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ROBUSTNESS OF TWO-PHASE REGRESSION TESTS

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Abstract:

- This article studies the robustness of different likelihood ratio tests proposed by Quandt ([1]) and ([2]), (Q-Test), Kim and Siegmund ([3]), (KS-Test), and Kim ([4]), (K-Test), to detect a change in simple linear regression models. These tests are evaluated and compared with respect to their performance taking into account different scenarios, such as, different error distributions, different sample sizes, different locations of the change point and departure from the homoscedasticity. Two different alternatives are considered: i) with a change in the intercept from one model to the other with the same slope and ii) with a change in both the intercept and slope.

  The simulation results reveal that the KS-Test is superior to the Q-Test for both models considered while the K-Test is more powerful than the other two tests for nonhomogeneous models with a known variance.

Key-Words:

- *segmented regression models; likelihood ratio tests; robustness.*

AMS Subject Classification:

- 62J02, 62F03.

1. INTRODUCTION

The use of models involving a sequence of submodels has been widely applied in areas such as economics, medicine and biology, among others. These types of models, denoted by segmented (or switching or multi-phase) regression models, are useful when it is believed that the model parameters change after an unknown time or in some region of the domain of the predictor variables.

A simple segmented regression model, in the case that a sequence of observations (x_i, y_i), $i=1,2,...,n$, is considered, can be written in the following way

$$(1.1) \qquad y_i = \begin{cases} \alpha_1 + \beta_1 x_i + \varepsilon_i, & \text{if } x_i \leqslant r \\ \alpha_2 + \beta_2 x_i + \varepsilon_i, & \text{if } x_i > r \ , \end{cases}$$

where α_1, α_2, β_1, β_2 and r are unknown parameters and the errors ε_i have distribution $N(0,\sigma^2)$. The submodels in (1.1) are referred to as regimes or segments and the point r as the change point.

Segmented regression models are divided into two types. One where the model is assumed to be continuous at the change point, and the other where it is not. The inferential theory is completely different for each type of model (Hawkins ([5])). The emphasis in this article is on the discontinuous model.

The linear-linear segmented regression model proposed by Quandt ([1]) is similar to model (1.1), except that the change point is identified by the observation order instead of the observation value as above. Moreover, the model (1.1) assumes homoscedasticity while the Quandt model assumes heteroscedasticity.

Considering a sequence of observations (x_i, y_i), $i = 1,2,...,n$, the Quandt two-phase regression model is given by

$$(1.2) \qquad y_i = \begin{cases} \alpha_1 + \beta_1 x_i + \varepsilon_i, & \text{if } i = 1,...,k \\ \alpha_2 + \beta_2 x_i + \varepsilon_i, & \text{if } i = k+1,...,n \ , \end{cases}$$

where the ε_i have independent normal distributions with mean zero and variance σ_1^2 if $i \leqslant k$ and variance σ_2^2 if $i > k$. The parameters α_1, α_2, β_1, β_2 and k are all unknown.

Various tests for the presence of a change point based on the likelihood ratio are discussed in the statistics literature. Quandt ([1], [2]) was the first to propose a likelihood ratio test to detect the presence of a change point in a simple linear regression model. Hinkley ([6]) derived the asymptotic distribution of the maximum likelihood estimate of a change point and the asymptotic distribution of the likelihood ratio statistics for testing hypotheses of no change in (1.2), where the independent observations $x_1, ..., x_n$, are ordered, the change point k is unknown and the errors ε_i are considered uncorrelated $N(0, \sigma^2)$. Furthermore, it is assumed that $x_k \leqslant \gamma \leqslant x_{k+1}$, where $\gamma = (\alpha_1 - \alpha_2)/(\beta_2 - \beta_1)$ is the intersection of the two regression lines. Hinkley ([7]) discussed inference concerning a change point considering the hypothesis $H_0\colon\ k = k_0$ versus the one-sided alternative $H_1\colon\ k > k_0$ or versus the two-sided alternative $H_2\colon\ k \neq k_0$. Brown *et al.* ([8]) described tests for detecting departures from constancy of regression relationships over time and illustrated the application of these tests with three sets of real data. Maronna and Yohai ([9]) derived the likelihood ratio test for the hypothesis that a systematic change has occurred after some point in the intercept alone. Worsley ([10]) found exact and approximate bounds for the null distributions of likelihood ratio statistics for testing hypotheses of no change in the two-phase multiple regression model

$$(1.3) \qquad y_i = \begin{cases} \mathbf{x}_i^{'} \beta + \varepsilon_i, & \text{if } \ i = 1, ..., k \\ \mathbf{x}_i^{'} \beta^* + \varepsilon_i, & \text{if } \ i = k+1, ..., n \ , \end{cases}$$

where $p \leqslant k \leqslant n-p$, $\mathbf{x}_i$ is a p-component vector of independent variables and β and β^* are p-component vectors of unknown parameters. His numerical results indicated that the accuracy of the approximation of the upper bound is very good for small samples. Kim and Siegmund ([3]) consider likelihood ratio tests for change point problems in simple linear regression models. They also present a review on segmented regression models and some real problems that motivated research in this area. Some of these problems are examined using change point methods by Worsley ([11]) and Raferty and Akman ([12]). Kim ([4]) derived the likelihood ratio tests for a change in simple linear regression with unequal variances. Kim and Cai ([13]) examined the distributional robustness of the likelihood ratio tests discussed by Kim and Sigmund ([3]) in a simple linear regression. They showed that these statistics converge to the same limiting distributions regardless of the underlying distribution. Through simulation the observed distributional insensitivity of the test statistics is observed when the errors follow a lognormal, a Weibull, or a contaminated normal distribution. Kim ([4]), using some numerical examples, examined the robustness to heteroscedasticity of these tests.

In this paper different likelihood ratio tests (Quandt ([1]) and ([2]), Kim and Siegmund ([3]) and Kim ([4])) to detect a change on a simple linear regression, are presented. The tests are evaluated and compared regarding their performance in different scenarios. Our main concern is to assist the user of such tests to decide which test is preferable to use and under which circumstances. The article is organized as follows. In Section 2, the likelihood ratio tests proposed by Quandt ([1]) and ([2]), Kim and Siegmund ([3]) and Kim ([4]) will be described. In Section 3, via Monte Carlo simulations, the performance of the tests discussed in Section 2 will be assessed and compared. Final comments on the results, presented in Section 4, will conclude the paper.

2. TEST STATISTICS

In this Section the likelihood ratio tests proposed by Quandt ([1]) and ([2]), Kim and Siegmund ([3]) and Kim ([4]) are described in more detail. In all the cases the model (1.2) is considered.

2.1. Likelihood Ratio Test by Quandt (Q-Test)

The test described by Quandt ([1]) and ([2]) is used for testing the hypothesis that no change has occurred against the alternative that a change took place. That is, $H_0\colon \alpha_1 = \alpha_2,\ \beta_1 = \beta_2,\ \sigma_1 = \sigma_2$ against $H_1\colon \alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$ or $\sigma_1 \neq \sigma_2$. The error terms ε_i are independently and normally distributed $N(0,\sigma_1^2)$ for $i = 1,...,k$ and $N(0,\sigma_2^2)$ for $i = k+1,...,n$.

The likelihood ratio λ is defined as

$$\lambda = \frac{l(k)}{l(n)} ,$$

where $l(n)$ is the maximum of the likelihood function over only a single phase and $l(k)$ is the maximum of the likelihood function over the presence of a change point. That is,

$$\begin{aligned} \lambda &= \frac{\exp\left[-\log(2\pi)^{\frac{n}{2}} - \log\hat{\sigma}^n - \frac{n}{2}\right]}{\exp\left[-\log(2\pi)^{\frac{n}{2}} - \log\hat{\sigma}_1^k - \log\hat{\sigma}_2^{n-k} - \frac{n}{2}\right]} \\ &= \frac{\hat{\sigma}_1^k\,\hat{\sigma}_2^{n-k}}{\hat{\sigma}^n} , \end{aligned}$$

where $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the estimates of the standard errors of the two regression lines, $\hat{\sigma}$ is the estimate of the standard error of the overall regression based on all observations and the constant k is chosen in order to minimize λ. On the basis of empirical distributions resulting from sampling experiments, Quandt ([1]) concluded that the distribution of $-2\log\lambda$ can not be assumed to be χ^2 distribution with 4 degrees of freedom.

2.2. Likelihood Ratio Tests by Kim and Siegmund (KS-Test)

Kim and Siegmund ([3]), assuming the model (1.2) with homoscedasticity, consider tests of the following hypotheses:

H_0: $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2$ against the alternatives

(i) H_1: $\beta_1 = \beta_2$ and there exists a k $(1 \leqslant k < n)$ such that $\alpha_1 \neq \alpha_2$ or

(ii) H_2: there exists a k $(1 \leqslant k < n)$ such that $\beta_1 \neq \beta_2$ and $\alpha_1 \neq \alpha_2$.

The alternative (i) specifies that a change has occurred after some point in the intercept alone and alternative (ii) specifies that a change has occurred after some point in both intercept and slope.

The likelihood ratio test of H_0 against H_1 rejects H_0 for large values of

$$\max_{n_0 \leqslant i \leqslant n_1} |U_n(i)| / \hat{\sigma} \ ,$$

where $n_j = nt_j$, $j = 0, 1$, for $0 < t_0 < t_1 < 1$, and

$$U_n(i) \ = \ (\widehat{\alpha}_i - \widehat{\alpha}_i^*) \left(\frac{i\,(1 - i/n)}{1 - i(\bar{x}_i - \bar{x}_n)^2 / \{Q_{xxn}(1 - i/n)\}} \right)^{1/2} .$$

The likelihood ratio test of H_0 against H_2 rejects H_0 for large values of

$$\sigma^{-2} \max_{n_0 \leqslant i \leqslant n_1} \left\{ \frac{n\, i\, (\bar{y}_i - \bar{y}_n)^2}{n - i} + \frac{Q^2_{xyi}}{Q_{xxi}} + \frac{Q^{*2}_{xyi}}{Q^*_{xxi}} - \frac{Q^2_{xyn}}{Q_{xxn}} \right\} ,$$

where, following Kim and Siegmund ([3]) notation,

$$\bar{x}_i = i^{-1} \sum_{k=1}^{i} x_k \ , \quad \bar{y}_i = i^{-1} \sum_{k=1}^{i} y_k \ , \quad \widehat{\alpha}_i = \bar{y}_i - \widehat{\beta}\, \bar{x}_i \ , \quad \widehat{\alpha}^*_i = \bar{y}^*_i - \widehat{\beta}\, \bar{x}^*_i \ ,$$

$$\bar{x}^*_i = (n-i)^{-1} \sum_{k=i+1}^{n} x_k \ , \quad \bar{y}^*_i = (n-i)^{-1} \sum_{k=i+1}^{n} y_k \ , \quad Q_{xyi} = \sum_{k=1}^{i} (x_k - \bar{x}_i)\,(y_k - \bar{y}_i) \ ,$$

$$Q_{xxi} = \sum_{k=1}^{i} (x_k - \bar{x}_i)^2 \ , \qquad Q^*_{xxi} = \sum_{k=i+1}^{n} (x_k - \bar{x}^*_i)^2 \ , \qquad \ldots \ ,$$

$$Q_{xxn} = \sum_{k=1}^{n} (x_k - \bar{x}_n)^2 \ , \qquad Q_{xyn} = \sum_{k=1}^{n} (x_k - \bar{x}_n)\,(y_k - \bar{y}_n) \ ,$$

$$\widehat{\beta} = Q_{xyn}/Q_{xxn} \qquad \text{and} \qquad \hat{\sigma}^2 = n^{-1} \left(Q_{yyn} - Q^2_{xyn}/Q_{xxn} \right) .$$

In these tests and in the tests by Kim, the values for t_0 and t_1 depend on the feeling we have concerning the location of the change point. This impression comes from a scatterplot of y and x. In this study we will use $t_0 = 0.1$ and $t_1 = 0.9$.

2.3. Likelihood Ratio Tests by Kim (K-Test)

Kim ([4]) studied likelihood ratio tests for a change in a simple linear regression model considering the two types of alternatives presented in the previous subsection. It is assumed that the error variance is non-homogeneous, that is, the error terms follow $N(0,\sigma_i^2)$, where $\sigma_i^2=\sigma^2/w_i$ and the w_i's are positive constants. The likelihood ratio statistics is denoted by *weighted likelihood ratio statistics*.

The weighted likelihood ratio statistics to test H_0 against H_1, $(WLRS_1)$, is given by

$$\hat{\sigma}^{-1} \max_{n_0 \leqslant i \leqslant n_1} |U_{w,n}(i)| \tag{2.1}$$

where

$$U_{w,n}(i) = \left(\frac{\sum_{k=1}^{i} w_k \sum_{k=1}^{n} w_k}{\sum_{k=i+1}^{n} w_k} \right)^{\frac{1}{2}} \left(\frac{\bar{y}_{w,i} - \bar{y}_{w,n} - \widehat{\beta}_w \cdot (\bar{x}_{w,i} - \bar{x}_{w,n})}{\left[1 - \left\{ \sum_{k=1}^{i} w_k \sum_{k=1}^{n} w_k \Big/ \sum_{k=i+1}^{n} w_k \right\} (\bar{x}_{w,i} - \bar{x}_{w,n})^2 \big/ Q_{xxn} \right]^{\frac{1}{2}}} \right) .$$

The weighted likelihood ratio statistics to test H_0 against H_2, $(WLRS_2)$, is given by

$$\hat{\sigma}^{-2} \max_{n_0 \leqslant i \leqslant n_1} |V_{w,n}(i)| \ , \tag{2.2}$$

where

$$V_{w,n}(i) \;=\; \frac{\sum_{k=1}^{i} w_k \sum_{k=1}^{n} w_k}{\sum_{k=i+1}^{n} w_k} \, (\bar{y}_{w,i} - \bar{y}_{w,n})^2 + \frac{Q_{xyi}^2}{Q_{xxi}} + \frac{Q_{xyi}^{*2}}{Q_{xxi}^*} - \frac{Q_{xyn}^2}{Q_{xxn}} \ .$$

In both tests $n_j = n t_j$, $j=0,1$, for $0 < t_0 < t_1 < 1$, and following Kim ([4]) notation,

$$\bar{x}_{w,i} \;=\; \left(\sum_{k=1}^{i} w_k \right)^{-1} \sum_{k=1}^{i} w_k \, x_k \ , \qquad \bar{x}_{w,i}^* \;=\; \left(\sum_{k=i+1}^{n} w_k \right)^{-1} \sum_{k=i+1}^{n} w_k \, x_k \ ,$$

$$\bar{y}_{w,i} \;=\; \left(\sum_{k=1}^{i} w_k \right)^{-1} \sum_{k=1}^{i} w_k \, y_k \ , \qquad \bar{y}_{w,i}^* \;=\; \left(\sum_{k=i+1}^{n} w_k \right)^{-1} \sum_{k=i+1}^{n} w_k \, y_k \ ,$$

$$Q_{xxi} = \sum_{k=1}^{i} w_k \left(x_k - \bar{x}_{w,i}\right)^2 \ , \qquad Q^*_{xxi} = \sum_{k=i+1}^{n} w_k \left(x_k - \bar{x}^*_{w,i}\right)^2 \ ,$$

$$Q_{yyi} = \sum_{k=1}^{i} w_k \left(y_k - \bar{y}_{w,i}\right)^2 \ , \qquad Q^*_{yyi} = \sum_{k=i+1}^{n} w_k \left(y_k - \bar{y}^*_{w,i}\right)^2 \ ,$$

$$Q_{xyi} = \sum_{k=1}^{i} w_k \left(x_k - \bar{x}_{w,i}\right)\left(y_k - \bar{y}_{w,i}\right) \ ,$$

$$Q_{xxn} = \sum_{k=1}^{n} w_k \left(x_k - \bar{x}_{w,n}\right)^2 \ , \qquad Q_{xyn} = \sum_{k=1}^{n} w_k \left(x_k - \bar{x}_{w,n}\right)\left(y_k - \bar{y}_{w,n}\right) \ ,$$

$$\widehat{\beta}_w = Q_{xyn}/Q_{xxn} \qquad \text{and} \qquad \hat{\sigma}^2 = n^{-1}\left(Q_{yyn} - Q^2_{xyn}/Q_{xxn}\right) \ .$$

Kim ([4]) also presents approximations of the p-values of the *WLRS* (2.1) and (2.2).

3. PERFORMANCE OF THE TESTS

To assess and compare the performance of the tests described in the previous section we carried out a Monte Carlo simulation taking into account different scenarios, such as, different error distributions, different sample sizes, different locations of the change point and departure from the homoscedasticity assumption. In all the cases two different alternative hypotheses were considered: i) with a change in the intercept from one model to the other with the same slope, and ii) with a change in both intercept and slope.

In the simulation process, for each model, the sequence of observations $(x_1, y_1), ..., (x_n, y_n)$, where $x_i = i/n$, $i = 1, ..., n$ and y_i satisfying the model (1.2), were generated 5,000 times. To calculate the distributional insensitivity of the tests the following distributions for the errors were considered: $N(0,1)$, $N(0,1)$ for one regime and $N(0,4)$ for the other, normal with variance given by $1/w_i$, where $w_i = (1 + i/n)^2$, contaminated normal using the mixture distribution $0.95\,N(0,1) + 0.5\,N(0,9)$, exponential(1), Weibull(α,γ), with $\alpha = 1.5$ and $\gamma = 1/(2^{1/2\alpha}\{\Gamma(2/\alpha+1) - \Gamma^2(1/\alpha+1)\}^{0.5})$, and lognormal($\alpha,\gamma$), with $\alpha = 0.1$ and $\gamma = \{\exp(2\alpha) - \exp(\alpha)\}^{-0.5}$. For more details for the choice of these parameters interested readers can see Kim and Cai (1993).

To generate the Weibull and lognormal random errors we use the methods presented in Kim and Cai (1993). The Weibull random errors are generated by the transformation $\varepsilon = \gamma(Z_1^2 + Z_2^2)^{1/\alpha}$, where Z_1 and Z_2 are independent and follow standard normal distributions. The lognormal random errors are generated considering $\varepsilon = \gamma \exp(\sqrt{\alpha}\, Z_1)$. The exponential and the normal random variables are generated using SAS/IML functions. The $N(0, 1/w_i)$ distribution for the errors implies that each observation has a different variance determined by the value w_i; it is used to compare the K-Test with the other tests. The K-Test is applied only in models in which a change can occur in the intercept alone.

Following Zhang and Boss ([14]), Monte Carlo estimates of critical values are used to create adjusted power estimates. It allows for comparisons among the competing tests under the same scenarios.

3.1. A Change in the Intercept

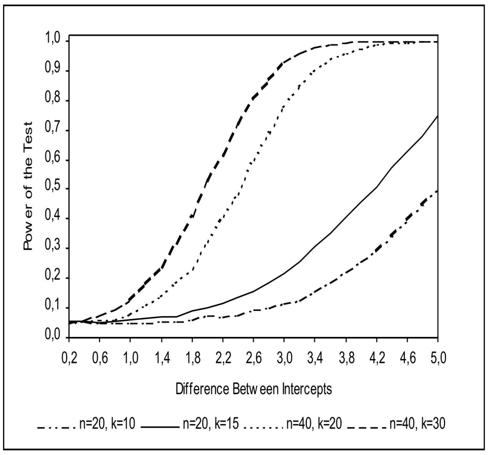
We start with the analysis of the performance of the Q-Test for models with a change in the intercept alone. Figure 1 – panel (a) shows the power of the Quandt test for different sample sizes and different locations of the change point considering models with errors following normal distributions. In all the cases the critical values are the 95^{th} percentiles under H_0, estimated by Monte Carlo simulations. The best performance of the test occurs when the change point is not at the central position, that is, when the number of observations in a regime is much smaller than the number of observations in the other, and also when the sample size increases.

Figure 1 – panel (b) shows the performance of the Q-Test for homogeneous error variance models with different distributions for the errors. In these cases the sample size is 20 and the change point is located at $k = 15$. It can be concluded that the likelihood ratio test of Quandt achieves almost the same power for the four different distributions for the errors. Similar results are reached for other different sample sizes and different locations of the change point. The performance of the Q-Test for non-homogeneous error variance models is showen in Figure 1 – panel (c), for $n = 20$ and $k = 15$, when the errors have normal distributions with different variances from one regime to the other and when the errors have contaminated normal distribution with each observation having a different variance. The standard normal distribution model is also presented. The power of the likelihood ratio test of Quandt achieves almost the same power for the three distributions when the difference between the intercepts of the two regimes is less than 2, after that the behavior of the power functions are different. Other sample sizes and locations of the change point were explored and the results were similar.

The performance of the KS-Test for models with a change in the intercept alone is now analyzed. Figure 2 − panel (a) shows the power of the KS-Test for models with errors following $N(0,1)$ considering different sample sizes and different locations of the change point in both regimes. This test performs well in those cases where the changes occurred far from the center. This is in broad agreement with the results of Kim and Cai ([13]). The Figure 2 − panel (b) shows the robustness of the KS-Test concerning different distributions of the errors but with homogeneous variances. However, for non-homogeneous models, as shown in Figure 2 − panel (c), the performance of the KS-Test depends on the distribution of the errors. For the cases where the errors follow non-homogeneous normal (different variances for each regime) and contaminated normal distribution the performance of the test is clearly inferior to the performance when the errors follow homoscedastic distributions. When the considered distribution is the heteroscedastic normal the performance of the test is worse than in the contaminated normal distribution case.

The non-homogeneous models considered in this study were the models in which the variances of the error terms are proportional to the square of a linear combination of the regressor variables. That is, the models have errors following a $N(0,1/w_i)$, with $w_i = (1+i/n)^2$. Models with this type of heteroscedasticity, known as additive heteroscedastic error models, have been discussed by Rutemiller and Bowers ([15]) and Harvey ([16]). These models were submitted to the Q, KS and K-Tests. The test powers are shown in Figure 3. Note that the Q-Test does not present a good performance in these cases. The KS-Test is better than the Q-Test but sensibly worse than the K-Test. Comparing such results with the results presented previously, the non robustness of the Q and KS-Tests is evident when applied to non-homogeneous models, mainly when the variances of the errors are different from an observation to another and not only from a regime to another. The K-Test is more powerful in the case where the variances of the error terms are different but known. In the cases of contaminated normal and heteroscedastic normal distributions, where the w_i is unknown, the application of the K-Test can be accomplished by taking $w_i = 1$, for $i = 1, 2, ..., n$, which would make such a test equivalent to the KS-Test, or by estimating these weights from data.

Comparatively, the Q-Test and KS-Test presented very different results for samples of size 20. The performance of the KS-Test is superior to the performance of the Q-Test. That superiority is noticed practically in all the considered differences between intercepts. It is important to point out that even when we considered the non-homogeneous error variance models submitted to the KS-Test, the results are better than the one of the Q-Test in homogeneous error variance models. Analyzing the Figures 1 and 2, a similar performance of the Q-Test and KS-Test when the samples are of size $n = 40$ is noticed. The K-Test is more efficient than the others in non-homogeneous error variance models with known variance.



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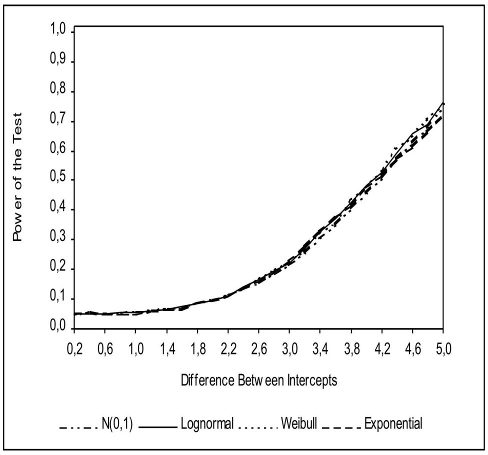

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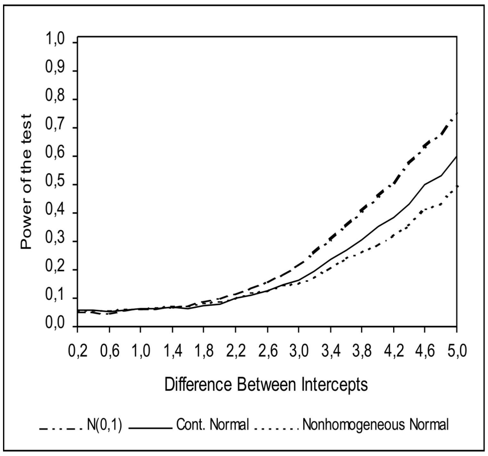

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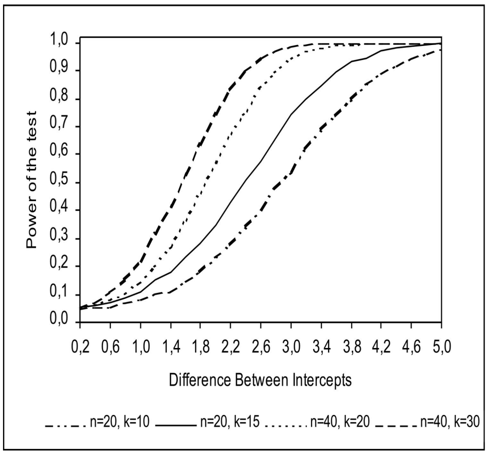
Figure 1: Power of the Q-Test in three scenarios:

Panel (*a*) for models with error terms following $N(0, 1)$ considering different sample sizes and different locations of the change point. In this case the best performance of the test occurs when the change point is not at the central position.

Panel (*b*) for homogeneous error variance models supposing the error terms have normal, lognormal, Weibull and exponential distribution, with $n = 20$ and $k = 15$. Note that the likelihood ratio test of Quandt achieves almost the same power for the four different distributions.

Panel (*c*) for models with error terms following N(0,1), nonhomogenous normal and contaminated normal distribution, with $n = 20$ and $k = 15$. The power of the Q-Test achieves almost the same power for the three distributions when the difference between the intercepts of the two regimes is less than 2. Afterwards that, the behavior of the power functions are different.



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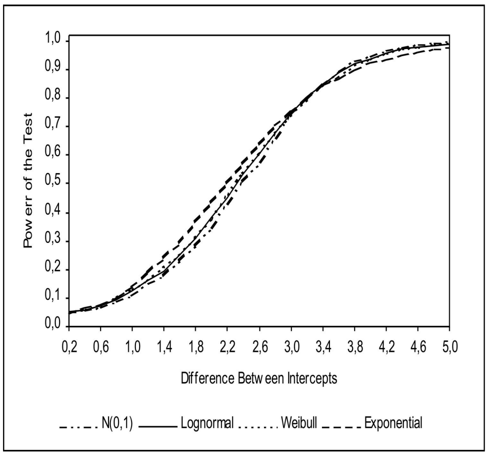

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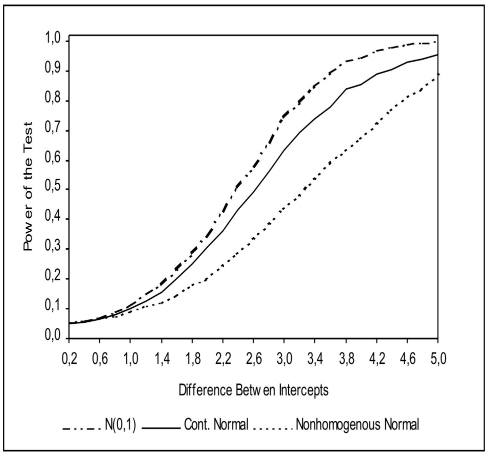

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Figure 2: Power of the KS-Test in three scenarios:

Panel (a) for models with error terms following N(0,1) considering different sample sizes and different locations of the change point. This test performs well in those cases where the changes occurred far from the center.

Panel (b) for homogeneous error variance models supposing the error terms follow N(0,1), lognormal, Weibull or exponential distribution, with $n = 20$ and $k = 15$. It shows the robustness of the KS-Test regarding the four different distributions.

Panel (c) for models with error terms following N(0,1), non-homogenous normal and contaminated normal distributions, with $n = 20$ and $k = 15$. In this case the performance of the KS-Test depends on the distribution of the errors.

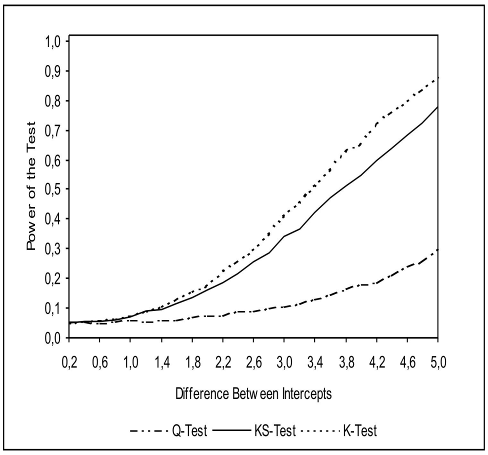


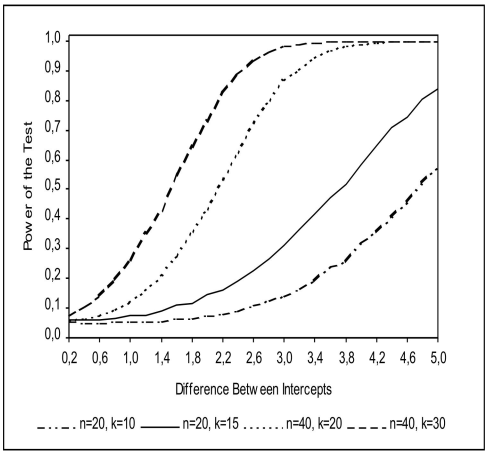
Figure 3: Power of the Q, KS and K-Tests for models with errors following $N(0, 1/w_i)$, where $w_i = (1 + i/n)^2$, for $i = 1, 2, ..., n$. The Q-Test does not present a good performance in these cases. The KS-Test is better than the Q-Test but sensibly worse than the K-Test.

3.2. Change in both Intercepts and Slope

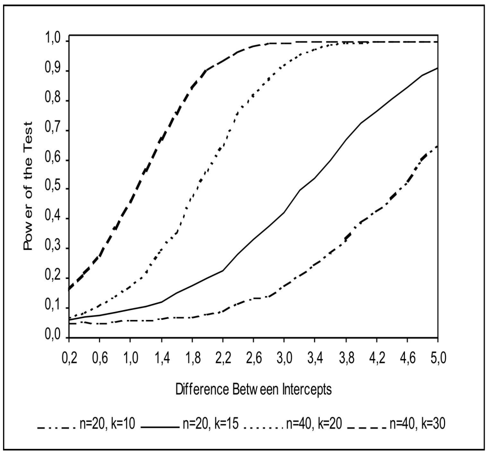
The cases considered in the previous section are analyzed here but with different slopes on the first and second regimes of each model. The difference between these slopes considered in the simulations are 0.5 and 1.0. The tests will continue to be denominated Q-Test and KS-Test but taking into account the versions that consider models with different intercepts and slopes between the regimes. Recall that the K-Test is applied only to models in which a change can occur in the intercept alone.

The Figures 4 − panel (a) and 4 − panel (b) present the results obtained with the Q-Test in models with distribution $N(0, 1)$, considering the differences between the slopes 0.5 and 1.0, respectively. The test is sensitive to the change in the difference between such slopes. Besides, as it happened with models where only the intercepts change, the results are clearly better when the change point is on a non central region. Moreover in these cases, the performance of the Q-Test improves as the sample size increases from 20 for 40.

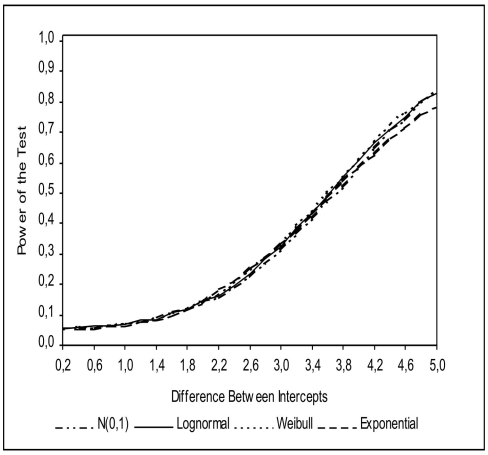
The Figures 4 − panel (c) and 4 − panel (d) present the results obtained with the Q-Test in homogeneous error variance models considering several distributions for the error terms with $n = 20$ and $k = 15$. Here it is also clear that the test is sensitive to the increase of the difference between the slopes. Both Figures evidence the robustness of the test in relation to the considered distributions of the errors.



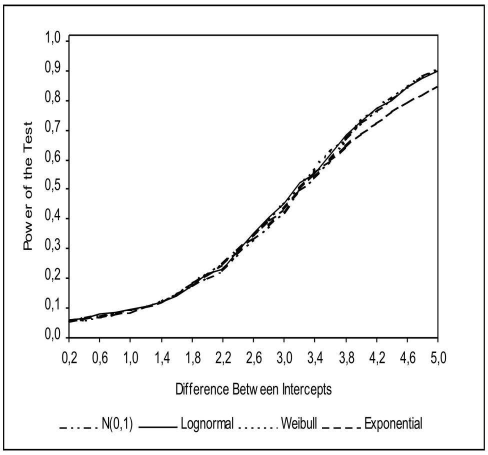
(*a*)



(*b*)



(*c*)



(*d*)

Figure 4: Power of the Q-Test in four scenarios:

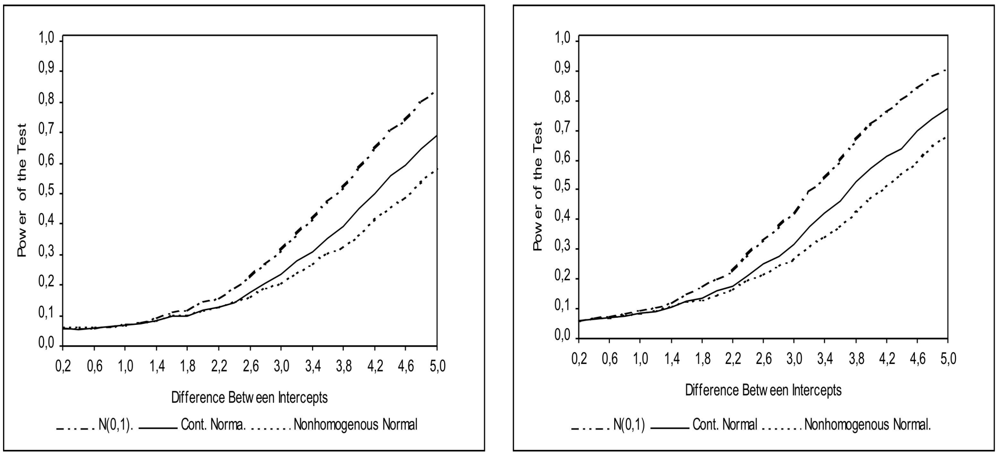
Panel (*a*) for models with error terms following $N(0,1)$ considering different sample sizes, different locations of the change point and $\beta_2 - \beta_1 = 0.5$.

Panel (*b*) for models with error terms following $N(0,1)$ considering different sample sizes, different locations of the change point and $\beta_2 - \beta_1 = 1.0$. In both cases the test is sensitive to the change in the difference between such slopes and improves as the sample size increases from 20 for 40.

Panel (*c*) for homogeneous error variance models supposing the error terms follow distribution $N(0,1)$, lognormal, Weibull and exponential, $n = 20$, $k = 15$ and $\beta_2 - \beta_1 = 0.5$.

Panel (*d*) for homogeneous error variance models supposing the error terms follow distributions $N(0,1)$, lognormal, Weibull and exponential, $n = 20$, $k = 15$ and $\beta_2 - \beta_1 = 1.0$. Both Figures evidence the robustness of the test in relation to the considered distributions of the errors.

The performance of the Q-Test in non-homogeneous error variance models in comparison to the case $N(0,1)$ can be seen in Figures 5 − panel (a) and 5 − panel (b), these also evidence that the test is sensitive to the increase of the difference between the slopes. Once again, the non robustness of the test studied in relation to the presence of heteroscedasticity can be clearly seen.



(a) (b)

Figure 5: Power of the Q-Test in two scenarios:
Panel (a) for models with error terms following $N(0,1)$, non-homogenous normal and contaminated normal distributions, $n = 20$, $k = 15$ and $\beta_2 - \beta_1 = 0.5$.
Panel (b) for models with error terms following $N(0,1)$, non-homogenous normal and contaminated normal distributions, $n = 20$, $k = 15$ and $\beta_2 - \beta_1 = 1.0$. In both cases there is evidence that the test is sensitive to the increase of the difference between the slopes.

The KS-Test when applied to models in which changes occurs in both intercept and slope is sensitive to the increase in the difference between the slopes and its best performance happens when the structural change occurs in a non central region. Another property of the KS-test is the non robustness concerning the heteroscedasticity of the model. Such properties were verified by the analysis of the plots (not shown here) which present the power of the KS-Test applied to the same models submitted to the Q-Test.

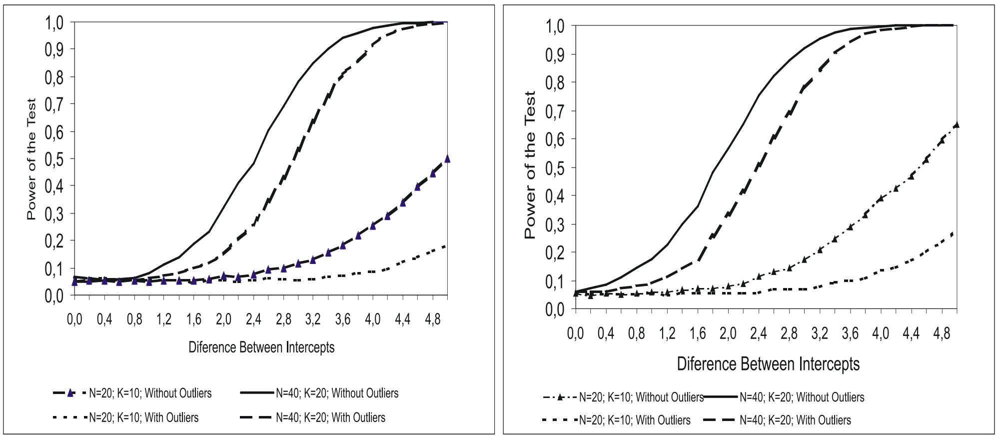
When the difference between intercepts is less than 1.0, the KS-Test presents superior performance, in both cases $\beta_2 - \beta_1 = 0.5$ and $\beta_2 - \beta_1 = 1.0$, in non-homogeneous normal models in relation to the $N(0,1)$ models. However, when the difference between intercepts is larger than 1.0, such superiority is total and completely reverted in favor of the cases $N(0,1)$, also in both cases $\beta_2 - \beta_1 = 0.5$ and $\beta_2 - \beta_1 = 1.0$.

Q-Test and KS-Test are sensitive to the alterations of differences between intercepts and differences between slopes, besides both tests present better performances when the structural change happens in a non central region. Another characteristic of the Q-Test and KS-Test is the non robustness of their performance in non-homogeneous error variance models in relation to the homogeneous models.

In all the presented cases the KS-Test performs better than the Q-Test. Both are shown to be robust regarding the different distributions considered in homogeneous models, except the KS-Test when applied in models with error following exponential distribution. However, even presenting inferior power in the exponential case, the KS-Test has more power than the Q-Test in all the homogeneous cases.

4. WORKING WITH OUTLIERS

In this section a small investigation of the sensitivity of the tests referring to outlying observations is presented. New data sets, contaminated with outliers, are simulated from the model (1.2) and the power results for the three tests, for some scenarios, are explored. This investigation involves the power of the Q-test, KS-Test and K-test considering data sets with and without outliers, different sample sizes and different locations of the change point, for errors following normal distributions. In all the cases the critical values are the 95$^{\text{th}}$ percentiles under H_0, estimated by Monte Carlo simulations.



(*a*) (*b*)

Figure 6: Power of the Q-Test for models with error terms following $N(0,1)$ considering data sets with and without outliers, different sample sizes and different locations of the change point. Panel (*a*) $\beta_2-\beta_1=0$. Panel (*b*) $\beta_2-\beta_1=1.0$.

For the Q-Test and KS-Test, if the outliers are clearly present in one of the regimes, inspection of the results reveals that the cases without outliers have slightly more power when compared to the cases with outliers, but for the K-test the power of cases with and without outliers are comparable.

For the three studied tests, the cases where the outliers are clearly in the change point region are slightly more powerful than those cases without outliers. The reason for that can reside in the fact that the simulated outliers reinforced the presence of change points.

5. CONCLUDING REMARKS

The robustness of different likelihood ratio tests was investigated under different scenarios via a simulation study as presented in Section 3. With the exception of the study by Kim and Cai ([13]) there has been little work done for a comprehensive discussion of the performance of these tests including the important question of deciding which test should be considered and under which circumstances. The simulation results suggested that the KS-Test is superior to the Q-Test when small to moderate sample sizes are considered for both homogeneous and non-homogeneous models with a change in the intercept alone. However, the K-Test is more powerful than the other two tests for non-homogeneous models with a known variance. The Q-Test and KS-Test are both robust regarding different distributions of the errors for homogeneous models. When there is a change point in both intercept and slope the KS-Test is superior to the Q-Test in all investigated scenarios.

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