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ON THE UPCROSSINGS OF TRIGONOMETRIC POLYNOMIALS WITH RANDOM COEFFICIENTS

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Abstract:

- Polynomials are one of the oldest and the most versatile classes of functions which are fundamental in approximating highly complex, deterministic as well as random nonlinear functions and systems. Their use has been acknowledged in every scientific field from physics to ecology. Just to emphasize their great use in many fields, we mention their fundamental role in linear and non-linear time series analysis. In this paper, we give a review of some of the results related to polynomials with random coefficients and highlight the Poisson character of high level upcrossings of certain random coefficient trigonometric polynomials which are used in spectral analysis of time series.

Key-Words:

- *random polynomials; trigonometric polynomials; extreme value theory.*

AMS Subject Classification:

- 60G70.

1. INTRODUCTION

Polynomials form the backbone of mathematics in general and approximation of complex deterministic and random nonlinear functions and dynamic processes in particular. Examples of their use are countless coming from diverse areas such as quantum chaotic dynamics to ecology. Their use repeatedly appears in asymptotic theory of statistics and in particular in time series analysis which will be our primary interest.

In its simplest form, algebraic characteristics of polynomials are very much used in statistics. The celebrated expansions related to central limit theorems such as the Edgeworth expansions, Berry–Esseen type theorems and the delta method all depend on polynomial expansions and form the basis of asymptotic theory of statistics. Polynomials and their algebraic properties are also used in constructing stationary, invertible finite parameter linear representations for time series. Wold decomposition theorem says that under fairly general conditions any stationary time series may be represented as a causal convergent sum

$$X_t = \sum_{j=0}^{\infty} \psi_j \, \epsilon_{t-j} \ ,$$

with uncorrelated finite variance random variables $\{\epsilon_t\}$ and real values $\{\psi_j\}$ such that $\sum_j \psi_j^2 < \infty$. As a class of models, such a representation is not particularly useful due to the infinite number of parameters, and finite parameter versions called the class of stationary and invertible ARMA models are found by using the backshift operator $B^j X_t = X_{t-j}$, then representing the series in the form

$$X_t = \left[\sum_{j=0}^{\infty} \psi_j \, B^j \right] \epsilon_t \ .$$

Under quite general conditions, the polynomial $\psi(B) = \sum_{j=0}^{\infty} \psi_j \, B^j$ can be written as a ratio of two finite order polynomials $\Phi_p(B)$ and $\Theta_q(B)$ of orders p and q respectively, thus permitting us to write $\Phi_p(B)\, X_t = \Theta_q(B)\, \epsilon_t$. The conditions of stationarity and invertibility of the process X_t are then given in terms of the roots of the polynomials $\Phi(B)$ and $\Theta(B)$. In these examples, the well known algebraic results on deterministic polynomials are used. However, in a more general set up, random polynomials are used for very general representations for stationary times series.

Let us start with a collection of standard Gaussian random variables $\{X_s, s \le t\}$ and consider the space of all measurable functions defined on this collection with the usual inner product

$$\langle f, g \rangle \ = \int_{-\infty}^{\infty} f(x)\, g(x) \, \frac{1}{\sqrt{2\pi}} \, \exp(-x^2/2) \ dx \ . \tag{1.1}$$

This space together with this inner product is a Hilbert space, and (random) Hermite polynomials form a closed and complete orthogonal system.

Hermite polynomials $H_n(x)$ of degree n are defined as

$$\int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, \frac{1}{\sqrt{2\pi}}\, \exp(-x^2/2)\; dx \;=\; n!\, \delta_{n,m}\;, \qquad n,m=0,1,2,... \tag{1.2}$$

where

$$\delta_{n,m} = \begin{cases} 1\,, & n=m\,; \\ 0\,, & n\neq m\,. \end{cases} \tag{1.3}$$

Hence, every Borel measurable function g with finite variance (with respect to the Gaussian density) such that

$$\int_{-\infty}^{\infty} g^2(x)\, \frac{1}{\sqrt{2\pi}}\, \exp(-x^2/2)\; dx \;<\; \infty\;,$$

can be written as a linear combination (or as a limit) of these Hermite polynomials

$$g(x) \;=\; \lim_{N\to\infty} \sum_{n=0}^{N} \frac{g_n}{n!}\, H_n(x)\;, \tag{1.4}$$

where, the coefficients g_n are given by

$$g_n \;=\; \int_{-\infty}^{\infty} g(x)\, H_n(x)\, \frac{1}{\sqrt{2\pi}}\, \exp(-x^2/2)\; dx\;.$$

The convergence of (1.4) is in the mean square sense

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left(g(x) - \sum_{n=0}^{N} \frac{g_n}{n!}\, H_n(x) \right)^2 \frac{1}{\sqrt{2\pi}}\, \exp(-x^2/2)\; dx \;=\; 0\;.$$

Note that the inner product is a integral with respect to the standard Gaussian density and hence the Hermite polynomials are orthogonal with respect to the standard normal probability distribution. Instead of Hermite polynomials, we can define Hermite functions

$$\psi_n(x) \;=\; \frac{1}{\sqrt{n!\, 2^n \sqrt{2\pi}}}\, \exp(-x^2/2)\, H_n(x)\;.$$

Hermite functions are normalized versions of the Hermite polynomials, therefore they form an closed and complete orthonormal basis. The closed linear span of Hermite polynomials is the space of all polynomials, and any element of this space can be represented as sums of products of polynomials given in the form

$$\sum_{p=1}^{\infty} \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} a_{i_1 i_2 \cdots i_p} \prod_{v=1}^{p} X_{i_v}\;. \tag{1.5}$$

Here we will not enter into further details, which can be found in Terdik (1999).

The following remarkable result due to Nisio (1964) extends this polynomial representation to any strictly stationary time series.

Let ϵ_t be independent, standard Gaussian random variables. The polynomial representation

$$
\begin{aligned}
Y_t^{(m)} &= \sum_{p=1}^{m} \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_m=-\infty}^{\infty} g_{i_1 i_2 \cdots i_m} \prod_{v=1}^{p} \epsilon_{t-i_v} \\
&= \sum_{i_1=-\infty}^{\infty} g_{i_1} \epsilon_{t-i_1} \\
&\quad + \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} g_{i_1 i_2} \epsilon_{t-i_1} \epsilon_{t-i_2} \\
&\quad + \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \sum_{i_3=-\infty}^{\infty} g_{i_1 i_2 i_3} \epsilon_{t-i_1} \epsilon_{t-i_2} \epsilon_{t-i_3} \\
&\quad + \cdots \\
&\quad + \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_m=-\infty}^{\infty} g_{i_1 i_2 \cdots i_m} \epsilon_{t-i_1} \epsilon_{t-i_2} \cdots \epsilon_{t-i_m}
\end{aligned} \tag{1.6}
$$

is called a Volterra series of order m. We will call

$$
Y_t = \sum_{p=1}^{\infty} \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_p=-\infty}^{\infty} g_{i_1 i_2 \cdots i_p} \prod_{v=1}^{p} \epsilon_{t-i_v} , \tag{1.7}
$$

the Volterra series expansion.

Theorem 1.1. *Let X_t be any strictly stationary time series. Then there exists a sequence of Volterra series $Y_t^{(m)}$ such that*

$$
\lim_{m \to \infty} Y_t^{(m)} \overset{d}{=} X_t ,
$$

in the sense that for any n and for any $\theta_j, |j| \le n$ as $m \to \infty$,

$$
\begin{aligned}
\Big| E \exp\big(i\,\theta_{-n} X_{-n} + \cdots + i\,\theta_n X_n\big) \\
- E \exp\big(i\,\theta_{-n} Y_{-n}^{(m)} + \cdots + i\,\theta_n Y_n^{(m)}\big) \Big| \to 0 .
\end{aligned} \tag{1.8}
$$

If further X_t is Gaussian, then X_t can be represented by

$$
X_t = \sum_{j=-\infty}^{\infty} g_j \,\epsilon_{t-j} ,
$$

for some real numbers $\{g_j\}$, such that $\sum_{=-\infty}^{\infty} g_j^2 < \infty$.

Hence random polynomials are basic functions with which we construct very general random processes. There is also a relationship between solutions of random difference equations and random polynomials.

Consider a stochastic difference equation

$$X_t = A_t X_{t-1} + B_t \ , \tag{1.9}$$

where $\{A_t, B_t\}$ is a sequence of random variables. We will call (1.9) a stochastic recurrence equation. It is possible to define (1.9) in a more general form

$$\mathbf{X}_t = \mathbf{A}_t \mathbf{X}_{t-1} + \mathbf{B}_t \ , \tag{1.10}$$

where $\mathbf{X}_t$ and $\mathbf{B}_t$ are random vectors in $\mathcal{R}^d$, $\mathbf{A}_t$ are $d \times d$ random matrices and $\{\mathbf{A}_t, \mathbf{B}_t\}_{n=0}^{\infty}$ is a strictly stationary ergodic process.

Many well known classes of nonlinear time series models such as bilinear, ARCH, GARCH, random coefficient autoregressive models (RCA) as well as threshold models can be represented in this form. Theorems due to Vervaat(1979) and Brant(1986) show that under fairly general conditions on $\{A_t, B_t\}$, stochastic difference equations of the form (1.9) have solutions given by

$$R = \sum_{k=1}^{\infty} \prod_{j=1}^{k-1} A_j B_k \ . \tag{1.11}$$

It is clear that the solution (1.11) are algebraic polynomials of random variables.

Extremal behavior of these polynomial expansions have played important role in understanding the oscillatory behavior of nonlinear processes, and many results on the point processes of upcrossings or exceedances of such polynomial expansions exist. See Turkman and Amaral Turkman(1997) and Scotto and Turkman(2002,2005) and de Haan *et al.*(1989).

Random polynomials of different nature given in the form

$$F_n(\omega, x) = a_0(\omega) F_1(x) + a_1(\omega) F_2(x) + ... + a_n(\omega) F_n(x) \ , \tag{1.12}$$

where $\{a_i(\omega)\}_{i=}^{n}$ are a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\{F_i(x)\}_{i=1}^{n}$ are deterministic functions of x, have found significant applications in many fields involving the reliability of complex random physical systems. When $F_i(x) = x^i$, then the solution (1.12) is an algebraic polynomial with random coefficients, taking the form

$$F_n(\omega, x) = \sum_{j=0}^{n} a_j(\omega)\, x^j \ ,$$

whereas when $F_i(x)$ are trigonometric functions then (1.12) is a trigonometric polynomial of order n with random coefficients, often given in the form

$$F_n(\omega, x) = a_0(\omega) + \sum_{j=0}^{n} a_j(\omega) \cos jx + \sum_{j=1}^{n} b_j(\omega) \sin jx \ .$$

Here, contrary to polynomials given in (1.6), the type of polynomials we consider in (1.12) are deterministic in its argument, having random coefficients. We refer the reader to Bharucha-Reid and Sambandham(1986) and Farahmand(1998) for the general treatment of such random polynomials. We also refer the reader to Zygmund(2002) for a full account of developments on trigonometric polynomials.

Polynomials with random coefficients have many interesting characteristics, but the level crossing properties are particularly important and useful. Describing the reliability of a complex physical system subject to random inputs depends on understanding the oscillatory behavior of its sample paths. Level crossing problems of stochastic processes and the related random variable, the number of times the trajectory of a stochastic process crosses an arbitrary level u during the time interval $[0, T]$ has considerable importance and forms the basis of extreme value theory for stochastic processes. We refer the reader to Cramer and Leadbetter(1962), Leadbetter *et al.*(1983) and Albin(1990, 2001) for the general treatment of extreme value theory for stochastic processes and level crossing problems.

Let $X(t)$, $t \geq 0$ be a continuous time, strictly stationary stochastic process with almost surely continuous sample paths $x(t)$. $x(t)$ is said to have an upcrossing of u at the point $t_0 > 0$, if for some $\epsilon > 0$, $x(t) \leq u$ in the interval $(t_0 - \epsilon, t_0)$ and $x(t) > u$ in $(t_0, t_0 + \epsilon)$. Here we assume that the sample paths are not identically equal to u in any subinterval with probability 1. Downcrossings of the level u can similarly be defined with obvious changes. We denote by the random variable $N_u(I)$, the number of upcrossings of the level u by the process $X(t)$ in the time interval I. We will also write $N_u(T) = N_u((0, T])$, and in particular $N_u(1) = N_u((0, 1])$. This random variable plays the fundamental role in studying the level crossing problems of stochastic processes. Much is known on the random variable $N_u(I)$, particularly for Gaussian processes. For example, under general conditions, the mean number of upcrossings of the level u in the unit interval $(0, 1]$ is given by

$$E(N_u(1)) = \lim_{q\to 0} J_q(u) = \int_0^\infty z\, p(u, z)\, dz \ , \tag{1.13}$$

where for arbitrarily small $q > 0$, $J_q(u) = \frac{1}{q} P\big(X(0) \leq u < X(q)\big)$ and $g_q(u, z)$ is the joint density of $X(0)$ and $\frac{1}{q}\big(X(q) - X(0)\big)$ such that $p(u, z) = \lim_{q\to 0} g_q(u, z)$. In most cases the limiting density $p(u, z)$ is the joint density of $X(t)$ and its derivative $X'(t)$ calculated at $t = 0$. In this case,

$$E(N_u(1)) = p(u) \int_0^\infty z\, p(z|u)\, dz = p(u)\, E\big(\max\{0, X'(0)\} \,|\, X(0) = u\big) \ , \tag{1.14}$$

where $p(u)$ and $p(z|u)$ are respectively the density of $X(t)$ and the conditional density of the derivative $X'(t)$ given $X(t) = u$, again calculated at $t = 0$. Hence, the expected number of upcrossings is given in terms of the density of $X(0)$ and the average positive slope of the sample path at u.

The expected number of upcrossings of a Gaussian process is totally characterized by the behavior of its covariance function at origin. If $r(\tau)$ is the covariance function of the process $X(t)$ such that as $\tau \to 0$,

$$r(\tau) \;=\; 1 - \frac{\lambda_2 \tau^2}{2} + o(\tau^2) \;,$$

then the expected number of upcrossings is given

$$E(N_u(1)) \;=\; \frac{1}{2\pi}\, \lambda_2^{1/2} \exp\!\left(-\frac{u^2}{2}\right) . \tag{1.15}$$

Here, $\lambda_2 = r''(0)$ is called the second spectral moment, assumed to be finite and (1.15) is the well known Rice formula. We note that in extreme value theory, a more general class of Gaussian processes with covariance function of the type

$$r(\tau) = 1 - C|\tau|^\alpha = O(|\tau|^\alpha) \;, \qquad \tau \to 0 \;,$$

where $0 < \alpha \le 2$ are considered. This class includes the Gaussian processes with differentiable sample paths, that is, Gaussian process with finite second spectral moments and consequently with finite number of upcrossings. In general, when $\alpha < 2$, the process is nondifferentiable and consequently, has infinitely many upcrossings in any finite interval.

Although the expected number of upcrossings gives quite a lot of information on the oscillatory behavior of the process, more can be learned from the higher moments of upcrossings or indeed from its asymptotic probability distribution. Second upcrossing moment given by

$$E\big(N_u(I)\,(N_u(I)-1)\big) = \int_0^\infty\!\!\int_0^\infty z_1 z_2\, p(u,u,z_1,z_2)\; dz_1\, dz_2 \;, \tag{1.16}$$

where $p(u,u,z_1,z_2)$ is the joint density of $(X(t_1), X(t_2), X'(t_1), X'(t_2))$ calculated at $X(t_1)=u$, $X(t_2)=u$, plays particularly important role in obtaining limiting results for the extremal behavior of the process. For example, as $u \to \infty$ it can be shown that

$$\begin{aligned} 1 - E\big(N_u(T)\big) + o(1) \;&\le\; P\Big(\max_{t\in(0,T]} X(t) \le u\Big) \\ &\le\; 1 - E\big(N_u(T)\big) + E\big(N_u(T)\,(N_u(T)-1)\big), \end{aligned}$$

from which one can obtain the asymptotic expression for the maximum of a stochastic process over fixed and increasing intervals. See for example, Leadbetter(1978) and Turkman and Walker(1984). It is possible to obtain more complete results on upcrossing events other than their moments. For a given level u, let $\mu(u) = E(N_u(1))$ be the finite mean number of u-crossings per unit time by the process $X(t)$. If we look at the number of u-upcrossings of the process over an interval $[0,T]$ as $T \to \infty$, then almost surely this number would diverge to ∞.

However, if we increase the level u as a function of the the increasing time interval T in such a way that that $T\mu(u) \to \tau$, for some fixed $0 < \tau < \infty$, as $T \to \infty$, then it may be possible to obtain many nice limiting properties. Indeed, Let $N_T(\cdot)$ be the time normalized point process of u-crossings defined by

$$N_T(B) \;=\; N_u(TB) \;=\; \sharp\Big\{u\text{-crossings by } X(t);\;\; t/T \in B\Big\}$$

for any Borel set in $[0,1]$. Thus, N_T has a point at t, if $X(t)$ has an u-upcrossing at tT. Then it is known that under quite general conditions, this point process converges to a homogeneous Poisson process with intensity τ. These results are called complete convergence theorems, since one can obtain many interesting asymptotic results from this basic convergence. For example, the asymptotic distribution of the maximum of the process over increasing intervals as well as the asymptotic distribution of the upper order statistics of the process can be recovered from such basic results. See for example Leadbetter *et al.*(1983) and Resnick(1987, 2007) for convergence of point processes related to exceedances and upcrossings.

The corresponding results for Gaussian processes are well known. Let $X(t)$ be a stationary Gaussian process with covariance function $r(\tau)$ such that

1. as $h \to 0,\; r(h) = 1 - \frac{\lambda_2}{2}h^2 + o(h^2)$;

2. as $h \to \infty,\; r(h)\log h \to 0$.

Let u and T tend to infinity in such a way that $T\mu \sim \tau$, where, $\mu = \frac{1}{2\pi}\,\lambda_2^{1/2}\exp(-u^2/2)$ is the expected number of upcrossings in the unit interval. Then the time normalized point process N_T of u-upcrossings converges in distribution to a Poisson process with intensity τ on the positive real line. For processes other than Gaussian processes, asymptotic results of similar type are very difficult to obtain. We refer the reader to Albin(2001) on asymptotic results on upcrossings by many non-Gaussian processes such as Markov jump processes, α-stable processes and quadratic functionals of Gaussian processes. For specific results on streams of upcrossings by random coefficient polynomials, see Farahmand(1998). See Scotto and Turkman(2005) for similar weak convergence of point processes of u-upcrossings of finite order Volterra series expansions, although such polynomials are quite different in nature than the random coefficient polynomials defined in (1.12).

In section 2, we will look at the point processes of u-upcrossings of certain types of trigonometric polynomials and show Poisson nature of the limiting process.

2. u-UPCROSSINGS OF RANDOM TRIGONOMETRIC POLYNOMIALS

Assume that x_t, $t=1,2,...n$ are n consecutive observations generated by a stationary time series X_t with 0 mean and finite variance. The periodogram of the observations defined by

$$I_{n,X}(\omega) \;=\; \frac{2}{n}\left|\sum_{t=1}^{n} x_t\, e^{i\omega t}\right|^2$$

plays an important role in the inference for spectral distribution function. In particular, crucial tests of hypotheses regarding jumps in the spectral distribution function depend on the statistics

$$M_{n,I} \;=\; \max_{\omega\in[0,\pi]} I_{n,X}(\omega) \ ,$$

and

$$M_{n,K} \;=\; \max_{\omega\in[0,\pi]} K_{n,X}(\omega) \ ,$$

where,

$$K_{n,X}(\omega) \;=\; \frac{I_{n,X}}{2\pi\,\hat{f}(\omega)} \ ,$$

and $\hat{f}(\omega)$ is a suitable estimator of the spectral density function. Hence the asymptotic distribution of $M_{n,I}$ and $M_{n,K}$ have considerable interest. Since,

$$I_{n,X} \;=\; X_n^2(\omega) + Y_n^2(\omega) \ ,$$

where

$$X_n(\omega) \;=\; \sqrt{\frac{n}{2}}\,\sum_{t=1}^{n} x_t\,\cos\omega t \ ,$$

and

$$Y_n(\omega) \;=\; \sqrt{\frac{n}{2}}\,\sum_{t=1}^{n} x_t\,\sin\omega t \ ,$$

it is clear that the study of the asymptotic distribution of the maximum periodogram ordinate in $\omega\in[0,\pi]$ can be done by studying similar asymptotic results for $X_n(\omega)$ and $Y_n(\omega)$. Both $X_n(\omega)$ and $Y_n(\omega)$ are trigonometric polynomials with random coefficients. Periodogram is also a trigonometric polynomial since it can be written in the form

$$I_{n,X}(\omega) \;=\; 2\sum_{k=-n}^{n} c_k\, e^{ik\omega} \ ,$$

where $c_k = \frac{1}{n}\sum_{t=1}^{n-|k|} x_t\, x_{t+|k|}$.

The asymptotic distributions for $M_{n,X} = \max_{\omega\in[0,\pi]} X_n(\omega)$ as well as of $M_{n,Y} = \max_{\omega\in[0,\pi]} Y_n(\omega)$ and $M_{n,I}$ are given in Turkman and Walker(1984), under the assumption that x_t are iid, normal random variables. These results are then extended to $M_{n,K}$, when X_t is a stationary time series.

Note that if X_t is a Gaussian time series, then both $X_n(\omega)$ and $Y_n(\omega)$ are continuous parameter Gaussian processes defined over the fixed interval $\omega \in [0,\pi]$. As such, it may be tempting to obtain all desired results on u-upcrossings based on the well known theory for Gaussian processes. However, the second spectral moments of the processes $X_n(\omega)$ and $Y_n(\omega)$ are given by

$$r''(0) \;=\; -\frac{n^2}{3}\big(1+O(1/n)\big)\;,$$

and for finite n, both processes are differentiable having finite number of upcrossings in $\omega \in [0,\pi]$. However, as the sample size n increases, these processes have sample paths that oscillate wildly, having infinitely many upcrossing of any finite level u in any finite subset of $\omega \in [0,\pi]$ with probability one. This is the fundamental reason why periodogram appears as an inconsistent estimator of the spectral density function. Hence, known results on u-upcrossings for Gaussian processes cannot be used in a straightforward fashion. In order to get meaningful asymptotic results for the level crossings of the polynomials $X_n(\omega)$ and $Y_n(\omega)$ as $n \to \infty$, one has to study the u-upcrossings for levels u which increase to infinity in a controlled fashion as $n \to \infty$. We refer the reader to Turkman and Walker(1984, 1991) for details in obtaining the first two moments of the u-crossings by such processes for appropriately chosen level u and the consequent asymptotic distribution of the maxima of these polynomials in the interval $[0,\pi]$. Here, we will derive the asymptotic Poisson character of the u-upcrossings of these trigonometric polynomials, for suitably increasing levels $u = u(n)$, as $n \to \infty$.

2.1. Poisson character of u-upcrossings

Let

$$X_n(\omega) \;=\; \sqrt{\frac{n}{2}}\,\sum_{t=1}^{n} x_t \cos\omega t\;,$$

and

$$Y_n(\omega) \;=\; \sqrt{\frac{n}{2}}\,\sum_{t=1}^{n} x_t \sin\omega t\;,$$

be trigonometric polynomials, where x_t is a realization of iid standard Gaussian random variables. Let $N_X(I) = N_{u_n,X}(I)$ and $N_Y(I) = N_{u_n,Y}(I)$ be respectively the number of upcrossings of a suitable chosen level u_n by the processes $X_n(\omega)$ and $Y_n(\omega)$ in the interval $I \subset [0,\pi]$. In this section we prove the following theorem:

Theorem 2.1. *Let*

$$u_n = \frac{x}{\sqrt{2\log n}} + \sqrt{2\log n} - \frac{\log 12}{2\sqrt{2\log n}} \ ,$$

and let

$$\tau = \tau(x) = e^{-x} \ .$$

Then

$$\lim_{n\to\infty} P\big(N_X[0,\pi]=s\big) = \frac{e^{-\tau}\tau^s}{s!} \ , \tag{2.1}$$

$$\lim_{n\to\infty} P\big(N_Y[0,\pi]=s\big) = \frac{e^{-\tau}\tau^s}{s!} \ . \tag{2.2}$$

We will give the proof only for (2.1). The proof for (2.2) is similar. For ease in notation, we write

$$N(I) = N_{u_n,X}(I) \ .$$

The proof of Theorem (2.1) is quite long and we give an outline of the proof.

Let $k>0$ be a fixed but arbitrarily large integer and divide the interval $[0,\pi]$ into subintervals I_j, $j=1,2,...,k$ such that

$$I_j = \left[\frac{\pi(j-1)}{k}, \frac{\pi j}{k}\right), \qquad j=1,2,..,k-1 \ ,$$

and

$$I_k = \left[\frac{\pi(k-1)}{k}, \pi\right] .$$

For any $\beta\in(0,1/2)$ arbitrarily small, for every $j=2,...,k$ divide every I_j further into two disjoint subintervals

$$I_{j,1} = \left[\frac{\pi(j-1)}{k}, \frac{\pi j}{k}\right) \ ,$$

$$I_{j,2} = \left[\frac{\pi(j-\beta)}{k}, \frac{\pi j}{k}\right), \qquad 2\le j\le k \ ,$$

$$I_{k,2} = \left[\frac{\pi(k-\beta)}{k}, \pi\right], \qquad 2\le j\le k \ .$$

Special attention is paid to the interval I_1 and we divide I_1 as

$$I_{1,0} = \left[0, \frac{\pi\beta}{k}\right), \qquad I_{1,1} = \left[\frac{\pi\beta}{k}, \frac{\pi(1-\beta)}{k}\right), \qquad I_{1,2} = \left[\frac{\pi(1-\beta)}{k}, \frac{\pi}{k}\right).$$

The proof is based on first showing that number of upcrossings over the intervals I_{j_1} are asymptotically independent and that number of upcrossings over the intervals $I_{j,2}$ are asymptotically negligible. Thus the outline of the proof is as follows:

1. For any s, approximate $P\big(N[0,\pi] \geq s\big)$ by $P\big(N\big(\bigcup_j I_{j,1}\big) \geq s\big)$.

2. Approximate $P\big(N\big(\bigcup_j I_{j,1}\big) \geq s\big)$ by $P(A_{n,s})$, where $A_{n,s}$ is the event that $N(I_{j,1}) \geq 1$ for at least s values of $j = 1, ..., k$.

3. Approximate $P(A_{n,s})$ by $P(D_{n,s})$ where $D_{n,s}$ is the event that in exactly s of the intervals $I_{j,1}$ we have $X_n(\omega) \geq u_n$ for some $\omega \in I_{j,1}$, so that $P\big(N[0,\pi] = s\big)$ is approximated by $P(D_{n,s})$.

4. Let

$$p \;=\; p_{k,\beta,\tau} = \lim_{n\to\infty} P\big(M_n(I_{j,1}) \leq u_n\big) \;,$$

show that

$$p \;=\; \exp\!\left(-\frac{(1-\beta)\,\tau}{k}\right) \,,$$

and then approximate $P(D_{n,s})$ by the binomial probability

$$\binom{k}{s}\,(1-p)^s\,p^{k-s} \;.$$

5. Let $\beta \to 0$, then $k \to \infty$ and use the Poisson approximation to the binomial probability to obtain the desired result.

We now give proofs for these assertions in terms of series of Lemmas.

Lemma 2.1. *For any* $s = 0, 1, 2, ...,$ *as* $n \to \infty$,

$$0 \;\leq\; P\Big(N[0,\pi] \geq s\Big) - P\left(N\left(\bigcup_{j=1}^{k} I_{j,1}\right) \geq s\right) \;\leq\; \beta\tau + o_n(1) \;. \tag{2.3}$$

Proof: The event $\big\{N[0,\pi] \geq s\big\}$ contains the event $\big\{N\big(\bigcup_{j=1}^k I_{j,1}\big) \geq s\big\}$ and the difference is the event

$$\left\{\bigcup_{j=1}^{k} \big(N(I_{j,2} \geq 1)\big) \,\cup\, \big(N(I_{1,0} \geq 1)\big)\right\} .$$

Hence

$$
\begin{aligned}
0 \;&\le\; P\Big(N[0,\pi] \ge s\Big) - P\bigg(N\Big(\bigcup_{j=1}^{k} I_{j,1}\Big) \ge s\bigg) \\
&=\; P\bigg[\bigcup_{j=1}^{k} \big(N(I_{j,2} \ge 1)\big) \,\cup\, \big(N(I_{1,0} \ge 1)\big)\bigg] \\
&\le\; \sum_{j=1}^{k} P\big(N(I_{j,2} \ge 1)\big) + P\big(N(I_{1,0} \ge 1)\big) \\
&\le\; \sum_{j=1}^{k-1} E\big(N(I_{j,2})\big) + E\big(N(I_{1,0})\big) + E\big(N(I_{k,2})\big)\;.
\end{aligned} \tag{2.4}
$$

It can be shown that (see Turkman and Walker, 1984) as $n \to \infty$, for every $j = 1, ..., k-1$

$$E\big(N(I_{j,2})\big) = \frac{\tau\beta}{k}\;.$$

However, $E(N(I_{1,0}))$ and $E(N(I_{k,2}))$ need special attention in calculations. The reason for this extra complication is that the expected number of upcrossings are calculated as an integral with respect to the joint density of the vector $\big(X_n(\omega), X'(\omega)\big)$ and this vector has a normal density with mean 0 and covariance function given by

$$
\begin{pmatrix} 1 + r_n(2\omega) & r'_n(2\omega) \\ r'_n(2\omega) & \frac{n^2}{3} + r''_n(2\omega) \end{pmatrix}, \tag{2.5}
$$

where $r_n(\omega) = \frac{1}{n}\sum_{j=1}^{n} \cos j\omega$ and $r'_n(\omega), r''_n(\omega)$ are respectively first and second order derivatives of $r_n(\omega)$ respectively. This covariance matrix tends to be singular as ω gets arbitrarily close to 0 or π. Hence $E(N(I_{j,0}))$ needs to be calculated separately over regions

$$
\begin{aligned}
R_{n,1} &= \Big\{\omega \in I_{1,0}\colon\; \omega \ge \frac{\log n}{n}\Big\}\,, \\
R_{n,2} &= \Big\{\omega \in I_{1,0}\colon\; \frac{1}{nk} \le \omega \le \frac{\log n}{n}\Big\}\,, \\
R_{n,3} &= \Big\{\omega \in I_{1,0}\colon\; 0 \le \omega \le \frac{1}{nk}\Big\}\,.
\end{aligned}
$$

It is shown in Turkman and Walker (1984) that

$$
\lim_{n\to\infty} E\big(N(I_{j,0})\big) = \begin{cases} \dfrac{\tau\beta}{k}\,, & \omega \in R_{n,1}\,, \\[2mm] 0\,, & \omega \in R_{n,2} \cup R_{n,3}\,. \end{cases} \tag{2.6}
$$

Similar expression can be found for $E(N(I_{k,2}))$ and hence from (2.4) for arbitrarily large k and arbitrarily small β,

$$\sum_{j=1}^{k} E\big(N(I_{j,2})\big) + E\big(N(I_{j,0})\big) \;=\; \tau\beta\Big(1+\frac{1}{k}\Big) .$$

This proves the Lemma. □

Define $A_{n,s}$ to be the event that $\{N(I_{j,1}) \geq 1\}$ for at least s values of $j = 1, .., k$. Then

Lemma 2.2. *As* $n \to \infty$,

$$0 \;\leq\; P\big(N[0,\pi] \geq s\big) - P\big(A_{n,s}\big) \;\leq\; \beta\tau + \sum_{j=1}^{k} P\big(N(I_{j,1}) \geq 2\big) + o_n(1) \ , \tag{2.7}$$

and

$$\limsup_{n\to\infty} \sum_{j=1}^{k} P\big(N(I_{j,1}) \geq 2\big) \;\leq\; (1-\beta)\tau - k\Big(1 - \exp\Big(-(1-\beta)\frac{\tau}{k}\Big)\Big) . \tag{2.8}$$

Proof:

$$\begin{aligned} 0 \;&\leq\; P\Big(N\Big(\bigcup_{j=1}^{k} I_{j,1}\Big) \geq s\Big) - P(A_{n,s}) \\ &\leq\; P\Big(\bigcup_{j=1}^{k} \big(N(I_{j,1}) \geq 2\big)\Big) \\ &\leq\; \sum_{j=1}^{k} P\big(N(I_{j,1}) \geq 2\big) \ . \end{aligned} \tag{2.9}$$

Now combining this inequality with the inequality (2.4), we get (2.7). To prove (2.8), we proceed as follows: First note that for any $j = 1, ..., k$

$$P\big(N(I_{j,1}) \geq 2\big) \;\leq\; E\big(N(I_{j,1})\big) - P\big(N(I_{j,1}) \geq 1\big) \ . \tag{2.10}$$

also

$$0 \;\leq\; P\big(M_n(I_{j,1}) > u_n\big) - P\big(N(I_{j,1}) \geq 1\big) \;\leq\; P\Big(X_n\Big(\frac{\pi(j-1)}{k}\Big) \geq u_n\Big) ,$$

which implies that

$$P\big(M_n(I_{j,1}) > u_n\big) - P\Big(X_n\Big(\frac{\pi(j-1)}{k}\Big) \geq u_n\Big) \;\leq\; P\big(N(I_{j,1}) \geq 1\big) \ ,$$

so that from (2.10),

$$P\big(N(I_{j,2}) \geq 2\big) \;\leq\; (1-\beta)\frac{\tau}{k} - P\big(M_n(I_{j,1} > u_n)\big) + P\Big(X_n\Big(\frac{\pi(j-1)}{k}\Big) \geq u_n\Big). \tag{2.11}$$

Now, for any $\omega \in I_{j,1}$, $X_n(\omega) \sim N(0, 1+r_n(2\omega))$, thus as $n \to \infty$,

$$P\left(X_n\left(\frac{\pi(j-1)}{k}\right) \geq u_n\right) \;=\; o_n(1) \;,$$

since, from Turkman and Walker(1984) we have

$$\lim_{n\to\infty} P\big(M_n(I_{j,1}) > u_n\big) \;=\; 1 - \exp\left(-\frac{(1-\beta)\tau}{k}\right) .$$

Now the proof is complete by combining (2.10) with (2.11). □

Denote by $C_{n,s}$, the event that $X_n(\omega) > u_n$ in at least $s = 1, ..., k$ of the intervals $I_{j,1}$ for some $\omega \in I_{j,1}$.

Then

Lemma 2.3. *As $n \to \infty$,*

$$0 \;\leq\; P(C_{n,s}) - P(A_{n,s}) \;=\; o_n(1) \;. \tag{2.12}$$

Proof: The event

$$A = \Big\{N(I_{j,1}) \geq 1\Big\}$$

is contained in the event

$$B = \Big\{X_n(\omega) \geq u_n, \text{ for some } \omega \in I_{j,1}\Big\}$$

and the difference of these events are given by

$$B - A \;=\; A^c \cap B \;=\; \left\{X_n\left(\frac{\pi(j-1)}{k}\right) > u_n\right\} .$$

Hence it follows from the definitions of the events $C_{n,s}$ and $A_{n,s}$ that as $n \to \infty$,

$$\begin{aligned} 0 \;&\leq\; P(C_{n,s}) - P(A_{n,s}) \\ &\leq\; P\left(\bigcup_{k=1}^{k}\left(X_n\left(\frac{\pi(j-1)}{k}\right) > u_n\right)\right) \\ &\leq\; \sum_{k=1}^{k} P\left(X_n\left(\frac{\pi(j-1)}{k}\right) > u_n\right) \;=\; o_n(1) \;. \end{aligned} \tag{2.13}$$

Clearly for any $s < k$, $C_{n,s+1} \subset C_{n,s}$. Let $D_{n,s} = C_{n,s} - C_{n,s+1} = C^c_{n,s+1} \cap C_{n,s}$. $D_{n,s}$ is the event that $X_n(\omega) > u_n$ in exactly s of the k intervals and

$$P(D_{n,s}) \;=\; P(C_{n,s}) - P(C_{n,s+1}) \;.$$

□

Lemma 2.4.

$$\limsup_{n\to\infty} \Big|P\big(N[0,\pi]=s\big)-P\big(D_{n,s}\big)\Big| \;\le\; \tau-k\bigg(1-\exp\Big(-\frac{(1-\beta)\,\tau}{k}\Big)\bigg)\,. \tag{2.14}$$

Proof: From (2.12), for any $s<k$, $0\le P(C_{n,s})-P(A_{n,s})=o_n(1)$, therefore, as $n\to\infty$,

$$\Big|P(A_{n,s})-P(A_{n,s+1})-P(D_{n,s})\Big| \;\le\; o_n(1)\ .$$

Hence from (2.7), for any $s<k$,

$$0\;\le\; P\big(N[0,\pi]\ge s\big)-P\big(D_{n,s}\big) \;\le\; \beta\tau+\sum_{j=1}^{k} P\big(N(I_{j,1})\ge 2\big)+o_n(1)\ , \tag{2.15}$$

and the lemma follows from (2.8).

Let $M_j=\{M_n(I_{j,1})\le u_n\}$ and M_j^c be the compliment of M_j. Let

$$P_{n,j} \;=\; P\big(M_n(I_{j,1})\le u_n\big)\ .$$

We know from Turkman and Walker(1984) that

$$\lim_{n\to\infty} P_{n,j} \;=\; \exp\bigg(-\frac{(1-\beta)\,\tau}{k}\bigg) \;=\; p\,, \qquad \text{say}\,. \qquad \square$$

Lemma 2.5.

$$\limsup_{n\to\infty} \bigg|P(D_{n,s})-\binom{k}{s}\,(1-p)^s\,p^{k-s}\bigg| \;=\; 0\ . \tag{2.16}$$

Proof:

$$D_{n,s} \;=\; \bigcup\big(M_{i_1}^c\,.\,M_{i_2}^c\ldots M_{i_s}^c\,.\,M_{i_s+1}\ldots M_{i_k}\big)\ ,$$

where the union is taken over all combinations of distinct integers with $i_1<i_2<\ldots<i_k$. Here, we omit the intersection signs, replacing them with ".". We first start by looking at the probability

$$P\big(M_{i_1}^c\,.\,M_{i_2}^c\ldots M_{i_m}^c\,.\,M_{i_1}\ldots M_{i_t}\big)\ , \tag{2.17}$$

where m and t are integers such that $m+t\le k$. When $m=0$, (2.17) is equal to

$$P\big(M_{i_1}\ldots M_{i_t}\big) \;=\; P\bigg(\bigcap_{k=1}^{t}\big(M_n(I_{j,1})\le u_n\big)\bigg).$$

It follows from Lemma 2.6 of Turkman and Walker (1984) that for any $t\le k$

$$\limsup_{n\to\infty} \Big|P\big(M_{i_1}\ldots M_{i_t}\big)-p^t\Big| \;=\; 0\ .$$

Now assume that for an $m \geq 1$ and for all $t \leq k-(m-1)$

$$\limsup_{n\to\infty} \Big| P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_1} \dots M_{i_t}\big) - (1-p)^{m-1} p^t \Big| = 0 \;. \tag{2.18}$$

We now show that

$$\limsup_{n\to\infty} \Big| P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_m}.\, M_{i_1} \dots M_{i_t}\big) - (1-p)^{m} p^t \Big| = 0 \;, \tag{2.19}$$

and the proof will be complete by induction:

$$\begin{aligned}
\Big\{M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_1} \dots M_{i_t}\Big\} &- \Big\{M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_m}.\, M_{i_1} \dots M_{i_t}\Big\} = \\
&= \Big\{M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_1} \dots M_{i_t}\Big\} \cap \Big\{M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_m}.\, M_{i_1} \dots M_{i_t}\Big\} \\
&= \Big\{M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_m}.\, M_{i_1} \dots M_{i_t}\Big\} \;.
\end{aligned}$$

Hence,

$$\begin{gathered}
P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_m}.\, M_{i_1} \dots M_{i_t}\big) = \\
= P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_m}.\, M_{i_1} \dots M_{i_t}\big) - P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_m}.\, M_{i_1} \dots M_{i_t}\big)
\end{gathered}$$

From the assumption (2.18) we have

$$\limsup_{n\to\infty} \Big| P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_1} \dots M_{i_t}\big) - (1-p)^{m-1} p^t \Big| = 0 \;,$$

and

$$\limsup_{n\to\infty} \Big| P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_{m-1}}.\, M_{i_m}.\, M_{i_1} \dots M_{i_t}\big) - (1-p)^{m-1} p^{t+1} \Big| = 0 \;,$$

so that (2.19) follows immediately. Choosing $t = m-k$, we get

$$\limsup_{n\to\infty} \Big| P\big(M^c_{i_1}.\, M^c_{i_2} \dots M^c_{i_m}.\, M_{i_{m+1}}.\, M_{i_1} \dots M_{i_k}\big) - (1-p)^{m} p^{k-m} \Big| = 0$$

and the lemma follows immediately from induction. □

The proof of the theorem 2.1 now follows from lemmas 1–5 by first letting $\beta \to 0$, then $k \to \infty$. First note that

$$\limsup_{n\to\infty} \bigg| P\big(N[0,\pi]=s\big) - \binom{k}{s} (1-p)^s p^{k-s} \bigg| \leq \tau - k\bigg(1 - \exp\Big(-\frac{(1-\beta)\,\tau}{k}\Big)\bigg) \;, \tag{2.20}$$

where,

$$p = \exp\bigg(-\frac{(1-\beta)\,\tau}{k}\bigg) \;.$$

Thus,

$$\limsup_{n\to\infty} P\big(N[0,\pi]=s\big) \leq \binom{k}{s} (1-p)^s p^{k-s} + \tau - k\bigg(1 - \exp\Big(-\frac{(1-\beta)\tau}{k}\Big)\bigg) \;, \tag{2.21}$$

so that letting $\beta \to 0$,

$$\limsup_{n\to\infty} P\big(N[0,\pi]=s\big) \;\le \tag{2.22}$$
$$\le \binom{k}{s}\big(1-\exp(-\tau/k)\big)^s\big(\exp(-\tau/k)\big)^{k-s}+\tau-k\Big(1-\exp\Big(-\frac{(1-\beta)\tau}{k}\Big)\Big)\;.$$

Now let $k\to\infty$. Then $k(1-\exp(-\tau/k))\to\tau$, and by Poisson approximation to binomial we get

$$\limsup_{n\to\infty} P\big(N[0,\pi]=s\big) \;\le\; \frac{e^{-\tau}\tau^s}{s!}\;.$$

Similarly we can show that

$$\liminf_{n\to\infty} P\big(N[0,\pi]=s\big) \;\ge\; \frac{e^{-\tau}\tau^s}{s!}\;,$$

and this completes the proof.

It is possible to obtain the following similar asymptotic result for the periodogram, although proofs are slightly more tedious and we omit the proof.

Theorem 2.2. *Let $u_n = 2(x+\log n+\frac{1}{2}\log\log n-\frac{1}{2}\log\frac{3}{\pi})$. Then the number of u_n-upcrossings $N_{u_n,I}[0,\pi]$ of the periodogram in the interval $[0,\pi]$ is asymptotically Poisson, in the sense that*

$$\lim_{n\to\infty} P\big(N_{u_n,I}[0,\pi]=s\big) \;=\; \frac{e^{-\tau}\tau^s}{s!}\;, \qquad s=0,1,\dots\;,$$

where $\tau=\tau(x)=e^{-x}$.

Asymptotic results given in Theorems 2.1 and 2.2 are very useful and many convergence results for upper order statistics can be recovered from these basic results. For example, if

$$M_n[0,\pi] = \max_{\omega\in[0,\pi]} X_n(\omega)\;,$$

then

$$\Big\{M_n(0,\pi]\le u_n\Big\} \;=\; \Big\{N_X[0,\pi]=0\Big\}\;,$$

and consequently,

$$\lim_{n\to\infty} P\bigg(M_n\le\frac{x}{\sqrt{2\log n}}+\sqrt{2\log n}-\frac{\log 12}{2\sqrt{2\log n}}\bigg) \;=\; e^{-e^{-x}}\;,$$

which was proved in Turkman and Walker(1984) based on calculating the first two moments of the u-upcrossings.

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