nagaraja, H.N. (1992). *A First Course in Order Statistics*, John Wiley & Sons Inc.

[2] Barakat, H.M. (2007). Limit theory of generalized order statistics, *Journal of Statistical Planning and Inference*, **137**(1), 1–11.

[3] Barakat, H.M. (2007). Measuring the asymptotic dependence between generalized order statistics, *Journal of Statistical Theory and Applications*, **6**(2), 106–117.

[4] Chibisov, D.M. (1964). On limit distributions for order statistics, *Theory of Probabilty and Its Applications*, **9**, 142–148.

[5] Cramer, E. (2003). *Contribuions to Generalized Order Statistics*, Habililationsschrift, Reprint, University of Oldenburg.

[6] Galambos, J. (1987). *The Asymptotic Theory of Extreme Order Statistics*, Krieger, FL (2nd ed.).

[7] Kamps, U. (1995). *A Concept of Generalized Order Statistics*, Order Statistics, Teubner, Stuttgart.

[8] Lebedev, N.N. (1995). *Special Functions and Their Applications*, Prentice-Hall, Inc.

[9] Mason, D.M. (1982). Laws of large numbers for sums of extreme values, *The Annals of Probabilty*, **10**, 750–764.

[10] Nasri-Roudsari, D. (1996). Extreme value theory of generalized order statistics, *Journal of Statistical Planning and Inference*, **55**, 281–297.

[11] Nasri-Roudsari, D. and Cramer, E. (1999). On the convergence rates of extreme generalized order statistics, *Extremes*, **2**, 421–447.

[12] Pickands, J. III. (1975). Statistical inference using extreme order statistics, *The Annals of Statistics*, **3**, 119–131.

[13] Smirnov, N.V. (1952). Limit distribution for terms of a variational series, *American Mathematical Society – Translation Ser.*, **1**(11), 82–143.

[14] Teugels, J.L. (1981). Limit theorems on order statistics, *The Annals of Probabilty*, **9**, 868–880.

[15] Wu, C.Y. (1966). The types of limit distributions for terms of variational series, *Scientia Sinica*, **15**, 749–762.